Curvature Inequalities for Operators in the Cowen-Douglas Class of a Planar Domain

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Let $\Omega$ be a bounded domain in $\mathbb{C}$. Assume that $\partial \Omega$ consists of $n + 1$ analytic Jordan curves. Set $\Omega^* = \{ \bar{z} \mid z \in \Omega \}$. Here we study operators in $B_1(\Omega^*)$, first introduced by Cowen and Douglas.

**Definition 1 (Cowen-Douglas class of operators: (1978))**

An operator $T$ acting on a complex separable Hilbert space $\mathcal{H}$, is said to be in the class $B_1(\Omega^*)$ if it meets the following requirements:

1. $\text{ran}(T - w) = \mathcal{H}$, $w \in \Omega^*$,
2. $\bigvee_{w \in \Omega^*} \ker(T - w) = \mathcal{H}$ and
3. $\dim(\ker(T - w)) = 1$, $w \in \Omega^*$.
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These conditions ensure the existence of a rank 1 Hermitian holomorphic vector bundle $E_T$ over $\Omega^*$, that is,

$$E_T := \{(w, v) \in \Omega^* \times \mathcal{H} : v \in \ker(T - w)\}, \pi(w, v) = w$$

and there exist a holomorphic frame $w \to \gamma(w)$ with the property $\ker(T - w) = \text{span} \{\gamma(w)\}$. 
The Hermitian structure of the vector bundle $E_T$ at the point $w$ with respect to the frame $\gamma(w)$ is obtained from that of the subspace $\text{ker}(T - w)$ of the Hilbert space $\mathcal{H}$ and we denote it by $h(w) = \langle \gamma(w), \gamma(w) \rangle_{\mathcal{H}}$. 

The curvature $K_{\gamma}(w)$ of the bundle $E_T$ w.r.t the frame $\gamma$ is given by the following formula:

$$K_{\gamma}(w) = \frac{\partial}{\partial \overline{w}} \left( h^{-1}(w) \frac{\partial}{\partial w} h(w) \right) d\overline{w} \wedge dw = \frac{\partial}{\partial w} \frac{\partial}{\partial \overline{w}} \log(h(w)) d\overline{w} \wedge dw = K_{\gamma}(w) dw \wedge d\overline{w}.$$

The expression of the curvature $K_{\gamma}(w)$ is independent of the choice of the frame $\gamma$. So, we call it curvature of the operator $T$ and denote it by $K_T(w)$. 

$$K_T(w) = -\frac{\partial}{\partial w} \frac{\partial}{\partial \overline{w}} \log(h(w)) = -\|\gamma(w)\|^2 \|\gamma'(w)\|^2 - |\langle \gamma(w), \gamma'(w) \rangle|^2 \|\gamma(w)\|^4.$$
The Hermitian structure of the vector bundle $E_T$ at the point $w$ with respect to the frame $\gamma(w)$ is obtained from that of the subspace $\ker(T - w)$ of the Hilbert space $\mathcal{H}$ and we denote it by $h(w) = \langle \gamma(w), \gamma(w) \rangle_{\mathcal{H}}$.

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$$\mathcal{K}_\gamma(w) = - \frac{\partial}{\partial w} \frac{\partial}{\partial \bar{w}} \log(h(w)) = - \frac{\|\gamma(w)\|^2 \|\gamma'(w)\|^2 - \langle \gamma(w), \gamma'(w) \rangle^2}{\|\gamma(w)\|^4}.$$
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Theorem 2 (Cowen-Douglas)

Two operators $T_1$ and $T_2$ in $B_1(\Omega^*)$ are unitarily equivalent if and only if the associated bundle $E_{T_1}$ and $E_{T_2}$ are equivalent as Hermitian holomorphic vector bundle if and only if $\mathcal{K}_{T_1}(w) = \mathcal{K}_{T_2}(w)$ for every $w$ in $\Omega^*$. 

Every $T \in B_1(\Omega^*)$ is unitarily equivalent to the adjoint $M^*$ of the operator of multiplication $M$ by the coordinate function on some Hilbert space $H_K$ consisting of holomorphic function on $\Omega$ possessing a reproducing kernel $K$.

The kernel $K$ is complex valued function defined on $\Omega \times \Omega$, which is holomorphic in the first and anti-holomorphic in the second variable and is positive semi-definite in the sense that $((K(z_i, z_j)))$ is positive semi-definite for every subset $\{z_1, \ldots, z_n\}$ of $\Omega$. 


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The curvature $\mathcal{K}_T$ of the operator $T$ is then equal to

$$\mathcal{K}_T(\bar{w}) = - \frac{\partial^2}{\partial z \partial \bar{z}} \log K_T(z, z)|_{z=\bar{w}}$$

$$= - \frac{\| K_w \|^2 \| \bar{\partial} K_w \|^2 - |\langle K_w, \bar{\partial} K_w \rangle|^2}{(K(w, w))^2}, \quad w \in \Omega.$$
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$$

• The space $\ker(M^* - \bar{w})^2 = \text{span}\{K_w, \bar{\partial}K_w\}$ is an invariant subspace for $M^*$.

• Representing the restriction of the operator $M^*$ to this subspace with respect to an orthonormal basis as a $2 \times 2$ matrix, we have

$$
M^*|_{\ker(M^* - \bar{w})^2} = \begin{pmatrix} \bar{w} & 1 \\ 0 & \sqrt{-\mathcal{K}_T(\bar{w})} \end{pmatrix}.
$$
Definition 3 (Spectral Set)

A compact subset $X \subseteq \mathbb{C}$ is said to be a spectral set for an operator $A$ in $\mathcal{L}(\mathcal{H})$, if $\sigma(A) \subseteq X$ and the homomorphism $\rho_A : \text{Rat}(X) \to \mathcal{L}(\mathcal{H})$ defined by $\rho_A(r) = r(A)$ is contractive.

where $\text{Rat}(X)$ denotes the algebra of rational functions whose poles are off $X$, equipped with sup norm on $X$. 
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- If $X = \overline{D}$, then by Von Neuman inequality $X$ is a spectral set for an operator $T$ if and only if $T$ is a contraction.
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Now assume $\overline{\Omega}^*$, the closure of $\Omega^*$, is a spectral set for an operator $M^*$ in $B_1(\Omega^*)$. For an arbitrary fixed point $w \in \Omega$ and $r \in \text{Rat}(\overline{\Omega}^*)$, we have

$$r(M^*)|_{\ker(M^* - \bar{w})^2} = \begin{pmatrix} r(\bar{w}) & r'(\bar{w}) \sqrt{-K_T(\bar{w})} \\ 0 & r(\bar{w}) \end{pmatrix}.$$
\[
\sup\left\{ \frac{|r'(\bar{w})|}{1 - |r(\bar{w})|^2} : \|r\|_{\infty} \leq 1, \ r \in \text{Rat}(\overline{\Omega}^*) \right\} = 2\pi \left( S_{\Omega^*}(\bar{w}, \bar{w}) \right),
\]

where \( S_{\Omega^*}(z, w) \), the Szegő kernel of \( \Omega^* \), is the reproducing kernel for the Hardy space \( (H^2(\Omega^*), ds) \).
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- Contractivity condition gives a curvature inequality:

**Curvature Inequality 1**

\[
K_T(\bar{w}) \leq -4\pi^2 (S_{\Omega^*}(\bar{w}, \bar{w}))^2, \quad \bar{w} \in \Omega^*.
\]

Equivalently, since \( S_{\Omega}(z, w) = S_{\Omega^*}(\bar{w}, \bar{z}) \), the curvature inequality takes the form

**Curvature Inequality 2**

\[
\frac{\partial^2}{\partial \bar{w} \partial \bar{w}} \log K_T(w, w) \geq 4\pi^2 (S_{\Omega}(w, w))^2, \quad w \in \Omega.
\]
The reproducing kernel of the Hardy space, as is well-known, is the Szego kernel $S_D$ of the unit disc $D$. It is given by the formula $S_D(z, w) = \frac{1}{2\pi(1-z\bar{w})}$, for all $z, w$ in $D$.

\[ \mathcal{K}_{M^*}(w) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log S_D(w, w) = -4\pi^2(S_D(w, w))^2, \quad w \in D. \]
The reproducing kernel of the Hardy space, as is well-known, is the Szegő kernel $S_\mathbb{D}$ of the unit disc $\mathbb{D}$. It is given by the formula $S_\mathbb{D}(z, w) = \frac{1}{2\pi(1-z\bar{w})}$, for all $z, w$ in $\mathbb{D}$.

$$K_{\mathcal{M}^*}(w) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log S_\mathbb{D}(w, w) = -4\pi^2(S_\mathbb{D}(w, w))^2, \quad w \in \mathbb{D}.$$  

If the region $\Omega$ is simply connected, then using the Riemann map and the transformation rule for the Szegő kernel together with the chain rule for composition, we see that

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log S_\Omega(w, w) = 4\pi^2(S_\Omega(w, w))^2, \quad w \in \Omega.$$
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• For simply connected domain, $M^*$ on $(H^2(\Omega), ds)$ is an extremal operator.
Pointwise equality and a uniqueness question of Douglas

For a contraction $T$ in $B_1(D)$, if $\mathcal{K}_T(\zeta) = -4\pi^2 S_D(\zeta, \zeta)^2$ for some fixed $\zeta$ in $D$, then does it follow that $T$ must be unitarily equivalent to $M^*$ on $(H^2(D), ds)$?

Answer: No. Misra('84) provided a counterexample.
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For a contraction $T$ in $B_1(\mathbb{D})$, if $K_T(\zeta) = -4\pi^2 S_D(\zeta, \zeta)^2$ for some fixed $\zeta$ in $\mathbb{D}$, then does it follow that $T$ must be unitarily equivalent to $M^*$ on $(H^2(\mathbb{D}), ds)$?

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However, he proved that

**Theorem 4**

*For a homogeneous contraction $T$ in $B_1(\mathbb{D})$, if $K_T(\zeta) = -4\pi^2 S_\mathbb{D}(\zeta, \zeta)^2$ for some fixed $\zeta$ in $\mathbb{D}$, then $T$ must be unitarily equivalent to $M^*$ on $(H^2(\mathbb{D}), ds)$.*
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**Theorem 4**

For a homogeneous contraction $T$ in $B_1(\mathbb{D})$, if $K_T(\zeta) = -4\pi^2 S_\mathbb{D}(\zeta, \zeta)^2$ for some fixed $\zeta$ in $\mathbb{D}$, then $T$ must be unitarily equivalent to $M^*$ on $(H^2(\mathbb{D}), ds)$.

**Theorem 5**

Let $T$ be an operator in $B_1(\mathbb{D})$ whose adjoint is a subnormal contraction. Let $\zeta$ be a fixed but arbitrary point in $\mathbb{D}$. Assume that polynomials are dense in $\mathcal{H}_{\phi_\zeta}(T)$ and that $K_T(\zeta) = -\frac{1}{(1-|\zeta|^2)^2}$, then $T$ is unitarily equivalent to $U^*_+$, the standard unilateral backward shift operator. (where $\mathcal{H}_{\phi_\zeta}(T)$ is the normalized reproducing kernel associated to the operator $\phi_\zeta(T)$.)
On the other hand, if the region is not simply connected, then Suita(’76) has shown that

\[ \frac{\partial^2}{\partial w \partial \bar{w}} \log S_\Omega(w, w) > 4\pi^2(S_\Omega(w, w))^2, \quad w \in \Omega. \]

- If \( \Omega \) is not simply connected, then the operator \( M^* \) on \((H^2(\Omega), ds)\) fails to be extremal.
- We don’t know if there exists an operator \( T \) in \( B_1(\Omega^*) \), admitting \( \overline{\Omega}^* \) as a spectral set for which

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• The question of equality at just one fixed but arbitrary point $\bar{\zeta}$ in $\Omega^*$ was answered by Misra (’84).

• For an arbitrary fixed point $\zeta$ in $\Omega^*$, he has shown the existence of a subnormal operator $M$ such that $\mathcal{K}_M(\zeta) = -4\pi^2(S_{\Omega^*}(\zeta, \zeta))^2$. 
Fix a point $\zeta$ in $\Omega^*$. We have already seen the existence of a co-subnormal operator $M_\zeta^*$ such that $K_{M_\zeta^*}(\zeta) = -4\pi^2(S_{\Omega^*}(\zeta, \zeta))^2$.

- In fact $M_\zeta$ is a pure, rationally cyclic, subnormal operator whose spectrum equal to $\overline{\Omega}$ and normal spectrum equal to $\partial \Omega$. 

Abrahamse and Douglas (’76) has characterized the unitary equivalence class of such operator. These are called bundle shift of multiplicity one. Their results also proves that adjoint of each of the bundle shift lies in $B_1(\Omega^*)$ having $\Omega^*$ as a spectral set.
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Let $\alpha$ be an element in $\text{Hom}(\pi_1(\Omega), \mathbb{T})$. Such a homomorphism is also called a character. Each of these character induces a flat unitary bundle $E_\alpha$ of rank 1 on $\Omega$. 

**Theorem 6**

Two rank one flat unitary vector bundle $E_\alpha$ and $E_\beta$ are equivalent as a flat unitary vector bundle if and only if their inducing characters are equal that is $\alpha = \beta$. 

Fix a character $\alpha$. Let $H^2_{E_\alpha}$ be the linear space of those holomorphic sections $f$ of $E_\alpha$ such that the subharmonic function $\|f(z)\|^2$ on $\Omega$ is majorized by a harmonic function on $\Omega$. 


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Fix a point $p \in \Omega$. Then the norm of a section $f$ in $H^2_\alpha(\Omega)$ is defined by

$$\|f\|^2 = \int_{\partial \Omega} \|f(z)\|^2 \, ds.$$ 

- The linear space $(H^2_{E\alpha}, ds)$ is complete with respect to this norm making it into a Hilbert space.
- A bundle shift $T_{E\alpha}$ is simply the operator of multiplication by the coordinate function on $(H^2_{E\alpha}, ds)$. 
Theorem 7 (Abrahamse and Douglas’76)

Let $E_\alpha$ and $E_\beta$ be two rank one flat unitary vector bundles induced by the homomorphisms $\alpha$ and $\beta$ respectively. Then the bundle shift $T_{E_\alpha}$ is unitarily equivalent to the bundle shift $T_{E_\beta}$ if and only if $E_\alpha$ and $E_\beta$ are equivalent as flat unitary vector bundles.

- It is not very hard to verify that $T_{E_\alpha}$ is a pure cyclic subnormal operator with spectrum $\bar{\Omega}$ and normal spectrum $\partial\Omega$. 
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Theorem 8 (Abrahamse and Douglas ’76)

Every pure cyclic subnormal operator with spectrum $\overline{\Omega}$ and normal spectrum contained in $\partial \Omega$ is unitarily equivalent to a bundle shift $T_{E_\alpha}$ for some character $\alpha$. 
Let $\lambda$ be a positive continuous function on $\partial \Omega$. Define an equivalent norm on $H^2(\Omega)$ in the following way

$$
\| f \|_{\lambda ds}^2 = \int_{\partial \Omega} |f(z)|^2 \lambda(z) ds(z).
$$

Let $(H^2(\Omega), \lambda ds)$ denote the linear space $H^2(\Omega)$ endowed with the norm $\lambda ds$. 

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Let $(H^2(\Omega),\lambda ds)$ denote the linear space $H^2(\Omega)$ endowed with the norm $\lambda ds$.

- the operator $M$ on it is cyclic, pure subnormal, its spectrum is equal to $\overline{\Omega}$ and finally its normal spectrum is equal to $\partial \Omega$. 
The operator $M$ on $(H^2(\Omega), \lambda ds)$ must be unitarily equivalent to the bundle shift $T_{E_\alpha}$ on $(H^2_{E_\alpha}(\Omega), ds)$ for some character $\alpha$. 

The character $\alpha$ is determined by the following $n$-tuple of numbers:

$$c_j(\lambda) = -\int_{\partial \Omega} \partial \partial \eta z(u_\lambda(z)) ds(z),$$

for $j = 1, 2, \ldots, n$, where $u_\lambda$ is the harmonic function on $\Omega$ with continuous boundary value $\frac{1}{2} \log \lambda$. 

$\alpha = (\exp(c_1(\lambda), \ldots, \exp(c_n(\lambda)))$.
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The character $\alpha$ is determined by the following $n$-tuple of numbers:

$$c_j(\lambda) = -\int_{\partial\Omega} \frac{\partial}{\partial \eta_z} (u_\lambda(z)) \, ds(z), \quad \text{for } j = 1, 2, \ldots, n,$$

where $u_\lambda$ is the harmonic function on $\Omega$ with continuous boundary value $\frac{1}{2} \log \lambda$.

$$\alpha = (\exp (ic_1(\lambda)), \ldots, \exp (ic_n(\lambda)))$$
Theorem 9

Let $\lambda, \mu$ be two positive continuous function on $\partial \Omega$. Then the operators $M$ on the Hilbert spaces $(H^2(\Omega), \lambda ds)$ and $(H^2(\Omega), \mu ds)$ are unitarily equivalent if and only if

$$\exp(icc_j(\lambda)) = \exp(icc_j(\mu)), \quad j = 1, \ldots, n.$$
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$$\exp (ic_j(\lambda)) = \exp (ic_j(\mu)), \quad j = 1, \ldots, n.$$ 

Following a result of Abrahamse, it follows that, given any character $\alpha$, there exists a positive continuous function $\lambda$ defined on $\partial \Omega$ such that the operator $M$ on $(H^2(\Omega), \lambda ds)$ is unitarily equivalent to the bundle shift $M$ on $(H^2_{E\alpha}(\Omega), ds)$. 

Weighted kernel and extremal operator at a point

- \((H^2(\Omega), \lambda ds)\) is a reproducing kernel Hilbert space.

Let \(K^{(\lambda)}(z, w)\) denote the kernel function for \((H^2(\Omega), \lambda ds)\).
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Let \(K^{(\lambda)}(z, w)\) denote the kernel function for \((H^2(\Omega), \lambda ds)\).

The case \(\lambda \equiv 1\) gives us the Szego kernel \(S(z, w)\) for the domain \(\Omega\). Associated to the Szego kernel, there exists a conjugate kernel \(L(z, w)\), called the Garabedian kernel, which is related to the Szego kernel via the following identity.

\[
\overline{S(z, w)} ds = \frac{1}{i} L(z, w) dz, \; w \in \Omega \text{ and } z \in \partial \Omega
\]

For each fixed \(w\) in \(\Omega\), the function \(S_w(z)\) is holomorphic in a neighbourhood of \(\Omega\) and \(L_w(z)\) is holomorphic in a neighbourhood of \(\Omega - \{w\}\) with a simple pole at \(w\). \(L_w(z)\) is non vanishing on \(\overline{\Omega} - \{w\}\). The function \(S_w(z)\) is non vanishing on \(\partial \Omega\) and has exactly \(n\) zeros in \(\Omega\).
Nehari has extended these result for the kernel $K^{(\lambda)}(z, w)$.

**Theorem 10 (Nehari'50)**

Let $\lambda$ be a positive continuous function on $\partial \Omega$. Then there exist two analytic function $K^{(\lambda)}(z, w)$ and $L^{(\lambda)}(z, w)$ with the following properties: for each fixed $w$ in $\Omega$, the function $K^{(\lambda)}_w(z)$ and $L^{(\lambda)}_w(z) - (2\pi(z - w))^{-1}$ are holomorphic in $\Omega$; $|K^{(\lambda)}_w(z)|$ is continuous on $\bar{\Omega}$ and $|L^{(\lambda)}_w(z)|$ is continuous in $\bar{\Omega} - C_\epsilon$, where $C_\epsilon$ denotes a small open disc about $w$; $K^{(\lambda)}_w(z)$ and $L^{(\lambda)}_w(z)$ are connected by the identity

$$K^{(\lambda)}_w(z)\lambda(z)ds = \frac{1}{i}L^{(\lambda)}_w(z)dz, \quad w \in \Omega \text{ and } z \in \partial\Omega$$

These properties determine both functions uniquely.
Let \( \{a_1, \ldots, a_n\} \) be the zeros of the Szegö kernel \( S_\zeta(z) \) in \( \Omega \).

**Theorem 11**

The operator \( M^* \) on the Hilbert space \( (H^2(\Omega), \lambda ds) \) is extremal at \( \bar{\zeta} \), that is, \( \frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w, w) \big|_{w=\zeta} = 4\pi^2 (S_\Omega(\zeta, \zeta))^2 \) if and only if \( L_\zeta^{(\lambda)}(z) \) has \( \{a_1, a_2, \ldots, a_n\} \) as the zero set.
Consider the closed convex set $M_1$ in $(H^2(\Omega), \lambda(z)ds)$ defined by

$$M_1 := \{ f \in (H^2(\Omega), \lambda(z)ds) : f(\zeta) = 0, f'(\zeta) = 1 \}.$$ 

Now consider the extremal problem of finding

$$\inf \{ \| f \|^2 : f \in M_1 \}.$$ 

The unique function $F$ in $(H^2(\Omega), \lambda(z)ds)$ is a solution to the extremal problem iff $F \in M_1$ and $F$ is orthogonal to the subspace

$$H_1 = \{ f \in (H^2(\Omega), \lambda(z)ds) : f(\zeta) = 0, f'(\zeta) = 0 \}$$ 

$$= (\text{Span}\{ K_\zeta^{(\lambda)}, \bar{\partial}K_\zeta^{(\lambda)} \})^\perp.$$ 

\[ \inf \{ \| f \|^2 : f \in M_1 \} = \frac{1}{K^{(\lambda)}(\zeta, \zeta)} \left( \frac{\partial^2}{\partial w \partial \overline{w}} \log K^{(\lambda)}(w, w) \big|_{w=\zeta} \right)^{-1}. \]

Now consider the function \( g \) in \( M_1 \) defined by

\[ g(z) := \frac{K^{(\lambda)}(\zeta)F^{(\lambda)}(z)}{2\pi S(\zeta, \zeta)K^{(\lambda)}(\zeta, \zeta)}, \quad z \in \Omega, \]

where \( F^{(\lambda)}(z) = \frac{S^{(\lambda)}(z)}{L^{(\lambda)}(z)} \) denote the Ahlfors map for the domain \( \Omega \) at the point \( \zeta \).

Using the reproducing property for the kernel function \( K^{(\lambda)} \) and the fact that \( |F^{(\lambda)}(z)| \equiv 1 \) on \( \partial \Omega \), it is straightforward to verify that

\[ \| g \|^2_{\lambda ds} = \left( K^{(\lambda)}(\zeta, \zeta) 4\pi^2 S(\zeta, \zeta)^2 \right)^{-1}. \]
Consequently we have,

\[
\frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w, w) \big|_{w=\zeta} \geq 4\pi^2 (S_{\Omega}(\zeta, \zeta))^2.
\]

So equality holds iff \( g \) solve the extremal problem iff \( g \) is orthogonal to the subspace \( H_1 \).

\( g \) is orthogonal to \( H_1 \) iff \( I_f \) vanishes for all \( f \in H_1 \).

\[
I_f = \int_{\partial \Omega} f(z) K^{(\lambda)}_{\zeta}(z) F_{\zeta}(z) \lambda(z) \, ds = \frac{1}{i} \int_{\partial \Omega} f(z) F_{\zeta}(z) L^{(\lambda)}_{\zeta}(z) \, dz
\]

\[
= \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(z) L^{(\lambda)}_{\zeta}(z) (2\pi L_{\zeta}(z))}{S_{\zeta}(z)} \, dz
\]

Using residue theorem, we get that \( I_f \) vanishes for all \( f \in H_1 \) iff the set of zeros of the function \( L^{(\lambda)}_{\zeta}(z) \) in \( \Omega \) is \( \{a_1, a_2, \ldots, a_n\} \).
Example of extremal operator at a point

\[ \lambda(z) := \prod_{k=1}^{n} |z - a_k|^2, \quad z \in \partial \Omega. \]

Then, for \( z \in \partial \Omega \), we have

\[ \frac{\overline{S_{\zeta}(z)}}{\prod_{j=1}^{n}(\overline{z} - \overline{a_j})(\overline{\zeta} - a_j)} \lambda(z) ds = \frac{1}{i} \frac{\prod_{k=1}^{n}(z - a_k)}{\prod_{k=1}^{n}(\zeta - a_k)} L_{\zeta}(z) dz \]
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Hence using the uniqueness part of the Nehari’s Theorem, we get

\[ K_{\zeta}^{(\lambda)}(z) = \frac{S_\zeta(z)}{\prod_{j=1}^{n} (z - a_j)(\bar{z} - \bar{a}_j)}, \quad L_{\zeta}^{(\lambda)}(z) = \frac{\prod_{k=1}^{n} (z - a_k)}{\prod_{k=1}^{n} (\zeta - a_k)} L_\zeta(z), \]

Clearly, \( \{a_1, a_2, \ldots, a_n\} \) is the zero set of the function \( L_{\zeta}^{(\lambda)}(z) \).
Another example of extremal operator at a point

\[ \lambda(z) ds = \frac{|S_\zeta(z)|^2}{S(\zeta, \zeta)} ds, \quad z \in \partial \Omega, \]

Using the reproducing property of the Szegő kernel, it is easy to verify that

\[ \langle f, 1 \rangle_{(H^2(\Omega), \lambda ds)} = f(\zeta) \]
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\[
\langle f, 1 \rangle_{(H^2(\Omega), \lambda ds)} = f(\zeta)
\]

This gives us \(K_\zeta^{(\lambda)}(z) \equiv 1\). Now for \(z \in \partial \Omega\), we have,

\[
\lambda(z)ds = \frac{S_\zeta(z)}{S(\zeta, \zeta)} S_\zeta(z) ds = \frac{1}{i} \frac{S_\zeta(z)}{S(\zeta, \zeta)} L_\zeta(z) dz,
\]

Again, using the uniqueness in Nehari’s Theorem we get

\[
L_\zeta^{(\lambda)}(z) = S_\zeta(z)L_\zeta(z)(S(\zeta, \zeta))^{-1}, \quad z \in \overline{\Omega} - \{\zeta\}.
\]

Again, the zero set of the function \(L_\zeta^{(\lambda)}(z)\) is \(\{a_1, a_2, \ldots, a_n\}\).
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In both cases, the character is determined by the following $n$-tuple of number

$$\left\{ \exp\left(2\pi i(1 - \omega_1(\zeta))\right), \ldots, \exp\left(2\pi i(1 - \omega_n(\zeta))\right) \right\}.$$ 

where $\omega_i(z)$ is the harmonic function on $\Omega$ with boundary value equal to 1 when $z \in \partial\Omega_i$ and equal to 0, when $z \in \partial\Omega / \partial\Omega_i$. 
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In fact, we have proved the following uniqueness theorem.
Theorem 12 (uniqueness of extremal bundle shift)

If the operator $M^*$ on the Hilbert space $(H^2(\Omega), \lambda ds)$ is extremal at $\bar{\zeta}$, then the operator $M$ is unitarily equivalent to the bundle shift $T_{E\alpha}$ on $(H^2_E(\Omega), ds)$, where $\alpha$ is uniquely determined by the following $n$-tuple of complex number of unit modulus:

$$\{ \exp(2\pi i(1 - \omega_1(\zeta))), \ldots, \exp(2\pi i(1 - \omega_n(\zeta))) \}.$$ 

$$\sum_{j=1}^{n+1} \omega_j \equiv 1 \quad \text{and} \quad 0 < \omega_{n+1}(z) < 1, \quad z \in \Omega.$$ 

Therefore we get that $(n - 1) < \sum_{j=1}^{n} (1 - \omega_j(\zeta)) < n.$
Theorem 12 (uniqueness of extremal bundle shift)

If the operator $M^*$ on the Hilbert space $(H^2(\Omega), \lambda ds)$ is extremal at $\bar{\zeta}$, then the operator $M$ is unitarily equivalent to the bundle shift $T_{E_\alpha}$ on $(H^2_{E_\alpha}(\Omega), ds)$, where $\alpha$ is uniquely determined by the following $n$-tuple of complex number of unit modulus:

$$\{ \exp(2\pi i(1 - \omega_1(\zeta))), \ldots, \exp(2\pi i(1 - \omega_n(\zeta))) \}.$$

\[\sum_{j=1}^{n+1} \omega_j \equiv 1 \quad \text{and} \quad 0 < \omega_{n+1}(z) < 1, \quad z \in \Omega.\]

- Therefore we get that $$(n - 1) < \sum_{j=1}^{n} (1 - \omega_j(\zeta)) < n.$$  
- For $n \geq 2$, the set of extremal operators does not include the adjoint of many of the bundle shifts.
Let \( \Omega = A(0; R, 1) \).

In this case, \( \omega_1(\zeta) = \frac{\log |\zeta|}{\log R} \).
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So, we have $\omega_1(\Omega) = (0, 1)$. In this case adjoint of every bundle shift, except the trivial one, is an extremal operator at some point $\bar{\zeta}$ in $\Omega^*$. 
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So, we have $\omega_1(\Omega) = (0, 1)$. In this case adjoint of every bundle shift, except the trivial one, is an extremal operator at some point $\bar{\zeta}$ in $\Omega^*$.

In fact this is true of any doubly connected bounded domain $\Omega$ with Jordan analytic boundary. Since using connectedness argument, in such cases, again we will have $\omega_1(\Omega) = (0, 1)$. 
Thank You