What is Brownian motion on a noncommutative manifold?

Uwe Franz (Université de Bourgogne Franche-Comté)

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- Das&Goswami: BM on NC manifolds (2012)

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We tried to combine and develop the NC and CQG approaches and find new (explicit) examples:

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The question

What is a NC manifold????

remains interesting.

So far we have limited ourselves to CQG and their homogeneous spaces...

Main results

We obtained:

- Classif. of central convolution semigroups on CQG's O_N^+, S_N^+

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- Laplace and Dirac operators, their spectrum, their spectral dimension
- Ultra- and hypercontractivity: F&Hong&Lemeux&Ulrich&Zhang 2017, see also Brannan&Vergnioux&Youn 2019

Outline

[1]

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What is BM?

What are Lévy processes?

- Stochastic processes with independent and stationary increments. This requires a semigroup structure on the state space.
- Equivalently, time- and space-homogeneous Markov processes. This requires that all points of the state space "look the same".
- Arise in many models of random phenonema

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What is Brownian motion?

- A very nice Lévy process: continuous paths, isotropic, etc.
- On compact simple connected Lie groups: a Markov process with continuous paths and bi-invariant generator (Laplace-Beltrami operator).
- On a Riemann manifold: A process whose Markov semigroup is generated by the Laplacian (defined via the metric)
- Arise in many models of random phenonema

What is a Compact Quantum Group?

- A possibly noncommutative analog of the algebra of continuous functions on a compact group.
- A CQG algebra (ie., a particularly nice involutive bialgebra! (has an antipode and a Haar state, spanned by the coefficients of unitary corepresentations)

The orthogonal group O_N

Theorem (Weyl)

The C*-algebra $C(O_N)$ of continuous functions on the orthogonal group O_N is the universal commutative C*-algebra generated by

$$x_{jk} \qquad 1 \leq j,k \leq N$$

with the relations

$$x_{jk}^* = x_{jk}$$

$$\sum_{\ell=1}^{N} x_{j\ell}x_{k\ell} = \delta_{jk} = \sum_{\ell=1}^{N} x_{\ell j}x_{\ell k}$$

The free orthogonal quantum group O_N^+

Definition (Wang)

The (universal or full) C*-algebra $C_u(O_N^+)$ (also denoted $A_o(I_N)$ or $A_o(N)$) of "continuous functions" on the free orthogonal quantum group O_N^+ is defined as the universal C*-algebra generated by

$$x_{jk} \qquad 1 \leq j, k \leq N$$

with the relations

$$x_{jk}^* = x_{jk}$$

$$\sum_{\ell=1}^{N} x_{j\ell} x_{k\ell} = \delta_{jk} = \sum_{\ell=1}^{N} x_{\ell j} x_{\ell k}$$

Compact Quantum Groups: definition

Definition (Woronowicz)

A compact quantum group is a pair $\mathbb{G} = (\mathsf{A}, \Delta)$, where A is a unital C^*-algebra, $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ is a unital, $*$-homomorphism which is coassociative (i.e. $(\Delta \otimes \mathrm{id}_\mathsf{A}) \circ \Delta = (\mathrm{id}_\mathsf{A} \otimes \Delta) \circ \Delta$) such that the quantum cancellation rules are satisfied

$$\overline{\mathrm{Lin}}((1 \otimes \mathsf{A})\Delta(\mathsf{A})) = \overline{\mathrm{Lin}}((\mathsf{A} \otimes 1)\Delta(\mathsf{A})) = \mathsf{A} \otimes \mathsf{A}.$$

A is called the algebra of "continuous functions" on $\mathbb{G}$ and denoted by $C(\mathbb{G})$.

O_N^+ *is a compact quantum group*

Remark

There exists a unique unital $*$-algebra homomorphism
$\Delta : C_u(O_N^+) \to C_u(O_N^+) \otimes C_u(O_N^+)$ with

$$\Delta(x_{jk}) = \sum_{\ell=1}^{N} x_{j\ell} \otimes x_{\ell k}.$$

$O_N^+ = (C_u(O_N^+), \Delta)$ is a compact quantum group.

The Haar state

Theorem (Woronowicz)

Let (A, Δ) be a compact quantum group. There exists unique state (called the Haar state) h on A such that

$$a \star h := (h \otimes \mathrm{id}) \circ \Delta(a) = h \star a = h(a)I, \quad a \in \mathrm{A}.$$

In general, h is not a trace. If it is, we say $\mathbb{G} = (A, \Delta)$ is of Kac type.
h need not be faithful, either.

Two more algebras of "functions" on O_N^+

The reduced C-algebra $C_r(O_N^+)$ of "cont. functions" on O_N^+*

For $N \geq 3$ the Haar state of O_N^+ is not faithful on $C_u(O_N^+)$. One defines the reduced C*-algebra $C_r(O_N^+)$ of "cont. functions" on O_N^+ as the image of the GNS representation of $C_u(O_N^+)$ w.r.t. h.

$\Rightarrow$ By construction h is faithful on $C_r(O_N^+)$.

The $*$-Hopf algebra $\mathrm{Pol}(O_N^+)$ of "polynomials" on O_N^+

$\mathrm{Pol}(O_N^+)$ is the $*$-subalgebra of $C_u(O_N^+)$ or $C_r(O_N^+)$ generated by x_{jk}, $1 \leq j, k \leq N$. It has a natural $*$-Hopf algebra structure.

O_N^+ is of Kac type, i.e. the Haar state h is a trace and $S^2 = \mathrm{id}$.

Another example: $SU_q(2)$

For $q \in \mathbb{R}\backslash\{0\}$ the universal C*-algebra generated by α, γ and the relations

$$\alpha^*\alpha + \gamma^*\gamma = 1 \qquad \alpha\alpha^* + q^2\gamma\gamma^* = 1$$
$$\gamma\gamma^* = \gamma^*\gamma \qquad \alpha\gamma = q\gamma\alpha \qquad \alpha\gamma^* = q\gamma^*\alpha$$

can be turned into a compact quantum group, with the comultiplication

$$\Delta\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \otimes \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}.$$

- For $q = 1$: $C(SU_1(2)) = C(SU(2)) = \{$continuous functions on the special unitary group $SU(2)\}$;
- $SU_q(2)$ is coamenable, i.e., $C_u(SU(2)) \cong C_r(SU(2))$, type I, etc.

More examples: $SU_q(N)$

For $q \in (0,1)$ and $N \in \mathbb{N}$ the universal unital C*-algebra $\mathsf{A} = C\big(SU_q(N)\big)$ is generated by $u = (u_{jk})_{j,k=1}^N$ with the relations

a) (unitarity condition):

$$\sum_{s=1}^{N} u_{js} u_{ks}^* = \delta_{jk}\mathbf{1} = \sum_{s=1}^{N} u_{sj}^* u_{sk} \tag{U}$$

b) (twisted determinant condition): for all $\tau \in S_N$,

$$\sum_{\sigma \in S_N} (-q)^{i(\sigma)} u_{\sigma(1),\tau(1)} u_{\sigma(2),\tau(2)} \cdots u_{\sigma(N),\tau(N)} = (-q)^{i(\tau)}\mathbf{1} \tag{TD}$$

($i(\tau)$ = number of inversions) and equipped with the coproduct

$$\Delta(u_{jk}) = \sum_{s=1}^{N} u_{js} \otimes u_{sk}.$$

We have

$$SU_q(N-1) \subseteq SU_q(N).$$

i.e. there exist surjective quantum group morphisms

$$C(SU_q(N)) \to C(SU_q(N-1)).$$

The morphism is $s_{N-1} : C(SU_q(N)) \to C(SU_q(N-1))$,

$$\begin{pmatrix} u_{11} & \cdots & u_{1,N-1} & u_{1N} \\ \vdots & \ddots & \vdots & \vdots \\ u_{N-1,1} & \cdots & u_{N-1,N-1} & u_{N-1,N} \\ u_{N1} & \cdots & u_{N,N-1} & u_{NN} \end{pmatrix} \mapsto \begin{pmatrix} u_{11} & \cdots & u_{1,N-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ u_{N-1,1} & \cdots & u_{N-1,N-1} & 0 \\ 0 & \cdots & 0 & \mathbf{1} \end{pmatrix}.$$

The compact quantum groups $SU_q(N)$ are coamenable, their C*-algebras are type I.

Theorem

Let $(\varphi_t)_{t\geq 0}$ be a continuous convolution semigroup of states on $\mathrm{Pol}(\mathbb{G})$, i.e.

$$\forall s,t\geq 0,\quad \varphi_{s+t}=\varphi_s\star\varphi_t:=(\varphi_s\otimes\varphi_t)\circ\Delta,$$
$$\forall a\in\mathrm{Pol}(\mathbb{G}),\quad \lim_{t\searrow 0}\varphi_t(a)=\varphi_0(a)=\varepsilon(a).$$

The semigroup $(T_t)_{t\geq 0}$,

$$T_t=(\mathrm{id}\otimes\varphi_t)\circ\Delta:\mathrm{Pol}(\mathbb{G})\to\mathrm{Pol}(\mathbb{G})$$

extends continuously to $C_u(\mathbb{G})$ and $C_r(\mathbb{G})$.
The T_t are translation invariant in the sense that

$$\Delta\circ T_t=(\mathrm{id}\otimes T_t)\circ\Delta.$$

Theorem

Let $\mathbb{G} = (\mathsf{A}, \Delta)$ be a compact quantum group and $(T_t)_{t\geq 0}$ a Markov semigroup on $C(\mathbb{G})$.
Then $(T_t|_{\mathrm{Pol}(\mathbb{G})})_{t\geq 0}$ is of the form

$$T_t|_{\mathrm{Pol}(\mathbb{G})} = (\mathrm{id} \otimes \varphi_t) \circ \Delta$$

if and only if T_t is translation invariant for all $t \geq 0$.

Corollary

One-to-one correspondence between translation invariant Markov semigroups on $C_r(\mathbb{G})$ and convolution semigroups (and Lévy processes in the sense of Schürmann) on $\mathrm{Pol}(\mathbb{G})$.

Lévy processes on compact quantum groups

We have one-to-one correspondences between the following objects:

- Lévy processes $(j_{st})_{0\leq s\leq t}$ on $\mathrm{Pol}(\mathbb{G})$
- Translation invariant Markov semigroups $(T_t)_{t\geq 0}$ on $C_r(\mathbb{G})$ or $C_u(\mathbb{G})$
- (Weak-*) cont. convolutions semigroups $(\varphi_t)_{t\geq 0}$ of states on $\mathrm{Pol}(\mathbb{G})$
- Generating functionals $L : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$
  - $L(\mathbf{1}) = 0$
  - $\forall a \in \mathrm{Pol}(\mathbb{G}),\ L(a^*) = \overline{L(a)}$
  - $\forall a \in \ker\varepsilon,\ L(a^*a) \geq 0$

Remark

$$L = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi_t \quad \longleftrightarrow \quad \varphi_t = \exp_\star tL, \quad t \geq 0$$

Definition

A linear functional $L \in \mathrm{Pol}(\mathbb{G})'$ is called central, if $L \star \phi = \phi \star L$ for all $\phi \in \mathrm{Pol}(\mathbb{G})'$.

Proposition

If $\mathbb{G}$ is of Kac type, then $\mathbb{E} : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})_0$ defined by

$$\mathbb{E}(a) = h\big(a_{(1)} S(a_{(3)})\big) a_{(2)}$$

satisfies preserves positivity. Furthermore, it is a conditional expectation onto

$$\mathrm{Pol}(\mathbb{G})_0 = \{a \in \mathrm{Pol}(\mathbb{G}); \tau \circ \Delta(a) = \Delta(a)\}.$$

Classifyng central convolution semigroups

Important observation

In order to classify central generating functionals on a compact quantum group $\mathbb{G}$ of Kac type, it is sufficient to classify the generating functionals on its algebra $\mathrm{Pol}(\mathbb{G})_0$ of central polynomial functions.

For $n \geq 2$, we have

$$\mathrm{Pol}(O_n^+)_0 \cong \mathrm{Pol}([-n, n]).$$

and $\varepsilon(f) = f(n)$ for $f \in \mathrm{Pol}(O_n^+)_0 \cong \mathrm{Pol}([-n, n])$.

The generating functionals on $\mathrm{Pol}(O_n^+)_0 \cong \mathrm{Pol}([-n, n])$ are of the form

$$L_{b,\nu} f = -bf'(n) + \int_{-n}^{n} \big(f(x) - f(n)\big) \frac{\mathrm{d}\nu(x)}{n - x}$$

where $b > 0$ is a real number and ν a finite measure on $[-n, n]$.

Spectrum of the generator

In the case $b = 1$, $\nu = 0$, the eigenvalues of the generator are

$$\lambda_s = -\frac{U_s'(N)}{U_s(N)} \sim -\frac{s}{N} \qquad \text{for } s \in \mathbb{N}.$$

This gives spectral dimension

$$d_N = \begin{cases} 3 & \text{if } N = 2, \\ \infty & \text{if } N \geq 3. \end{cases}$$

Using norm estimates

$$\|a\|_\infty \leq D(s+1)\|a\|_2 \qquad \text{for } a \in V_s$$

due to Vergnioux, we can prove ultra- and hypercontractivity.

See F&Hong&Lemeux&Ulrich&Zhang 2017, or Brannan&Vergnioux&Youn 2019 for lower bounds and improved estimates.

We can define the free sphere via its algebra of "continuous functions@

$$C_u(S_+^{N-1}) = C^*\left(x_1, \cdots, x_N \middle| x_i = x_i^*, \sum_i x_i^2 = 1\right)$$

It has an action of the free orthogonal group defined by

$$\alpha : C_u(S_+^{N-1}) \to C_u(O_N^+) \otimes C_u(S_+^{N-1}), \qquad \alpha(x_i) = \sum_{j=1}^{N} x_j \otimes x_{ji}.$$

Similar procedure (use bi-inv. functions and functionals instead of central ones) yields

$$\lambda_s = -bq_s'(1) + \int_{-1}^{1} \frac{q_s(x) - 1}{x - 1} \mathrm{d}\nu(x)$$

for the eigenvalues gen. of inv. Markov semigroups on the free sphere, with $b \geq 0$ and ν a finite measure on $[-1, 1]$. The $(q_s)_{s \in \mathbb{N}}$ are a family of orthogonal polynomials on $[-1, 1]$, for the distribution of x_{11} w.r.t. to the Haar state on O_N^+.

For $b = 1$, $\nu = 0$, we get

$$d_L = \begin{cases} 2 & \text{if } N = 2, \\ +\infty & \text{if } N \geq 3, \end{cases}$$

for the spectral dimension.

General approach

To classify Lévy processes, translation invariant Markov semigroups, etc., we can classify

Schürmann triples (π, η, L)

- $\pi : \mathrm{Pol}(\mathbb{G}) \to L(H)$ is a unital $*$-representation of $\mathrm{Pol}(\mathbb{G})$ on some (pre-)Hilbert space H
- $\eta : \mathrm{Pol}(\mathbb{G}) \to H$ is a π-ε-cocycle, i.e.

$$\eta(ab) = \pi(a)\eta(b) + \eta(a)\varepsilon(b)$$

- $L : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ is a hermitian linear functional, whose ε-ε-coboundary is

$$\mathrm{Pol}(\mathbb{G}) \otimes \mathrm{Pol}(\mathbb{G}) \ni (a, b) \mapsto -\langle \eta(a^*), \eta(b) \rangle$$

i.e.

$$-\langle \eta(a^*), \eta(b) \rangle = \varepsilon(a)L(b) - L(ab) + L(a)\varepsilon(b)$$

Gaussian generating functionals

Definition

A generating functional $L : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ is called Gaussian, if one (and all) of the following equivalent conditions are satisfied:

- $L|_{K_3} = 0$
- $\eta|_{K_2} = 0$
- η is an ε-ε-derivation
- $\pi|_{K_1} = 0$
- $\pi = \varepsilon(\cdot)\mathrm{id}_H$

Here we denote

$$K_n = \mathrm{span}\{a_1 \cdots a_n; a_1, \ldots, a_n \in \ker \varepsilon\}.$$

Generating functionals on $SU_q(N)$

General strategy: "$\pi \rightsquigarrow \eta \rightsquigarrow L$"

- Step 1: It is not difficult to classify the Gaussian generating functionals on $SU_q(N)$, they correspond to classical Gauss processes on the "classical torus" $\mathbb{T}^{N-1}$, resp.
- Step 2: decompose representation and cocycle (π, η, L) according to

$$H_{\mathrm{Gauss}} = \bigcap_{j=1}^{N} \ker\big(\pi(u_{jj} - \mathrm{id}_H)\big), \qquad H_N = \ker\big(\pi(u_{NN} - \mathrm{id}_H)\big)^{\perp},$$

$$H = H_{\mathrm{Gauss}} \oplus H_N \oplus H_{\mathrm{Rest}}$$

  and show that $\eta_N = P_{H_N}\eta$ can be approximated by coboundaries, and that $\eta - \eta_{\mathrm{Gauss}} - \eta_N$ "lives" on $SU_q(N-1)$, resp.
- Step 3: Induction

Generating functionals on $SU_q(N)$

We get a decomposition of the triple

$$(\pi, \eta, L) = (\pi|_{H_N}, \eta_N, L_N) \oplus \cdots \oplus (\pi|_{H_2}, \eta_2, L_2) \oplus (\pi|_{H_{\text{Gauss}}}, \eta_{\text{Gauss}}, L_{\text{Gauss}})$$

where $\eta_N, \ldots, \eta_2$ are limits of coboundaries.

Corollary

Any non-Gaussian cocycle on $\mathrm{Pol}(SU_q(N))$ admits a generating functional, i.e., these algebras the property NGC introduced by F&Gerhold&Thom (2015).

Open Problems

- Identify "nice" (ie. central) generating functionals on $SU_q(N)$ ($\rightsquigarrow$ Brownian motion, Laplace operator)

  We know (in principle) the central generating functionals on $SU_q(2)$, thanks to De Commer&Freslon&Yamashita(2014)'s work on the CCAP

- Do we have similar results for the q-deformation $\mathbb{G}_q$ of the other simple compact Lie groups, e.g., $O_q(N)$ or $Sp_q(N)$ (cf. Rosso, Klimyk&Schmüdgen)?

  (We did also $U_q(N)$)

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