Multiparameter E_0-semigroups

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E_0-semigroups

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Definition

By an E_0-semigroup over P on $B(\mathcal{H})$, we mean a family $\alpha := \{\alpha_x\}_{x \in P}$ such that

(1) *for every $x \in P$, α_x is a unital normal endomorphism of $B(\mathcal{H})$,*

(2) *for $x, y \in P$, $\alpha_x \circ \alpha_y = \alpha_{x+y}$, and*

(3) *for $A \in B(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$, the map $P \ni x \to \langle \alpha_x(A)\xi | \eta \rangle \in \mathbb{C}$ is continuous.*

We identify E_0-semigroups acting on different Hilbert spaces if they are unitarily equivalent.

cocycle conjugacy

The basic equivalence relation in the theory of E_0-semigroups is that of cocycle conjugacy.

Definition

Let $\alpha := \{\alpha_x\}_{x \in P}$ *be an* E_0*-semigroup on* $B(\mathcal{H})$*. By an* α*-cocycle we mean a strongly continuous family of unitaries* $\{U_x\}_{x \in P}$ *such that* $U_x \alpha_x(U_y) = U_{x+y}$*. Let* $U := \{U_x\}_{x \in P}$ *be an* α*-cocycle. Define for* $x \in P$*,*

$$\beta_x(.) = U_x \alpha_x(.) U_x^*.$$

Then $\beta := \{\beta_x\}_{x \in P}$ *is an* E_0*-semigroup on* $B(\mathcal{H})$*. Such an* E_0*-semigroup is called a cocycle perturbation of* α*.*

Cocycle conjugacy is an equivalence relation.

Examples

Let us recall the symmetric Fock space and the Weyl operators. For a Hilbert space $\mathcal{H}$, $\Gamma(\mathcal{H})$ denotes the symmetric Fock space. For $u \in \mathcal{H}$, let

$$e(u) := \sum_{n=0}^{\infty} \frac{u^{\otimes n}}{\sqrt{n!}}.$$

We have the following.

1. The set $\{e(u) : u \in \mathcal{H}\}$ is linearly independent and total in $\Gamma(\mathcal{H})$.
2. For $u, v \in \mathcal{H}$, $\langle e(u)|e(v)\rangle = e^{\langle u|v\rangle}$.

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For $u \in \mathcal{H}$, there exists a unique unitary operator denoted $W(u)$ on $\Gamma(\mathcal{H})$ such that

$$W(u)e(v) := e^{-\frac{||u||^2}{2} - \langle u|v\rangle} e(u+v).$$

An important fact is that the von-Neumann algebra generated by $\{W(u) : u \in \mathcal{H}\}$ is σ-weak dense in $B(\Gamma(\mathcal{H}))$.

CCR flows

Proposition

Let $V := (V_x)_{x \in P}$ *be a strongly continuous semigroup of isometries on* $\mathcal{H}$*. Then there exists a unique* E_0*-semigroup* $\alpha^V := \{\alpha_x\}_{x \in P}$ *such that*

$$\alpha_x(W(u)) = W(V_x u)$$

for $x \in P$ *and* $u \in \mathcal{H}$*.*

The E_0-semigroup α^V is called the CCR flow associated to the isometric representation V.

Product systems

The basic theory stays intact yet there are significant differences.
Let $\alpha := \{\alpha_x\}_{x \in P}$ be an E_0-semigroup. For $x \in P$, let

$$E(x) := \{T \in B(\mathcal{H}) : \alpha_x(A)T = TA \ \ \forall A \in B(\mathcal{H})\}.$$

Then

(1) For $x \in P$, $E(x)$ is a separable Hilbert space where the inner product is given by $\langle S|T\rangle = T^*S$.

(2) The map $E(x) \otimes E(y) \ni S \otimes T \to ST \in E(x+y)$ is a unitary.

The bundle of Hilbert spaces $\coprod_{x \in P} E(x)$ with its associative product is called the product system of α and is denoted $\mathcal{E}_\alpha$.

Murugan's thesis

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Theorem

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Theorem

(1) *Two E_0-semigroups α and β are cocycle conjugate if and only if $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are isomorphic.*

(2) *Any "abstract product system" is isomorphic to a product system of an E_0-semigroup.*

Differences

The following are some of the differences between the 1-parameter theory and the multiparameter theory.

(1) Decomposable product systems need not necessarily have a unit.

(2) The opposite of a CCR flow need not be cocycle conjugate to itself.

(3) The CAR and CCR flows need not be the same. (R. Srinivasan)

Decomposable product systems

Let $\alpha := \{\alpha_x\}_{x \in P}$ be an E_0-semigroup and $E := \{E(x)\}_{x \in P}$ be its product system. For $x, y \in \mathbb{R}^d$, we write $x \leq y$ if $y - x \in P$.

Definition

(1) *Let $x \in P$. A vector $u \in E(x)$ is called* **decomposable** *if given $y \leq x$ there exist $v \in E(y)$, $w \in E(x-y)$ such that $u = vw$. We denote the set of decomposable vectors by $D(x)$.*

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(2) *The product system E is said to be* **decomposable** *if*
- *for $x, y \in P$, $D(x)D(y) \subset D(x+y)$, and*
- *for $x \in P$, $D(x)$ is total in $E(x)$.*

Arveson's characterisation

Suppose $P = [0, \infty)$. Arveson's ground breaking work says that the CCR functor

$$V \to \alpha^V$$

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Thus in the 1-parameter case a decomposable product system have units in abundance. By a unit of a product system E, we mean a **nowhere vanishing multiplicative measurable cross section** of E.

This phenomenon is no longer true in the multiparameter context.

Arveson's characterisation of CCR flows

In the multiparameter case, we have the following.

Theorem (SS)

(1) *The CCR functor $V \to \alpha^V$ is injective.*

(2) *The range of the above functor is precisely the set of decomposable E_0-semigroups which have a unit.*

(3) *There are indeed uncountably many examples of decomposable E_0-semigroups which do not have a unit.*

Uncountably many examples

Assume $d \geq 2$. Let $\Omega^* := \{\mu \in \mathbb{R}^d : \langle \mu | x \rangle > 0, x \in P \backslash \{0\}\}$. Fix $\lambda, \mu \in \Omega^*$ be given. Let $\{S_t\}_{t \geq 0}$ be the 1-parameter shift semigroup on $L^2(0, \infty)$. For $a \in P$, let $V_a := S_{\langle \mu | a \rangle}$. Then $V := (V_a)_{a \in P}$ is an isometric representation of P on $L^2(0, \infty)$.
Let E be the product system associated to the CCR flow α^V. Set $\overline{E} := E$ and define a new product rule as follows: for $S \in \overline{E}(x)$, $T \in \overline{E}(y)$, let

$$S.T = SW(\langle \lambda | x \rangle 1_{(0, \langle \mu | y \rangle)})T.$$

Then $\overline{E}$ with the product defined above is a decomposable product system. We denote this by $\overline{E}_{\lambda, \mu}$ to stress the dependence of λ and μ.

Theorem

The following are equivalent.

(1) *The product system* $\overline{E}_{\lambda,\mu}$ *has a unit.*

(2) λ *and* μ *are scalar multiplies of each other.*

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Theorem

The product systems $\overline{E}_{\lambda,\mu_1}$ *and* $\overline{E}_{\lambda,\mu_2}$ *are isomorphic if and only if* μ_1 *and* μ_2 *are scalar multiples of each other.*

Thus there are uncountably many examples of decomposable E_0-semigroups which do not admit a unit.