Composition operators on some analytic reproducing kernel Hilbert spaces

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Operator Theory and Operator Algebras 2016
December 13-22, 2016 (Tuesday, December 20)
Indian Statistical Institute, Bangalore
By an **operator** in a complex Hilbert space $\mathcal{H}$ we mean a linear mapping $A \colon \mathcal{H} \supseteq \mathcal{D}(A) \to \mathcal{H}$ defined on a vector subspace $\mathcal{D}(A)$ of $\mathcal{H}$, called the **domain** of $A$;

We say that a densely defined operator $A$ in $\mathcal{H}$ is

- **positive** if $\langle A\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{D}(A)$; then we write $A \geq 0$,
- **selfadjoint** if $A = A^*$,
- **hyponormal** if $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|A^*\xi\| \leq \|A\xi\|$ for all $\xi \in \mathcal{D}(A)$,
- **cohyponormal** if $\mathcal{D}(A^*) \subseteq \mathcal{D}(A)$ and $\|A\xi\| \leq \|A^*\xi\|$ for all $\xi \in \mathcal{D}(A^*)$,
- **normal** if $A$ is hyponormal and cohyponormal,
- **subnormal** if there exist a complex Hilbert space $\mathcal{M}$ and a normal operator $N$ in $\mathcal{M}$ such that $\mathcal{H} \subseteq \mathcal{M}$ (isometric embedding), $\mathcal{D}(A) \subseteq \mathcal{D}(N)$ and $Af = Nf$ for all $f \in \mathcal{D}(A)$,
- **seminormal** if $A$ is either hyponormal or cohyponormal.
By an **operator** in a complex Hilbert space $\mathcal{H}$ we mean a linear mapping $A : \mathcal{H} \supseteq D(A) \to \mathcal{H}$ defined on a vector subspace $D(A)$ of $\mathcal{H}$, called the **domain** of $A$; we say that a densely defined operator $A$ in $\mathcal{H}$ is

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- **seminormal** if $A$ is either hyponormal or cohyponormal.
The class $\mathcal{F}$ stands for the class of all entire functions $\Phi$ of the form

$$\Phi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C},$$

such that $a_k \geq 0$ for all $k \geq 0$ and $a_n > 0$ for some $n \geq 1$.

If $\Phi \in \mathcal{F}$, then, by Liouville’s theorem,

$$\limsup_{|z| \to \infty} |\Phi(z)| = \infty.$$
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If $\Phi \in \mathcal{F}$ is as in (1), we set
\[ \mathcal{Z}_\Phi = \{ n \in \mathbb{N} : a_n > 0 \} \]
and define the multiplicative group $\mathcal{G}_\Phi$ by
\[ \mathcal{G}_\Phi = \bigcap_{n \in \mathcal{Z}_\Phi} G_n, \]
where $G_n$ is the multiplicative group of $n$th roots of 1, i.e.,
\[ G_n := \{ z \in \mathbb{C} : z^n = 1 \}, \quad n \geq 1. \]
The order of the group $\mathcal{G}_\Phi$ can be calculated explicitly.
The RKHS $\Phi(\mathcal{H})$

- $\mathcal{H}$ is a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$.
  If $\Phi \in \mathcal{F}$, then by the Schur product theorem, the kernel $K^\Phi : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ defined by
  \[
  K^\Phi(\xi, \eta) = K^\Phi,\mathcal{H}(\xi, \eta) = \Phi(\langle \xi, \eta \rangle), \quad \xi, \eta \in \mathcal{H},
  \]
  is positive definite.

- $\Phi(\mathcal{H})$ stands the reproducing kernel Hilbert space with the reproducing kernel $K^\Phi$;
  $\Phi(\mathcal{H})$ consists of holomorphic functions on $\mathcal{H}$.

- Reproducing property of $\Phi(\mathcal{H})$:
  \[
  f(\xi) = \langle f, K^\Phi_\xi \rangle, \quad \xi \in \mathcal{H}, \ f \in \Phi(\mathcal{H}),
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  \[
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- $\mathcal{H}^\Phi = \text{the linear span of } \{K^\Phi_\xi : \xi \in \mathcal{H}\}$ is dense in $\Phi(\mathcal{H})$. 
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Some examples - I

- Frankfurt spaces [1975/6/7];
  Multidimensional generalizations - Szafraniec [2003].

- For $\nu$, a positive Borel measure on $\mathbb{R}_+$ such that
  
  \[ \int_{\mathbb{R}_+} t^n \, d\nu(t) < \infty \text{ and } \nu((c, \infty)) > 0 \text{ for all } n \in \mathbb{Z}_+ \text{ and } c > 0. \]

  we define the positive Borel measure $\mu$ on $\mathbb{C}$ by

  \[ \mu(\Delta) = \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}_+} \chi_{\Delta}(r e^{i\theta}) \, d\nu(r) \, d\theta, \quad \Delta \text{ - Borel subset of } \mathbb{C}. \]

- Then we define the function $\Phi \in \mathcal{H}$ by

  \[ \Phi(z) = \sum_{n=0}^{\infty} \frac{1}{\int_{\mathbb{R}_+} t^{2n} \, d\nu(t)} \, z^n, \quad z \in \mathbb{C}. \]
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Frankfurt proved that $\Phi(\mathbb{C})$ can be described as follows

$$\Phi(\mathbb{C}) = \left\{ f : f \text{- entire function } \& f \in L^2(\mu) \right\}; \tag{2}$$

hence the right-hand side of (2) is a reproducing kernel Hilbert space with the reproducing kernel

$$\mathbb{C} \times \mathbb{C} \ni (\xi, \eta) \mapsto \sum_{n=0}^{\infty} \frac{1}{\int_{\mathbb{R}^+} t^{2n} d\nu(t)} \xi^n \bar{\eta}^n \in \mathbb{C}.$$ 

If $\int_{\mathbb{R}^+} t^{2n} d\nu(t) = n!$ for all $n \in \mathbb{Z}_+$, then $\Phi = \exp$, $\mu$ is the Gaussian measure on $\mathbb{C}$ and $\Phi(\mathbb{C})$ is the Segal-Bargmann space $\mathcal{B}_1$ of order $1$. 
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If $\int_{\mathbb{R}^+} t^{2n} \, d\nu(t) = n!$ for all $n \in \mathbb{Z}_+$, then $\Phi = \exp$, $\mu$ is the Gaussian measure on $\mathbb{C}$ and $\Phi(\mathbb{C})$ is the Segal-Bargmann space $\mathcal{B}_1$ of order 1.
Given a holomorphic mapping $\varphi : \mathcal{H} \to \mathcal{H}$, we define the operator $C_\varphi$ in $\Phi(\mathcal{H})$, called a \textit{composition operator} with a \textit{symbol} $\varphi$, by

$$D(C_\varphi) = \{ f \in \Phi(\mathcal{H}) : f \circ \varphi \in \Phi(\mathcal{H}) \},$$

$$C_\varphi f = f \circ \varphi, \quad f \in D(C_\varphi).$$

- $C_\varphi$ is always closed.
- If $\Phi(0) \neq 0$ and $C_\varphi \in B(\Phi(\mathcal{H}))$, then $r(C_\varphi) \geq 1$ and thus $\|C_\varphi\| \geq 1$. 
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Theorem

Let $\Phi \in F$ and $\varphi, \psi : \mathcal{H} \to \mathcal{H}$ be holomorphic mappings. Assume that the operators $C_\varphi$ and $C_\psi$ are densely defined in $\Phi(\mathcal{H})$. Then the following conditions are equivalent:

1. $C_\varphi \subseteq C_\psi$,
2. $C_\varphi = C_\psi$,
3. there exists $\alpha \in \mathcal{G}_\Phi$ such that $\varphi(\xi) = \alpha \cdot \psi(\xi)$ for every $\xi \in \mathcal{H}$. 

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Composition operators on some analytic reproducing kernel Hilbe
Proposition

Suppose $\Phi \in \mathcal{T}$, $\varphi : \mathcal{H} \to \mathcal{H}$ is a holomorphic mapping and $\mathcal{D}(C_\varphi) = \Phi(\mathcal{H})$. Then $C_\varphi$ is bounded and there exists a unique pair $(A, b) \in \mathcal{B}(\mathcal{H}) \times \mathcal{H}$ such that $\varphi = A + b$, i.e., $\varphi(\xi) = A\xi + b$, $\xi \in \mathcal{H}$.

The Segal-Bargmann space over $\mathbb{C}^d$ [B. J. Carswell, B. D. MacCluer, A. Schuster 2003]
Fock’s type model for $C_A$

**Theorem**

Suppose $\Phi \in \mathcal{F}$, $Q$ is a conjugation on $\mathcal{H}$ and $A \in B(\mathcal{H})$. Then there exists a unitary isomorphism $U = U_{\Phi,Q} : \Phi(\mathcal{H}) \to \bigoplus_{n \in \mathbb{Z}} \mathcal{H}^\otimes n$ such that

$$C_A^* = U^{-1} \Gamma_{\Phi}(\Xi_Q(A))U,$$

where $\Xi_Q(A) = QAQ$, $\Gamma_{\Phi}(T) = \bigoplus_{n \in \mathbb{Z}} T^\otimes n$ and $T^\otimes n$ is the $n$th symmetric tensor power of $T \in B(\mathcal{H})$. 
The adjoint of $C_A^*$

**Theorem**

Suppose $\Phi \in \mathcal{F}$ and $A \in B(\mathcal{H})$. Then

(i) $C_A^* = C_{A^*}$,

(ii) $\mathcal{H}\Phi$ is a core for $C_A$.
Suppose $\Phi \in \mathcal{F}$ and $A \in B(\mathcal{H})$. Set $m = \min \mathcal{Z}_\Phi$ and $n = \sup \mathcal{Z}_\Phi$. Then

1. if $n < \infty$, then $C_A \in B(\Phi(\mathcal{H}))$,

2. if $n = \infty$, then $C_A \in B(\Phi(\mathcal{H}))$ if and only if $\|A\| \leq 1$.

3. Moreover, if $C_A \in B(\Phi(\mathcal{H}))$, then

\[ \|C_A\| = q_{m,n}(\|A\|) \text{ and } r(C_A) = q_{m,n}(r(A)). \]

\[ a \text{ Note that } 0 \text{ is a zero of } \Phi \text{ of multiplicity } m \text{ and } \infty \text{ is a pole of } \Phi \text{ of order } n. \]

If $m \in \mathbb{Z}_+$ and $n \in \mathbb{Z}_+ \cup \{\infty\}$, then

\[ q_{m,n}(\vartheta) = \vartheta^m \max\{1, \vartheta^{n-m}\}, \quad \vartheta \in [0, \infty), \]

where $\vartheta^0 = 1$ for $\vartheta \in [0, \infty)$, $\vartheta^\infty = \infty$ for $\vartheta \in (1, \infty)$, $\vartheta^\infty = 0$ for $\vartheta \in [0, 1)$ and $1^\infty = 1$. 
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When is $C_A$ an isometry, ..., a partial isometry?

**Proposition**

Suppose $\Phi \in \mathcal{F}$ and $A \in B(\mathcal{H})$. Then $C_A$ is an isometry (resp.: a coisometry, a unitary operator) if and only if $A$ is a coisometry (resp.: an isometry, a unitary operator).

**Proposition**

Let $\Phi \in \mathcal{F}$ and $P \in B(\mathcal{H})$. Then $C_P$ is an orthogonal projection if and only if there exists $\alpha \in \mathcal{G}_\Phi$ such that $\alpha P$ is an orthogonal projection.

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Let $\Phi \in \mathcal{F}$ and $A \in B(\mathcal{H})$. Then $C_A$ is a partial isometry if and only if $A$ is a partial isometry.
Theorem

Suppose \( \Phi \in \mathcal{F} \) and \( A \in B(\mathcal{H}) \). Then the following conditions are equivalent:

1. \( C_A \geq 0 \),
2. there exists \( \alpha \in \mathcal{G}_\Phi \) such that \( \alpha A \geq 0 \),
3. there exists \( B \in B(\mathcal{H}) \) such that \( B \geq 0 \) and \( C_A = C_B \).
4. Moreover, if \( A \geq 0 \), then \( C_A \) is selfadjoint and \( C_A = C_{A^{1/2}} C_{A^{1/2}} \).
Theorem

Let $\Phi \in \mathcal{H}$, $A \in B(\mathcal{H})$ and $t \in (0, \infty)$. Suppose $A \geq 0$. Then

1. $C_A$ is selfadjoint and $C_A \geq 0$,
2. $C_A^t = C_{At}$,
3. $\mathcal{D}(C_A^t) \subseteq \mathcal{D}(C_{As})$ for every $s \in (0, t)$. 

Powers of a positive $C_A$
Theorem

Suppose that $\Phi \in \mathcal{F}$ and $A \in \mathcal{B}(\mathcal{H})$. Let $A = U|A|$ be the polar decomposition of $A$. Then $C_A = C_U C_{|A^*|}$ is the polar decomposition of $C_A$. In particular, $|C_A| = C_{|A^*|}$. 
Seminormality of $C_A$

**Theorem**

If $\Phi \in \mathcal{F}$ and $A, B \in B(\mathcal{H})$, then the following conditions are equivalent:

1. $\mathcal{D}(C_B) \subseteq \mathcal{D}(C_A)$ and $\|C_A f\| \leq \|C_B f\|$ for all $f \in \mathcal{D}(C_B)$,
2. $\|C_A f\| \leq \|C_B f\|$ for all $f \in \mathcal{H}^\Phi$,
3. $\|A^* \xi\| \leq \|B^* \xi\|$ for all $\xi \in \mathcal{H}$.

**Theorem**

If $\Phi \in \mathcal{F}$ and $A \in B(\mathcal{H})$, then the following conditions are equivalent:

1. $C_A$ is cohyponormal (resp., hyponormal),
2. $A$ is hyponormal (resp., cohyponormal).
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If \( \Phi \in \mathcal{F} \) and \( A, B \in B(\mathcal{H}) \), then the following conditions are equivalent:

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2. \( \| C_A f \| \leq \| C_B f \| \) for all \( f \in \mathcal{H}^\Phi \),

3. \( \| A^* \xi \| \leq \| B^* \xi \| \) for all \( \xi \in \mathcal{H} \).

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If \( \Phi \in \mathcal{F} \) and \( A \in B(\mathcal{H}) \), then the following conditions are equivalent:

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Theorem

Let $\Phi \in \mathcal{F}$ and let $A, B \in B_+(\mathcal{H})$. Then the following conditions are equivalent:

1. $C_A \preccurlyeq C_B$,
2. $\langle C_A f, f \rangle \leq \langle C_B f, f \rangle$ for all $f \in \mathcal{H}^\Phi$,
3. $A \preccurlyeq B$. 
Suppose $A \in B(\mathcal{H})$ is selfadjoint. It is well-known (and easy to verify) that $A|_{\overline{\mathcal{R}(A)}}: \overline{\mathcal{R}(A)} \to \mathcal{R}(A)$ is a bijection.

Hence, we may define a generalized inverse $A^{-1}$ of $A$ by

$$A^{-1} = (A|_{\overline{\mathcal{R}(A)}})^{-1}.$$ 

$A^{-1}$ is an operator in $\mathcal{H}$ (not necessarily densely defined) such that

$$\mathcal{D}(A^{-1}) = \mathcal{R}(A), \quad \mathcal{R}(A^{-1}) = \overline{\mathcal{R}(A)},$$

$$AA^{-1} = I_{\mathcal{R}(A)} \quad \text{and} \quad A^{-1}A = P,$$

where $I_{\mathcal{R}(A)}$ is the identity operator on $\mathcal{R}(A)$ and $P$ is the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathcal{R}(A)}$.

If $A \in B_+(\mathcal{H})$, then we write

$$A^{-t} = (A^t)^{-1}, \quad t \in (0, \infty).$$
Suppose $A \in \mathcal{B}(\mathcal{H})$ is selfadjoint. It is well-known (and easy to verify) that $A|_{\overline{\mathcal{R}(A)}}: \overline{\mathcal{R}(A)} \to \mathcal{R}(A)$ is a bijection.

Hence, we may define a generalized inverse $A^{-1}$ of $A$ by

$$A^{-1} = (A|_{\overline{\mathcal{R}(A)}})^{-1}.$$ 

$A^{-1}$ is an operator in $\mathcal{H}$ (not necessarily densely defined) such that

$$\mathcal{D}(A^{-1}) = \mathcal{R}(A), \quad \mathcal{R}(A^{-1}) = \overline{\mathcal{R}(A)},$$

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If $A \in \mathcal{B}_+(\mathcal{H})$, then we write

$$A^{-t} = (A^t)^{-1}, \quad t \in (0, \infty).$$
Suppose $A \in B(\mathcal{H})$ is selfadjoint. It is well-known (and easy to verify) that $A|_{\overline{\mathcal{R}(A)}} : \overline{\mathcal{R}(A)} \to \mathcal{R}(A)$ is a bijection.

Hence, we may define a generalized inverse $A^{-1}$ of $A$ by

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$A^{-1}$ is an operator in $\mathcal{H}$ (not necessarily densely defined) such that

$$\mathcal{D}(A^{-1}) = \mathcal{R}(A), \quad \mathcal{R}(A^{-1}) = \overline{\mathcal{R}(A)},$$

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$$A^{-1} = (A|_{\overline{\mathcal{R}(A)}})^{-1}.$$ 

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where $I_{\mathcal{R}(A)}$ is the identity operator on $\mathcal{R}(A)$ and $P$ is the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathcal{R}(A)}$.

If $A \in B_+(\mathcal{H})$, then we write

$$A^{-t} = (A^t)^{-1}, \quad t \in (0, \infty).$$
The partial order \(\preceq\)

Given two operators \(A, B \in B_+(\mathcal{H})\), we write \(B^{-1} \preceq A^{-1}\) if

\[
\mathcal{D}(A^{-1/2}) \subseteq \mathcal{D}(B^{-1/2}),
\]

\[
\|B^{-1/2}f\| \leq \|A^{-1/2}f\|, \quad f \in \mathcal{D}(A^{-1/2}).
\]

If \(\mathcal{R}(A) = \mathcal{R}(B) = \mathcal{H}\) (\(\iff A^{-1}, B^{-1} \in B(\mathcal{H})\)), then \(B^{-1} \preceq A^{-1}\) if and only if \(B^{-1} \leq A^{-1}\) (i.e., \(\langle B^{-1}f, f \rangle \leq \langle A^{-1}f, f \rangle\) for all \(f \in \mathcal{H}\)).

**Lemma**

If \(A, B \in B_+(\mathcal{H})\) and \(\varepsilon \in (0, \infty)\), then TFAE:

(i) \(B^{-1} \preceq A^{-1}\),

(ii) \(A \leq B\),

(iii) \((\varepsilon + B)^{-1} \leq (\varepsilon + A)^{-1}\).
The partial order \( \preceq \)

- Given two operators \( A, B \in B_+(\mathcal{H}) \), we write \( B^{-1} \preceq A^{-1} \) if
  
  \[
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  \]
  
  \[
  \| B^{-1/2} f \| \leq \| A^{-1/2} f \|, \quad f \in \mathcal{D}(A^{-1/2}).
  \]

- If \( \mathcal{R}(A) = \mathcal{R}(B) = \mathcal{H} \) (\( \iff \) \( A^{-1}, B^{-1} \in B(\mathcal{H}) \)), then \( B^{-1} \preceq A^{-1} \) if and only if \( B^{-1} \preceq A^{-1} \) (i.e., \( \langle B^{-1} f, f \rangle \leq \langle A^{-1} f, f \rangle \) for all \( f \in \mathcal{H} \)).

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The partial order \( \preceq \)

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\|B^{-1/2} f\| \leq \|A^{-1/2} f\|, \quad f \in \mathcal{D}(A^{-1/2}).
\]

If \( R(A) = R(B) = \mathcal{H} \) (\( \iff A^{-1}, B^{-1} \in B(\mathcal{H}) \)), then

\( B^{-1} \preceq A^{-1} \) if and only if \( B^{-1} \leq A^{-1} \)

(i.e., \( \langle B^{-1} f, f \rangle \leq \langle A^{-1} f, f \rangle \) for all \( f \in \mathcal{H} \)).

**Lemma**

If \( A, B \in B_+ (\mathcal{H}) \) and \( \varepsilon \in (0, \infty) \), then TFAE:

(i) \( B^{-1} \preceq A^{-1} \),

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Lemma

Assume \( \{ A_P \}_{P \in \mathcal{P}} \subseteq \mathcal{B}_+(\mathcal{H}) \) is a monotonically decreasing net which converges in WOT to \( A \in \mathcal{B}_+(\mathcal{H}) \). If \( \xi \in \mathcal{H} \), then TFAE:

(i) \( \xi \in \mathcal{R}(A^{1/2}) \),

(ii) for every \( P \in \mathcal{P} \), \( \xi \in \mathcal{R}(A^{1/2}_P) \) and \( c := \sup_{P \in \mathcal{P}} \| A^{-1/2}_P \xi \| < \infty \).

Moreover, if \( \xi \in \mathcal{R}(A^{1/2}) \), then \( c = \| A^{-1/2} \xi \| \).

Apply

Theorem (Mac Nerney-Shmul’yan theorem)

If \( A \in \mathcal{B}_+(\mathcal{H}) \) and \( \xi \in \mathcal{H} \), then TFAE:

(i) \( \xi \in \mathcal{R}(A^{1/2}) \),

(ii) there exists \( c \in \mathbb{R}_+ \) such that \( |\langle \xi, h \rangle| \leq c \| A^{1/2} h \| \) for all \( h \in \mathcal{H} \).

Moreover, if \( \xi \in \mathcal{R}(A^{1/2}) \), then the smallest \( c \in \mathbb{R}_+ \) in (ii) is equal to \( \| A^{-1/2} \xi \| \).
Ranges of WOT limits

Lemma

Assume \( \{A_P\}_{P \in P} \subseteq B_+(\mathcal{H}) \) is a monotonically decreasing net which converges in WOT to \( A \in B_+(\mathcal{H}) \). If \( \xi \in \mathcal{H} \), then TFAE:

(i) \( \xi \in \mathcal{R}(A^{1/2}) \),

(ii) for every \( P \in P \), \( \xi \in \mathcal{R}(A_P^{1/2}) \) and \( c := \sup_{P \in P} \|A_P^{-1/2}\xi\| < \infty \).

Moreover, if \( \xi \in \mathcal{R}(A^{1/2}) \), then \( c = \|A^{-1/2}\xi\| \).

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Theorem (Mac Nerney-Shmul'yan theorem)

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Moreover, if \( \xi \in \mathcal{R}(A^{1/2}) \), then the smallest \( c \in \mathbb{R}_+ \) in (ii) is equal to \( \|A^{-1/2}\xi\| \).
**Theorem (main)**

Let \( \Phi = \exp, \varphi : \mathcal{H} \to \mathcal{H} \) be a holomorphic mapping and \( \mathcal{P} \subseteq \mathcal{B}(\mathcal{H}) \) be an upward-directed partially ordered set of orthogonal projections of finite rank such that \( \bigvee_{P \in \mathcal{P}} \mathcal{R}(P) = \mathcal{H} \). Then the following conditions are equivalent:

(i) \( C_\varphi \in \mathcal{B}(\exp(\mathcal{H})) \),

(ii) \( \varphi = A + b, \) where \( A \in \mathcal{B}(\mathcal{H}), \|A\| \leq 1, \) \( b \in \mathcal{R}(I - |A^*|P|A^*|) \) for every \( P \in \mathcal{P} \) and

\[
S(A, b) := \sup \{ \langle (I - |A^*|P|A^*|)^{-1} b, b \rangle : P \in \mathcal{P} \} < \infty,
\]

(iii) \( \varphi = A + b, \) where \( A \in \mathcal{B}(\mathcal{H}), \|A\| \leq 1 \) and \( b \in \mathcal{R}((I - AA^*)^{1/2}) \).

Moreover, if \( C_\varphi \in \mathcal{B}(\exp(\mathcal{H})) \), then

\[
\|C_\varphi\|^2 = \exp(\|(I - AA^*)^{-1/2}b\|^2) = \exp(S(A, b)).
\]
The case of $\mathcal{H} = \mathbb{C}^n$ was proved by Carswell, MacCluer and Schuster in 2003 (of course without (ii)).

In fact, our statement differs from the above, however they are equivalent if dim $\mathcal{H} < \infty$.

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In fact, our statement differs from the above, however they are equivalent if $\dim \mathcal{H} < \infty$.

Trieu Le
We begin with the following proposition.

**Proposition**

If $\Phi \in \mathcal{F}$, $\varphi : \mathcal{H} \rightarrow \mathcal{H}$ is a holomorphic mapping and $D(C_\varphi) = \Phi(\mathcal{H})$, then $C_\varphi$ is bounded and there exists a unique pair $(A, b) \in B(\mathcal{H}) \times \mathcal{H}$ such that $\varphi = A + b$.

In view of the above proposition, there is no loss of generality in assuming that $\varphi = A + b$, where $A \in B(\mathcal{H})$ and $b \in \mathcal{H}$, i.e., $\varphi$ is an affine mapping.
We begin with the following proposition.

**Proposition**

If $\Phi \in \mathcal{F}$, $\varphi : \mathcal{H} \to \mathcal{H}$ is a holomorphic mapping and $\mathcal{D}(C_\varphi) = \Phi(\mathcal{H})$, then $C_\varphi$ is bounded and there exists a unique pair $(A, b) \in B(\mathcal{H}) \times \mathcal{H}$ such that $\varphi = A + b$.

In view of the above proposition, there is no loss of generality in assuming that $\varphi = A + b$, where $A \in B(\mathcal{H})$ and $b \in \mathcal{H}$, i.e., $\varphi$ is an affine mapping.
Sketch of the proof 2

An idea the proof of the Proposition. Noting that for all $\xi \in \mathcal{H} \setminus \{0\}$,

$$\frac{\Phi(\|\varphi(\xi)\|^2)}{\Phi(\|\xi\|^2)} = \frac{\|K_{\varphi(\xi)}\|^2}{\|K_{\varphi}^\xi\|^2} = \left\| C_{\varphi}^* \left( \frac{K_{\varphi(\xi)}}{\|K_{\varphi}^\xi\|} \right) \right\|^2 \leq \|C_{\varphi}\|^2,$$

and using

**Lemma (The cancellation principle)**

*If $\Phi \in \mathcal{F}$ and $f, g : \mathcal{H} \to [0, \infty)$ are such that $\liminf_{\|\xi\| \to \infty} g(\xi) > 0$ and $\limsup_{\|\xi\| \to \infty} \frac{\Phi(f(\xi))}{\Phi(g(\xi))} < \infty$, then $\limsup_{\|\xi\| \to \infty} \frac{f(\xi)}{g(\xi)} < \infty$.  

we see that $\limsup_{\|\xi\| \to \infty} \frac{\|\varphi(\xi)\|}{\|\xi\|} < \infty$. Since $\varphi$ is an entire function, we conclude that $[!] \varphi$ is of the form $\varphi = A + b$.\)
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- An idea the proof of the Proposition.
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$$\frac{\Phi(\|\varphi(\xi)\|^2)}{\Phi(\|\xi\|^2)} = \frac{\|K_{\varphi(\xi)}\|^2}{\|K_{\xi}\|^2} = \left\| C_{\varphi} \left( \frac{K_{\varphi(\xi)}}{K_{\xi}} \right) \right\|^2 \leq \|C_{\varphi}\|^2,$$

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Lemma (The cancellation principle)

*If $\Phi \in \mathcal{F}$ and $f, g: \mathcal{H} \rightarrow [0, \infty)$ are such that
\[ \lim\inf_{\|\xi\| \rightarrow \infty} g(\xi) > 0 \text{ and } \lim\sup_{\|\xi\| \rightarrow \infty} \frac{\Phi(f(\xi))}{\Phi(g(\xi))} < \infty, \text{ then}
\]
\[ \lim\sup_{\|\xi\| \rightarrow \infty} \frac{f(\xi)}{g(\xi)} < \infty. \]

we see that $\lim\sup_{\|\xi\| \rightarrow \infty} \frac{\|\varphi(\xi)\|}{\|\xi\|} < \infty$. Since $\varphi$ is an entire function, we conclude that $[!] \varphi$ is of the form $\varphi = A + b$. 

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Sketch of the proof 2

- An idea the proof of the Proposition.
  Noting that for all \( \xi \in \mathcal{H} \setminus \{0\} \),

\[
\frac{\Phi(\| \varphi(\xi) \|^2)}{\Phi(\| \xi \|^2)} = \frac{\| K_{\varphi(\xi)} \|^2}{\| K_{\xi} \|^2} = \left\| C^*_\varphi \left( \frac{K_{\xi}}{\| K_{\xi} \|^2} \right) \right\|^2 \lesssim \| C_\varphi \|^2,
\]

and using

Lemma (The cancellation principle)

If \( \Phi \in \mathcal{F} \) and \( f, g : \mathcal{H} \to [0, \infty) \) are such that
\[\lim \inf_{\| \xi \| \to \infty} g(\xi) > 0 \text{ and } \lim \sup_{\| \xi \| \to \infty} \frac{\Phi(f(\xi))}{\Phi(g(\xi))} < \infty, \text{ then}\]
\[\lim \sup_{\| \xi \| \to \infty} \frac{f(\xi)}{g(\xi)} < \infty.\]

we see that \( \lim \sup_{\| \xi \| \to \infty} \frac{\| \varphi(\xi) \|}{\| \xi \|} < \infty. \) Since \( \varphi \) is an entire function, we conclude that [!] \( \varphi \) is of the form \( \varphi = A + b \).
Sketch of the proof 3

The proof of (i) ⇔ (ii) and $\|C_\varphi\|^2 = \exp(S(A, b))$:

**Proposition**

If $\Phi \in \mathcal{T}$, $\varphi = A + b$ and $\psi = |A^*| + b$ ($A \in B(\mathcal{H})$, $b \in \mathcal{H}$), then

(i) $C_\varphi \in B(\Phi(\mathcal{H}))$ if and only if $C_\psi \in B(\Phi(\mathcal{H}))$,

(ii) $\|C_\varphi\| = \|C_\psi\|$ provided $C_\varphi \in B(\Phi(\mathcal{H}))$.

**Lemma (A version of (i) ⇔ (ii) of the main result when $A \geq 0$)**

Suppose $A \in B_+(\mathcal{H})$, $b \in \mathcal{H}$ and $\mathcal{P} \subseteq B(\mathcal{H})$ is an upward-directed partially ordered set of finite rank orthogonal projections such that $\bigvee_{P \in \mathcal{P}} R(P) = \mathcal{H}$. Then TFAE:

(i) $C_{A+b} \in B(\exp(\mathcal{H}))$,

(ii) $\|A\| \leq 1$, $b \in R(I-APA)$ for every $P \in \mathcal{P}$ and

$$S(A, b) := \sup \{ \langle (I-APA)^{-1} b, b \rangle : P \in \mathcal{P} \} < \infty.$$ 

Moreover, if (ii) holds, then $\|C_{A+b}\|^2 = \exp(S(A, b))$. 

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Sketch of the proof 3

- The proof of (i) ⇔ (ii) and \( \| C_\varphi \|^2 = \exp(S(A, b)) \):

Proposition

If \( \Phi \in \mathcal{F} \), \( \varphi = A + b \) and \( \psi = |A^*| + b \) (\( A \in \mathcal{B}(\mathcal{H}) \), \( b \in \mathcal{H} \)), then

(i) \( C_\varphi \in \mathcal{B}(\Phi(\mathcal{H})) \) if and only if \( C_\psi \in \mathcal{B}(\Phi(\mathcal{H})) \),

(ii) \( \| C_\varphi \| = \| C_\psi \| \) provided \( C_\varphi \in \mathcal{B}(\Phi(\mathcal{H})) \).

Lemma (A version of (i) ⇔ (ii) of the main result when \( A \geq 0 \))

Suppose \( A \in \mathcal{B}_+(\mathcal{H}) \), \( b \in \mathcal{H} \) and \( \mathcal{P} \subseteq \mathcal{B}(\mathcal{H}) \) is an upward-directed partially ordered set of finite rank orthogonal projections such that \( \bigvee_{P \in \mathcal{P}} \mathcal{R}(P) = \mathcal{H} \). Then TFAE:

(i) \( C_{A+b} \in \mathcal{B}(\exp(\mathcal{H})) \),

(ii) \( \| A \| \leq 1 \), \( b \in \mathcal{R}(I - APA) \) for every \( P \in \mathcal{P} \) and

\[
S(A, b) := \sup\{ \langle (I - APA)^{-1} b, b \rangle : P \in \mathcal{P} \} < \infty.
\]

Moreover, if (ii) holds, then \( \| C_{A+b} \|^2 = \exp(S(A, b)) \).
Sketch of the proof 3

- The proof of (i)⇔(ii) and \( \| C_\varphi \|^2 = \exp(S(A, b)) \):

**Proposition**

If \( \Phi \in \mathcal{T} \), \( \varphi = A + b \) and \( \psi = |A^*| + b \) (\( A \in B(\mathcal{H}) \), \( b \in \mathcal{H} \)), then

(i) \( C_\varphi \in B(\Phi(\mathcal{H})) \) if and only if \( C_\psi \in B(\Phi(\mathcal{H})) \),

(ii) \( \| C_\varphi \| = \| C_\psi \| \) provided \( C_\varphi \in B(\Phi(\mathcal{H})) \).

**Lemma (A version of (i)⇔(ii) of the main result when \( A \geq 0 \))**

Suppose \( A \in B_+(\mathcal{H}) \), \( b \in \mathcal{H} \) and \( \mathcal{P} \subseteq B(\mathcal{H}) \) is an upward-directed partially ordered set of finite rank orthogonal projections such that \( \bigvee_{P \in \mathcal{P}} R(P) = \mathcal{H} \). Then TFAE:

(i) \( C_{A+b} \in B(\exp(\mathcal{H})) \),

(ii) \( \| A \| \leq 1 \), \( b \in R(I - APA) \) for every \( P \in \mathcal{P} \) and

\[
S(A, b) := \sup \{ \langle (I - APA)^{-1} b, b \rangle : P \in \mathcal{P} \} < \infty.
\]

Moreover, if (ii) holds, then \( \| C_{A+b} \|^2 = \exp(S(A, b)) \).
Proof of the Lemma.

(i)⇒(ii) One can show \([!]\) that \(C_{A+b} \in B(\exp(\mathcal{H}))\) implies that \(\|A\| \leq 1\) and \(b \in \mathcal{R}((I - A^2)^{1/2})\). Take \(P \in \mathcal{P}\). Since \(APA \leq A^2\), we see that \(I - APA \geq I - A^2 \geq 0\). By the Douglas theorem we have

\[
b \in \mathcal{R}((I - A^2)^{1/2}) \subseteq \mathcal{R}((I - APA)^{1/2}) = \mathcal{R}(I - APA).
\]

This, the fact that \(\dim \mathcal{R}((APA)^{1/2}) < \infty\) and

Proposition

Suppose \(A \in B_+(\mathcal{H})\), \(b \in \mathcal{H}\) and \(\dim \mathcal{R}(A) < \infty\). Then \(C_{A+b} \in B(\exp(\mathcal{H}))\) if and only if \(\|A\| \leq 1\) and \(b \in \mathcal{R}(I - A^2)\). Moreover, if \(C_{A+b} \in B(\exp(\mathcal{H}))\), then

\[
\|C_{A+b}\|^2 = \exp(\langle (I - A^2)^{-1}b, b \rangle).
\]

yield \(C_{(APA)^{1/2}+b} \in B(\exp(\mathcal{H}))\).
Proof of the Lemma.

(i) ⇒ (ii) One can show [!] that $C_{A+b} \in B(\exp(\mathcal{H}))$ implies that $\|A\| \leq 1$ and $b \in \mathcal{R}((I - A^2)^{1/2})$. Take $P \in \mathcal{P}$. Since $APA \leq A^2$, we see that $I - APA \geq I - A^2 \geq 0$. By the Douglas theorem we have

$$b \in \mathcal{R}((I - A^2)^{1/2}) \subseteq \mathcal{R}((I - APA)^{1/2}) = \mathcal{R}(I - APA).$$

This, the fact that $\dim \mathcal{R}((APA)^{1/2}) < \infty$ and

Proposition

Suppose $A \in B_+(\mathcal{H})$, $b \in \mathcal{H}$ and $\dim \mathcal{R}(A) < \infty$. Then $C_{A+b} \in B(\exp(\mathcal{H}))$ if and only if $\|A\| \leq 1$ and $b \in \mathcal{R}(I - A^2)$. Moreover, if $C_{A+b} \in B(\exp(\mathcal{H}))$, then

$$\|C_{A+b}\|^2 = \exp(\langle (I - A^2)^{-1} b, b \rangle).$$

yield $C_{(APA)^{1/2} + b} \in B(\exp(\mathcal{H}))$. 
Proof of the Lemma.

(i)⇒(ii) One can show [!] that $C_{A+b} \in B(\exp(\mathcal{H}))$ implies that $\|A\| \leq 1$ and $b \in \mathcal{R}((I - A^2)^{1/2})$. Take $P \in \mathcal{P}$. Since $APA \leq A^2$, we see that $I - APA \geq I - A^2 \geq 0$. By the Douglas theorem we have

$$b \in \mathcal{R}((I - A^2)^{1/2}) \subseteq \mathcal{R}((I - APA)^{1/2}) = \mathcal{R}(I - APA).$$

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Proposition

Suppose $A \in B_+(\mathcal{H})$, $b \in \mathcal{H}$ and $\dim \mathcal{R}(A) < \infty$. Then $C_{A+b} \in B(\exp(\mathcal{H}))$ if and only if $\|A\| \leq 1$ and $b \in \mathcal{R}(I - A^2)$. Moreover, if $C_{A+b} \in B(\exp(\mathcal{H}))$, then

$$\|C_{A+b}\|^2 = \exp(\langle (I - A^2)^{-1}b, b \rangle).$$

yield $C_{(APA)^{1/2}+b} \in B(\exp(\mathcal{H}))$. 
Sketch of the proof 5

Since $C_P$ is an orthogonal projection [], $C_{AP+b} = C_PC_{A+b} \in B(\exp(H))$ and $\|C_{B+b}\| = \|C|B^*|+b\|$, one can deduce that (with $B = AP$)

$$\exp(\langle (I - APA)^{-1} b, b \rangle) = \|C_{(APA)^{1/2}+b}\|^2$$

$$= \|C_{AP+b}\|^2 = \|C_PC_{A+b}\|^2 \leq \|C_{A+b}\|^2.$$

This implies that $\exp(S(A, b)) \leq \|C_{A+b}\|^2$.

(ii) \(\Rightarrow\) (i) Take $P \in \mathcal{P}$. Using the Proposition from the previous slide, we see that $C_{AP+b} \in B(\exp(H))$, $C_{(APA)^{1/2}+b} \in B(\exp(H))$, $\|C_{AP+b}\| = \|C_{(APA)^{1/2}+b}\|$ and

$$\|C_PC_{A+b}f\|^2 = \|C_{AP+b}f\|^2$$

$$\leq \|C_{(APA)^{1/2}+b}\|^2 \|f\|^2$$

$$= \exp(\langle (I - APA)^{-1} b, b \rangle) \|f\|^2$$

$$\leq \exp(S(A, b)) \|f\|^2, \quad f \in D(C_{A+b}).$$
Sketch of the proof 5

Since $C_P$ is an orthogonal projection $[!$, 
$C_{AP+b} = C_P C_{A+b} \in \mathcal{B}(\exp(\mathcal{H}))$ and $\|C_{B+b}\| = \|C_{B^*+b}\|$, 
one can deduce that (with $B = AP$) 
\[
\exp(\langle (I - APA)^{-1} b, b \rangle) = \|C_{(APA)^{1/2}+b}\|^2 \\
= \|C_{AP+b}\|^2 = \|C_P C_{A+b}\|^2 \leq \|C_{A+b}\|^2.
\]

This implies that $\exp(S(A, b)) \leq \|C_{A+b}\|^2$.

(ii) $\Rightarrow$ (i) Take $P \in \mathcal{P}$. Using the Proposition from the 
previous slide, we see that $C_{AP+b} \in \mathcal{B}(\exp(\mathcal{H}))$, 
$C_{(APA)^{1/2}+b} \in \mathcal{B}(\exp(\mathcal{H}))$, $\|C_{AP+b}\| = \|C_{(APA)^{1/2}+b}\|$ and 
\[
\|C_P C_{A+b}f\|^2 = \|C_{AP+b}f\|^2 \\
\leq \|C_{(APA)^{1/2}+b}\|^2 \|f\|^2 \\
= \exp(\langle (I - APA)^{-1} b, b \rangle) \|f\|^2 \\
\leq \exp(S(A, b)) \|f\|^2, \quad f \in \mathcal{D}(C_{A+b}).
\]
Now applying

**Proposition**

If $\Phi \in \mathcal{F}$ and $\mathcal{P} \subseteq \mathcal{B}(\mathcal{H})$ is an upward-directed partially ordered set of orthogonal projections, then

$$\lim_{P \in \mathcal{P}} C_P f = C_Q f, \quad f \in \Phi(\mathcal{H}),$$

where $Q$ is the orthogonal projection of $\mathcal{H}$ onto $\bigvee_{P \in \mathcal{P}} \mathcal{R}(P)$.

we deduce that

$$\|C_{A+b}f\|^2 \leq \exp(S(A,b))\|f\|^2, \quad f \in \mathcal{D}(C_{A+b}).$$

Since composition operators are closed and $C_{A+b}$ is densely defined [!], this implies that $C_{A+b} \in \mathcal{B}(\exp(\mathcal{H}))$ and $\|C_{A+b}\|^2 \leq \exp(S(A,b))$, which completes the proof of the Lemma.
Now applying Proposition

If $\Phi \in \mathcal{F}$ and $\mathcal{P} \subseteq \mathcal{B}(\mathcal{H})$ is an upward-directed partially ordered set of orthogonal projections, then

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The proof of (ii)⇔(iii) of the main result.
Without loss of generality we may assume that $A$ is a contraction. Set $A_P = I - |A^*|P|A^*|$ for $P \in \mathcal{P}$. Then $A_P \in B_+(\mathcal{H})$ for all $P \in \mathcal{P}$. Since $\bigvee_{P \in \mathcal{P}} R(P) = \mathcal{H}$, we see that $\{P\}_{P \in \mathcal{P}}$ is a monotonically increasing net which converges in the SOT to the identity operator $I$.

This implies that $\{A_P\}_{P \in \mathcal{P}} \subseteq B_+(\mathcal{H})$ is a monotonically decreasing net which converges in the WOT to $I - |A^*|^2$. Since $\dim R(|A^*|P|A^*|) < \infty$ for all $P \in \mathcal{P}$, one can show [!] that $R(A_P)$ is closed and $R(A_P) = R(A_P^{1/2})$ for all $P \in \mathcal{P}$.

Hence, by our first lemma in this presentation, $\langle A_P^{-1} \xi, \xi \rangle = \|A_P^{-1/2} \xi\|^2$ for all $\xi \in R(A_P)$ and $P \in \mathcal{P}$.
The proof of (ii) ⇔ (iii) of the main result.
Without loss of generality we may assume that $A$ is a contraction. Set $A_P = I - |A^*|P|A^*|$ for $P \in \mathcal{P}$. Then $A_P \in B_+(\mathcal{H})$ for all $P \in \mathcal{P}$. Since $\bigvee_{P \in \mathcal{P}} \mathcal{R}(P) = \mathcal{H}$, we see that $\{P\}_{P \in \mathcal{P}}$ is a monotonically increasing net which converges in the SOT to the identity operator $I$.

This implies that $\{A_P\}_{P \in \mathcal{P}} \subseteq B_+(\mathcal{H})$ is a monotonically decreasing net which converges in the WOT to $I - |A^*|^2$. Since $\dim \mathcal{R}(|A^*|P|A^*|) < \infty$ for all $P \in \mathcal{P}$, one can show [!] that $\mathcal{R}(A_P)$ is closed and $\mathcal{R}(A_P) = \mathcal{R}(A_P^{1/2})$ for all $P \in \mathcal{P}$. Hence, by our first lemma in this presentation,

$$\langle A_P^{-1} \xi, \xi \rangle = \|A_P^{-1/2} \xi\|^2 \text{ for all } \xi \in \mathcal{R}(A_P) \text{ and } P \in \mathcal{P}.$$
Now applying

Lemma

Assume \( \{A_P\}_{P \in \mathcal{P}} \subseteq B_+(\mathcal{H}) \) is a monotonically decreasing net which converges in WOT to \( A \in B_+(\mathcal{H}) \). If \( \xi \in \mathcal{H} \), then TFAE:

(i) \( \xi \in \mathcal{R}(A^{1/2}) \),

(ii) for every \( P \in \mathcal{P} \), \( \xi \in \mathcal{R}(A^{1/2}_P) \) and

\[
c := \sup_{P \in \mathcal{P}} \|A^{-1/2}_P \xi\| < \infty.
\]

Moreover, if \( \xi \in \mathcal{R}(A^{1/2}) \), then

\[
c = \|A^{-1/2} \xi\|.
\]

we deduce that the conditions (ii) and (iii) are equivalent and

\[
\exp(\|(I - AA^*)^{-1/2} b\|^2) = \exp(S(A, b)).
\]

This completes the proof of the main result.
Now applying

**Lemma**

Assume \( \{ A_P \}_{P \in \mathcal{P}} \subseteq B_+(\mathcal{H}) \) is a monotonically decreasing net which converges in WOT to \( A \in B_+(\mathcal{H}) \). If \( \xi \in \mathcal{H} \), then TFAE:

(i) \( \xi \in \mathcal{R}(A^{1/2}) \),

(ii) for every \( P \in \mathcal{P} \), \( \xi \in \mathcal{R}(A_P^{1/2}) \) and \( c := \sup_{P \in \mathcal{P}} \| A_P^{-1/2} \xi \| \) < \( \infty \).

Moreover, if \( \xi \in \mathcal{R}(A^{1/2}) \), then \( c = \| A^{-1/2} \xi \| \).

we deduce that the conditions (ii) and (iii) are equivalent and \( \exp(\| (I - AA^*)^{-1/2} b \|^2) = \exp(S(A, b)) \). This completes the proof of the main result.
Theorem (Carswell, MacCluer, Schuster)

Let \( \varphi : \mathbb{C}^d \to \mathbb{C}^d \) be a holomorphic mapping \((d \in \mathbb{N})\). Then \( C_\varphi \in B(B_d) \) if and only if there exist \( A \in B(\mathbb{C}^d) \) and \( b \in \mathbb{C}^d \) such that \( \varphi = A + b \), \( \|A\| \leq 1 \) and \( b \in \mathcal{R}(I - AA^*) \). Moreover, if \( C_\varphi \in B(B_d) \), then

\[
\| C_\varphi \|^2 = \exp(\langle (I - AA^*)^{-1}b, b \rangle).
\]

Proof
First we reduce the proof to the case of \( d = 1 \) (skipped).
Theorem (Carswell, MacCluer, Schuster)

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\[
\|C_\varphi\|^2 = \exp(\langle (I - AA^*)^{-1} b, b \rangle).
\]

**Proof**

First we reduce the proof to the case of \( d = 1 \) (skipped).
The case of $d = 1$

Lemma

*Fix* $\alpha \in [0, 1)$ and $b \in \mathbb{C}$. Let $D$ be an operator in $\mathcal{B}_1$ given by

$$(Df)(z) = f(\alpha z + b) \exp(z \bar{b}), \quad z \in \mathbb{C}, \ f \in \mathcal{B}_1.$$  

*Then* $D \in \mathcal{B}(\mathcal{B}_1)$ and

$$\|D\| \leq \frac{\exp\left(\frac{|b|^2}{1-\alpha}\right)}{\sqrt{1-\alpha^2}}.$$
The case of $d = 1$

**Proof of the Lemma:**

\[
\pi \int_{\mathbb{C}} |Df|^2 \, d\mu_1 = \int_{\mathbb{C}} |f(\alpha z + b)|^2 e^{2\Re(z\bar{b})} e^{-|z|^2} \, dV_1(z)
\]
\[
\leq \|f\|^2 \int_{\mathbb{C}} e^{\alpha z + b + 2\Re(z\bar{b}) - |z|^2} \, dV_1(z)
\]
\[
= \|f\|^2 \exp\left(\frac{2|b|^2}{1 - \alpha}\right) \int_{\mathbb{C}} e^{-(1 - \alpha^2)|z - \frac{b}{1 - \alpha}|^2} \, dV_1(z)
\]
\[
= \|f\|^2 \exp\left(\frac{2|b|^2}{1 - \alpha}\right) \int_{\mathbb{C}} e^{-(1 - \alpha^2)|z|^2} \, dV_1(z)
\]
\[
= \pi \|f\|^2 \frac{\exp\left(\frac{2|b|^2}{1 - \alpha}\right)}{1 - \alpha^2}, \quad f \in B_1,
\]
The case of $d = 1$

Lemma

If $D$ is as in the previous Lemma, then

$$(D^nf)(z) = f\left(\alpha^n z + b_n\right) e^{z b_n} \exp\left(\frac{|b|^2}{1 - \alpha} \left(n - 1 - \frac{\alpha - \alpha^n}{1 - \alpha}\right)\right),$$

for all $z \in \mathbb{C}$, $f \in B_1$ and $n \in \mathbb{N}$, where $b_n = \frac{1-\alpha^n}{1-\alpha} b$ for $n \in \mathbb{N}$. 
Combining the previous two Lemmata with Gelfand’s formula for the spectral radius, one can prove the following.

**Lemma**

Let $A \in \mathbb{C}$ be such that $|A| < 1$ and let $b \in \mathbb{C}$. Set $\varphi(z) = Az + b$ for $z \in \mathbb{C}$. Then $C_{\varphi} \in B(B_1)$ and

$$
\| C_{\varphi} \|^2 = \exp \left( \frac{|b|^2}{1 - |A|^2} \right).
$$
Powers of $C_{A+b}$

- If $C_{A+b} \in B(\exp(\mathcal{H}))$, then (with $\varphi = A + b$)
  
  \[ \|C^n\| = \|C_{A^n+b_n}\|^2 = \exp(\|I - A^nA^*n\|^{1/2}b_n^2), \quad n \in \mathbb{Z}_+. \]

  where $b_n = (I + \ldots + A^{n-1})b$ for $n \in \mathbb{N}$.

- The rate of growth of $\{\|I - A^nA^*n\|^{1/2}b_n\}^\infty_{n=1}$.

**Proposition**

Suppose $C_\varphi \in B(\exp(\mathcal{H}))$, where $\varphi = A + b$ with $A \in B(\mathcal{H})$ and $b \in \mathcal{H}$. Then the following holds:

(i) $\varphi^n = A^n + b_n$ and $b_n \in \mathcal{R}(\|I - A^nA^*n\|^{1/2})$ for all $n \in \mathbb{N}$,

(ii) there exists a constant $M \in (0, \infty)$ such that

\[ \|(I - A^nA^*n)^{-1/2}b_n\| \leq M\sqrt{n}, \quad n \in \mathbb{N}. \]

The proof of (ii) depends on our main theorem and Gelfand’s formula for the spectral radius.
If $C_{A+b} \in B(\exp(H))$, then (with $\varphi = A + b$)

$$\|C^n_{\varphi}\|_2^2 = \|C^{n+b}_{A^n+b_n}\|_2^2 = \exp(\| (I - A^n A^{*n})^{-1/2} b_n \|_2^2), \quad n \in \mathbb{Z}_+.$$ 

where $b_n = (I + \ldots + A^{n-1})b$ for $n \in \mathbb{N}$.

The rate of growth of $\{\| (I - A^n A^{*n})^{-1/2} b_n \|}\}_{n=1}^\infty$.

### Proposition

Suppose $C_\varphi \in B(\exp(H))$, where $\varphi = A + b$ with $A \in B(H)$ and $b \in \mathcal{H}$. Then the following holds:

(i) $\varphi^n = A^n + b_n$ and $b_n \in \mathcal{R}((I - A^n A^{*n})^{1/2})$ for all $n \in \mathbb{N}$,

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The proof of (ii) depends on our main theorem and Gelfand’s formula for the spectral radius.
Powers of $C_{A+b}$

- If $C_{A+b} \in B(\exp(\mathcal{H}))$, then (with $\varphi = A + b$)
  \[
  \|C_\varphi^n\|^2 = \|C_{A^n+b_n}\|^2 = \exp(\| (I - A^nA^*n)^{-1/2} b_n \|^2), \quad n \in \mathbb{Z}_+.
  \]
  where $b_n = (1 + \ldots + A^{n-1})b$ for $n \in \mathbb{N}$.
- The rate of growth of \( \{ \| (I - A^nA^*n)^{-1/2} b_n \| \} \) for all $n \in \mathbb{N}$.

**Proposition**

Suppose $C_\varphi \in B(\exp(\mathcal{H}))$, where $\varphi = A + b$ with $A \in B(\mathcal{H})$ and $b \in \mathcal{H}$. Then the following holds:

(i) $\varphi^n = A^n + b_n$ and $b_n \in \mathcal{R}((I - A^nA^*n)^{1/2})$ for all $n \in \mathbb{N}$,

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  \[
  \|(I - A^nA^*n)^{-1/2} b_n\| \leq M \sqrt{n}, \quad n \in \mathbb{N}.
  \]

- The proof of (ii) depends on our main theorem and Gelfand’s formula for the spectral radius.
Proposition

Suppose \( \varphi = A + b \), where \( A \in B(\mathcal{H}) \), \( b \in \mathcal{H} \) and \( \|A\| < 1 \). Then \( C_\varphi \in B(\exp(\mathcal{H})) \) and \( r(C_\varphi) = 1 \). Moreover, if \( b \neq 0 \), then \( C_\varphi \) is not normaloid.

Proof.

It follows from our main theorem that \( C_\varphi \in B(\exp(\mathcal{H})) \). Since \( \|A\| < 1 \), we deduce from C. Neumann’s theorem that \( (I - A)^{-1} \in B(\mathcal{H}) \) and

\[
b_n = (I - A^n)(I - A)^{-1}b, \quad n \in \mathbb{N}.
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Proposition

Suppose $\varphi = A + b$, where $A \in B(\mathcal{H})$, $b \in \mathcal{H}$ and $\|A\| < 1$. Then $C_\varphi \in B(\exp(\mathcal{H}))$ and $r(C_\varphi) = 1$. Moreover, if $b \neq 0$, then $C_\varphi$ is not normaloid.

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$$b_n = (I - A^n)(I - A)^{-1}b, \quad n \in \mathbb{N}.$$
Applying C. Neumann’s theorem again, we see that 
\((I - A^nA^*)^{-1} \in \mathcal{B}(\mathcal{H})\) for all \(n \in \mathbb{N}\) and

\[
\| (I - A^nA^*)^{-1/2}b_n \|^2 = \langle (I - A^nA^*)^{-1}b_n, b_n \rangle 
\leq \frac{\| (I - A^n)(I - A)^{-1}b \|^2}{1 - \|A\|^{2n}} 
\leq \frac{4\|b\|^2}{(1 - \|A\|^{2n})(1 - \|A\|)^2}, \quad n \in \mathbb{N}.
\]

This, together with Gelfand’s formula for the spectral radius

\[
r(C_{\varphi}) = \lim_{n \to \infty} \| C_{\varphi}^n \|^{1/n} = \lim_{n \to \infty} \exp \left( \frac{1}{2n} \| (I - A^nA^*)^{-1/2}b_n \|^2 \right).
\]

gives \(r(C_{\varphi}) = 1\). As \(\mathcal{H} \neq \{0\}\), we infer from the equality \(\|C_{\varphi}\|^2 = \exp(\| (I - AA^*)^{-1/2}b \|^2)\) that \(\|C_{\varphi}\| > 1\) whenever \(b \neq 0\). Hence, \(r(C_{\varphi}) \neq \|C_{\varphi}\|\), which means that \(C_{\varphi}\) is not normaloid.
Applying C. Neumann’s theorem again, we see that 
\((I - A^nA^{*n})^{-1} \in B(\mathcal{H})\) for all \(n \in \mathbb{N}\) and 
\[
\|(I - A^nA^{*n})^{-1/2}b_n\|^2 = \langle (I - A^nA^{*n})^{-1}b_n, b_n \rangle \\
\leq \frac{\|(I - A^n)(I - A)^{-1}b\|^2}{1 - \|A\|^{2n}} \\
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Spectral radius: \( \dim \mathcal{H} < \infty \)

**Theorem**

If \( \varphi: \mathbb{C}^d \to \mathbb{C}^d \) is a holomorphic mapping \((d \in \mathbb{N})\) such that \( C_\varphi \in B(\mathcal{B}_d) \), then \( r(C_\varphi) = 1 \).

The proof of this theorem is more subtle.

**Theorem**

Assume \( \varphi = A + b \) with \( A \in B(\mathbb{C}^d) \) and \( b \in \mathbb{C}^d \), and \( C_\varphi \in B(\mathcal{B}_d) \) \((d \in \mathbb{N})\). Then the following conditions are equivalent:

(i) \( C_\varphi \) is normaloid,

(ii) \( b = 0 \).

Moreover, if \( C_\varphi \) is normaloid, then \( r(C_\varphi) = \|C_\varphi\| = 1 \).

Hence there are no bounded seminormal composition operators on the Bargmann-Segal space \( \mathcal{B}_d \) of finite order \( d \) whose symbols have nontrivial translation part \( b \).
Theorem

If $\varphi: \mathbb{C}^d \to \mathbb{C}^d$ is a holomorphic mapping ($d \in \mathbb{N}$) such that $C\varphi \in B(B_d)$, then $r(C\varphi) = 1$.

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Theorem

Assume $\varphi = A + b$ with $A \in B(\mathbb{C}^d)$ and $b \in \mathbb{C}^d$, and $C\varphi \in B(B_d)$ ($d \in \mathbb{N}$). Then the following conditions are equivalent:

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Hence there are no bounded seminormal composition operators on the Bargmann-Segal space $B_d$ of finite order $d$ whose symbols have nontrivial translation part $b$. 
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If $\varphi : \mathbb{C}^d \to \mathbb{C}^d$ is a holomorphic mapping ($d \in \mathbb{N}$) such that $C\varphi \in B(B_d)$, then $r(C\varphi) = 1$.

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Theorem

Assume \( \varphi = A + b \) with \( A \in B(\mathbb{C}^d) \) and \( b \in \mathbb{C}^d \), and \( C_\varphi \in B(B_d) \) \((d \in \mathbb{N})\). Then TFAE:

(i) \( C_\varphi \) is seminormal,

(ii) \( C_\varphi \) is normal,

(iii) \( A \) is normal and \( b = 0 \).
Example
Let $\mathcal{H}$ be an infinite dimensional Hilbert space, $V \in B(\mathcal{H})$ be an isometry and $b \in \mathcal{H}$. Set $\varphi = V + b$. By our main theorem, we see that

$$C_{\varphi} \in B(\exp(\mathcal{H})) \iff b \in \mathcal{N}(V^*).$$

Suppose $V$ is not unitary, i.e., $\mathcal{N}(V^*) \neq \{0\}$. Take $b \in \mathcal{N}(V^*) \setminus \{0\}$. Then $\{V^n b\}_{n=0}^{\infty}$ is an orthogonal sequence, $\mathcal{R}((I - V^n V^*n)^{1/2}) = \mathcal{N}(V^n)$ for all $n \in \mathbb{N}$ and

$$\|(I - V^n V^*n)^{-1/2} b_n\|^2 = \|b_n\|^2 = \|b + \ldots + V^{n-1} b\|^2 = \|b\|^2 n,$$

which means that the inequality in

$$\|(I - A^n A^*n)^{-1/2} b_n\| \leq M\sqrt{n}, \quad n \in \mathbb{N},$$

becomes an equality with $M = \|b\|$. 
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$$\| (I - A^n A^*n)^{-1/2} b_n \| \leq M \sqrt{n}, \quad n \in \mathbb{N},$$

becomes an equality with $M = \| b \|$.
One can show that $e^{-\|b\|^2/2} C_\varphi$ is a coisometry. In particular, $C_\varphi$ is cohyponormal. Hence, $C_\varphi$ is normaloid and consequently, by our main theorem, we have

$$r(C_\varphi) = \|C_\varphi\| = e^{\|b\|^2/2}.$$  \hspace{1cm} (3)

Note that $C_\varphi$ is not normal (because if $C_\varphi$ is hyponormal, then $b = \varphi(0) = 0$).

In other words, if $\dim \mathcal{H} \geq \aleph_0$, then there always exists bounded non-normal cohyponormal composition operators in $\exp(\mathcal{H})$. 
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