Wold decomposition for doubly commuting isometric covariant representations

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Let $V$ be an **isometry** on a Hilbert space $\mathcal{H}$, that is, $V^*V = I_{\mathcal{H}}$.

A closed subspace $\mathcal{W} \subseteq \mathcal{H}$ is said to be a **wandering subspace** for $V$ if $V^k\mathcal{W} \perp V^l\mathcal{W}$ for all $k, l \in \mathbb{N}$ with $k \neq l$, or equivalently, if $V^m\mathcal{W} \perp \mathcal{W}$ for all $m \geq 1$.

An isometry $V$ on $\mathcal{H}$ is said to be a **unilateral shift** or **shift** if

$$\mathcal{H} = \bigoplus_{m \geq 0} V^m\mathcal{W},$$

for some wandering subspace $\mathcal{W}$ for $V$. 
For a shift $V$ on $\mathcal{H}$ with a wandering subspace $\mathcal{W}$ we have

$$\mathcal{H} \ominus V\mathcal{H} = \bigoplus_{m \geq 0} V^m \mathcal{W} \ominus V(\bigoplus_{m \geq 0} V^m \mathcal{W})$$

$$= \bigoplus_{m \geq 0} V^m \mathcal{W} \ominus \bigoplus_{m \geq 1} V^m \mathcal{W} = \mathcal{W}.$$ 

The wandering subspace of a shift is unique and is given by

$$\mathcal{W} = \mathcal{H} \ominus V\mathcal{H}.$$
Theorem (Wold, 1938)

Let $V$ be an isometry on $\mathcal{H}$. Then $\mathcal{H}$ admits a unique decomposition $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_u$, where $\mathcal{H}_s$ and $\mathcal{H}_u$ are $V$-reducing subspaces of $\mathcal{H}$ and $V|_{\mathcal{H}_s}$ is a shift and $V|_{\mathcal{H}_u}$ is unitary. Moreover,

$$
\mathcal{H}_s = \bigoplus_{m=0}^{\infty} V^m \mathcal{W} \quad \text{and} \quad \mathcal{H}_u = \bigcap_{m=0}^{\infty} V^m \mathcal{H},
$$

where $\mathcal{W} = \text{ran}(I - VV^*) = \ker V^*$ is the wandering subspace for $V$. 
Let $V = (V_1, \ldots, V_n)$ be an $n$-tuple ($n \geq 2$) of commuting isometries on $\mathcal{H}$. Then $V$ is said to doubly commute if

$$ V_i V_j^* = V_j^* V_i, $$

for all $1 \leq i < j \leq n$.

**Theorem (M. Slocinski, 1980)**

Let $V = (V_1, V_2)$ be a pair of doubly commuting isometries on a Hilbert space $\mathcal{H}$. Then there exists a unique decomposition

$$ \mathcal{H} = \mathcal{H}_{ss} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{uu}, $$

where $\mathcal{H}_{ij}$ are joint $V$-reducing subspace of $\mathcal{H}$ for all $i, j = s, u$. Moreover, $V_1$ on $\mathcal{H}_{i,j}$ is a shift if $i = s$ and unitary if $i = u$ and that $V_2$ is a shift if $j = s$ and unitary if $j = u$. 
Let \((T_1, \ldots, T_n)\) be an \(n\)-tuple of commuting operators on a Hilbert space \(\mathcal{H}\) and \(1 \leq m \leq n\). Let \(A = \{i_1, \ldots, i_l\} \subseteq I_m\) and \(1 \leq l \leq m\). We denote by \(T_A\) the \(|A|\)-tuple of commuting operators \((T_{i_1}, \ldots, T_{i_l})\) and \(\mathbb{N}^A := \{k = (k_{i_1}, \ldots, k_{i_l}) : k_{i_j} \in \mathbb{N}, 1 \leq j \leq l\}\). We also denote \(T_{i_1}^{k_{i_1}} \cdots T_{i_l}^{k_{i_l}}\) by \(T_A^k\) for all \(k \in \mathbb{N}^A\).
Theorem (J. Sarkar, 2014)

Let \( V = (V_1, \ldots, V_n) \) be an \( n \)-tuple \((n \geq 2)\) of doubly commuting isometries on \( \mathcal{H} \) and \( m \in \{2, \ldots, n\} \). Let \( I_m = \{1, 2, \ldots, m\} \). Then there exists \( 2^m \) joint \((V_1, \ldots, V_m)\)-reducing subspaces \( \{\mathcal{H}_A : A \subseteq I_m\} \) (counting the trivial subspace \( \{0\} \)) such that

\[
\mathcal{H} = \bigoplus_{A \subseteq I_m} \mathcal{H}_A, 
\]

(1)

where for each \( A \subseteq I_m \),

\[
\mathcal{H}_A = \bigoplus_{k \in \mathbb{N}^A} V_A^k \left( \bigcap_{j \in \mathbb{N}^{I_m \setminus A}} V_{I_m \setminus A}^j \right). 
\]

(2)
In particular, there exists $2^n$ orthogonal joint $V$-reducing subspaces $\{\mathcal{H}_A : A \subseteq I_n\}$ such that

$$\mathcal{H} = \sum_{A \subseteq I_n} \bigoplus \mathcal{H}_A,$$

and for each $A \subseteq I_n$ and $\mathcal{H}_A \neq \{0\}$, $V_i|\mathcal{H}_A$ is a shift if $i \in A$ and unitary if $i \in I_n \setminus A$ for all $i = 1, \ldots, n$. Moreover, the above decomposition is unique, in the sense that

$$\mathcal{H}_A = \bigoplus_{k \in \mathbb{N}^A} V^k_A \left( \bigcap_{j \in \mathbb{N}^{I_n \setminus A}} V^j_{I_n \setminus A} \mathcal{W}_A \right),$$

for all $A \subseteq I_n$.

- This decomposition is stronger in the sense that the orthogonal decomposition works for any $m \in \{2, \ldots, n\}$ with $(2 < n)$. 
Definition
Let $\mathcal{M}$ be a $C^*$-algebra and $E$ be a vector space which is a right $\mathcal{M}$-module satisfying $\alpha(xm) = (\alpha x)m = x(\alpha m)$ for $x \in E, m \in \mathcal{M}, \alpha \in \mathbb{C}$. The module $E$ is called an (right) inner-product $\mathcal{M}$-module if there exists a map $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{M}$ satisfying:

(i) $\langle x, x \rangle \geq 0$ for $x \in E$ and $\langle x, x \rangle = 0$ only if $x = 0$,
(ii) $\langle x, ym \rangle = \langle x, y \rangle m$ for $x, y \in E$ and for $m \in \mathcal{M}$,
(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for $x, y \in E$,
(iv) $\langle x, \mu y + \nu z \rangle = \mu \langle x, y \rangle + \nu \langle x, z \rangle$ for $x, y, z \in E$ and for $\mu, \nu \in \mathbb{C}$.

Definition
A (right) Hilbert $\mathcal{M}$-module is an inner-product $\mathcal{M}$-module $E$ which is complete w.r.t. $\|x\| := \|\langle x, x \rangle\|^{1/2}$ for $x \in E$. 
Let \( E \) be a Hilbert \( \mathcal{M} \)-modules. A map \( T : E \rightarrow E \) is called \textbf{adjointable} if there exists a map \( S : E \rightarrow E \) such that
\[
\langle T(x), y \rangle = \langle x, S(y) \rangle \text{ for all } x, y \in E.
\]

Notation: \( \mathcal{L}(E) \).

The module \( E \) is said to be a \textbf{\( C^* \)-correspondence over} \( \mathcal{M} \) if it has a left \( \mathcal{M} \)-module structure induced by a non-zero \( * \)-homomorphism \( \phi : \mathcal{M} \rightarrow \mathcal{L}(E) \) in the sense
\[
a\xi := \phi(a)\xi \quad (a \in \mathcal{M}, \xi \in E).
\]

If \( F \) is another \( C^* \)-correspondence over \( \mathcal{M} \), then \textbf{tensor product} \( F \otimes \phi E \) satisfy the following properties: for all \( \zeta_1, \zeta_2 \in F, \xi_1, \xi_2 \in E \) and \( a \in \mathcal{M} \)
\[
(\zeta_1 a) \otimes \xi_1 = \zeta_1 \otimes \phi(a)\xi_1,
\]
\[
\langle \zeta_1 \otimes \xi_1, \zeta_2 \otimes \xi_2 \rangle = \langle \xi_1, \phi(\langle \zeta_1, \zeta_2 \rangle)\xi_2 \rangle.
\]
Definition
Let $E$ be a $C^*$-correspondence over $\mathcal{M}$ and $\mathcal{H}$ be a Hilbert space. Assume $\sigma : \mathcal{M} \rightarrow B(\mathcal{H})$ to be a representation and $T : E \rightarrow B(\mathcal{H})$ to be a linear map. The tuple $(T, \sigma)$ is called a **covariant representation** of $E$ on $\mathcal{H}$ if

$$T(m\xi m') = \sigma(m) T(\xi) \sigma(m') \quad (\xi \in E, m, m' \in \mathcal{M}).$$

(3)

The covariant representation is called **completely contractive** if $T$ is completely contractive. The covariant representation $(T, \sigma)$ is called **isometric** if

$$T(\xi_1)^* T(\xi_2) = \sigma(\langle \xi_1, \xi_2 \rangle) \quad (\xi_1, \xi_2 \in E).$$
Lemma (Muhly and Solel, 1998)

The map \((T, \sigma) \mapsto \tilde{T}\) provides a bijection between the collection of all completely contractive, covariant representations \((T, \sigma)\) of \(E\) on \(\mathcal{H}\) and the collection of all contractive linear maps \(\tilde{T} : E \otimes \sigma \mathcal{H} \to \mathcal{H}\) defined by

\[
\tilde{T}(\xi \otimes h) := T(\xi)h \quad (\xi \in E, h \in \mathcal{H}),
\]

and such that \(\tilde{T}(\phi(a) \otimes I_{\mathcal{H}}) = \sigma(a)\tilde{T}\), \(a \in \mathcal{M}\). Moreover, \(\tilde{T}\) is isometry if and only if \((T, \sigma)\) is isometric.
Let $E$ be a $C^*$-correspondence over a $C^*$-algebra $\mathcal{M}$. Then for each $n \in \mathbb{N}$, $E^\otimes n := E \otimes \phi \cdots \otimes \phi E$ ($n$ times) is the $C^*$-correspondence over the $C^*$-algebra $\mathcal{M}$, where the left action of $\mathcal{M}$ on $E^\otimes n$ is given by

$$\phi^n(a)(\xi_1 \otimes \cdots \otimes \xi_n) := \phi(a)\xi_1 \otimes \cdots \otimes \xi_n.$$ 

Denote $E^\otimes 0 := \mathcal{M}$. The Fock space $\mathcal{F}(E) := \bigoplus_{n \geq 0} E^\otimes n$ is the $C^*$-correspondence over a $C^*$-algebra $\mathcal{M}$, with left action of $\mathcal{M}$ on $\mathcal{F}(E)$ is given by $\phi_\infty : \mathcal{M} \longrightarrow L(\mathcal{F}(E))$ where

$$\phi_\infty(a)(\bigoplus_{n \geq 0} \xi_n) := \bigoplus_{n \geq 0} a \cdot \xi_n, \; \xi_n \in E^\otimes n.$$ 

Let $\xi \in E$, we define the creation operator $T_\xi$ on $\mathcal{F}(E)$ by

$$T_\xi(\eta) := \xi \otimes \eta, \; \eta \in E^\otimes n.$$
Let $\pi$ be a representation of $\mathcal{M}$ on the Hilbert space $\mathcal{H}$. The isometric covariant representation $(\rho, S)$ of $E$ on the Hilbert space $\mathcal{F}(E) \otimes_{\pi} \mathcal{H}$ defined by

$$\rho(a) : = \phi_{\infty}(a) \otimes I_{\mathcal{H}}, \ a \in \mathcal{M}$$

$$S(\xi) : = T_{\xi} \otimes I_{\mathcal{H}}, \ \xi \in E.$$ 

is called an **induced representation** (induced by $\pi$).
Definition (L. Helmer, 2016)

Let $E$ be a $C^*$-correspondence over a $C^*$-algebra $\mathcal{M}$. Let $(\sigma, V)$ be an isometric covariant representation of $E$ on a Hilbert space $\mathcal{H}$. For a closed $\sigma(\mathcal{M})$-invariant subspace $\mathcal{W}$, we define

$$\mathcal{L}^n(\mathcal{W}) := \bigvee \{ V(\xi_1)V(\xi_2)\ldots V(\xi_n)h : \xi_i \in E, h \in \mathcal{W} \},$$

for $n \in \mathbb{N}$ and $\mathcal{L}^0(\mathcal{W}) := \mathcal{W}$. Then $\mathcal{W}$ is called wandering for $(\sigma, V)$, if the subspaces $\mathcal{L}^n(\mathcal{W})$, $n \in \mathbb{N}_0$ are mutually orthogonal.
Theorem (Muhly-Solel, 1999)

Let \((\sigma, V)\) be an isometric covariant representation of \(E\) on a Hilbert space \(\mathcal{H}\). Then the representation \((\sigma, V)\) decomposes into a direct sum \((\sigma_1, V_1) \oplus (\sigma_2, V_2)\) on \(\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2\) where \((\sigma_1, V_1) = (\sigma, V)|_{\mathcal{H}_1}\) is an induced covariant representation and \((\sigma_2, V_2) = (\sigma, V)|_{\mathcal{H}_2}\) is fully coisometric. The above decomposition is unique in the sense that if \(\mathcal{K}\) reduces \((\sigma, V)\), and if the restriction \((\sigma, V)|_{\mathcal{K}}\) is induced (resp. fully coisometric), then \(\mathcal{K} \subset \mathcal{H}_1\) (resp. \(\mathcal{K} \subset \mathcal{H}_2\)). Moreover, \(\mathcal{H}_2 := \bigoplus_{k \geq 0} \mathcal{L}^k(\mathcal{W})\), and hence

\[
\mathcal{H}_1 := \left( \bigoplus_{k \geq 0} \mathcal{L}^k(\mathcal{W}) \right) ^\perp = \bigcap_{k=0}^{\infty} (\mathcal{L}^k(\mathcal{H})).
\]
Let $k \in \mathbb{N}$ and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We require product system $\mathbb{E}$ of $C^*$-correspondences over $\mathbb{N}_0^k$ [Fowler, 2002]: Consider $\mathbb{E}$ to be a family of $k$ $C^*$-correspondences $\{E_1, \ldots, E_k\}$ together with the unitary isomorphisms $t_{i,j} : E_i \otimes E_j \to E_j \otimes E_i$ ($i > j$). Thus we identify for all $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}_0^k$ the correspondence $\mathbb{E}(\mathbf{n})$ with $E_1^\otimes n_1 \otimes \cdots \otimes E_k^\otimes n_k$. Indeed, we use $t_{i,j} = \text{id}_{E_i \otimes E_j}$, $t_{i,j} = t_{j,i}^{-1}$ for $i < j$. 
Definition
Assume $E$ to be a product system over $\mathbb{N}_0^k$. By a **completely contractive covariant representation** of $E$ on a Hilbert space $\mathcal{H}$ we mean a tuple $(\sigma, T^{(1)}, \ldots, T^{(k)})$, where $(\sigma, \mathcal{H})$ is a representation of $\mathcal{M}$, and $T^{(i)} : E_i \to B(\mathcal{H})$ are linear completely contractive maps satisfying

$$T^{(i)}(a\xi_ib) = \sigma(a)T^{(i)}(\xi_i)\sigma(b), \quad a, b \in \mathcal{M}, \xi_i \in E_i,$$

as well as $\tilde{T}^{(i)}(I_{E_i} \otimes \tilde{T}^{(j)}) = \tilde{T}^{(j)}(I_{E_j} \otimes \tilde{T}^{(i)})(t_{i,j} \otimes I_{\mathcal{H}})$ with $i, j \in \{1, \ldots, k\}$. 
Moreover, the representation is called **isometric** if each \((\sigma, T^{(i)})\) is isometric as a representation of \(E_i\), and **fully coisometric** if each \((\sigma, T^{(i)})\) is fully coisometric.

**Definition**

A representation \((\sigma, T^{(1)}, \ldots, T^{(k)})\) of \(E\) on a Hilbert space \(\mathcal{H}\) is called **doubly commuting** if for each \(i, j \in \{1, \ldots, k\}\), \(i \neq j\) implies

\[
\tilde{T}(j)^* \tilde{T}(i) = (l_{E_j} \otimes \tilde{T}(i))(t_{i,j} \otimes l_{\mathcal{H}})(l_{E_i} \otimes \tilde{T}(j)^*).
\] (4)
Definition
Let \( \mathcal{K} \) be a closed subspace of a Hilbert space \( \mathcal{H} \). The subspace \( \mathcal{K} \) is called **reducing** for a doubly commuting representation \((\sigma, T^{(1)}, \ldots, T^{(k)})\) on \( \mathcal{H} \), if it reduces \( \sigma(\mathcal{M}) \) (this means that the projection onto \( \mathcal{K} \), will be denoted by \( P_{\mathcal{K}} \), lies in \( \sigma(\mathcal{M})' \)), and both \( \mathcal{K}, \mathcal{K}^\perp \) are left invariant by each operator \( T^{(i)}(\xi_i) \) for \( \xi_i \in E_i \), \( i \in \{1, \ldots, k\} \). Then it is evident that the ‘restriction’ of this representation provides a new representation of \( \mathcal{E} \) on \( \mathcal{K} \), which is called a **summand** of \((\sigma, T^{(1)}, \ldots, T^{(k)})\) and will be denoted by \((\sigma, T^{(1)}, \ldots, T^{(k)})|_{\mathcal{K}}\).

Remark
To check \( \mathcal{K} \) reduces \((\sigma, T^{(j)})\), it is enough to check \( \mathcal{K} \) reduces \( \sigma(\mathcal{M}) \), and \( P_{\mathcal{K}} \) commutes with \( \tilde{T}(j) \tilde{T}(j)^* \).
For a closed subspace $\mathcal{K}$, we use notation $\mathcal{L}^i_j(\mathcal{K})$ for the closed subspace generated by

$$\{ T^{(i)}(\xi_1) \cdots T^{(i)}(\xi_l) k : \xi_1, \ldots \xi_l \in E_i, k \in \mathcal{K} \}.$$ 

When $l = 1$, we denote it by $\mathcal{L}^i_j(\mathcal{K})$.

For $\mathbf{n} = (n_1, \cdots, n_k) \in \mathbb{N}^k$, we define $\tilde{T}_n : \mathbb{E}(\mathbf{n}) \otimes_\sigma \mathcal{H} \rightarrow \mathcal{H}$ by

$$\tilde{T}_n := \tilde{T}^{(1)}_{n_1} \left( I_{E_1^{\otimes n_1}} \otimes \tilde{T}^{(2)}_{n_2} \right) \cdots \left( I_{E_1^{\otimes n_1} \otimes \cdots \otimes E_{k-1}^{\otimes n_{k-1}}} \otimes \tilde{T}^{(k)}_{n_k} \right).$$

Let $A = \{ i_1, \cdots, i_p \} \subset \{ 1, 2, \cdots, k \}$, denote

$$\mathbb{N}^A_0 := \{ \mathbf{m} = (m_{i_1}, \cdots, m_{i_p}) : m_{i_j} \in \mathbb{N}_0, 1 \leq j \leq p \}. \text{ Let}$$

$$\mathbf{m} = (m_{i_1}, \cdots, m_{i_p}) \in \mathbb{N}^A_0, \text{ define } \tilde{T}_m^A : \mathbb{E}(\mathbf{m}) \otimes_\sigma \mathcal{H} \rightarrow \mathcal{H} \text{ by}$$

$$\tilde{T}_m^A = \tilde{T}^{(i_1)}_{m_{i_1}} \left( I_{E_{i_1}^{\otimes m_{i_1}}} \otimes \tilde{T}^{(i_2)}_{m_{i_2}} \right) \cdots \left( I_{E_{i_1}^{\otimes m_{i_1}} \otimes \cdots \otimes E_{i_{p-1}}^{\otimes m_{i_{p-1}}} \otimes \tilde{T}^{(i_p)}_{m_{i_p}} \right).$$

Moreover, for a given closed subspace $\mathcal{K}$, we use symbol

$$\mathcal{L}^A_{\mathbf{m}}(\mathcal{K}) := \bigvee \{ T^{(i_1)}_{m_{i_1}}(\eta_1) \cdots T^{(i_p)}_{m_{i_p}}(\eta_{i_p}) h : \eta_{i_j} \in E_{i_j}^{\otimes m_{i_j}}, 1 \leq j \leq p, h \in \mathcal{K} \}.$$ 

Clearly $\mathcal{L}^A_{\mathbf{m}}(\mathcal{K}) = \tilde{T}_m^A(\mathbb{E}(\mathbf{m}) \otimes_\sigma \mathcal{K}).$
Theorem (H.-Shankar V.)

Let $\mathbb{E}$ be a product system of $C^*$-correspondences over $\mathbb{N}_0^k$. Let $(\sigma, T^{(1)}, \ldots, T^{(k)})$ be a doubly commuting isometric, covariant representation of $\mathbb{E}$ on a Hilbert space $\mathcal{H}$. Then for $2 \leq m \leq k$, there exists $2^m (\sigma, T^{(1)}, \ldots, T^{(m)})$-reducing subspaces \{\mathcal{H}_A : A \subseteq I_m\} such that

$$\mathcal{H} := \bigoplus_{A \subseteq I_m} \mathcal{H}_A,$$

where

$$\mathcal{H}_A = \bigoplus_{n \in \mathbb{N}_0^A} \mathcal{L}_n^A \left( \bigcap_{j \in \mathbb{N}_0^{I_m \setminus A}} \mathcal{L}_j^{I_m \setminus A} (\mathcal{W}_A) \right).$$
Corollary (Theorem 2.4, Skalski-Zacharias, 2008)

In particular, there exist $2^k$ orthogonal $(\sigma, T^{(1)}, \ldots, T^{(k)})$-reducing subspaces $\{\mathcal{H}_A : A \subset I_k\}$ such that

$$\mathcal{H} := \bigoplus_{A \subset I_k} \mathcal{H}_A,$$

and for each $A \subset I_k$ and $\mathcal{H}_A \neq \{0\}$; $(\sigma, T^{(i)})|_{\mathcal{H}_A}$ is an induced representation whenever $i \in A$ and $(\sigma, T^{(i)})|_{\mathcal{H}_A}$ is fully coisometric whenever $i \in I_n \setminus A$. Moreover, the above decomposition is unique.
References:


THANK YOU.