Periodic Jacobi Matrices on Trees

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Joint Work with Nir Avni (Northwestern) and Jonathan Breuer (HUJI)

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and addenda also with Gil Kalai, Jacob Christensen and Maxim Zinchenko

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It is obvious that $\text{spec}(H) = \text{supp}(d\nu)$.

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3. ***(gap labelling)* k in any gap of the spectrum is a multiple of $1/\mathrm{p}$**. Again this goes back to the dawn of quantum mechanics although its important extension to the almost periodic case goes back to Johnson–Moser, Avron–Simon and Bellisard in the early 1980s.

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7. **These functions are hyperelliptic and, in particular, have only square root branch points and the surface is two sheeted.**

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These results follow by writing down an explicit quadratic equation for the m-functions and analyzing it using, in part, the monotonicity of G in gaps and the fact that poles of m correspond to zeros of G.

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Both Borg (1946) and Hochstadt (1984) proved their results for Hill's equation (i.e. continuum Schrödinger operators) but it is known to hold for the Jacobi case.

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We will most often consider regular graphs.

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Pick a base point, $j_0 \in \mathcal{T}$ and define the ball, Λ_r, as the set of all vertices in $\mathcal{T}$ with distance at most r from j_0.

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The definition of the DOS, $d\nu$ (and so IDS, k) is obvious. For each vertex, $J \in \mathcal{G}$, the spectral measure for H, $d\mu_j$ is the same for all $j \in \mathcal{T}$ with $\Xi(j) = J$. So the DOS is defined by picking one $d\mu_j$ for each $J \in \mathcal{G}$, summing over J and dividing by the number of vertices in $\mathcal{G}$.

Pick a base point, $j_0 \in \mathcal{T}$ and define the ball, Λ_r, as the set of all vertices in $\mathcal{T}$ with distance at most r from j_0. Because the number of boundary points in Λ_r is comparable to the total number of points in Λ_r, you **cannot** get $d\nu$ as a limit eigenvalue counting measures with free boundary conditions but

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We have another way involving suitable normal subgroups of the fundamental group to construct highly symmetric periodic BC objects whose eigenvalue counting measure converges. The details of both these results cannot be described in this brief talk!

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$$k(E) = \mathrm{Tr}(P_{(-\infty,E)}(H))$$

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Because in this discrete case, the trace can be normalized, he gets a full gap labelling result although nothing is noted explicitly.

Projections in $\mathcal{C}(\mathcal{T})$

Theorem (Sunada) *For a period p periodic Jacobi matrix on a tree, $k(E)$ in any gap has a value which is a multiple of $1/p$. This implies the spectrum has at most p bands.*

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Given the above formula for k, this result is a corollary of

Theorem (Sunada) *If Ξ is a covering map from a tree to a graph with p vertices, then for any projection, $P \in \mathcal{C}(\mathcal{T}, \Xi)$, its normalized trace, $\mathrm{Tr}(P)$, is a multiple of $1/p$.*

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G and M

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$$G_j(z) = \langle \delta_j, (H-z)^{-1}\delta_j\rangle \qquad m_j^\alpha = \langle \delta_j, (H(\mathcal{T}_j^\alpha)-z)^{-1}\delta_j\rangle$$

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Schur Complements

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$$L = \begin{pmatrix} \mathbf{1} & 0 \\ -Y^{-1}Z^* & \mathbf{1} \end{pmatrix} \text{ so } L^{-1} = \begin{pmatrix} \mathbf{1} & 0 \\ Y^{-1}Z^* & \mathbf{1} \end{pmatrix}$$

Schur Complements

A simple calculation shows that

$$L^*NL = \begin{pmatrix} S & 0 \\ 0 & Y \end{pmatrix} \tag{4.1}$$

so

$$\begin{aligned} N^{-1} &= L \begin{pmatrix} S^{-1} & 0 \\ 0 & Y^{-1} \end{pmatrix} L^* \\ &= \begin{pmatrix} S^{-1} & -S^{-1}ZY^{-1} \\ -Y^{-1}Z^*S^{-1} & Y^{-1} + Y^{-1}Z^*S^{-1}ZY^{-1} \end{pmatrix} \end{aligned}$$

which proves Banachiewicz' formula $(N^{-1})_{11} = S^{-1}$

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Similarly, if $\beta = (rj)$ is an edge in $\mathcal{T}$, we have that

$$m_j^\beta(z) = \frac{1}{-z + b_j - \sum_{\alpha=(jk);\, k\neq r} a_\alpha^2 m_k^\alpha(z)}$$

Note that if q is the number of edges in the underlying graph, $\mathcal{G}$, then there are $2q$ m–functions.

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an analog of a well known formula from the 1D case.

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the famed Kesten–McKay distribution, which arose first in random graph models.

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If $b \neq 0$, there is a single gap which is always open.

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If $c = 0$, the problem breaks into disjoint 1D chains. So as c varies from 0 to a, the DOS goes from $d = 2$ Kesten McKay (i.e. 1D free) to $d = 4$ Kesten McKay. By adjusting, a in a c dependent way, one can get degree 4 examples with spectrum $[-2, 2]$ and different DOS. So the lovely property in 1D that the spectrum determines the DOS does not extend to trees!!!

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so there is an eigenvalue at $z = 0$! Notice that since $r \neq g$, the red and green vertices have different degrees and the corresponding tree is not regular. Rather than rely on this argument of Aomoto, we can write eigenvectors explicitly.

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Problem 1 *Find a simpler proof of the above bound state theorem of Aomoto.*

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I want to explain our more explicit form of Green's function regularity.

A Theorem on Algebraic Varieties

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Most importantly. we need the following which can be found, for example, in Lang's book *Algebra*:

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Application to Periodic m–functions

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Absence of Singular Continuous Spectrum

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Corollary *Periodic Jacobi matrices on trees have no singular continuous spectrum.*

Green's Functions

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Conjecture 2. *The periodic eigenvalue with positive eigenfunction is an anti-bound state, i.e. pole on a non-principle sheet*

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Problem 4 *Prove a Hochstadt type theorem for general periodic trees with this improved definition of period.*

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We at least know the set is non-empty, for if all b are different and $\sum a < \min_{i\neq j} |b_i - b_j|$, then all gaps are open.

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Problem 5 *Find an effective description of gap edges.*

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Problem 6 *Classify the possible DOS allowed for a given tree, period and set.*

IsoDOS sets

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Problem 8 *Is there an natural flow on the IsoDOS set?*

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Relevant Representations

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Problem 11 *Determine if the direct integral decomposition is of any use in spectral analysis. In particular, do gap edges have to do with particular irreps?*

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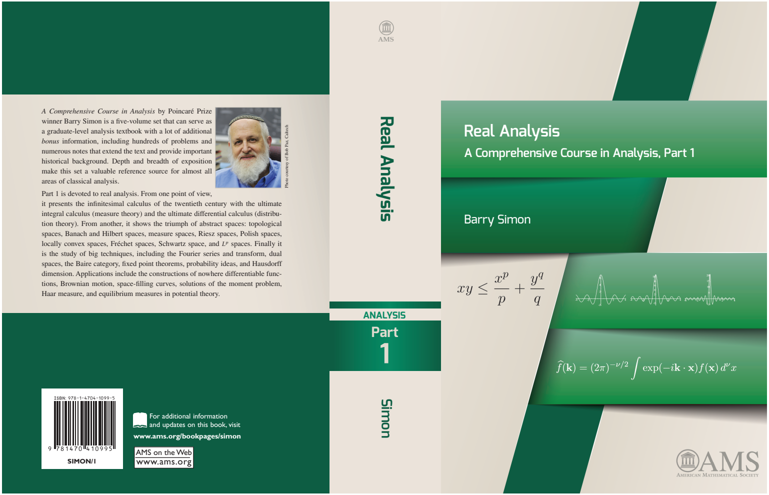

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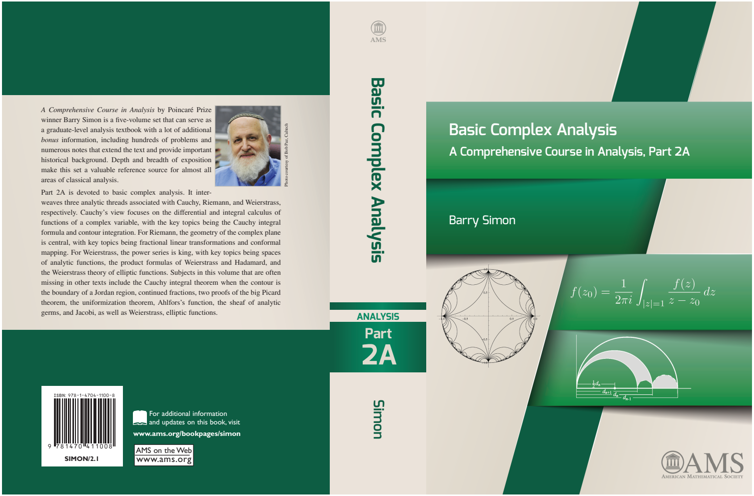

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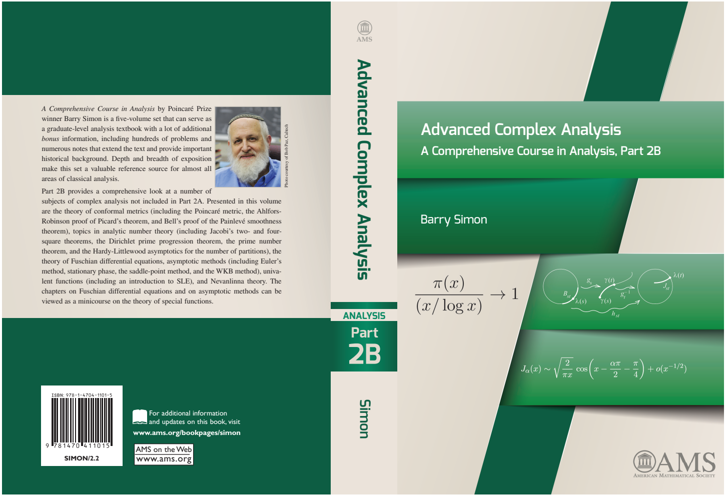

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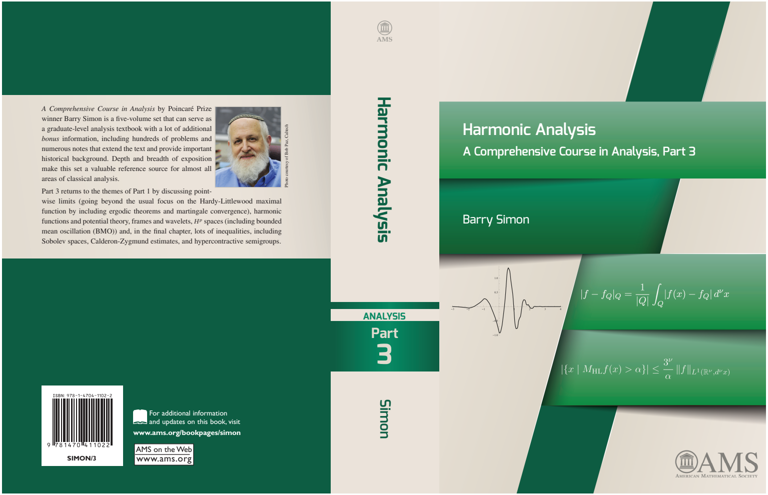

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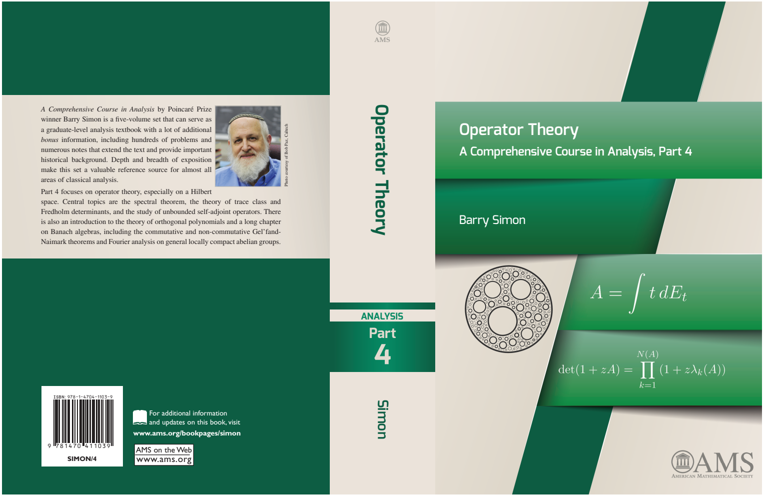

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