Some recent results on eigenvalue statistics for random Schrödinger operators

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Part 1: The set-up for random Schrödinger operators

Basic random Schrödinger operator:

$$H_\omega = H_0 + V_\omega$$

Hilbert space: lattice $\ell^2(\mathbb{Z}^d)$ or continuum $L^2(\mathbb{R}^d)$

- H_0 deterministic (fixed) self-adjoint operator: $H_0 = -\Delta$, Laplacian
- V_ω random potential:
  - $(V_\omega f)(k) = \omega_k f(k)$, on $\ell^2(\mathbb{Z}^d)$
  - $(V_\omega f)(x) = \sum_{k \in \mathbb{Z}^d} \omega_k u(x-k) f(x)$, on $L^2(\mathbb{R}^d)$

Each ω represents a configuration of the potential function.

The set-up for random Schrödinger operators

Randomness: The coupling constants $\{\omega_j \mid j \in \mathbb{Z}^d\}$

- family of independent, identically distributed random variables (iid)
- random variable ω_0 distributed according to a probability measure ρ on $\mathbb{R}$

Single-site potential

- $V_\omega(k) = \omega_k \Pi_k$, where $(\Pi_k)f)(n) = f(k)\delta_{nk}$, for lattice $\mathbb{Z}^d$
- $V_\omega(x) = \sum_{k \in \mathbb{Z}^d} \omega_k u(x-k)$, where $u(x) \geq 0$ is a single-site bump function, for continuum $\mathbb{R}^d$

Deterministic spectrum: $\Sigma \subset \mathbb{R}$ (fixed) equals $\sigma(H_\omega)$ almost surely

Finite-volume operators

It is sometimes convenient to work with operators with discrete spectrum:

Restriction to cubes:

Λ_L cube of side-length $L > 0$

$H_\omega^\Lambda := H_\omega | \Lambda_L$ plus boundary conditions

Spectrum of H_ω^Λ is discrete: $\{E_j^\Lambda(\omega)\}_{j=1}^N$,

- $N = |\Lambda|$ for lattice $\mathbb{Z}^d$
- $N = \infty$ for continuum $\mathbb{R}^d$

Density of States

Let $I \subset \mathbb{R}$ be a bounded interval and let $E_{H_\omega^\Lambda}(I)$ denote the local spectral projector.

Average number of eigenvalues per unit volume:

$$n_\Lambda(I) := \frac{1}{|\Lambda|} \mathbb{E}\{\mathrm{Tr} E_{H_\omega^\Lambda}(I)\}$$

Infinite-volume limit gives the density of states measure (DOSm):

$$\lim_{|\Lambda| \to \infty} n_\Lambda(I) = n(I) = \int_I d\mu_\rho.$$

Density of States

Two functions associated with the DOSm μ_ρ for the probability density ρ:

Integrated density of states (IDS): The right-continuous cumulative distribution of the DOSm,

$$N_\rho(E) := \mu_\rho^{(\infty)}((-\infty, E]) = \int_{-\infty}^{E} d\mu_\rho(s),$$

Density of states function (DOSf)

$$n_\rho(E) := \frac{dN_\rho}{dE}(E), \quad \text{so} \;\; d\mu_\rho(E) = n_\rho(E)\; dE,$$

when it exists.
We drop the subscript ρ.

Part 2: Local eigenvalue statistics (LES) overview

Local eigenvalue statistics

Recall the finite-volume restriction: $H_\omega^\Lambda := H_\omega|\Lambda_L$, plus boundary conditions

Spectrum of H_ω^Λ is discrete: $\{E_j^\Lambda(\omega)\}_{j=1}^N$,

Local eigenvalue statistics: Fix $E_0 \in \Sigma$:

$$d\xi_\omega^\Lambda(s) = \sum_{j=1}^N \delta(|\Lambda_L|(E_j^\Lambda(\omega) - E_0) - s)\ ds$$

$\xi_\omega^\Lambda(s)$ is point process on $\mathbb{R}$ measuring the average rescaled eigenvalue spacing around E_0.

Local eigenvalue statistics (LES) overview

LES $\xi_\omega^\Lambda(s)$: a local point process built from rescaled eigenvalues :

$$\widetilde{E}_j^\Lambda(\omega) := |\Lambda_L|(E_j^\Lambda(\omega) - E_0)$$

Questions:

1. Does ξ_ω^Λ converge to a point process as $|\Lambda| \to \infty$?
2. How does one characterize the limiting process?

Answers:

1. Does ξ_ω^Λ converge to a point process as $|\Lambda| \to \infty$? **YES**
2. How does one characterize the limiting process? **Depends on E_0 and the dimension d**

LES overview

CONJECTURES:

1. If $E_0 \in \Sigma$ lies in a region for which the localization length γ_L of eigenfunctions for $H_\omega^{\Lambda_L}$ is small compared to L,

$$\frac{\gamma_L}{L} \to 0, \qquad L \to \infty,$$

then the limiting point process ξ_ω is a **Poisson point process**.

2. If $E_0 \in \Sigma$ lies in a region for which the localization length γ_L of eigenfunctions for $H_\omega^{\Lambda_L}$ is large compared to L,

$$\frac{\gamma_L}{L} > 0, \qquad L \to \infty,$$

then the limiting point process ξ_ω is the same as **random matrix theory** ***GOE*.**

LES overview

LSD: Level spacing distribution

Order eigenvalues of H_ω^Λ: $E_1^\Lambda(\omega) \leq E_2^\Lambda(\omega) \leq \cdots \leq E_N^\Lambda(\omega)$

For $E_0 \in \Sigma$, set $I_\Lambda = [E_0 - \frac{1}{|\Lambda|^{1-\epsilon}}, E_0 + \frac{1}{|\Lambda|^{1-\epsilon}}]$ and $n(E_0)$ is the DOSf.

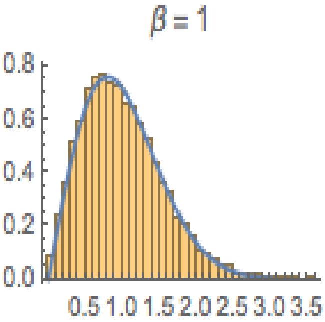
$$LSD_\omega^\Lambda(x; I_\Lambda) = \frac{\#\{j \mid E_j^\Lambda(\omega) \in I_\Lambda, \ \ |\Lambda| n(E_0)(E_{j+1}^\Lambda(\omega) - E_j^\Lambda(\omega)) \geq x\}}{\#\{j \mid E_j^\Lambda(\omega) \in I_\Lambda\}}$$

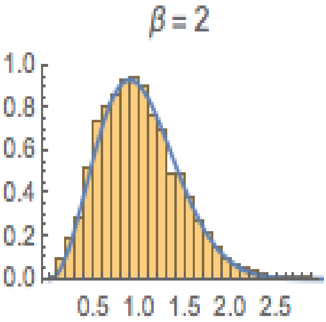
$$LSD(x) = \lim_{|\Lambda| \to \infty} LSD_\omega^\Lambda(x; I_\Lambda).$$

Behavior of $LSD(x)$ depends on E_0 in *localized* or *delocalized* regime.

LES overview

Poisson: Density of the $LSD(s)$ is exponential: $P(s) = e^{-s}$.
Random mstrix theory: Density of $LSD(s)$ follows the Wigner surmise:
$P(s) = As^{\beta}e^{-Bs^2}$, $\quad A, B > 0$ ($\beta = 1$ GOE, $\beta = 2$ GUE).





LES Results

Random Schrödinger operators on $\mathbb{Z}^d$ with $E_0 \in \Sigma^{\mathrm{CL}}$

- Minami: LES ξ_ω is a Poisson point process.
- Germinet-Klopp: LES for eigenvalues is Poisson and LSD with exponential density.

Random Schrödinger operators on $\mathbb{R}^d$ with $E_0 \in \Sigma^{\mathrm{CL}}$

- Hislop-Krishna: LES always have limit points ξ_ω that are compound Poisson processes.
- Hislop, Kirsch, Krishna: random Schrödinger operators with δ-interactions, LES is Poisson.
- Dietlein-Elgart: Anderson-type random Schrödinger operators has LES Poisson at the bottom of the spectrum.

Example 1: Scaled disorder in one-dimension

Scaled disorder random Anderson model:

$$H_\omega^{(n)} = L^{(n)} + \sum_{j=-n}^{n} \frac{\sigma \omega_j}{\langle n \rangle^\alpha} \Pi_j, \quad \mathcal{H} = \ell^2([-n, n]).$$

$L^{(n)}$: Finite difference Laplacian on $[-n, n]$ with simple boundary conditions

$(\Pi_j f)(k) = f(j) \delta_{jk}$ and $\sigma > 0$

Localization length: $\gamma_n \sim \frac{n^{2\alpha}}{\sigma^2}$

Scaling ratio: $\frac{\gamma_n}{n} = \frac{n^{2\alpha - 1}}{\sigma^2}$

Scaled disorder in one dimension

Transition in LES depending on α (Kritchevski, Valkò, Viràg, Kotani, Nakano, H-Klopp)

Scaling regimes:

$$0 \leq \alpha < \frac{1}{2} \qquad \frac{\gamma_n}{n} \to 0 \qquad LES = Poisson$$

$$\alpha = \frac{1}{2} \qquad \frac{\gamma_n}{n} = 1 \qquad critical$$

$$\frac{1}{2} < \alpha \quad 1 \leq \frac{\gamma_n}{n} \to \infty \quad LES = Clock$$

Clock is the LES of the Laplacian L on $\ell^2(\mathbb{Z})$.

Scaled disorder: Basic ideas in the localization regime

Localization regime: $0 \leq \alpha < \frac{1}{2}$

Scales: $0 \leq \alpha'' < \alpha < \alpha' < \frac{1}{2}$

Construction of local Hamiltonians $H_\omega^{(n')}$ on scale $n' = n^{2\alpha'}$: Domain decomposition: Localization length $\gamma_n \sim n^{2\alpha}$

Divide $\Lambda_n = [-n, n]$ into N' disjoint cells $\Lambda_{n'}(j)$, with $j = 1, \ldots, N'$, of length $n^{2\alpha'}$, capturing localized wave functions.

Conclusion: With probability close to one, each EV of $H_\omega^{(n)}$ in I is close to an EV of $H_\omega^{(n')}(j)$, for some j. EVs of $H_\omega^{(n')}(j)$ are **independent** of the EVs of $H_\omega^{(n')}(j')$, for $j \neq j'$.

Scaled disorder: Localization estimates on $H_\omega^{(n')}$

Decompose Λ_n into disjoint subcubes $\Lambda_{n'}(j)$

Center of Localization (COL): A point $j \in \Lambda$ is a COL for an eigenfunction φ of H_ω^Λ if $|\varphi(j)| = \max_{k \in \Lambda} |\varphi(k)|$.

1. Each COL of EV of $H_\omega^{(n)}$ in some $\Lambda_{n'}(j)$
2. At most one COL in each $\Lambda_{n'}(j)$
3. If COL in $\Lambda_{n'}(j)$ then $H_\omega^{(n')}$ has EV in $\tilde{I}_n$ near EV of Λ_n

This holds for a set of configurations $\mathcal{Z}_n$ so that

$$\mathbb{P}\{\mathcal{Z}_n\} \geq 1 - n^{-\beta}, \quad \beta \in (0,1)$$

Scaled disorder: localization estimates on $H_\omega^{(n')}$

The eigenvalues $E_j(\omega; \Lambda_{(n')}(j))$ are independent for different j and close to the EVs of $H_\omega^{(n)}$:

$$|E_{j_k}(\omega; n) - E_j(\omega; \Lambda_{n'}(j))| \leq e^{-n''}$$

Conclusion: So with good probability, all the eigenvalues of $H_\omega^{(n)}$ are described in terms of eigenvalues of the local Hamiltonians $H_\omega^{(n')}(j)$.

The LES of $H_\omega^{(n)}$ can be approximated by those of the independent family $H_\omega^{(n')}(j)$.

Scaled disorder: Basic idea in the localization regime

Let $\{X_j \mid j = 1, \ldots, N\}$ = family of independent Bernoulli random variables

- $\mathbb{P}\{X_j = 1\} = p_N$, for $0 < p_N < 1$;
- $\mathbb{P}\{X = 0\} = 1 - p_N$.

By choosing k points j from $\{1, \ldots, N\}$ we get

$$\mathbb{P}\{\#\{j \mid X_j = 1\} = k\} = \left(\begin{array}{c} N \\ k \end{array} \right) p_N^k (1 - p_N)^{N-k}.$$

Poisson Limit theorem: If $\lim_{N \to \infty} p_N N = \lambda > 0$ then

$$\lim_{N \to \infty} \left(\begin{array}{c} N \\ k \end{array} \right) p_N^k (1 - p_N)^{N-k} = e^{-\lambda} \left(\frac{\lambda^k}{k!} \right).$$

Scaled disorder: LES for $H_\omega^{(n)}$

Local Hamiltonians: $H_\omega^{(n')} = H_\omega^{(n)} | \Lambda_{n'}$

Definition: Bernoulli random variable:

$$X_{I, \Lambda_{n'}(j)} := \begin{cases} 1 & \mathrm{Tr} E_{H_\omega^{n'}(j)}(I) = 1 \\ 0 & \text{other} \end{cases}$$

This family is *independent*. Main calculation:

$$\mathbb{P}\{\#\{j \mid X_{I, \Lambda_{n'}(j)} = 1\} = k\}$$

requires

$$\lim_{n \to \infty} \left(\frac{n}{n^{2\alpha'}} \right) \mathbb{P}\{X_{I, \Lambda_{n'}(j)} = 1\} = n_0(E_0)$$

where $n_0(E_0)$ is the DOSf of the Laplacian at energy E_0.
Conclusion: Apply Poisson Limit Theorem to get Poisson statistics with intensity measure $n_0(E_0) ds$.

Example 2: Random band matrices (RBM)

Random band matrices (RBM): Interpolation between Wigner matrices of the Gaussian Orthogonal Ensemble (GOE) and RSO.

Finite-size RBM: H_L^N, $\quad (2N+1) \times (2N+1)$ real random band matrix

Bandwidth: $W = 2L + 1$.

H_L^N has matrix elements:

$$\langle e_i, H_L^N e_j \rangle \quad = \quad \frac{1}{\sqrt{W}} \left\{ \begin{array}{lll} v_{ij} & \text{if} & |i-j| \leq L \\ 0 & \text{if} & |i-j| > L \end{array} \right. ,$$

with

$$-N \leq i, j \leq N.$$

Overview of RBM

Random variables : $v_{ij} = v_{ji}$, within the band, independent and identically distributed (up to symmetry).

Example: v_{ij} Gaussian distributed, $\mathbb{E}\{v_{ij}\} = 0$ and $\mathrm{Var}\{v_{ij}\} = \mathbb{E}\{v_{ij}^2\} = 1$

Two extremes:

1. $W = 3$. Disorder only on the diagonal–1D lattice random Schrödinger operator: *Complete localization*
2. $W = 2N + 1$. GOE: *Complete delocalization*

Overview of RBM

Interpolating models: RBM $W = N^{\alpha}$, with $0 \leq \alpha \leq 1$

CONJECTURES:

1. For $0 \leq \alpha < \frac{1}{2}$: then
   1. The limiting point process ξ_{ω} is a Poisson point process;
   2. The eigenvectors are exponentially localized and the spectrum is pure point almost surely pure point.

2. For $\frac{1}{2} \leq \alpha \leq 1$: then
   1. The limiting point process ξ_{ω} is that of the GOE;
   2. The eigenvectors are spatially extended.

The model exhibits a *Localization–Delocalization transition*

RBM: Delocalization regime

Delocalization regime better understood than Localization regime

Main results: $W > N^{\frac{3}{4}}$ (Bourgade, Yau, Yin)

1. Eigenvectors extended: If $H_W^N \psi = E\psi$, then $\|\psi_E\|_\infty^2 \leq N^{-1+\tau}$ for most eigenvectors with probability $> 1 - N^{-D}$

2.Semi-circle law: IDS $N(E) \sim \frac{1}{2\pi} \int_{-2}^{E} \sqrt{(4-s^2)_+}\, ds$, with good probability

3. GOE eigenvalue statistics : k-point correlation functions $\rho_N^{(k)}$ converge to $\rho_{\mathrm{GOE}}^{(k)}$

RBM: Localization regime

$L \sim N^{\alpha}$, for $0 \leq \alpha < \frac{1}{2}$. Since $d = 1$, there is localization at all energies.

3 finite-N estimates required for H_L^N (Peled, Schenker, Shamis, Sodin, Brodie-H):

1. Wegner estimate
2. Minami estimate
3. Localization bounds

Basic Technique: Average over diagonal random variables v_{jj}, $-N \leq j \leq N$.

RBM: Localization regime

For $m = 1$, Wegner estimate:

$$\mathbb{E}\,\{\mathrm{tr}\, E_H(I)\} \leq C_W L^{1/2}(2N+1)|I|$$

For $m = 2$, Minami estimate:

$$\mathbb{E}\,\{\mathrm{tr}\, E_H(I)\,(\mathrm{tr}\, E_H(I) - 1)\} \leq C_M (L^{1/2}(2N+1)|I|)^2$$

Equivalent:

$$\mathbb{P}\{\mathrm{tr}\, E_H(I) \geq 2\} \leq C_M (L^{1/2}(2N+1)|I|)^2$$

Remainder: Focus on fixed-band width random matrices: L independent of N.

RBM: Spectral averaging

Warm-up: a priori bounds on resolvents via *Spectral Averaging*

Let H_ω self-adjoint matrix with its diagonal entries $[H_\omega]_{jj} := \omega_j$ *iid*, probability density function ρ with rapid decay

1. For any $0 < s < 1$, there exists a finite constant $C_{\rho,s} > 0$, independent of N, so that for any $z \in \mathbb{C}$, we have
$$\mathbb{E}\left\{ \left| \left\langle e_j, (H_\omega - z)^{-1} e_\ell \right\rangle \right|^s \right\} \leq C_{\rho,s}. \quad (1)$$
2. There exists a finite constant $C_\rho > 0$, independent of N, so that for any $z \in \mathbb{C}$,
$$\mathbb{E}\left\{ \Im \left\langle e_j, (H_\omega - z)^{-1} e_j \right\rangle \right\} \leq C_\rho. \quad (2)$$

RBM: Localization bounds

Theorem (Localization at all energies)

For any $s \in (0,1)$, and for all $z \in \mathbb{C}$, there exist finite constants $C_{s,L} > 0$ and $\alpha_{s,L} > 0$, depending on L and s (but uniform in $z \in \mathbb{C}$), such that

$$\mathbb{E}\left\{\left|\left\langle e_j, \left(H_L^N - z\right)^{-1} e_k\right\rangle\right|^s\right\} \leq C_{s,L} e^{-\alpha_{s,L}|j-k|}.$$

Proof combines Schenker's bound for $\lambda \in [-r, r] \subset \mathbb{R}$, localization bounds for $|\lambda| > R$ via Aizenman-Molchanov, and subharmonicity to extend to $\mathbb{C}$.

RBM:Density of states

Local density of states function $n_L^N(E)$:

$$n_L^N(E) = \frac{1}{\pi} \frac{1}{2N+1} \sum_{j=-N}^{N} \lim_{\epsilon \to 0} \mathbb{E}\{\Im \langle e_j, (H_L^N - E - i\epsilon)^{-1} e_j \rangle\}$$

Infinite N density of states function $n_L^\infty(E)$:
Iinfinite N operatorH_L^∞ is ergodic under translation in $\mathbb{Z}$ so:

$$n_L^\infty(E) = \lim_{\epsilon \to 0} \mathbb{E}\{\Im \langle e_0, (H_L^\infty - E - i\epsilon)^{-1} e_0 \rangle\}.$$

Theorem

Let H_L^N be a random symmetric band matrix with fixed bandwidth L and random iid entries (up to symmetry) having finite moments and a density satisfying $\hat{\rho}(\lambda) \sim \lambda^{-2}$. Then, for each $E \in \mathbb{R}$, we have

$$n_L^N(E) \longrightarrow n_L^\infty(E).$$

RBM: Poisson point process

Local eigenvalue statistics (LES): Rescaled eigenvalue point process

$$d\xi_L^N(s) := \sum_{j=-N,\ldots,N} \delta(N(E_j^{(N,L)}) - E) - s)\ ds$$

Let $I_N = E + \frac{1}{N}[-1.1]$

Main technical result:

$$\lim_{N\to\infty} \mathbb{P}\{\mathrm{tr}\, E_{I_N}(H_L^N)\} = n_L^\infty(E)|I|.$$

Given this, localization, Wegner and Minami:

Theorem

The LES for the fixed-width random band matrices H_N^L in the limit $N \to \infty$ is a Poisson point process with intensity measure $n_L^\infty(E)ds$.

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