Homogeneous Operators in the Cowen-Douglas Class

Subrata Shyam Roy

Thesis submitted to Indian Statistical Institute
in partial fulfilment of the requirements
for the award of the degree of
Doctor of Philosophy
Bangalore
2008

ACKNOWLEDGEMENT

The work on this thesis was done under the supervision of Professor Gadadhar Misra. I take this opportunity to thank him for his continuous guidance, active participation and stimulating discussions on various aspects of operator theory.

I am grateful to the Stat-Math faculty at the Indian Statistical Institute, especially to Professor S. C. Bagchi, Professor Rajarama Bhat, Professor T. S. S. R. K Rao, Professor A. Sitaram, Professor T. Bhattachayya, Professor Maneesh Thakur and Professor R. P. Sarkar for their constant encouragement.

My note of thanks goes to ISI and DST for providing me financial support, otherwise, it would have been very difficult for me to complete my thesis.

I specially thank Dr. Swagato Ray, Dr. Sudipta Dutta and Dr. Amit Kulshrestha for their encouragement and cooperation. I would also like to thank Shibanada Biswas, Arnab Mitra and Mithun Mukherjee for helping me to survive in this isolated place. I thank all my friends as well as the members of Stat-Math Unit in Bangalore Centre for their cooperation.

Finally, I thank my teacher B. K. Berry at Presidency College, Calcutta for motivating me to do mathematics and giving me constant mental support. I cannot express my gratitude to him in words.

I dedicate my thesis to my parents and Suparna.

CONTENTS

NOTATION

$\mathbb{N}$	the set of positive integers
$\mathbb{Z}$	the set of all integers
$\mathbb{Z}^+$	the set of non-negative integers
$\mathbb{C}$	the complex plane
$\mathbb{T}$	the group $\{z \in \mathbb{C} : \lvert z\rvert = 1\}$
$\mathbb{D}$	the unit disc in $\mathbb{C}$
$(x)_n$	$(x)_0 = 1, (x)_n = x(x+1)\dots(x+n-1), n \geq 1$ is the Pochhammer symbol
$f^{(k)}$	k-th order derivative of the function f
$\mathcal{M}_n, \mathbb{C}^{n\times n}$	the set of $n \times n$ complex matrices
$\mathcal{M}_{m,n}$	the set of $m \times n$ complex matrices
X^{tr}	transpose of the matrix X
X^*	conjugate transpose of the matrix X
$\varphi_{t,a}$	$\varphi_{t,a}(z) = t\frac{z-a}{1-\bar{a}z}$ for $(t,a) \in \mathbb{T}\times\mathbb{D}, z \in \mathbb{D}$
Möb	$\{\varphi_{t,a} : (t,a) \in \mathbb{T}\times\mathbb{D}\}$, the group of biholomorphic automorphisms of $\mathbb{D}$
$\mathrm{B}_n(\Omega)$	the class of Cowen-Douglas operators, $n \geq 1$
E_T	the Hermitian holomorphic vector bundle associated with an operator $T \in \mathrm{B}_n(\Omega)$
$\partial, \bar{\partial}$	$\partial = \frac{\partial}{\partial z}, \bar{\partial} = \frac{\partial}{\partial \bar{z}}$
$\partial_i, \bar{\partial}_i$	$\partial_i = \frac{\partial}{\partial z_i}, \bar{\partial}_i = \frac{\partial}{\partial \bar{z}_i}$, $i = 1,2,3$
$\mathcal{K}_T$	curvature of the bundle E_T
$\mathcal{K}_h$	curvature of E_T with respect to the metric h, $\mathcal{K}_T(z) = \frac{\partial}{\partial \bar{z}}(h^{-1}\partial h)(z)$
$(\mathcal{K}_T)_{\bar{z}}$	covariant derivative of curvature of order $(0,1)$
$(\mathcal{K}_T)_{z\bar{z}}$	covariant derivative of curvature of order $(1,1)$
$\tilde{K}$	normalization of the reproducing kernel K
$\tilde{a}_{mn}$	coefficient of power series for $\tilde{K}$ around the point of normalization
$\tilde{\mathcal{K}}$	curvature with respect to the metric $\tilde{h}$, where $\tilde{h}(z) = \tilde{K}(z,z)^{tr}$
$\mathcal{A}(\Omega)$	natural function algebra over Ω, for Ω open, connected, bounded subset of $\mathbb{C}^m$
$\mathbb{A}^{(\alpha)}(\mathbb{D})$	Hilbert space of holomorphic functions with reproducing kernel $(1-z\bar{w})^{-\alpha}, \alpha > 0$
$M^{(\alpha)}$	multiplication operator on $\mathbb{A}^{(\alpha)}(\mathbb{D})$
D_α^+	the discrete series representation of Möb which lives on $\mathbb{A}^{(\alpha)}(\mathbb{D})$
$\triangle$	$\{(z,z) : z \in \mathbb{D}\} \subseteq \mathbb{D}^2$ or $\{(z,z,z) : z \in \mathbb{D}\} \subseteq \mathbb{D}^3$
$\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$	$\mathbb{A}^{(\alpha)}(\mathbb{D}) \otimes \mathbb{A}^{(\beta)}(\mathbb{D})$ identified as space of functions of two variables

$\mathbb{A}_k^{(\alpha,\beta)}(\mathbb{D}^2)$	$\{f \in \mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) : \partial_2^i f_{\|\triangle} = 0 \text{ for } 0 \le i \le k\}, k \ge 1$
$\mathbb{A}_{k\ \mathrm{res}}^{(\alpha,\beta)}(\mathbb{D}^2)$	$\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) \ominus \mathbb{A}_k^{(\alpha,\beta)}(\mathbb{D}^2)$
$J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{\|\mathrm{res}\triangle}$	realization of $\mathbb{A}_{k\ \mathrm{res}}^{(\alpha,\beta)}(\mathbb{D}^2)$ as a Hilbert space of $\mathbb{C}^{k+1}$-valued functions on $\mathbb{D}$
$B_k^{(\alpha,\beta)}$	reproducing kernel for the Hilbert space $J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{\|\mathrm{res}\triangle}$
$M_k^{(\alpha,\beta)}$	multiplication operator on the Hilbert space $J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{\|\mathrm{res}\triangle}$
$\mathcal{W}_k$	$\{M_k^{(\alpha,\beta)} : \alpha, \beta > 0\}$, $k \ge 1$, the “generalized Wilkins’ operators”
$\mathbb{S}(z)$	$(1-\|z\|^2)^{-1}$
$A(i,j)$	(i,j)-th entry of the $m \times n$ matrix A
$\boldsymbol{v}(i)$	i-th component of the vector $\boldsymbol{v}$ in $\mathbb{C}^k$
$c(\varphi^{-1}, z)$	$(\varphi^{-1})'(z)$ for $\varphi = \varphi_{t,a} \in$ Möb, $z \in \mathbb{D}$
$p(\varphi^{-1}, z)$	$\frac{\overline{ta}}{1+\overline{ta}z}$; for $\varphi = \varphi_{t,a} \in$ Möb, $z \in \mathbb{D}$
$\boldsymbol{\mu}$	column vector with $\boldsymbol{\mu}(j) = \mu_j$ and $1 = \mu_0, \mu_1, \ldots \mu_m > 0$
$\delta_{i,j}$	Kronecker Delta
$S((c_i)_{i=1}^m)$	$S((c_i)_{i=1}^m)(\ell, p) = c_\ell \delta_{p+1,\ell}, c_i \in \mathbb{C}$ for $0 \le p, \ell \le m$
S_β	$S(i(\beta+i-1)_{i=1}^m)$
$\mathbb{S}_m$	$S((i)_{i=1}^m)$
$\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$	Hilbert space depending on $\lambda > \frac{m}{2}$ and $\boldsymbol{\mu}$ for $m \in \mathbb{N}$
$\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$	reproducing kernel for the Hilbert space $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$
$M^{(\lambda,\boldsymbol{\mu})}$	multiplication operator on the Hilbert space $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$
$E^{(\lambda,\boldsymbol{\mu})}$	bundle associated with the operator $M^{(\lambda,\boldsymbol{\mu})*}$
$\mathcal{K}^{(\lambda,\boldsymbol{\mu})}(z)$	curvature of $E^{(\lambda,\boldsymbol{\mu})}$ with respect to the metric $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(z,z)^{\mathrm{tr}}$
$\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(z)$	curvature of $E^{(\lambda,\boldsymbol{\mu})}$ with respect to the metric $\tilde{\mathbf{B}}^{(\lambda,\boldsymbol{\mu})}(z,z)^{\mathrm{tr}}$
S_n	symmetric group of degree n
ρ, τ	$\rho, \tau \in S_3$ such that $\rho(1) = 2, \rho(2) = 1, \rho(3) = 3$ and $\tau(1) = 1, \tau(2) = 3, \tau(3) = 2$

0. OVERVIEW

The classification of bounded linear operators up to unitary equivalence is not an entirely tractable problem. However, the spectral theorem provides a complete set of unitary invariants for normal operators. There are only a few other instances where such a complete classification is possible. In a foundational paper [18], Cowen and Douglas initiated the study of a class of operators T possessing an open set Ω of eigenvalues. Such an operator cannot be normal on a separable Hilbert space. The class of all such operators is denoted by $\mathrm{B}_n(\Omega)$, where the dimension of the kernel of $T - w$ for $w \in \Omega$, which is assumed to be constant, is n. They associate a Hermitian holomorphic vector bundle E_T on Ω to the operator T in $\mathrm{B}_n(\Omega)$. One of the main results of [18] says that T and $\tilde{T}$ in $\mathrm{B}_n(\Omega)$ are unitarily equivalent if and only if E_T and $E_{\tilde{T}}$ are equivalent as Hermitian holomorphic vector bundles. Moreover, they provide a complete set of unitary invariants for an operator T in $\mathrm{B}_n(\Omega)$, namely, the simultaneous unitary equivalence class of the curvature and its covariant derivatives up to a certain order of the corresponding bundle E_T. While these invariants are not easy to compute in general, it may be reasonable to expect that they are tractable for some appropriately chosen family of operators. Over the last few years, it has become evident that one such family is the class of homogeneous operators. Several constructions of homogeneous operators are known. One such construction is via the *jet construction* of [24], see also, [46, 29]. It was observed in [9] that all the homogeneous operators in $\mathrm{B}_2(\mathbb{D})$ which were described in [51] arise from the jet construction. This naturally leads to a two parameter family of "generalized Wilkins operators" in $\mathrm{B}_k(\mathbb{D})$ for $k \geq 2$. We show that it is possible to construct, starting with the jet construction, a much larger class of homogeneous operators via a simple similarity. Indeed, the class of homogeneous operators obtained this way coincides with the homogeneous operators which were recently constructed in [31]. Using the explicit description of these operators and the homogeneity, we answer, in part, a question of Cowen and Douglas [18, page. 214].

Let $\text{Möb} := \{\varphi_{t,\alpha} : t \in \mathbb{T} \text{ and } \alpha \in \mathbb{D}\}$ be the group of bi-holomorphic automorphisms of the unit disc $\mathbb{D}$, where

$$\varphi_{t,\alpha}(z) = t\frac{z-\alpha}{1-\bar{\alpha}z}, \quad z \in \mathbb{D}.$$

As a topological group (with the topology of locally uniform convergence) it is isomorphic to $\mathrm{PSU}(1,1)$ and to $\mathrm{PSL}(2,\mathbb{R})$.

An operator T from a Hilbert space into itself is said to be *homogeneous* if $\varphi(T)$ is unitarily equivalent to T for all φ in Möb which are analytic on the spectrum of T. The spectrum of a homogeneous operator T is either the unit circle $\mathbb{T}$ or the closed unit disc $\bar{\mathbb{D}}$, so that, actually,

$\varphi(T)$ is unitarily equivalent to T for all φ in Möb.

We say that a projective unitary representation U of Möb is *associated* with an operator T if

$$\varphi(T) = U_\varphi^* T U_\varphi$$

for all φ in Möb. If T has an associated representation then it is homogeneous. Conversely, if a homogeneous operator T is irreducible then it has an associated representation U (cf. [10, Theorem 2.2]). It is not hard to see that U is uniquely determined up to unitary equivalence. The ongoing research of B. Bagchi and G. Misra has established that the associated representation, in case of an irreducible cnu (completely non unitary) contraction, lifts to the dilation space and intertwines the dilations of T and $\varphi(T)$. What is more, they have found an explicit formula for this lift and for a cnu irreducible homogeneous contraction, they have also found a product formula for the Sz.- Nagy–Foias characteristic function. A related question involves Möbius invariant function spaces [1, 2, 3, 43, 44, 45].

The first examples of homogeneous operators were given in [32, 34]. These examples also appeared in the the work of Berezin in describing what is now known as "Berezin quantization" [11]. This was followed by a host of examples [7, 10, 35, 51]. The homogeneous scalar shifts were classified in [10]. However, the classification problem, in general, remains open.

Many examples (unitarily inequivalent) of homogeneous operators are known [9]. Since the direct sum (more generally direct integral) of two homogeneous operators is again homogeneous, a natural problem is the classification (up to unitary equivalence) of *atomic homogeneous operators*, that is, those homogeneous operators which cannot be written as the direct sum of two homogeneous operators. However, the irreducible homogeneous operators in the Cowen-Douglas class $\mathrm{B}_1(\mathbb{D})$ and $\mathrm{B}_2(\mathbb{D})$ have been classified (cf. [34] and [51]) and all the scalar shifts (not only the irreducible ones) which are homogeneous are known. Clearly, irreducible homogeneous operators are atomic. Therefore, it is important to understand when a homogeneous operator is irreducible.

There are only two examples of atomic homogeneous operators known which are not irreducible. These are the multiplication operators – by the respective co-ordinate functions – on the Hilbert spaces $L^2(\mathbb{T})$ and $L^2(\mathbb{D})$. Both of these examples happen to be normal operators. We do not know if all atomic homogeneous operators possess an associated projective unitary representation. However, to every homogeneous operator in $\mathrm{B}_k(\mathbb{D})$, there exists an associated representation of the universal covering group of Möb [30, Theorem 4].

It turns out that an irreducible homogeneous operator in $\mathrm{B}_2(\mathbb{D})$ is the compression of an operator of the form $T \otimes I$, for some homogeneous operator T in $\mathrm{B}_1(\mathbb{D})$ (cf. [9]) to the ortho-complement of a suitable invariant subspace of $T \otimes I$. In the language of Hilbert modules, this is the statement that every homogeneous module in $\mathrm{B}_2(\mathbb{D})$ is obtained as quotient of a homogeneous modules in $\mathrm{B}_1(\mathbb{D}^2)$ by the sub-module of functions vanishing to order 2 on $\triangle \subseteq \mathbb{D}^2$, where $\triangle = \{(z,z) : z \in \mathbb{D}\}$. However, beyond the case of rank 2, the situation is more complicated. The question of classifying homogeneous modules in the class $\mathrm{B}_k(\mathbb{D})$ amounts to not only classifying

Hermitian holomorphic vector bundles of rank k on the unit disc which are homogeneous but also deciding that when they correspond to modules in $\mathrm{B}_k(\mathbb{D})$. Classification problems such as this one are well known in the representation theory of locally compact second countable groups. However, in that context, there is no Hermitian structure present which makes the classification problem entirely algebraic. A complete classification of homogeneous modules in $\mathrm{B}_k(\mathbb{D})$ may still be possible using techniques from the theory of unitary representations of the Möbius group. Leaving aside, the classification problem of the homogeneous operators in $\mathrm{B}_k(\mathbb{D})$, we proceed to show that the "generalized Wilkins examples" (cf. [9]) are irreducible in section 2.1 and [37]. A trick involving a simple change of inner product in the "generalized Wilkins examples", we construct a huge family of homogeneous modules in $\mathrm{B}_{k+1}(\mathbb{D})$. These are shown to be exactly the same family given in the recent paper of Koranyi and Misra [31].

Many of these results can be recast, following R. G. Douglas and V. I. Paulsen [25], in the language of Hilbert modules. A Hilbert module is just a Hilbert space on which a natural action of an appropriate function algebra is given.

Let $\mathcal{M}$ be a complex and separable Hilbert space. Let $\mathcal{A}(\Omega)$ be the natural function algebra consisting of functions holomorphic in a neighborhood of the closure $\bar{\Omega}$ of some open, connected and bounded subset Ω of $\mathbb{C}^m$. The Hilbert space $\mathcal{M}$ is said to be a *Hilbert module* over $\mathcal{A}(\Omega)$ if $\mathcal{M}$ is a module over $\mathcal{A}(\Omega)$ and

$$\|f \cdot h\|_{\mathcal{M}} \leq C\|f\|_{\mathcal{A}(\Omega)}\|h\|_{\mathcal{M}} \text{ for } f \in \mathcal{A}(\Omega) \text{ and } h \in \mathcal{M},$$

for some positive constant C independent of f and h. It is said to be *contractive* if we also have $C \leq 1$.

Fix an inner product on the algebraic tensor product $\mathcal{A}(\Omega) \otimes \mathbb{C}^n$. Let the completion of $\mathcal{A}(\Omega) \otimes \mathbb{C}^n$ with respect to this inner product be the Hilbert space $\mathcal{M}$. A Hilbert module is obtained if this action

$$\mathcal{A}(\Omega) \times \big(\mathcal{A}(\Omega) \otimes \mathbb{C}^n\big) \to \mathcal{A}(\Omega) \otimes \mathbb{C}^n$$

extends continuously to $\mathcal{A}(\Omega) \times \mathcal{M} \to \mathcal{M}$.

The simplest family of modules over $\mathcal{A}(\Omega)$ corresponds to evaluation at a point in the closure of Ω. For $\boldsymbol{z}$ in the closure of Ω, we make the one-dimensional Hilbert space $\mathbb{C}$ into the Hilbert module $\mathbb{C}_{\boldsymbol{z}}$, by setting $\varphi v = \varphi(\boldsymbol{z})v$ for $\varphi \in \mathcal{A}(\Omega)$ and $v \in \mathbb{C}$. Classical examples of contractive Hilbert modules are the Hardy and Bergman modules over the algebra $\mathcal{A}(\Omega)$.

Let G be a locally compact second countable group acting transitively on Ω. Let us say that the module $\mathcal{M}$ over the algebra $\mathcal{A}(\Omega)$ is homogeneous if

$$\varrho(f \circ \varphi) \cong \varrho(f) \text{ for all } \varphi \in G,$$

where $\cong$ stands for "unitary equivalence". (This is the imprimitivity relation of Mackey.) Here $\varrho : \mathcal{A}(\Omega) \to \mathcal{L}(\mathcal{M})$ is the homomorphism of the algebra $\mathcal{A}(\Omega)$ defined by $\varrho(f)h := f \cdot h$ for

$f \in \mathcal{A}(\Omega)$ and $h \in \mathcal{M}$. Here $\mathcal{L}(\mathcal{M})$ is the algebra of bounded linear operators on $\mathcal{M}$. In the particular case of the unit disc $\mathbb{D}$, it is easily seen that a Hilbert module $\mathcal{M}$ is homogeneous if and only if the multiplication by the coordinate function defining the module action for the function algebra $\mathcal{A}(\mathbb{D})$ is a homogeneous operator.

We point out that the notion of a "system of imprimitivity" which is due to Mackey is closely related to the notion of homogeneity – a system of imprimitivity corresponds to a homogeneous normal operator, or equivalently, a homogeneous Hilbert module over a C^* - algebra. As one may expect, if we work with a function algebras rather than a C^* - algebra, we are naturally lead to a homogeneous Hilbert module over this function algebra. A $*$ - homomorphism ϱ of a C^* - algebra $\mathcal{C}$ and a unitary group representation U of G on the Hilbert space $\mathcal{M}$ satisfying the condition as above were first studied by Mackey [33] and were called *Systems of Imprimitivity.* Mackey proved the Imprimitivity theorem which sets up a correspondence between induced representations of the group G and the Systems of Imprimitivity. The notion of homogeneity is obtained, for instance, by taking $\mathcal{C}$ to be the algebra of continuous functions on the boundary $\partial\Omega$. However, in this case, the homomorphism ϱ defines a commuting tuple of normal operators. More interesting examples are obtained by compressing these to a closed subspace $\mathcal{N} \subseteq \mathcal{M}$ invariant under the representation U:

$$P_{\mathcal{N}}\varrho(f \circ \varphi)_{|\mathcal{N}} = U(\varphi^{-1})^*_{|\mathcal{N}}\big(P_{\mathcal{N}}\varrho(f)_{|\mathcal{N}}\big)U(\varphi^{-1})_{|\mathcal{N}} \text{ for all } \varphi \in G,\ f \in \mathcal{A}(\Omega)$$

(cf. [6]). However, in the case of $\Omega = \mathbb{D}$, Clark and Misra [15] established the converse for a contraction as long as it is assumed to be irreducible. Clearly, a homogeneous operator T defines an imprimitivity over $\mathcal{A}(\Omega)$ via the map $\varrho(f) = f(T)$ for $f \in \mathcal{A}(\Omega)$ and vice-versa.

The notion of homogeneity is of interest not only in operator theory but it is also related to the inductive algebras of Steger-Vemuri [47], the Higher order Hankel forms [27, 28], the holomorphically induced representations and homogeneous holomorphic Hermitian vector bundles.

At a future date, we will consider a somewhat more general situation. Let $X \subseteq \mathbb{C}^m$ is a bounded connected open set. As usual, let $\mathcal{A}(X)$ be the function algebra consisting of continuous functions on the closure of X which are holomorphic on X. Let $\mathcal{M}$ be a Hilbert module over the algebra $\mathcal{A}(X)$ and Aut(X) be the group of bi-holomorphic automorphisms of X. It is easy to see that the systems of imprimitivity, as above, are in one-one correspondence with homogeneous Hermitian holomorphic vector bundles over X.

In the important special case that $X = G/K$ is a bounded symmetric domain (generalizing the disk and the ball), a number of examples of systems of imprimtivity $(\mathrm{Aut(X)}, \mathrm{X}, \mathcal{M})$ were given in [4, 7, 35, 46] and many of their properties are described in [4, 7]. In this case, the relationship between Hilbert quotient modules, Toeplitz C^*-algebras and harmonic analysis on the semi-simple Lie group $G = Aut\,(X)$ can be made quite explicit in the following way: Suppose Y is another bounded symmetric domain (of higher dimension) such that $X \subset Y$ is realized as the fixed point set under a reflection symmetry of Y (preserving the so-called Jordan structure).

An example is $Y = X \times X$, with $X \subset Y$ identified with the diagonal. The Hilbert quotient module $\mathcal{M}$ associated with this setting is induced by the ideal of holomorphic functions on Y which vanish (up to a certain order) on the linear subvariety X. This Hilbert module corresponds to a homogeneous vector bundle on X related to the so-called Jordan-Grassmann manifolds which are of current interest in algebraic geometry. Recent work along these lines [4, 7, 30, 31, 27, 28] shows the following features:

The Hilbert module $\mathcal{M}$ decomposes as a multiplicity-free sum of irreducible G - representations; moreover, the associated intertwining operators have an interesting combinatorial structure (related to multi-variate special functions).

In the paper [28], the explicit matrix representation for the two multiplication operators compressed to the quotient module is calculated. These are exactly the generalized Wilkins' operators discussed in [9]. One of the main points of this thesis is to construct a large family of new Hilbert modules from these quotient modules involving a simple modification of the inner product, which continue to be homogeneous. Moreover, for each generalized Wilkins' operator, there corresponds via this construction, a k-parameter family of homogeneous operators which are mutually similar but unitarily inequivalent. As result, a $(k+1)$-parameter family of mutually inequivalent homogeneous operator is produced.

In a recent preprint [31] Koranyi and Misra produce a large class of mutually inequivalent irreducible homogeneous operators all of which belong to the class $\mathrm{B}_n(\mathbb{D})$. The multiplier representation of the universal covering group of the Möbius group associated with such an operator is reducible and multiplicity free. A one-one correspondence between this class of operators and the $(k+1)$-parameter family of operators constructed above is established in this thesis.

It turns out that for $n = 2$ and 3, all the representations associated with an irreducible homogeneous operator in $\mathrm{B}_n(\mathbb{D})$ are multiplicity free. For $n = 4$, we construct an example of an irreducible homogeneous operator in $\mathrm{B}_4(\mathbb{D})$ such that the associated representation is not multiplicity free. In the decomposition of the associated representation of an irreducible homogeneous operator which irreducible representations occur and with what multiplicity appears to be an enticing problem.

Suppose T is a bounded linear operator on a Hilbert space $\mathcal{H}$ possessing an open set of eigenvalues, say Ω, with constant multiplicity 1. For $w \in \Omega$, let γ_w be the eigenvector for T with eigenvalue w. In a significant paper [18], Cowen and Douglas showed that for these operators T, under some additional mild hypothesis, one may choose the eigenvector γ_w to ensure that the map $w \mapsto \gamma_w$ is holomorphic. Thus the operator T gives rise to a holomorphic Hermitian vector bundle E_T on Ω. They proved that

(i) the equivalence class of the Hermitian holomorphic vector bundle E_T determines the unitary equivalence class of the operator T;

(ii) The operator T is unitarily equivalent to the adjoint of the multiplication by the coordinate function on a Hilbert space $\mathcal{H}$ of holomorphic functions on Ω^*. The point evaluation on $\mathcal{H}$ are shown to be bounded and locally bounded assuring the existence of a reproducing kernel function for $\mathcal{H}$.

From (i), as shown in [18], it follows that the curvature

$$\mathcal{K}(w) := \frac{\partial}{\partial w}\frac{\partial}{\partial \bar{w}} \log \|\gamma_w\|^2,\ w \in \Omega$$

of the line bundle E_T is a complete invariant for the operator T. On the other hand, following (ii), Curto and Salinas [21] showed that the normalized kernel

$$\tilde{K}(z,w) = K(w_0,w_0)^{1/2}K(z,w_0)^{-1}K(z,w)K(w_0,w)^{-1}K(w_0,w_0)^{1/2},\ z,\, w \in \Omega,$$

at $w_0 \in \Omega$ is a complete invariant for the operator T as well.

If the dimension of the eigenspace of the operator T at w is no longer assumed to be 1, then a complete set of unitary invariants for the operator T involves not only the curvature but a certain number of its covariant derivatives. The reproducing kernel, in this case, takes values in the $n \times n$ matrices $\mathcal{M}_n$, where n is the (constant) dimension of the eigenspace of the operator T at w. The normalized kernel, modulo conjugation by a fixed unitary matrix from $\mathcal{M}_n$, continues to provide a complete invariant for the operator T.

Unfortunately, very often, the computation of these invariants tend to be hard. However, there is one situation, where these computations become somewhat tractable, namely, if T is assumed to be homogeneous. One may expect that in the case of homogeneous operators, the form of the invariants, discussed above, at any one point will determine it completely. We illustrate this phenomenon throughout the section 4.1 and section 4.2. Homogeneous operators have been studied extensively over the last few years ([4, 7, 8, 9, 10, 12, 30, 31, 46, 51]). Some of these homogeneous operators correspond to a holomorphic Hermitian homogeneous bundle – as discussed above. Recall that a Hermitian holomorphic bundle E on the open unit disc $\mathbb{D}$ is homogeneous if every φ in Möb lifts to an isometric bundle map of E.

Although, the homogeneous bundles E on the open unit disc $\mathbb{D}$ have been classified in [12, 51], it is not easy to determine which of these homogeneous bundles E comes from a homogeneous operators. In [51], Wilkins used his classification to describe all the irreducible homogeneous operators of rank 2. In the paper [31], Koranyi and Misra gives an explicit description of a class of homogeneous bundles and the corresponding homogeneous operator. Thus making it possible for us to compute the curvature invariants for these homogeneous operators. Although, our main focus will be the computation of the curvature invariants, we will also compute the normalized kernel and explain the relationship between these two sets of invariants. Along the way, we give a partial answer to some questions raised in [18, 20].

For a bounded open connected set $\Omega \subseteq \mathbb{C}$ and $n \in \mathbb{N}$, let us recall that the class $\mathrm{B}_n(\Omega)$, introduced in [18], consists of bounded operators T with the following properties:

a) $\Omega \subset \sigma(T)$

b) $\operatorname{ran}(T - w) = \mathcal{H}$ for $w \in \Omega$

c) $\bigvee_{w\in\Omega} \ker(T - w) = \mathcal{H}$ for $w \in \Omega$

d) $\dim\ \ker(T - w) = n$ for $w \in \Omega$.

It was shown in [18, proposition 1.11] that the eigenspaces for each T in $\mathrm{B}_n(\Omega)$ form a Hermitian holomorphic vector bundle E_T over Ω, that is,

$$E_T := \{(w, x) \in \Omega \times \mathcal{H} : x \in \ker(T - w)\},\ \pi(w, x) = w$$

and there exists a holomorphic frame $w \mapsto \gamma(w) := (\gamma_1(w), \dots, \gamma_n(w))$ with $\gamma_i(w) \in \ker(T - w)$, $1 \leq i \leq n$. The Hermitian structure at w is the one that $\ker(T - w)$ inherits as a subspace of the Hilbert space $\mathcal{H}$. In other words, the metric at w is simply the grammian $h(w) = \big(\!\!\big(\langle\gamma_j(w), \gamma_i(w)\rangle\big)\!\!\big)_{i,j=1}^n$. The curvature $\mathcal{K}_T(w)$ of the bundle E_T is then defined to be $\frac{\partial}{\partial \bar{w}}\big(h^{-1}\frac{\partial}{\partial w}h\big)(w)$ for $w \in \Omega$ (cf. [50, pp. 78 – 79]).

Theorem 0.0.1. *[19, Page. 326] Two operators $T, \tilde{T}$ in $\mathrm{B}_1(\Omega)$ are unitarily equivalent if and only if $\mathcal{K}_T(w) = \mathcal{K}_{\tilde{T}}(w)$ for w in Ω.*

Thus, the curvature of the line bundle E_T is a complete set of unitary invariant for an operator T in $\mathrm{B}_1(\Omega)$. Although, more complicated, a complete set of unitary invariants for the operators in the class $\mathrm{B}_n(\Omega)$ is given in [18].

It is not hard to see (cf. [50, pp. 72]) that the curvature of a bundle E transforms according to the rule $\mathcal{K}(fg)(w) = (g^{-1}\mathcal{K}(f)g)(w)$, $w \in \Delta$, where $f = (e_1, ..., e_n)$ is a frame for E over an open subset $\Delta \subseteq \Omega$ and $g : \Delta \to GL(n, \mathbb{C})$ is a holomorphic change of frame. For a line bundle E, locally, the change of frame g is a scalar valued holomorphic function. In this case, it follows from the transformation rule for the curvature that it is independent of the choice of a frame. In general, the curvature of a bundle E of rank $n > 1$ depends on the choice of a frame. Thus the curvature $\mathcal{K}$ itself cannot be an invariant for the bundle E. However, the eigenvalues of $\mathcal{K}$ are invariants for the bundle E. More interesting is the description of a complete set of invariants given in [18, Definition 2.17 and Theorem 3.17] involving the curvature and the covariant derivatives

$$\mathcal{K}_{z^i\bar{z}^j},\ 0 \leq i \leq j \leq i + j \leq n, (i, j) \neq (0, n),$$

where rank of $E = n$. In a subsequent paper (cf. [20, page. 78]), by means of examples, they showed that fewer covariant derivatives of the curvature will not suffice to determine the class of the bundle E. These examples do not necessarily correspond to operators in the class $\mathrm{B}_n(\Omega)$. Recall that if a Hermitian holomorphic vector bundle E is the pullback of the tautological bundle defined over the Grassmannian $\mathcal{G}r(n, \mathcal{H})$ under the holomorphic map

$$t : \Omega \longrightarrow \mathcal{G}r(n, \mathcal{H}),\ t(w) = \ker(T - w),\ w \in \Omega$$

for some operator $T : \mathcal{H} \to \mathcal{H}$, $T \in \mathrm{B}_n(\Omega)$, then $E = E_T$ and we say that it corresponds to the operator T. On the other hand, for certain class of operators like the generalized Wilkins operators $\mathcal{W}_k := \{M_k^{(\alpha,\beta)} : \alpha,\, \beta > 0\} \subseteq \mathrm{B}_{k+1}(\mathbb{D})$ (cf. [9, page 428]) discussed in section 2.1 and [37], the unitary equivalence class of the curvature $\mathcal{K}$ (just at one point) determines the unitary equivalence class of these operators in $\mathcal{W}_k$. This is easily proved using the form of the curvature at 0 of the generalized Wilkins operators $M_k^{(\alpha,\beta)}$, namely, $\mathrm{diag}(\alpha, \cdots, \alpha, \alpha + (k+1)\beta + k(k+1))$ (cf. [37, Theorem 4.12]).

It is surprising that there are no known examples of operators $T \in \mathrm{B}_n(\Omega)$, $n > 1$, for which the set of eigenvalues of the curvature $\mathcal{K}_T$ is not a complete invariant. We construct some examples in [36] to show that one needs the covariant derivatives of the curvature as well to determine the unitary equivalence class of an operator $T \in \mathrm{B}_n(\Omega)$, $n > 1$. The inherent difficulty in finding such examples suggests the possibility that the complete set of invariants for an operator $T \in \mathrm{B}_n(\Omega)$ described in [18, 20] may not be the most economical. Although, in [18, 20], it is shown that for generic bundles, the set of complete invariants is much smaller and consists of the curvature and its covariant derivatives of order $(0,1)$ and $(1,1)$. However, even for generic bundles, it is not clear if this is the best possible. Indeed, we show that for a certain class of homogeneous operators corresponding to generic holomorphic Hermitian homogeneous bundles, the curvature along with its covariant derivative of order $(0,1)$ at 0 provides a complete set of invariants.

Here is a detailed description of the contents of the thesis:

Multiplication operators on functional Hilbert space and the Cowen-Douglas class

We discuss the multiplication operator on a Hilbert space $\mathcal{H}$ consisting of holomorphic functions on a bounded domain $\Omega \subseteq \mathbb{C}^m$. We assume that our Hilbert space $\mathcal{H}$ possesses a reproducing kernel K, that is, $K : \Omega \times \Omega \to \mathcal{M}_n$ which is

1. holomorphic in the first variable and anti-holomorphic in the second;
2. $K(\cdot, w)\xi$ is in $\mathcal{H}$ for $w \in \Omega$ and $\xi \in \mathbb{C}^n$;
3. it has the reproducing property:

$$\langle f, K(\cdot, w)\xi\rangle = \langle f(w), \xi\rangle, \text{ for } w \in \Omega,\ \xi \in \mathbb{C}^n.$$

In particular, the kernel K is positive definite. The important role that the kernel functions play in operator theory, representation theory, and theory of several complex variables is evident from the papers [5, 38, 41, 48, 49] which by no means a complete list. For most naturally occurring positive definite kernels the joint eigenspace of the m - tuple $\mathbf{M} = (M_1, \ldots, M_m)$ defines a holomorphic map, that is, the map $t : \Omega \to Gr(n, \mathcal{H})$

$$t : w \mapsto \cap_{k=1}^m \ker(M_k - w_k)^*,\ \ w \in \Omega,$$

is holomorphic. Here $Gr(n, \mathcal{H})$ denotes the Grassmannian of manifold of rank n, the set of all n-dimensional subspcaes of $\mathcal{H}$. Clearly, the holomorphy of the map t also defines a holomorphic Hermitian vector bundle E on Ω. A mild hypothesis on the kernel function [21] ensures that the commuting tuple of multiplication operators $\mathbf{M}$ is bounded. The adjoint $\mathbf{M}^*$ of the commuting tuple $\mathbf{M}$ is then said to be in the *Cowen-Douglas class* $\mathrm{B}_n(\Omega)$, where n is the dimension of the joint eigenspace $\cap_{k=1}^m \ker(M_k - w_k)^*$. One of the main theorems of [18] states that the equivalence class, as a Hermitian holomorphic vector bundle, of E and the unitary equivalence class of the operator $\mathbf{M}$ determines each other.

Quasi-invariant kernels, cocycle and unitary representations

We formulate the transformation rule for the kernel function under the action of the automorphism group of the domain Ω and the functional calculus for the operator $\mathbf{M}$ for automorphisms of the domain Ω. We assume that the action $z \mapsto g \cdot z$ of the automorphism group $\mathrm{Aut}(\Omega)$ is transitive. We show that if $\mathcal{H}$ is a Hilbert space possessing a reproducing kernel K then the following are equivalent. The positive definite kernel K transforms according to the rule

$$J(g, z)K(g.z, g.w)J(g, w)^* = K(z, w),\ z, w \in \Omega$$

and the map $U_g : f \mapsto J(g^{-1}, \cdot)f \circ g^{-1}$ is unitary. Furthermore, the map $g \mapsto U_g$ is a unitary representation if and only if J is a *cocyle*. Unitary representations of this form induced by a cocycle J are called *multiplier representations*. Recall that J is a cocyle if there exists a Borel map $J : \mathrm{Aut}(\Omega) \times \Omega \to \mathcal{M}_n$ satisfying the cocyle property:

$$J(g_1 g_2, z) = J(g_2, z)J(g_1, g_2.z) \text{ for } g_1, g_2 \in \mathrm{Aut}(\Omega) \text{ and } z \in \Omega.$$

A positive definite kernel K transforming according to the rule prescribed above with a cocycle J is said to be *quasi-invariant*.

In the body of the thesis, we will be forced to work with *projective unitary representations*. This involves some technical complications and nothing will be achieved by elaborating on them now.

Homogeneous operators and associated representations

It is not hard to see that if the kernel K is quasi-invariant then the operator tuple $\mathbf{M}$ is *homogeneous* in the sense that $g \cdot \mathbf{M}$ is unitarily equivalent to $\mathbf{M}$ for all g in $\mathrm{Aut}(\Omega)$. Here $g \cdot \mathbf{M}$ is defined using the usual holomorphic functional calculus and consequently, $g \cdot \mathbf{M}$ is the commuting tuple of multiplication operators $\mathbf{M}_g := (M_{g_1}, \ldots, M_{g_m})$, where

$$(M_{g_i} f)(z) = (g_i \cdot z)f(z),\ f \in \mathcal{H},\ z \in \Omega.$$

Indeed, it is easy to verify that $U_g^* \mathbf{M} U_g = g \cdot \mathbf{M}$. The representation U_g, in this case, is the *associated representation*.

The jet construction

For z, w in the unit disc $\mathbb{D}$, let $S(z,w) = (1 - z\bar{w})^{-1}$ be the Szëgo kernel. Among several other properties, the Szëgo kernel is characterized by its reproducing property for the Hardy space of the unit disc $\mathbb{D}$. Any positive ($\alpha > 0$) real power of the Szëgo kernel determines a Hilbert space, say, $\mathbb{A}^{(\alpha)}(\mathbb{D})$ whose reproducing kernel is S^α. A straightforward computation shows that not only the Szëgo kernel but all its positive real powers S^α, $\alpha > 0$ are quasi-invariant with respect to the group Möb, the automorphism group of the unit disc $\mathbb{D}$. A little more work shows that the corresponding multiplication operator $M^{(\alpha)}$ on the Hilbert space $\mathbb{A}^{(\alpha)}(\mathbb{D})$ is homogeneous and its adjoint belongs to the Cowen-Douglas class $\mathrm{B}_1(\mathbb{D})$. In fact, the associated representation is the familiar Discrete series representation $D^+_\alpha(\varphi^{-1}) : f \mapsto (\varphi')^{\alpha/2} f \circ \varphi$, $\varphi \in$ Möb . (We have to remember that unless α is an even integer, the map D^+_α is merely a projective representation.) It is not hard to see that $\{M^{(\alpha)} : \alpha > 0\}$ is the the complete list of homogeneous operators in $\mathrm{B}_1(\mathbb{D})$.

However, constructing homogeneous operators of rank > 1 seems to be somewhat difficult. There is no clear choice of a quasi-invariant kernel.

In a somewhat intriguing manner, Wilkins [51] was the first to construct explicit examples of all irreducible homogeneous operators in the Cowen-Douglas class $\mathrm{B}_2(\mathbb{D})$. (We observe that a homogeneous operator in the class $\mathrm{B}_2(\mathbb{D})$ is either the direct sum of two homogeneous operators from $\mathrm{B}_1(\mathbb{D})$ or it is irreducible completing the classification of homogeneous operators in $\mathrm{B}_2(\mathbb{D})$.)

In a later paper [9], using the jet construction of [24], a large family of homogeneous operators were constructed. We briefly recall the "jet construction". Let α, $\beta > 0$ be any two positive real numbers. The representation $D^+_\alpha \otimes D^+_\beta$ acts naturally (as a unitary representation of the group Möb) on the tensor product $\mathbb{A}^{(\alpha)}(\mathbb{D}) \otimes \mathbb{A}^{(\beta)}(\mathbb{D})$. Now, identify the Hilbert space $\mathbb{A}^{(\alpha)}(\mathbb{D}) \otimes \mathbb{A}^{(\beta)}(\mathbb{D})$ with the Hilbert space of holomorphic functions in two variables on the bi-disc $\mathbb{D}^2$ and call it $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$. One may now consider the subspace $\mathbb{A}^{(\alpha,\beta)}_k(\mathbb{D}^2) \subseteq \mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ of all functions which *vanish to order* $k+1$ on the diagonal $\triangle := \{(z,z) \in \mathbb{D}^2 : z \in \mathbb{D}\}$. It was pointed out in [9] that the compression of the operator $M^{(\alpha)} \otimes I$ to the ortho-complement $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) \ominus \mathbb{A}^{(\alpha,\beta)}_k(\mathbb{D}^2)$ is homogeneous.

A concrete realization of these operators is possible via the jet construction as follows. Let $J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) = \{Jf := \sum_{i=0}^k \partial_2^i f \otimes e_i : f \in \mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)\}$, where e_i, $0 \le i \le k$, denotes the standard unit vectors in $\mathbb{C}^{k+1}$. The vector space $J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ inherits a Hilbert space structure via the map J. Now, $\mathbb{A}^{(\alpha,\beta)}_k(\mathbb{D}^2)$ is realized in the Hilbert space $J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ as the largest subspace of functions in $J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ vanishing on the diagonal $\triangle$ which we denote by $J^{(k)}_0\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$. The main theorem of [23, 24] then states that the compression of $M^{(\alpha)} \otimes I$ to the orthocomplement of the subspace $J^{(k)}_0\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ is the multiplication operator on the space $J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\,\triangle} := \{f : f = g_{|\mathrm{res}\triangle}$ for some $g \in J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)\}$. We will denote this operator by $M^{(\alpha,\beta)}_k$. Also, the reproducing kernel $B^{(\alpha,\beta)}_k$ for the space $J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\,\triangle}$ can be written down explicitly (cf. [24, page. 376]). The operators $M_k^{(\alpha,\beta)}, \alpha, \beta, k \geq 1$ are called the "generalized Wilkins' operators" [9].

However, irreducibility of these operators was left open. In section 2.1 we show that all these operators are irreducible and mutually inequivalent [37]. Also, the transformation rule for the reproducing kernel obtained via the jet construction is given explicitly. In particular, the corresponding cocycle J is determined concretely. It is also pointed out that the *associated representation* is multiplicity free.

Although, this may appear to produce a large family of inequivalent irreducible homogeneous operators (a two parameter family in rank $k > 1$), it turns out that except in the case $k = 2$, there are many more of these [31].

We also point out that the notion of a quasi-invariant kernel occurs, although somewhat implicitly, in the work of Berezin [11]. A host of papers have appeared applying the notion of the Berezin transform to several areas of operator theory [3, 22, 48, 49], representation theory [38, 39, 40, 41, 42, 43] and several complex variables [16, 17] etc.

The jet construction applies with very little modification to the p-fold tensor product

$$\mathbb{A}^{(\alpha_1)}(\mathbb{D}) \otimes \cdots \otimes \mathbb{A}^{(\alpha_p)}(\mathbb{D}) \simeq \mathbb{A}^{(\alpha_1,\ldots,\alpha_p)}(\mathbb{D}^p)$$

thought of as a space of holomorphic functions on the polydisc $\mathbb{D}^p$. As before, we consider the submodule $\mathcal{M}_0$ of functions vanishing to order k on the diagonal $\{(z,\ldots,z) : z \in \mathbb{D}\} \subseteq \mathbb{D}^p$. Then it is not hard to see that the compression of the operator $M^{(\alpha_1)} \otimes I \ldots \otimes I$ to $\mathcal{M}_0^\perp$ is a homogeneous operator. Although, a systematic study of this class of operators for $p > 2$ is postponed to the future, here we show that for $p = 3$ and with an appropriate choice of $\mathcal{M}_0$ consisting of functions vanishing to order 3 on the diagonal, the corresponding homogeneous operator is irreducible and the associated representation is no longer multiplicity - free!

A different jet construction

Fix a positive integer m and a real $\lambda > m/2$. Let $\oplus_{j=0}^m D^+_{2\lambda-m+2j}$ be the direct sum of the usual Discrete series representations of the universal covering group of the group Möb acting on the Hilbert space $\oplus_{j=0}^m \mu_j \mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})$. The operator $\oplus_{j=0}^m M^{(2\lambda-m+2j)}$ acting on this Hilbert space is in $\mathrm{B}_{m+1}(\mathbb{D})$ and is homogeneous being the direct sum of the homogeneous operators $M^{(2\lambda-m+2j)}$, $j = 0, \ldots, m$. Starting from here, a $m+1$ parameter family of inequivalent irreducible homogeneous operators were constructed in [31] as described below.

Let $\mathrm{Hol}(\mathbb{D}, \mathbb{C}^{m+1})$ be the space of all holomorphic functions taking values in $\mathbb{C}^{m+1}$. Define the map $\Gamma_j : \mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D}) \longrightarrow \mathrm{Hol}(\mathbb{D}, \mathbb{C}^{m+1})$, $0 \leq j \leq m$, as in [31]:

$$(\Gamma_j f)(\ell) = \begin{cases} \binom{\ell}{j} \frac{1}{(2\lambda-m+2j)_{\ell-j}} f^{(\ell-j)} & \text{if } \ell \geq j \\ 0 & \text{if } 0 \leq \ell < j, \end{cases}$$

for $f \in \mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})$, $0 \leq j \leq m$, where $(x)_n := x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol. Here $(\Gamma_j f)(\ell)$ denotes the ℓ-th component of the function $\Gamma_j f$ and $f^{(\ell-j)}$ denotes the $(\ell-j)$-th derivative of the holomorphic function f.

We transport the inner product of $\mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})$ to the range of Γ_j making Γ_j a unitary and $\Gamma_j\big(\mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})\big)$ a Hilbert space. Let

$$\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D}) := \oplus_{j=0}^{m}\mu_j\Gamma_j\big(\mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})\big),\ 1=\mu_0,\mu_1,\ldots,\mu_m>0,$$

where $\mu_j\,\mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})$ is the same as a linear space $\mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})$ with the inner product $\frac{1}{\mu_j^2}$ times that of $\mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})$. The direct sum of the discrete series representations $\oplus_{j=0}^{m}D^{+}_{2\lambda-m+2j}$ acting on $\oplus_{j=0}^{m}\mu_j\mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})$ transforms into a multiplier representation on $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$ with the multiplier:

$$J(g,z) = \big(g'\big)^{\lambda-\frac{m}{2}}D(g,z)\exp(-c_g\mathbb{S}_m)D(g,z),\ g\in \text{Möb},\ z\in\mathbb{D},$$

where $D(g,z)$ is a diagonal matrix of size $m+1$ with $D(g,z)_{jj} = \big(g'\big)^{m-j}(z)$ and $c_g = -\frac{g''}{2(g')^{3/2}}$. It follows from [31, Proposition 2.1] that the reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ is quasi-invariance. Hence

$$\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(z,w) = (1-z\bar{w})^{-2\lambda-m}D(z\bar{w})\exp(\bar{w}\mathbb{S}_m)\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(0,0)\exp(z\mathbb{S}_m^*)D(z\bar{w}),$$

$z,\, w\in\mathbb{D}$.

It was shown in [31] that the multiplication operators $M^{(\lambda,\boldsymbol{\mu})}$ acting on the Hilbert space $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$ are mutually *bounded, homogeneous, unitarily inequivalent, irreducible and its adjoint belong to the Cowen-Douglas class* $\mathrm{B}_{m+1}(\mathbb{D})$. Finally, the associated representation is multiplicity-free by the construction.

The relationship between the two jet constructions

Although, it is not clear at the outset that there exists (α,β) and $(\lambda,\boldsymbol{\mu})$ such that the two homogeneous operators $M_m^{(\alpha,\beta)}$ and $M^{(\lambda,\boldsymbol{\mu})}$ are unitarily equivalent. We calculate those λ and $\boldsymbol{\mu}$ (for a fixed m) as a function of α,β explicitly for which $M_m^{(\alpha,\beta)}$ is unitarily equivalent to $M^{(\lambda,\boldsymbol{\mu})}$. We show in this chapter that the set of homogeneous operators that appear from the first jet construction, is a small subset of those appearing in the second one. However, there is an easy modification of the first construction that allows us to construct the entire family of homogeneous operators which were first exhibited in [31]. To do this, we start with the pair $\alpha,\beta>0$ and observe that the kernel $B_m^{(\alpha,\beta)}$ can be written as :

$$B_m^{(\alpha,\beta)}(z,w) = (1-z\bar{w})^{-\alpha-\beta-2m}D(z\bar{w})\exp(\bar{w}S_\beta)\mathrm{D}\exp(zS_\beta^*)D(z\bar{w}),\ z,w\in\mathbb{D},$$

where S_β is a forward shift on $\mathbb{C}^{m+1}$ with weights $(j(\beta+j-1))_{j=1}^m$ and D is a diagonal matrix with $\mathrm{D}_{jj} = j!(\beta)_j$, $0\leq j\leq m$. We therefore easily see that

$$\Phi B_m^{(\alpha,\beta)}\Phi^* = \mathbf{B}^{(\lambda,\boldsymbol{\mu})},\ \Phi_{jj} = \frac{1}{(\beta)_j},\ 0\leq j\leq m,$$

Φ is a diagonal matrix,

$$2\lambda = \alpha + \beta + m \text{ and } \mu_j^2 := \frac{j!(\alpha)_j}{(\alpha+\beta+j-1)_j(\beta)_j},\ 0 \le j \le m.$$

Thus, the two multiplication operators are unitarily equivalent as we have claimed.

Now, the family of these quasi-invariant kernels can be enlarged in a very simple manner. Clearly, if we replace the constants $\frac{j!(\alpha)_j}{(\alpha+\beta+j-1)_j(\beta)_j}$, $0 \le j \le m$ appearing in reproducing kernel $B_m^{(\alpha,\beta)}$ by arbitrary positive constants $\mu_j > 0$, $0 \le j \le m$ then the new kernel coincides with the kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ of [31]. However, now the multiplication operator $M^{(\lambda,\boldsymbol{\mu})}$ on this space is *similar* to the operator $M_m^{(\alpha,\beta)}$ that we had constructed earlier by the usual jet construction.

We point out that in our situation, if we start with a homogeneous operator corresponding to a quasi-invariant kernel, then there is a natural family of operators similar to it which are also homogeneous with the same associated representation. The similarity transformation is easily seen to be a direct sum of scalar operators using the Schur Lemma.

Complete invariants for operators in the Cowen-Douglas class $\mathrm{B}_{k+1}(\mathbb{D})$

We construct examples of operators T in $\mathrm{B}_2(\mathbb{D})$ and $\mathrm{B}_3(\mathbb{D})$ to show that the eigenvalues of the curvature for the corresponding bundle E_T does not necessarily determine the class of the bundle E_T. Our examples consisting of homogeneous bundles E_T show that the covariant derivatives of the curvature up to order $(1,1)$ cannot be dropped, in general, from the set of invariants described above. These verifications are somewhat nontrivial and use the homogeneity of the bundle in an essential way. It is not clear if for a homogeneous bundle the curvature along with its derivatives up to order $(1,1)$ suffices to determine its equivalence class. Secondly the original question of sharpness of [18, Page. 214] and [20, page. 39], remains open, although our examples provide a partial answer.

One of the main theorems we prove in section 4.2 and [36] involves the class of operators constructed in [31]. This construction provides a complete list (up to mutual unitary in-equivalence) of irreducible homogeneous operators in $\mathrm{B}_{k+1}(\mathbb{D}), k \ge 1$, whose associated representation is multiplicity free. It turns out that for $k = 1$, this is exactly the same list as that of Wilkins [51], namely, $\mathcal{W}_1$. However, for $k \ge 2$, the class of operators $\mathcal{W}_k \subseteq \mathrm{B}_{k+1}(\mathbb{D})$ is much smaller than the corresponding list from [31]. Now consider those homogeneous and irreducible operators from [31] for which the eigenvalues of the curvature are distinct and have multiplicity 1. The Hermitian holomorphic vector bundles corresponding to such operators are called generic (cf. [18, page. 226]). We show that for these operators, the simultaneous unitary equivalence class of the curvature and the covariant derivative of order $(0,1)$ at 0 determine the unitary equivalence class of the operator T. This is considerably more involved than the corresponding result for the class $\mathcal{W}_k$ of section 2.1 and [37, Theorem 4.12, page 187].

Although, we have used techniques developed in the paper of Cowen-Douglas [18, 20], a systematic account of Hilbert space operators using a variety of tools from several different areas of mathematics is given in the book [26]. This book provides, what the authors call, a sheaf model for a large class of commuting Hilbert space operators. It is likely that these ideas will play a significant role in the future development of the topics discussed here.

1. PRELIMINARIES

In this chapter we briefly describe reproducing kernel, the Cowen-Douglas class, quasi-invariant kernel and the jet construction.

1.1 Reproducing kernel

Let $\mathcal{L}(\mathbb{F})$ be the Banach space of all linear transformations on a Hilbert space $\mathbb{F}$ of dimension n for some $n \in \mathbb{N}$. Let $\Omega \subset \mathbb{C}^m$ be a bounded, open, connected set. A function $K : \Omega \times \Omega \to \mathcal{L}(\mathbb{F})$, satisfying

$$\sum_{i,j=1}^{p} \langle K(w^{(i)}, w^{(j)})\zeta_j, \zeta_i\rangle_{\mathbb{F}} \;\geq\; 0, \quad w^{(1)}, \ldots, w^{(p)} \in \Omega, \quad \zeta_1, \ldots, \zeta_p \in \mathbb{F}, \; p > 0 \tag{1.1.1}$$

is said to be a *non negative definite (nnd) kernel* on Ω. Given such an nnd kernel K on Ω, it is easy to construct a Hilbert space $\mathcal{H}$ of functions on Ω taking values in $\mathbb{F}$ with the property

$$\langle f(w), \zeta\rangle_{\mathbb{F}} = \langle f, K(\cdot, w)\zeta\rangle_{\mathcal{H}}, \text{ for } w \in \Omega, \; \zeta \in \mathbb{F}, \text{ and } f \in \mathcal{H}. \tag{1.1.2}$$

The Hilbert space $\mathcal{H}$ is simply the completion of the linear span of all vectors of the form $\mathcal{S} = \{K(\cdot, w)\zeta, \; w \in \Omega, \; \zeta \in \mathbb{F}\}$, where the inner product between two of the vectors from $\mathcal{S}$ is defined by

$$\langle K(\cdot, w)\zeta, K(\cdot, w')\eta\rangle = \langle K(w', w)\zeta, \eta\rangle, \text{ for } \zeta, \eta \in \mathbb{F}, \text{ and } w, w' \in \Omega, \tag{1.1.3}$$

which is then extended to the linear span $\mathcal{H}^\circ$ of the set $\mathcal{S}$. This ensures the reproducing property (1.1.2) of K on $\mathcal{H}^\circ$.

Remark 1.1.1. *We point out that although the kernel K is required to be merely* nnd*, the equation (1.1.3) defines a positive definite sesqui-linear form. To see this, simply note that $|\langle f(w), \zeta\rangle| = |\langle f, K(\cdot, w)\zeta\rangle|$ which is at most $\|f\|\langle K(w, w)\zeta, \zeta\rangle^{1/2}$ by the Cauchy - Schwarz inequality. It follows that if $\|f\|^2 = 0$ then $f = 0$.*

Conversely, let $\mathcal{H}$ be any Hilbert space of functions on Ω taking values in $\mathbb{F}$. Let $e_w : \mathcal{H} \to \mathbb{F}$ be the evaluation functional defined by $e_w(f) = f(w)$, $w \in \Omega$, $f \in \mathcal{H}$. If e_w is bounded for each $w \in \Omega$ then it admits a bounded adjoint $e_w^* : \mathbb{F} \to \mathcal{H}$ such that $\langle e_w f, \zeta\rangle = \langle f, e_w^*\zeta\rangle$ for all $f \in \mathcal{H}$ and $\zeta \in \mathbb{F}$. A function f in $\mathcal{H}$ is then orthogonal to $e_w^*(\mathbb{F})$ if and only if $f = 0$. Thus

$f = \sum_{i=1}^{p} e_{w^{(i)}}^*(\zeta_i)$ with $w^{(1)}, \ldots, w^{(p)} \in \Omega$, $\zeta_1, \ldots, \zeta_p \in \mathbb{F}$, and $p > 0$, form a dense set in $\mathcal{H}$. Therefore, we have

$$\|f\|^2 = \sum_{i,j=1}^{p} \langle e_{w^{(i)}} e_{w^{(j)}}^* \zeta_j, \zeta_i \rangle,$$

where $f = \sum_{i=1}^{p} e_{w^{(i)}}^*(\zeta_i)$, $w^{(i)} \in \Omega$, $\zeta_i \in \mathcal{F}$. Since $\|f\|^2 \geq 0$, it follows that the kernel $K(z,w) = e_z e_w^*$ is non-negative definite as in (1.1.1). It is clear that $K(z,w)\zeta \in \mathcal{H}$ for each $w \in \Omega$ and $\zeta \in \mathbb{F}$, and that it has the reproducing property (1.1.2).

Remark 1.1.2. *If we assume that the evaluation functional e_w is surjective then the adjoint e_w^* is injective and it follows that $\langle K(w,w)\zeta, \zeta \rangle > 0$ for all non-zero vectors $\zeta \in \mathbb{F}$.*

There is a useful alternative description of the reproducing kernel K in terms of the orthonormal basis $\{e_k : k \geq 0\}$ of the Hilbert space $\mathcal{H}$. We think of the vector $e_k(w) \in \mathbb{F}$ as a column vector for a fixed $w \in \Omega$ and let $e_k(w)^*$ be the row vector $(\overline{e_k^1(w)}, \ldots, \overline{e_k^n(w)})$. We see that

$$\begin{aligned}
\langle K(z,w)\zeta, \eta \rangle &= \langle K(\cdot,w)\zeta, K(\cdot,z)\eta \rangle \\
&= \sum_{k=0}^{\infty} \langle K(\cdot,w)\zeta, e_k \rangle \langle e_k, K(\cdot,z)\eta \rangle \\
&= \sum_{k=0}^{\infty} \overline{\langle e_k(w), \zeta \rangle} \langle e_k(z), \eta \rangle \\
&= \sum_{k=0}^{\infty} \langle e_k(z) e_k(w)^* \zeta, \eta \rangle,
\end{aligned}$$

for any pair of vectors $\zeta, \eta \in \mathbb{F}$. Therefore, we have the following very useful representation for the reproducing kernel K:

$$K(z,w) = \sum_{k=0}^{\infty} e_k(z) e_k(w)^*, \tag{1.1.4}$$

where $\{e_k : k \geq 0\}$ is any orthonormal basis in $\mathcal{H}$.

1.2 The Cowen-Douglas class

Let $\boldsymbol{T} = (T_1, \ldots, T_m)$ be a d-tuple of commuting bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. Define the operator $D_{\mathbf{T}} : \mathcal{H} \to \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ by $D_{\mathbf{T}}(x) = (T_1 x, \ldots, T_m x)$, $x \in \mathcal{H}$. Let Ω be a bounded domain in $\mathbb{C}^m$. For $w = (w_1, \ldots, w_m) \in \Omega$, let $\boldsymbol{T} - w$ denote the operator tuple $(T_1 - w_1, \ldots, T_m - w_m)$. Let n be a positive integer. The m-tuple $\boldsymbol{T}$ is said to be in the Cowen-Douglas class $\mathrm{B}_n(\Omega)$ if

1. ran $D_{\mathbf{T}-w}$ is closed for all $w \in \Omega$
2. span $\{\ker D_{\mathbf{T}-w} : w \in \Omega\}$ is dense in $\mathcal{H}$
3. $\dim \ker D_{\mathbf{T}-w} = n$ for all $w \in \Omega$.

This class was introduced in [19]. The case of a single operator was investigated earlier in the paper [18]. In this paper, it is pointed out that an operator T in $\mathrm{B}_1(\Omega)$ is unitarily equivalent to the adjoint of the multiplication operator M on a reproducing kernel Hilbert space, where $(Mf)(z) = zf(z)$. It is not very hard to see that, more generally, a m-tuple $\boldsymbol{T}$ in $\mathrm{B}_n(\Omega)$ is unitarily equivalent to the adjoint of the m-tuple of multiplication operators $\boldsymbol{M} = (M_1, \dots, M_m)$ on a reproducing kernel Hilbert space [18] and [21, Remark 2.6 a) and b)]. Also, Curto and Salinas [21] show that if certain conditions are imposed on the reproducing kernel then the corresponding adjoint of the m-tuple of multiplication operators belongs to the class $\mathrm{B}_n(\Omega)$.

To an m-tuple $\boldsymbol{T}$ in $\mathrm{B}_n(\Omega)$, on the one hand, one may associate a Hermitian holomorphic vector bundle $E_{\mathbf{T}}$ on Ω (cf. [18]), while on the other hand, one may associate a normalized reproducing kernel K (cf. [21]) on a suitable sub-domain of $\Omega^* = \{w \in \mathbb{C}^m : \bar{w} \in \Omega\}$. It is possible to answer a number of questions regarding the m-tuple of operators $\boldsymbol{T}$ using either the vector bundle or the reproducing kernel . For instance, in the two papers [18] and [20], Cowen and Douglas show that the curvature of the bundle $E_{\mathbf{T}}$ along with a certain number of covariant derivatives forms a complete set of unitary invariants for the operator $\boldsymbol{T}$ while Curto and Salinas [21] establish that the unitary equivalence class of the normalized kernel K is a complete unitary invariant for the corresponding m-tuple of multiplication operators. Also, in [18], it is shown that a single operator in $\mathrm{B}_n(\Omega)$ is reducible if and only if the associated Hermitian holomorphic vector bundle admits an orthogonal direct sum decomposition.

We recall the correspondence between an m-tuple of operators in the class $\mathrm{B}_n(\Omega)$ and the corresponding m-tuple of multiplication operators on a reproducing kernel Hilbert space on Ω.

Let $\boldsymbol{T}$ be an m-tuple of operators in $\mathrm{B}_n(\Omega)$. Pick n linearly independent vectors $\gamma_1(w), \dots, \gamma_n(w)$ in $\ker D_{\mathbf{T}-w}$, $w \in \Omega$. Define a map $\Gamma : \Omega \to \mathcal{L}(\mathbb{F}, \mathcal{H})$ by $\Gamma(w)\zeta = \sum_{i=0}^n \zeta_i \gamma_i(w)$, where $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{F}$, $\dim \mathbb{F} = n$. It is shown in [18, Proposition 1.11] and [21, Theorem 2.2] that it is possible to choose $\gamma_1(w), \dots, \gamma_n(w)$, w in some domain $\Omega_0 \subseteq \Omega$, such that Γ is holomorphic on Ω_0. Let $\mathcal{A}(\Omega, \mathbb{F})$ denote the linear space of all $\mathbb{F}$-valued holomorphic functions on Ω. Define $U_\Gamma : \mathcal{H} \to \mathcal{A}(\Omega_0^*, \mathbb{F})$ by

$$(U_\Gamma x)(w) = \Gamma(w)^* x, \quad x \in \mathcal{H}, \ w \in \Omega_0. \tag{1.2.5}$$

Define a sesqui-linear form on $\mathcal{H}_\Gamma = \operatorname{ran} U_\Gamma$ by $\langle U_\Gamma f, U_\Gamma g \rangle_\Gamma = \langle f, g \rangle$, $f, g \in \mathcal{H}$. The map U_Γ is linear and injective. Hence $\mathcal{H}_\Gamma$ is a Hilbert space of $\mathbb{F}$-valued holomorphic functions on Ω_0^* with inner product $\langle \cdot, \cdot \rangle_\Gamma$ and U_Γ is unitary. Then it is easy to verify the following (cf. [21, Remarks 2.6]).

a) $K(z, w) = \Gamma(\bar{z})^* \Gamma(\bar{w})$, $z, w \in \Omega_0^*$ is the reproducing kernel for the Hilbert space $\mathcal{H}_\Gamma$.

b) $M_i^* U_\Gamma = U_\Gamma T_i$, where $(M_i f)(z) = z_i f(z)$, $z = (z_1, \dots, z_m) \in \Omega$.

An nnd kernel K for which $K(z, w_0) = I$ for all $z \in \Omega_0^*$ and some $w_0 \in \Omega$ is said to be normalized at w_0.

For $1 \leq i \leq m$, suppose that the operators $M_i : \mathcal{H} \to \mathcal{H}$ are bounded. Then it is easy to verify that for each fixed $w \in \Omega$, and $1 \leq i \leq m$,

$$M_i^* K(\cdot, w)\eta = \bar{w}_i K(\cdot, w)\eta \text{ for } \eta \in \mathbb{F}. \tag{1.2.6}$$

Differentiating (1.1.2), we also obtain the following extension of the reproducing property:

$$\langle (\partial_i^j f)(w), \eta \rangle = \langle f, \bar{\partial}_i^j K(\cdot, w)\eta \rangle \text{ for } 1 \leq i \leq m, \;\; j \geq 0, \; w \in \Omega, \; \eta \in \mathbb{F}, \; f \in \mathcal{H}. \tag{1.2.7}$$

Let $\boldsymbol{M} = (M_1, \ldots, M_m)$ be the commuting m-tuple of multiplication operators and let $\boldsymbol{M}^*$ be the m-tuple $(M_1^*, \ldots, M_m^*)$. It then follows from (1.2.6) that the eigenspace of the m-tuple $\boldsymbol{M}^*$ at $w \in \Omega^* \subseteq \mathbb{C}^m$ contains the n-dimensional subspace $\mathrm{r}an\, K(\cdot, \bar{w}) \subseteq \mathcal{H}$.

One may impose additional conditions on K to ensure that $\boldsymbol{M}$ is in $\mathrm{B}_n(\Omega^*)$. Assume that $K(w, w)$ is invertible for $w \in \Omega$. Fix $w_0 \in \Omega$ and note that $K(z, w_0)$ is invertible for z in some neighborhood $\Omega_0 \subseteq \Omega$ of w_0. Let $K_{\rm res}$ be the restriction of K to $\Omega_0 \times \Omega_0$. Define a kernel function K_0 on Ω_0 by

$$K_0(z, w) = \varphi(z) K(z, w) \varphi(w)^*, \;\; z, w \in \Omega_0, \tag{1.2.8}$$

where $\varphi(z) = K_{\rm res}(w_0, w_0)^{1/2} K_{\rm res}(z, w_0)^{-1}$. The kernel K_0 is said to be *normalized* at w_0 and is characterized by the property $K_0(z, w_0) = I$ for all $z \in \Omega_0$. Let $\boldsymbol{M}_0$ denote the m-tuple of multiplication operators on the Hilbert space $\mathcal{H}$. It is not hard to establish the unitary equivalence of the two m - tuples $\boldsymbol{M}$ and $\boldsymbol{M}_0$ as in (cf. [21, Lemma 3.9 and Remark 3.8]). First, the restriction map res : $f \to f_{\rm res}$, which restricts a function in $\mathcal{H}$ to Ω_0 is a unitary map intertwining the m-tuple $\boldsymbol{M}$ on $\mathcal{H}$ with the m-tuple $\boldsymbol{M}$ on $\mathcal{H}_{\rm res} = \mathrm{ran\ res}$. The Hilbert space $\mathcal{H}_{\rm res}$ is a reproducing kernel Hilbert space with reproducing kernel $K_{\rm res}$. Second, suppose that the m-tuples $\boldsymbol{M}$ defined on two different reproducing kernel Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are in $\mathrm{B}_n(\Omega)$ and $X : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator intertwining these two operator tuples. Then X must map the joint kernel of one tuple in to the other, that is, $XK_1(\cdot, w)\boldsymbol{x} = K_2(\cdot, w)\Phi(w)\boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{C}^n$, for some function $\Phi : \Omega \to \mathbb{C}^{n \times n}$. Assuming that the kernel functions K_1 and K_2 are holomorphic in the first and anti-holomorphic in the second variable, it follows, again as in [21, pp. 472], that Φ is anti-holomorphic. An easy calculation then shows that X^* is the multiplication operator $M_{\bar{\Phi}^{\rm tr}}$. If the two operator tuples are unitarily equivalent then there exists an unitary operator U intertwining them. Hence U^* must be of the form M_Ψ for some holomorphic function Ψ. Also, the operator U must map the joint kernel of $(\boldsymbol{M} - w)^*$ acting on $\mathcal{H}_1$ isometrically onto the joint kernel of $(\boldsymbol{M} - w)^*$ acting on $\mathcal{H}_2$ for all $w \in \Omega$. The unitarity of U is equivalent to the relation $K_1(\cdot, w)\boldsymbol{x} = U^* K_2(\cdot, w)\overline{\Psi(w)}^{\rm tr}\boldsymbol{x}$ for all $w \in \Omega$ and $\boldsymbol{x} \in \mathbb{C}^n$. It then follows that

$$K_1(z, w) = \Psi(z) K_2(z, w) \overline{\Psi(w)}^{\rm tr}, \tag{1.2.9}$$

where $\Psi : \Omega_0 \subseteq \Omega \to \mathcal{GL}(\mathbb{F})$ is some holomorphic function. Here, $\mathcal{GL}(\mathbb{F})$ denotes the group of all invertible linear transformations on $\mathbb{F}$.

Conversely, if two kernels are related as above then the corresponding tuples of multiplication operators are unitarily equivalent since

$$M_i^* K(\cdot, w)\zeta = \bar{w}_i K(\cdot, w)\zeta, \quad w \in \Omega,\ \zeta \in \mathbb{F},$$

where $(M_i f)(z) = z_i f(z)$, $f \in \mathcal{H}$ for $1 \leq i \leq m$.

Remark 1.2.1. *We observe that if there is a self adjoint operator X commuting with the m-tuple $\boldsymbol{M}$ on the Hilbert space $\mathcal{H}$ then we must have the relation $\overline{\Phi(z)}^{\mathrm{tr}} K(z,w) = K(z,w)\Phi(w)$ for some anti-holomorphic function $\Phi : \Omega \to \mathbb{C}^{n\times n}$. Hence if the kernel K is normalized then any projection P commuting with the m-tuple $\boldsymbol{M}$ is induced by a constant function Φ such that $\Phi(0)$ is an ordinary projection on $\mathbb{C}^n$.*

In conclusion, what is said above shows that a m-tuple of operators in $\mathrm{B}_n(\Omega^*)$ admits a representation as the adjoint of a m-tuple of multiplication operators on a reproducing kernel Hilbert spaces of $\mathbb{F}$-valued holomorphic functions on Ω_0, where the reproducing kernel K may be assumed to be normalized. Conversely, the adjoint of the m-tuple of multiplication operators on the reproducing kernel Hilbert space associated with a normalized kernel K on Ω belongs to $\mathrm{B}_n(\Omega^*)$ if certain additional conditions are imposed on K (cf. [21]).

Our interest in the class $\mathrm{B}_n(\Omega)$ lies in the fact that the Cowen-Douglas theorem [18] provides a complete set of unitary invariants for operators which belong to this class. However, these invariants are somewhat intractable. Besides, often it is not easy to verify that a given operator is in the class $\mathrm{B}_n(\Omega)$. Although, we don't use the complete set of invariants that [18] provides, it is useful to ensure that the homogeneous operators that arise from the jet construction are in this class.

1.3 Quasi-invariant kernels, cocycle and unitary representations

Let G be a locally compact second countable (lcsc) topological group acting transitively on the domain $\Omega \subseteq \mathbb{C}^m$. Let $\mathbb{C}^{n\times n}$ denote the set of $n \times n$ matrices over the complex field $\mathbb{C}$. We start with a cocycle J, that is, a Borel map $J : G \times \Omega \longrightarrow \mathbb{C}^{n\times n}$, holomorphic on Ω, satisfying the cocycle relation

$$J(gh, z) = J(h, z)J(g, h\cdot z), \quad \text{for all } g, h \in G,\ z \in \Omega, \tag{1.3.10}$$

Let $\mathrm{Hol}(\Omega, \mathbb{C}^n)$ be the linear space consisting of all holomorphic functions on Ω taking values in $\mathbb{C}^n$. We then obtain a natural (left) action U of the group G on $\mathrm{Hol}(\Omega, \mathbb{C}^n)$:

$$(U_{g^{-1}} f)(z) = J(g, z) f(g\cdot z),\ f \in \mathrm{Hol}(\Omega, \mathbb{C}^n),\ z \in \Omega. \tag{1.3.11}$$

Let e be the identity element of the group G. Note that the cocycle condition (1.3.10) implies, among other things, $J(e, z) = J(e, z)^2$ for all $z \in \Omega$.

Let $\mathbb{K} \subseteq G$ be the compact subgroup which is the stabilizer of 0. For h, k in $\mathbb{K}$, we have $J(kh, 0) = J(h, 0)J(k, 0)$ so that $k \mapsto J(k, 0)^{-1}$ is a representation of $\mathbb{K}$ on $\mathbb{C}^n$.

A positive definite kernel K on Ω defines an inner product on some linear subspace of $\mathrm{Hol}(\Omega, \mathbb{C}^n)$. The completion of this subspace is then a Hilbert space of holomorphic functions on Ω (cf. [5]). The natural action of the group G described above is seen to be unitary for an appropriate choice of such a kernel. Therefore, we first discuss these kernels in some detail.

Let $\mathcal{H}$ be a functional Hilbert space consisting of holomorphic functions on Ω possessing a reproducing kernel K. We will always assume that the m-tuple of multiplication operators $\boldsymbol{M} = (M_1, \ldots, M_m)$ on the Hilbert space $\mathcal{H}$ is bounded. We also define the action of the group G on the space of multiplication operators – $g \cdot M_f = M_{f \circ g}$ for $f \in \mathcal{A}(\Omega)$ and $g \in G$. In particular, we have $g \cdot \boldsymbol{M} = \boldsymbol{M}_g$. We will say that the m-tuple $\boldsymbol{M}$ is *G-homogeneous* if the operator $g \cdot \boldsymbol{M}$ is unitarily equivalent to $\boldsymbol{M}$ for all $g \in G$. $g \mapsto U_{g^{-1}}$ defined in (1.3.11) leaves $\mathcal{H}$ invariant. The following theorem says that the reproducing kernel of such a Hilbert space must be *quasi invariant* under the G action.

A version of the following Theorem appears in [31] for the unit disc. However, the proof here, which is taken from [31], is for a more general domain Ω in $\mathbb{C}^m$.

Theorem 1.3.1. *Suppose that $\mathcal{H}$ is a Hilbert space which consists of holomorphic functions on Ω and possesses a reproducing kernel K on which the m-tuple $\boldsymbol{M}$ is irreducible and bounded. Then the following are equivalent.*

1. *The m-tuple $\boldsymbol{M}$ is G-homogeneous.*
2. *The reproducing kernel K of the Hilbert space $\mathcal{H}$ transforms, for some cocycle $J : G \times \Omega \longrightarrow \mathbb{C}^{n\times n}$, according to the rule*
$$K(z, w) = J(g, z)K(g \cdot z, g \cdot w)J(g, w)^*,\ z, w \in \Omega.$$
3. *The operator $U_{g^{-1}} : f \mapsto M_{J(g,.)} f \circ g$ for $f \in \mathcal{H}$ is unitary.*

Proof. Assuming that K is quasi-invariant, that is, K satisfies the transformation rule, we see that the linear transformation U defined in (1.3.11) is unitary. To prove this, note that

$$\begin{aligned}
\langle U_{g^{-1}} K(z, w)\boldsymbol{x}, U_{g^{-1}} K(z, w')\boldsymbol{y}\rangle &= \langle J(g, z)K(g \cdot z, w)\boldsymbol{x}, J(g, z)K(g \cdot z, w')\boldsymbol{y}\rangle \\
&= \langle K(z, \tilde{w})J(g, \tilde{w})^{*-1}\boldsymbol{x}, K(z, \tilde{w}')J(g, \tilde{w}')^{*-1}\boldsymbol{y}\rangle \\
&= \langle K(\tilde{w}', \tilde{w})J(g, \tilde{w})^{*-1}\boldsymbol{x}, J(g, \tilde{w}')^{*-1}\boldsymbol{y}\rangle \\
&= \langle J(g, \tilde{w}')^{-1}K(\tilde{w}', \tilde{w})J(g, \tilde{w})^{*-1}\boldsymbol{x}, \boldsymbol{y}\rangle \\
&= \langle K(g \cdot \tilde{w}', g \cdot \tilde{w})\boldsymbol{x}, \boldsymbol{y}\rangle,
\end{aligned}$$

where $\tilde{w} = g^{-1} \cdot w$ and $\tilde{w}' = g^{-1} \cdot w'$. Hence

$$\langle K(g \cdot \tilde{w}', g \cdot \tilde{w})\boldsymbol{x}, \boldsymbol{y}\rangle = \langle K(w', w)\boldsymbol{x}, \boldsymbol{y}\rangle.$$

It follows that the map $U_{g^{-1}}$ is isometric. On the other hand, if U of (1.3.11) is unitary then the reproducing kernel K of the Hilbert space $\mathcal{H}$ satisfies

$$K(z,w) = J(g,z)K(g\cdot z, g\cdot w)J(g,w)^*. \qquad (1.3.12)$$

This follows from the fact that the reproducing kernel has the expansion (1.1.4) for some orthonormal basis $\{e_\ell : \ell \geq 0\}$ in $\mathcal{H}$. The uniqueness of the reproducing kernel implies that the expansion is independent of the choice of the orthonormal basis. Consequently, we also have $K(z,w) = \sum_{\ell=0}^\infty (U_{g^{-1}}e_\ell)(z)(U_{g^{-1}}e_\ell)(w)^*$ which verifies the equation (1.3.12). Thus we have shown that U is unitary if and only if the reproducing kernel K transforms according to (1.3.12).

We now show that the m-tuple $\boldsymbol{M}$ is homogeneous if and only if $f \mapsto M_{J(g,.)} f\circ g$ is unitary. The eigenvector at w for $g.\boldsymbol{M}$ is clearly $K(\cdot, g^{-1}\cdot w)$. It is not hard, using the unitary operator U_Γ in (1.2.5), to see that that $g^{-1}\cdot\boldsymbol{M}$ is unitarily equivalent to $\boldsymbol{M}$ on a Hilbert space $\mathcal{H}_g$ whose reproducing kernel is $K_g(z,w) = K(g\cdot z, g\cdot w)$ and the unitary U_Γ is given by $f\mapsto f\circ g$ for $f\in\mathcal{H}$. However, the homogeneity of the m-tuple $\boldsymbol{M}$ is equivalent to the existence of a unitary operator intertwining the m-tuple of multiplication on the two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_g$. As we have pointed out in section 1.2, this unitary operator is induced by a multiplication operator $M_{J(g,.)}$, where $J(g,.)$ is a holomorphic function (depends on g) such that $K_g(z,w) = J(g,z)K(z,w)\overline{J(g,w)}^{\mathrm{tr}}$. The composition of these two unitaries is $f\mapsto M_{J(g,.)}f\circ g$ and is therefore a unitary. □

The discussion below and the Corollary following it is implicit in [31]. Let g_z be an element of G which maps 0 to z, that is $g_z\cdot 0 = z$. We could then try to define possible kernel functions $K:\Omega\times\Omega\to\mathbb{C}^{n\times n}$ satisfying the transformation rule (1.3.12) via the requirement

$$K(g_z\cdot 0, g_z\cdot 0) = (J(g_z,0))^{-1}K(0,0)(J(g_z,0)^*)^{-1}, \qquad (1.3.13)$$

choosing any positive operator $K(0,0)$ on $\mathbb{C}^n$ which commutes with $J_k(0)$ for all $k\in\mathbb{K}$. Then the equation (1.3.13) determines the function K unambiguously as long as $J(k,0)$ is unitary for $k\in\mathbb{K}$. Pick $g\in G$ such that $g\cdot 0 = z$. Then $g = g_z k$ for some $k\in\mathbb{K}$. Hence

$$\begin{aligned}
K(g_zk\cdot 0, g_zk\cdot 0) &= (J(g_zk,0))^{-1}K(0,0)(J(g_zk,0)^*)^{-1}\\
&= \big(J(k,0)J(g_z,k\cdot 0)\big)^{-1}K(0,0)\big(J(g_z,k\cdot 0)^*J(k,0)^*\big)^{-1}\\
&= (J(g_z,0))^{-1}(J(k,0))^{-1}K(0,0)(J(k,0)^*)^{-1}(J(g_z,0)^*)^{-1}\\
&= (J(g_z,0))^{-1}K(0,0)(J(g_z,0)^*)^{-1}\\
&= K(g_z\cdot 0, g_z\cdot 0)
\end{aligned}$$

Given the definition (1.3.13), where the choice of $K(0,0) = A$ involves as many parameters as the number of irreducible representations of the form $k\mapsto J(k,0)^{-1}$ of the compact group $\mathbb{K}$, one can polarize (1.3.13) to get $K(z,w)$. In this approach, one has to find a way of determining if K is non-negative definite, or for that matter, if $K(\cdot,w)$ is holomorphic on all of Ω for each fixed but arbitrary $w \in \Omega$. However, it is evident from the definition (1.3.13) that

$$\begin{aligned} K(h \cdot z, h \cdot z) &= J(h, g_z \cdot 0)^{-1} J(g_z, 0)^{-1} A J(g_z, 0)^{*-1} (J(h, g_z \cdot 0)^*)^{-1} \\ &= J(h, z)^{-1} K(z, z) J(h, z)^{*-1} \end{aligned}$$

for all $h \in G$. Polarizing this equality, we obtain

$$K(h \cdot z, h \cdot w) = J(h, z)^{-1} K(z, w) J(h, w)^{*-1}$$

which is the identity (1.3.12). It is also clear that the linear span of the set $\{K(\cdot, w)\zeta : w \in \Omega, \zeta \in \mathbb{C}^n\}$ is stable under the action (1.3.11) of G:

$$g \mapsto J(g, z) K(g \cdot z, w)\zeta = K(z, g^{-1} \cdot w) J(g, g^{-1}w)^{*-1}\zeta,$$

where $J(g, g^{-1}w)^{*-1}\zeta$ is a fixed element of $\mathbb{C}^n$.

Corollary 1.3.2. *If $J : G \times \Omega \to \mathbb{C}^{n\times n}$ is a cocycle and g_z is an element of G which maps 0 to z then the kernel $K : \Omega \times \Omega \to \mathbb{C}^{n\times n}$ defined by the requirement*

$$K(g_z \cdot 0, g_z \cdot 0) = (J(g_z, 0))^{-1} K(0, 0) (J(g_z, 0)^*)^{-1}$$

is quasi-invariant, that is, it transforms according to (1.3.12).

1.4 The jet construction

Let $\mathcal{M}$ be a Hilbert module over the algebra $\mathcal{A}(\Omega)$ for Ω a bounded domain in $\mathbb{C}^m$. Let $\mathcal{M}_k$ be the submodule of functions in $\mathcal{M}$ vanishing to order $(k+1)$, $k > 0$ on some analytic hyper-surface $\mathcal{Z}$ in Ω – the zero set of a holomorphic function φ in $\mathcal{A}(\Omega)$. A function f on Ω is said to vanish to order k on $\mathcal{Z}$ if it can be written $f = \varphi^{k+1} g$ for some holomorphic function g. The quotient module $\mathcal{Q} = \mathcal{M} \ominus \mathcal{M}_k$ has been characterized in [24]. This was done by a generalization of the approach in [5] to allow vector-valued kernel Hilbert modules. The basic result in [24] is that $\mathcal{Q}$ can be characterized as such a vector-valued kernel Hilbert space over the algebra $\mathcal{A}(\Omega)|_{\mathcal{Z}}$ of the restriction of functions in $\mathcal{A}(\Omega)$ to $\mathcal{Z}$ and multiplication by φ acts as a nilpotent operator of order k.

For a fixed integer $n > 0$, in this realization, $\mathcal{M}$ consists of $\mathbb{C}^n$-valued holomorphic functions, and there is an $\mathbb{C}^{n\times n}$-valued function $K(z, w)$ on $\Omega \times \Omega$ which is holomorphic in z and anti-holomorphic in w such that

(1) $K(\cdot, w)v$ is in $\mathcal{M}$ for w in Ω and v in $\mathbb{C}^n$;

(2) $\langle f, K(\cdot, w)v\rangle_{\mathcal{M}} = \langle f(w), v\rangle_{\mathbb{C}^n}$ for f in $\mathcal{M}$, w in Ω and v in $\mathbb{C}^n$; and

(3) $\mathcal{A}(\Omega)\mathcal{M} \subset \mathcal{M}$.

If we assume that $\mathcal{M}$ is in the class $\mathrm{B}_1(\Omega)$, then it is possible to describe the quotient module via a jet construction along the normal direction to the hypersurface $\mathcal{Z}$. The details are in [24]. In this approach, to every positive definite kernel $K : \Omega \times \Omega \to \mathbb{C}$, we associate a kernel $JK = \left(\!\!\left(\partial_1^i \bar{\partial}_1^{\,j} K\right)\!\!\right)_{i,j=0}^k$, where ∂_1 denotes differentiation along the normal direction to $\mathcal{Z}$. Then we may equip

$$J\mathcal{M} = \Big\{\mathbf{f} := \sum_{i=0}^{k} \partial_1^i f \otimes \varepsilon_i \in \mathcal{M} \otimes \mathbb{C}^{k+1} : f \in \mathcal{M}\Big\},$$

where $\varepsilon_0, \ldots, \varepsilon_{k-1}$ are standard unit vectors in $\mathbb{C}^k$, with a Hilbert space structure via the kernel JK. The module action is defined by $\mathbf{f} \mapsto \mathbb{J}\mathbf{f}$ for $\mathbf{f} \in J\mathcal{M}$, where $\mathbb{J}$ is the array –

$$\mathbb{J} = \begin{pmatrix} 1 & \ldots & \ldots & \ldots & \ldots & 0 \\ \partial_1 & 1 & & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & \binom{\ell}{j}\partial_1^{\ell-j} & & 1 & & \vdots \\ \vdots & & & & \ddots & 0 \\ \partial_1^k & \ldots & \ldots & \ldots & \ldots & 1 \end{pmatrix}$$

with $0 \le \ell, j \le k$. The module $J\mathcal{M}_{|\mathrm{res}\ \mathcal{Z}}$ which is the restriction of $J\mathcal{M}$ to $\mathcal{Z}$ is then shown to be isomorphic to the quotient module $\mathcal{M} \ominus \mathcal{M}_k$.

We illustrate these results by means of an example. Let $\mathbb{A}^{(\alpha)}(\mathbb{D})$ be the Hilbert module over $\mathcal{A}(\mathbb{D})$ with reproducing kernel $(1 - z\bar{w})^{-\alpha}$, $z, w \in \mathbb{D}$, $\alpha > 0$. Let $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) := \mathbb{A}^{(\alpha)}(\mathbb{D}) \otimes \mathbb{A}^{(\beta)}(\mathbb{D})$ be the Hilbert module which corresponds to the reproducing kernel

$$B^{(\alpha,\beta)}(\boldsymbol{z}, \boldsymbol{w}) = (1 - z_1\bar{w}_1)^{-\alpha}(1 - z_2\bar{w}_2)^{-\beta},$$

$\boldsymbol{z} = (z_1, z_2) \in \mathbb{D}^2$ and $\boldsymbol{w} = (w_1, w_2) \in \mathbb{D}^2$. Let $\mathbb{A}_1^{(\alpha,\beta)}(\mathbb{D}^2)$ be the subspace of all functions in $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ which vanish to order 2 on the diagonal $\triangle := \{(z, z) : z \in \mathbb{D}\} \subseteq \mathbb{D} \times \mathbb{D}$. The quotient module $\mathbb{A}_{1\ \mathrm{res}}^{(\alpha,\beta)}(\mathbb{D}^2) := \mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) \ominus \mathbb{A}_1^{(\alpha,\beta)}(\mathbb{D}^2)$ which is realized as $J^{(1)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\ \triangle}$ was described in [23] using an orthonormal basis for the quotient module $J^{(1)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\ \triangle}$. This includes the calculation of the compression of the two operators, $M_1 : f \mapsto z_1 f$ and $M_2 : f \mapsto z_2 f$ for $f \in \mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$, on the quotient module $J^{(1)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\ \triangle}$ (block weighted shift operators) with respect to this orthonormal basis. These are homogeneous operators in the class $\mathrm{B}_2(\mathbb{D})$ which were first discovered by Wilkins [51].

In [23], an orthonormal basis $\left\{e_p^{(1)}, e_p^{(2)}\right\}_{p=0}^{\infty}$ was constructed in the quotient module $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) \ominus \mathbb{A}_1^{(\alpha,\beta)}(\mathbb{D}^2)$. It was shown that the matrix

$$M_p^{(1)} = \begin{pmatrix} -\sqrt{\frac{p+1}{\alpha+\beta+p}} & 0 \\ \sqrt{\frac{\beta(\alpha+\beta+1)}{\alpha(\alpha+\beta+p)(\alpha+\beta+p+1)}} & -\sqrt{\frac{p}{\alpha+\beta+p+1}} \end{pmatrix}$$

represents the operator M_1 which is multiplication by z_1 with respect to the orthonormal basis $\{e_p^{(1)}, e_p^{(2)}\}_{p=0}^{\infty}$. Similarly,

$$M_p^{(2)} = \begin{pmatrix} -\sqrt{\frac{p+1}{\alpha+\beta+p}} & 0 \\ -\sqrt{\frac{\alpha(\alpha+\beta+1)}{\beta(\alpha+\beta+p)(\alpha+\beta+p+1)}} & -\sqrt{\frac{p}{\alpha+\beta+p+1}} \end{pmatrix}$$

represents the operator M_2 which is multiplication by z_2 with respect to the orthonormal basis $\{e_p^{(1)}, e_p^{(2)}\}_{p=0}^{\infty}$. Therefore, we see that $Q_1^{(p)} = \frac{1}{2}(M_1^{(p)} - M_2^{(p)})$ is a nilpotent matrix of index 2 while $Q_2^{(p)} = \frac{1}{2}(M_1^{(p)} + M_2^{(p)})$ is a diagonal matrix in case $\beta = \alpha$. These definitions naturally give a pair of operators Q_1 and Q_2 on the quotient module $J^{(1)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\text{res } \triangle}$. Let f be a function in the bi-disc algebra $\mathcal{A}(\mathbb{D}^2)$ and

$$f(u_1, u_2) = f_0(u_1) + f_1(u_1)u_2 + f_2(u_1)u_2^2 + \cdots$$

be the Taylor expansion of the function f with respect to the coordinates $u_1 = \frac{z_1+z_2}{2}$ and $u_2 = \frac{z_1-z_2}{2}$. Now, the module action for $f \in \mathcal{A}(\mathbb{D}^2)$ in the quotient module $J^{(1)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\text{res } \triangle}$ is then given by

$$\begin{aligned} f \cdot h &= f(Q_1, Q_2) \cdot h \\ &= f_0(Q_1) \cdot h + f_1(Q_1)Q_2 \cdot h \\ &\overset{\text{def}}{=} \begin{pmatrix} f_0 & 0 \\ f_1 & f_0 \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \end{aligned}$$

where $h = \binom{h_1}{h_2} \in J^{(1)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\text{res } \triangle}$ is the unique decomposition obtained from realizing the quotient module as the direct sum $J^{(1)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\text{res } \triangle} = \big(\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) \ominus \mathbb{A}_0^{(\alpha,\beta)}(\mathbb{D}^2)\big) \oplus \big(\mathbb{A}_0^{(\alpha,\beta)}(\mathbb{D}^2) \ominus \mathbb{A}_1^{(\alpha,\beta)}(\mathbb{D}^2)\big)$, where $\mathbb{A}_{i-1}^{(\alpha,\beta)}(\mathbb{D}^2)$, $i = 1, 2$, are the submodules in $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ consisting of all functions vanishing on $\triangle$ to order 1 and 2 respectively.

Following [23] the curvature $\mathcal{K}^{(\alpha,\beta)}$ for the bundle $E^{(\alpha,\beta)}$ corresponding to the metric $B^{(\alpha,\beta)}(\mathbf{u}, \mathbf{u})$, where $\mathbf{u} = (u_1, u_2) \in \mathbb{D}^2$ can be calculated as follows:

$$\mathcal{K}^{(\alpha,\beta)}(u_1, u_2) = (1 - |u_1 + u_2|^2)^{-2} \begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix} + (1 - |u_1 - u_2|^2)^{-2} \begin{pmatrix} \beta & -\beta \\ -\beta & \beta \end{pmatrix}.$$

The restriction of the curvature to the hyper-surface $\{u_2 = 0\}$ is

$$\mathcal{K}^{(\alpha,\beta)}(u_1, u_2)_{|u_2=0} = (1 - |u_1|^2)^{-2} \begin{pmatrix} \alpha+\beta & \alpha-\beta \\ \alpha-\beta & \alpha+\beta \end{pmatrix},$$

where $u_1 \in \mathbb{D}$. Thus we find that if $\alpha = \beta$, then the curvature is of the form $2\alpha(1 - |u_1|^2)^{-2} I_2$.

Also, the unitary map which is basic to the construction of the quotient module is easy to describe, namely,

$$h \mapsto \sum_{\ell=0}^{k} \partial_1^\ell h \otimes \varepsilon_\ell \Big|_{\text{res } \triangle}$$

for $h \in \mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$. For $k = 2$, it is enough to describe this map just for the orthonormal basis $\{e_p^{(1)}, e_p^{(2)} : p \geq 0\}$:

$$\begin{aligned} e_p^{(1)}(z_1, z_2) &\mapsto \begin{pmatrix} \binom{-(\alpha+\beta)}{p}^{1/2} z_1^p \\ \beta\sqrt{\frac{p}{\alpha+\beta}}\binom{-(\alpha+\beta+1)}{p-1}^{1/2} z_1^{p-1} \end{pmatrix} \\ e_p^{(2)}(z_1, z_2) &\mapsto \begin{pmatrix} 0 \\ \sqrt{\frac{\alpha\beta}{\alpha+\beta}}\binom{-(\alpha+\beta+2)}{p-1}^{1/2} z_1^{p-1} \end{pmatrix}. \end{aligned} \tag{1.4.14}$$

This allows the computation of the 2×2 matrix-valued kernel function [23]

$$K_{\mathcal{Q}}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{p=0}^{\infty} e_p^{(1)}(\boldsymbol{z}) e_p^{(1)}(\boldsymbol{w})^* + \sum_{p=0}^{\infty} e_p^{(2)}(\boldsymbol{z}) e_p^{(2)}(\boldsymbol{w})^*, \;\; \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^2$$

which restricted to $\triangle$ corresponds to the quotient module. Recall that $S(z, w) := (1 - z\bar{w})^{-1}$ is the Szegö kernel for the unit disc $\mathbb{D}$. We set $\mathbb{S}^r(z) := S(z, z)^r = (1 - |z|^2)^{-r}$, $r > 0$. A straight forward computation shows that

$$\begin{aligned} & K_{\mathcal{Q}}(\boldsymbol{z}, \boldsymbol{z})_{|\text{res} \;\; \triangle} \\ &= \begin{pmatrix} \mathbb{S}(z)^{\alpha+\beta} & \beta z \mathbb{S}(z)^{\alpha+\beta+1} \\ \beta\bar{z}\mathbb{S}(z)^{\alpha+\beta+1} & \frac{\beta^2}{\alpha+\beta}\frac{d}{d|z|^2}\left(|z|^2\mathbb{S}(z)^{\alpha+\beta+1}\right) + \frac{\beta\alpha}{\alpha+\beta}\mathbb{S}(z)^{\alpha+\beta+2} \end{pmatrix} \\ &= \left(\!\left(\mathbb{S}(z_1)^\alpha \partial_2^i \bar{\partial}_2^j \mathbb{S}(z_2)^\beta{}_{|\text{res} \;\; \triangle}\right)\!\right)_{i,j=0,1} \\ &= (JK)(\boldsymbol{z}, \boldsymbol{z})_{|\text{res} \;\; \triangle}, \;\; \boldsymbol{z} \in \mathbb{D}^2, \end{aligned}$$

where $\triangle = \{(z, z) \in \mathbb{D}^2 : z \in \mathbb{D}\}$. These calculations give an explicit illustration of one of the main theorems on quotient modules from [24, Theorem 3.4].

2. HOMOGENEOUS OPERATORS VIA THE JET CONSTRUCTION

Our main results on irreducibility of certain class of homogeneous operators is in Section 2.1. The kernel $B^{(\alpha,\beta)}(\boldsymbol{z},\boldsymbol{w}) = (1-z_1\bar{w}_1)^{-\alpha}(1-z_2\bar{w}_2)^{-\beta}$, $\boldsymbol{z}=(z_1,z_2)$, $\boldsymbol{w}=(w_1,w_2)\in\mathbb{D}^2$, determines a Hilbert module over the function algebra $\mathcal{A}(\mathbb{D}^2)$. We recall the computation of a matrix valued kernel on the unit disc $\mathbb{D}$ using the jet construction for this Hilbert module which consists of holomorphic functions on the unit disc $\mathbb{D}$ taking values in $\mathbb{C}^n$. The multiplication operator on this Hilbert space is then shown to be irreducible by checking that all of the coefficients of the "normalized" matrix valued kernel, obtained from the jet construction, cannot be simultaneously reducible.

In section 5, we show that the kernel obtained from the jet construction is quasi-invariant and consequently, the corresponding multiplication operator is homogeneous. This proof involves the verification of a cocycle identity, which in turn, depends on a beautiful identity involving binomial coefficients.

Finally, in section 6, we discuss some examples arising from the jet construction applied to a certain natural family of Hilbert modules over the algebra $\mathcal{A}(\mathbb{D}^3)$. Along the way we construct an example of an irreducible homogeneous operator in $\mathrm{B}_4(\mathbb{D})$ such that the associated representation is *not* multiplicity–free.

2.1 Irreducibility

In the section 1.4, we have already pointed out that any Hilbert space $\mathcal{H}$ of scalar valued holomorphic functions on $\Omega\subset\mathbb{C}^m$ with a reproducing kernel B determines a line bundle $\mathcal{E}$ on $\Omega^* := \{\bar{w} : w\in\Omega\}$. The fibre of $\mathcal{E}$ at $\bar{w}\in\Omega^*$ is spanned by $B(.,w)$. We can now construct a rank $(n+1)$ vector bundle $J\mathcal{E}$ over Ω^*. A holomorphic frame for this bundle is $\{\bar{\partial}_2^\ell B(.,w) : 0\le\ell\le n, w\in\Omega\}$, and as usual, this frame determines a metric for the bundle which we denote by JB, where

$$JB(w,w) = \big(\!\big(\langle\bar{\partial}_2^j B(.,w), \bar{\partial}_2^i B(.,w)\rangle\big)\!\big)_{i,j=0}^n = \big(\!\big(\partial_2^i\bar{\partial}_2^j B(w,w)\big)\!\big)_{i,j=0}^n, w\in\Omega.$$

Recall that $\mathbb{A}^{(\alpha)}(\mathbb{D})$ is the Hilbert space of holomorphic functions on $\mathbb{D}$ whose reproducing kernel is $(1-z\bar{w})^{-\alpha}, \alpha>0$ and the multiplication operator on $\mathbb{A}^{(\alpha)}(\mathbb{D})$ is denoted by $M^{(\alpha)}$. The reproducing kernel for the tensor product $\mathbb{A}^{(\alpha)}(\mathbb{D})\otimes\mathbb{A}^{(\beta)}(\mathbb{D})$ is

$$B^{(\alpha,\beta)}(\boldsymbol{z},\boldsymbol{w}) = (1-z_1\bar{w}_1)^{-\alpha}(1-z_2\bar{w}_2)^{-\beta},$$

for $\boldsymbol{z}=(z_1,z_2)\in\mathbb{D}^2$ and $\boldsymbol{w}=(w_1,w_2)\in\mathbb{D}^2$, $\alpha,\beta>0$.

Now, identify the Hilbert space $\mathbb{A}^{(\alpha)}(\mathbb{D})\otimes\mathbb{A}^{(\beta)}(\mathbb{D})$ with the Hilbert space of holomorphic functions in two variables on the bi-disc $\mathbb{D}^2$ and call it $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$. One may now consider the subspace $\mathbb{A}_n^{(\alpha,\beta)}(\mathbb{D}^2)\subseteq\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ of all functions which *vanish to order* $(n+1)$ on the diagonal $\triangle:=\{(z,z)\in\mathbb{D}^2: z\in\mathbb{D}\}$. Let us denote the ortho-complement $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)\ominus\mathbb{A}_n^{(\alpha,\beta)}(\mathbb{D}^2)$ by $\mathbb{A}_{n\ \mathrm{res}}^{(\alpha,\beta)}(\mathbb{D}^2)$.

A concrete realization of the Hilbert space $\mathbb{A}_{n\ \mathrm{res}}^{(\alpha,\beta)}(\mathbb{D}^2)$ is possible via the jet construction as follows. Let $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)=\{Jf:=\sum_{i=0}^n\partial_2^i f\otimes e_i : f\in\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)\}$, where e_i, $0\le i\le n$, denotes the standard unit vectors in $\mathbb{C}^{n+1}$. The vector space $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ inherits a Hilbert space structure via the map J. Now, $\mathbb{A}_n^{(\alpha,\beta)}(\mathbb{D}^2)$ is realized in the Hilbert space $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ as the largest subspace of functions in $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ vanishing on the diagonal $\triangle$ which we denote by $J_0^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$. The main theorem of [23, 24] then states that the compression of $M^{(\alpha)}\otimes I$ to the orthocomplement of the subspace $J_0^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ is the multiplication operator on the space $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\,\triangle}:=\{f: f=g_{|\mathrm{res}\triangle}$ for some $g\in J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)\}$. We will denote this operator by $M_n^{(\alpha,\beta)}$. Moreover, the Hilbert space $\mathbb{A}_{n\ \mathrm{res}}^{(\alpha,\beta)}(\mathbb{D}^2)$ is realized as $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\,\triangle}$. Here, $\Omega=\mathbb{D}^2$ and $B=B^{(\alpha,\beta)}$. The reproducing kernel $(JB^{(\alpha,\beta)})_{|\mathrm{res}\,\triangle}$ for the Hilbert space $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\,\triangle}$ can be written down explicitly (cf. [24, page. 376]). We write $B_n^{(\alpha,\beta)}$ for $(JB^{(\alpha,\beta)})_{|\mathrm{res}\,\triangle}$.

It follows from [24] that $h(z)=B_n^{(\alpha,\beta)}(z,z)$ is a metric for the Hermitian anti-holomorphic vector bundle $J\mathcal{E}_{|\mathrm{res}\,\triangle}$ over $\triangle=\{(z,z): z\in\mathbb{D}\}\subseteq\mathbb{D}^2$. However, $J\mathcal{E}_{|\mathrm{res}\,\triangle}$ is a Hermitian holomorphic vector bundle over $\triangle^*=\{(\bar z,\bar z): z\in\mathbb{D}\}$, that is, $\bar z$ is the holomorphic variable in this description. Thus $\partial f=0$ if and only if f is holomorphic on $\triangle^*$. To restore the usual meaning of ∂ and $\bar\partial$, we interchange the roles of z and $\bar z$ in the metric which amounts to replacing h by its transpose.

As shown in [24], this Hermitian anti-holomorphic vector bundle $J\mathcal{E}_{|\mathrm{res}\,\triangle}$ defined over the diagonal subset $\triangle$ of the bi-disc $\mathbb{D}^2$ gives rise to a reproducing kernel Hilbert space $J^{(n)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|_{\mathrm{res}\triangle}}$. The reproducing kernel for this Hilbert space is $B_n^{(\alpha,\beta)}(z,w)$ which is obtained by polarizing $B_n^{(\alpha,\beta)}(z,z)=h(z)^{\mathrm{tr}}$.

Lemma 2.1.1. *[9, Theorem 5.2] Let α,β be two positive real numbers and $n\ge 1$ be an integer. Let $\mathbb{A}_{n\ \mathrm{res}}^{(\alpha,\beta)}(\mathbb{D}^2)$ be the ortho-complement of the subspace of $\mathbb{A}^{(\alpha)}(\mathbb{D})\otimes\mathbb{A}^{(\beta)}(\mathbb{D})$ consisting of all the functions vanishing to order $(n+1)$ on the diagonally embedded unit disc $\triangle:=\{(z,z): z\in\mathbb{D}\}$. The compressions to $\mathbb{A}_{n\ \mathrm{res}}^{(\alpha,\beta)}(\mathbb{D}^2)$ of $M^{(\alpha)}\otimes I$ and $I\otimes M^{(\beta)}$ are homogeneous operators with a common associated representation.*

Proof. For each real number $\alpha>0$, let $\mathbb{A}^{(\alpha)}(\mathbb{D})$ be the Hilbert space completion of the inner product space spanned by $\{f_k: k\in\mathbb{Z}^+\}$ where the f_k's are mutually orthogonal vectors with norms given by

$$\|f_k\|^2=\frac{\Gamma(1+k)}{\Gamma(\alpha+k)},\ k\in\mathbb{Z}^+.$$

(Up to scaling of the norm, this Hilbert space may be identified, via non-tangential boundary values, with the Hilbert space of analytic functions on $\mathbb{D}$ with reproducing kernel $(z,w)\mapsto(1-z\bar{w})^{-\alpha}$.) The representation D^+_α lives on $\mathbb{A}^{(\alpha)}(\mathbb{D})$, and is given (at least on the linear span of the f_k's) by the formula

$$D^+_\alpha(\varphi^{-1})f=(\varphi')^{\alpha/2}f\circ\varphi,\ \varphi\in \text{Möb}.$$

Clearly, the subspace $\mathbb{A}^{(\alpha,\beta)}_{n\ \mathrm{res}}(\mathbb{D}^2)$ is invariant under the Discrete series representation $\pi:=D^+_\alpha\otimes D^+_\beta$ associated with both the operators $M^{(\alpha)}\otimes I$ and $I\otimes M^{(\beta)}$. It is also co-invariant under these two operators. An application of Proposition 2.4 in [8] completes the proof of the lemma. □

The subspace $\mathbb{A}^{(\alpha,\beta)}_n(\mathbb{D}^2)$ consists of those functions $f\in\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ which vanish on $\triangle$ along with their first n derivatives with respect to z_2. As it turns out, the compressions to $\mathbb{A}^{(\alpha,\beta)}_{n\ \mathrm{res}}(\mathbb{D}^2)=\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)\ominus\mathbb{A}^{(\alpha,\beta)}_n(\mathbb{D}^2)$ of $M^{(\alpha)}\otimes I$ is the multiplication operator on the Hilbert space $\mathbb{A}^{(\alpha,\beta)}_{n\ \mathrm{res}}(\mathbb{D}^2)$ which we denote $M^{(\alpha,\beta)}_n$. An application of [24, Proposition 3.6] shows that the adjoint of the multiplication operator $M^{(\alpha,\beta)}_n$ is in $\mathrm{B}_{n+1}(\mathbb{D})$.

Theorem 2.1.2. *The multiplication operator* $M:=M^{(\alpha,\beta)}_n$ *is irreducible.*

The proof of this theorem will be facilitated by a series of lemmas which are proved in the sequel. We first describe the notion of a normalized kernel which was introduced by Curto-Salinas and plays a significant role in this thesis.

Let $\widehat{K}(z,w)=K(0,0)^{-1/2}K(z,w)K(0,0)^{-1/2}$, so that $\widehat{K}(0,0)=I$. Also, let $\tilde{K}(z,w)=\widehat{K}(z,0)^{-1}\widehat{K}(z,w)\widehat{K}(0,w)^{-1}$. This ensures that $\tilde{K}(z,0)=I$ for $z\in\mathbb{D}$, that is, $\tilde{K}$ is a normalized kernel (cf . [21, Remark 4.7 (b)]). Each of the kernels K, $\widehat{K}$ and $\tilde{K}$ admit a power series expansion, say, $K(z,w)=\sum_{m,p\geq 0}a_{mp}\,z^m\bar{w}^p$, $\widehat{K}(z,w)=\sum_{m,p\geq 0}\widehat{a}_{mp}\,z^m\bar{w}^p$, and $\tilde{K}(z,w)=\sum_{m,p\geq 0}\tilde{a}_{mp}\,z^m\bar{w}^p$ for $z,\ w\in\mathbb{D}$, respectively. Here the coefficients a_{mp} and $\widehat{a}_{mp}$ and $\tilde{a}_{mp}$ are in $\mathcal{M}_{n+1}$ for $m,\ p\geq 0$. In particular, $\widehat{a}_{mp}=K(0,0)^{-1/2}a_{mp}\,K(0,0)^{-1/2}=a_{00}^{-1/2}a_{mp}\,a_{00}^{-1/2}$ for $m,p\geq 0$. Also, let us write $K(z,w)^{-1}=\sum_{m,p\geq 0}b_{mp}\,z^m\bar{w}^p$ and $\widehat{K}(z,w)^{-1}=\sum_{m,p\geq 0}\widehat{b}_{mp}\,z^m\bar{w}^p$, $z,w\in\mathbb{D}$. Again, the coefficients b_{mp} and $\widehat{b}_{mp}$ are in $\mathcal{M}_{n+1}$ for $m,\ p\geq 0$. However, $\tilde{a}_{00}=I$ and $\tilde{a}_{m0}=\tilde{a}_{0p}=0$ for $m,\ p\geq 1$.

We set $K=B^{(\alpha,\beta)}_n$ for simplicity of notation. The following Theorem is from [21, Theorem 3.7, Remark 3.8 and Lemma 3.9]. The proof was discussed in section 1.2.

Theorem 2.1.3. *The multiplication operators on Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *with reproducing kernels* $K_1(z,w)$ *and* $K_2(z,w)$ *respectively, are unitarily equivalent if and only if* $K_2(z,w)=\Psi(z)K_1(z,w)\overline{\Psi(w)}^{\mathrm{tr}}$*, where* Ψ *is an invertible matrix-valued holomorphic function.*

The proof of the lemma below appears in [31, Lemma 5.2] and is discussed in section 1.2, see Remark 1.2.1.

Lemma 2.1.4. *The multiplication operator* M *on the Hilbert space* $\mathcal{H}$ *with reproducing kernel* K *is irreducible if and only if there is no non-trivial projection* P *on* $\mathbb{C}^{n+1}$ *commuting with all the coefficients in the power series expansion of the normalized kernel* $\tilde{K}(z,w)$.

We will prove irreducibility of M by showing that only operators on $\mathbb{C}^{n+1}$ which commutes with all the coefficients of $\tilde{K}(z,w)$ are scalars. It turns out that the coefficients of $z^k\bar{w}$ for $2 \le k \le n+1$, that is, the coefficients $\tilde{a}_{k1}$ for $2 \le k \le n+1$ are sufficient to reach the desired conclusion.

Lemma 2.1.5. *The coefficient of $z^k\bar{w}$ is $\tilde{a}_{k1} = \sum_{s=1}^{k} \widehat{b}_{s0}\widehat{a}_{k-s,1} + \widehat{a}_{k1}$ for $1 \le k \le n+1$.*

Proof. Let us denote the coefficient of $z^k\bar{w}^\ell$ in the power series expansion of $\tilde{K}(z,w)$ by $\tilde{a}_{k\ell}$ for $k,\ell \ge 0$. We see that

$$\begin{aligned}\tilde{a}_{k\ell} &= \sum_{s=0}^{k}\sum_{t=0}^{\ell} \widehat{b}_{s0}\widehat{a}_{k-s,\ell-t}\widehat{b}_{0t}\\ &= \sum_{s=1}^{k}\sum_{t=1}^{\ell} \widehat{a}_{s0}\widehat{a}_{k-s,\ell-t}\widehat{b}_{0t} + \sum_{s=1}^{k} \widehat{b}_{s0}\widehat{a}_{k-s,\ell} + \sum_{t=1}^{\ell} \widehat{a}_{k,\ell-t}\widehat{b}_{0t} + \widehat{a}_{k\ell}\end{aligned}$$

as $\widehat{a}_{00} = \widehat{b}_{00} = I$. Also,

$$\begin{aligned}\tilde{a}_{k1} &= \sum_{s=1}^{k} \widehat{b}_{s0}\widehat{a}_{k-s,0}\widehat{b}_{01} + \sum_{s=1}^{k} \widehat{b}_{s0}\widehat{a}_{k-s,1} + \widehat{a}_{k0}\widehat{b}_{01} + \widehat{a}_{k1}\\ &= \big(\sum_{s=0}^{k} \widehat{b}_{s0}\widehat{a}_{k-s,0}\big)\widehat{b}_{01} + \sum_{s=1}^{k} \widehat{b}_{s0}\widehat{a}_{k-s,1} + \widehat{a}_{k1}\\ &= \sum_{s=1}^{k} \widehat{b}_{s0}\widehat{a}_{k-s,1} + \widehat{a}_{k1}\end{aligned}$$

as $\widehat{b}_{00} = I$ and coefficient of z^k in $\widehat{K}(z,w)^{-1}\widehat{K}(z,w) = \sum_{s=0}^{k} \widehat{b}_{s0}\widehat{a}_{k-s,0} = 0$ for $k \ge 1$. □

Now we compute some of the coefficients of $K(z,w)$ which are useful in computing $\tilde{a}_{k1}$. In what follows, we will compute only the non-zero entries of the matrices involved, that is, *all those entries which are not specified are assumed to be zero.*

Notation 2.1.6. *For a positive integer m, let $S(c_1,\dots,c_m)$ denote the forward shift on $\mathbb{C}^{m+1}$ with weight sequence $(c_1,\dots,c_m)$, $c_i \in \mathbb{C}$, that is,*

$$S(c_1,\dots,c_m)(\ell,p) = c_\ell\delta_{p+1,\ell} \quad \text{for} \quad 0 \le p,\ell \le m.$$

We set $\mathbb{S}_m := S(1,\dots,m)$. For A in $\mathcal{M}_{p,q}$, we let $A(i,j)$ denote the (i,j)-th entry of the matrix A for $1 \le i \le p$, $1 \le j \le q$ and $A(i,j)$ is understood to be zero if the ordered pair $(i,j) \notin \{1\dots p\} \times \{1,\dots q\}$. For a vector $\boldsymbol{v}$ in $\mathbb{C}^k$, let $\boldsymbol{v}(i)$ denote the i-th component of the vector $\boldsymbol{v}$, $1 \le i \le k$.

For $x \in \mathbb{C}$, $(x)_0 = 1$ and $(x)_n = x(x+1)\dots(x+n-1)$ for $n \ge 1$.

Lemma 2.1.7. *In the notation as above, we have*

$a_{00}(k,k) = k!(\beta)_k$ *for* $0 \le k \le n$.

$$a_{m0}(r, r+m) = \frac{(m+r)!}{m!}(\beta)_{m+r} \text{ for } 0 \le r \le n-m, 0 \le m \le n,$$

and

$$a_{m+1,1}(r, r+m) = \frac{(m+r)!}{m!}(\beta)_{m+r}\big(\alpha + (1 + \frac{r}{m+1})(\beta + m + r)\big)$$

for $0 \le r \le n-m, 0 \le m \le n$.

Proof. The coefficient of $z^p \bar{w}^q$ in $B_n^{(\alpha,\beta)}(z,w)$ is the same as the coefficient of $z^p \bar{z}^q$ in $B_n^{(\alpha,\beta)}(z,z)$. Recalling that $\mathbb{S}(z) = (1-|z|^2)^{-1}$ we have $a_{00}(k,k) =$ constant term in $\bar{\partial}_2^k \partial_2^k \big(\mathbb{S}(z_1)^\alpha \mathbb{S}(z_2)^\beta\big)|_\triangle$. Now,

$$\begin{aligned}
\bar{\partial}_2^k \partial_2^k \big(\mathbb{S}(z_1)^\alpha \mathbb{S}(z_2)^\beta\big)|_\triangle &= \bar{\partial}_2^k \partial_2^k \big(\mathbb{S}(z_1)^\alpha \mathbb{S}(z_2)^\beta\big)|_\triangle \\
&= \mathbb{S}(z_1)^\alpha (\beta)_k \bar{\partial}_2^k \big(\mathbb{S}(z_2)^{\beta+k} \bar{z}_2^k\big)|_\triangle \\
&= \big(\mathbb{S}(z_1)^\alpha (\beta)_k \sum_{\ell=0}^{k} \binom{k}{\ell} \bar{\partial}_2^{k-\ell}(\mathbb{S}(z_2)^{\beta+k}) \bar{\partial}_2^\ell(\bar{z_2}^k)\big)|_\triangle \\
&= \big(\mathbb{S}(z_1)^\alpha (\beta)_k \sum_{\ell=0}^{k} \binom{k}{\ell} (\beta+k)_{k-\ell} \mathbb{S}(z_2)^{\beta+k+(k-\ell)} {z_2}^{k-\ell} \ell! \binom{k}{\ell} \bar{z_2}^{k-\ell}\big)|_\triangle,
\end{aligned}$$

that is, $a_{00}(k,k) = k!(\beta)_k$ for $0 \le k \le n$.

We see that $a_{m0}(r, r+m)$ is the coefficient of z^m in $\bar{\partial}_2^{m+r} \partial_2^r \big(\mathbb{S}(z_1)^\alpha \mathbb{S}(z_2)^\beta\big)|_\triangle$. Thus

$$\begin{aligned}
\bar{\partial}_2^{m+r} \partial_2^r \big(\mathbb{S}(z_1)^\alpha \mathbb{S}(z_2)^\beta\big)|_\triangle &= \mathbb{S}(z_1)^\alpha (\beta)_r \bar{\partial}_2^{m+r} \big(\mathbb{S}(z_2)^{\beta+r} \bar{z}_2^r\big)|_\triangle \\
&= \big(\mathbb{S}(z_1)^\alpha (\beta)_r \sum_{\ell=0}^{m+r} \tbinom{m+r}{\ell} \bar{\partial}_2^{m+r-\ell}(\mathbb{S}(z_2)^{\beta+r}) \bar{\partial}_2^\ell(\bar{z_2}^r)\big)|_\triangle \\
&= \big(\mathbb{S}(z_1)^\alpha (\beta)_r \sum_{\ell=0}^{m+r} \tbinom{m+r}{\ell} (\beta+r)_{m+r-\ell} \mathbb{S}(z_2)^{\beta+2r+m-\ell} {z_2}^{m+r-\ell} \ell! \binom{r}{\ell} \bar{z_2}^{r-\ell}\big)|_\triangle.
\end{aligned}$$

Therefore, the term containing z^m occurs only when $\ell = r$ in the sum above, that is,

$$a_{m0}(r, r+m) = (\beta)_r \binom{m+r}{r} (\beta+r)_m r! = \frac{(m+r)!}{m!}(\beta)_{m+r}, \text{ for } 0 \le r \le n-m, 0 \le m \le n.$$

One observes that $a_{m+1,1}(r, r+m)$ is the coefficient of $z^{m+1}\bar{z}$ in $\bar{\partial}_2^{m+r} \partial_2^r \big(\mathbb{S}(z_1)^\alpha \mathbb{S}(z_2)^\beta\big)|_\triangle$. For any real analytic function f on $\mathbb{D}$, for now, let $\big(f(z,\bar{z})\big)_{(p,q)}$ denote the coefficient of $z^p \bar{z}^q$ in $f(z,\bar{z})$. We have

$$\begin{aligned}
a_{m+1,1}(r, r+m) &= \big(\bar{\partial}_2^{m+r} \partial_2^r \big(\mathbb{S}(z_1)^\alpha \mathbb{S}(z_2)^\beta\big)|_\triangle\big)_{(m+1,1)} \\
&= \Big((\beta)_r \sum_{\ell=0}^{m+r} \tbinom{m+r}{\ell} (\beta+r)_{m+r-\ell} \mathbb{S}(z)^{\alpha+\beta+r+(m+r-\ell)} z^{m+r-\ell} \ell! \binom{r}{\ell} \bar{z}^{r-\ell}\Big)_{(m+1,1)}
\end{aligned}$$

The terms containing $z^{m+1}\bar{z}$ occurs in the sum above, only when $\ell = r$ and $\ell = r-1$, that is,

$$\begin{aligned}
a_{m+1,1}(r, r+m) &= \Big((\beta)_r r!\Big(\tbinom{m+r}{r}(\beta+r)_m \mathbb{S}(z)^{\alpha+\beta+m+r} z^m \\
&\qquad + \tbinom{m+r}{r-1}(\beta+r)_{m+1}\mathbb{S}(z)^{\alpha+\beta+m+r+1} z^{m+1}\bar{z}\Big)\Big)_{(m+1,1)} \\
&= \Big((\beta)_r r!\Big(\frac{(m+r)!}{r!m!}(\beta+r)_m(1+(\alpha+\beta+m+r)|z|^2)z^m \\
&\qquad + \frac{(m+r)!r}{r!(m+1)!}(\beta+r)_{m+1}\mathbb{S}(z)^{\alpha+\beta+m+r+1} z^{m+1}\bar{z}\Big)\Big)_{(m+1,1)} \\
&= \frac{(m+r)!}{m!}(\beta)_{m+r}\big((\alpha+\beta+m+r) + \frac{r}{m+1}(\beta+m+r)\big) \\
&= \frac{(m+r)!}{m!}(\beta)_{m+r}\big(\alpha + (1+\frac{r}{m+1})(\beta+m+r)\big),
\end{aligned}$$

for $0 \le r \le n-m, 0 \le m \le n$, where we have followed the convention: $\binom{p}{q} = 0$ for a negative integer q. This completes the proof. ☐

Lemma 2.1.8. *Let c_{k0} denote $a_{00}^{1/2}\widehat{b}_{k0}a_{00}^{1/2}$. We have*

$$c_{k0}(r, r+k) = \frac{(-1)^k(r+k)!}{k!}(\beta)_{r+k} \text{ for } 0 \le r \le n-k,\ 0 \le k \le n.$$

Proof. Recall that

$$\widehat{K}(z,w)^{-1} = a_{00}^{1/2}K(z,w)^{-1}a_{00}^{1/2} = \sum_{m,n\ge 0}\big(a_{00}^{1/2}b_{mn}a_{00}^{1/2}\big)z^m\bar{w}^n.$$

Hence $\widehat{b}_{mn} = a_{00}^{1/2}b_{mn}a_{00}^{1/2}$ for $m, n \ge 0$. By invertibility of a_{00}, we see that $\widehat{b}_{k0}$ and c_{k0} uniquely determine each other for $k \ge 0$. Since $(\widehat{b}_{k0})_{k\ge 0}$ are uniquely determined as the coefficients of power series expansion of $\widehat{K}(z,w)^{-1}$, it is enough to prove that $\sum_{\ell=0}^{m}\widehat{a}_{m-\ell}\widehat{b}_{\ell 0} = 0$ for $1 \le m \le n$. Equivalently, we must show that $\sum_{\ell=0}^{m}(a_{00}^{-1/2}a_{m-\ell,0}a_{00}^{-1/2})(a_{00}^{-1/2}c_{\ell 0}a_{00}^{-1/2}) = 0$ which amounts to showing $a_{00}^{-1/2}\big(\sum_{\ell=0}^{m}a_{m-\ell,0}a_{00}^{-1}c_{\ell 0}\big)a_{00}^{-1/2} = 0$ for $1 \le m \le n$. It follows from Lemma 2.1.7 that $a_{m-\ell,0}(r, r+(m-\ell)) = \frac{(m-\ell+r)!}{(m-\ell)!}(\beta)_{m-\ell+r}$ and $a_{00}(r,r) = r!(\beta)_r$. Therefore

$$\begin{aligned}
(a_{m-\ell,0}a_{00}^{-1})(r, r+(m-\ell)) &= a_{m-\ell,0}(r, r+(m-\ell))a_{00}^{-1}(r+(m-\ell), r+(m-\ell)) \\
&= \frac{(m-\ell+r)!}{(m-\ell)!}(\beta)_{m-\ell+r}\big((m-\ell+r)!(\beta)_{m-\ell+r}\big)^{-1} \\
&= \frac{1}{(m-\ell)!}.
\end{aligned}$$

We also have

$$\begin{aligned}
&(a_{m-\ell,0}a_{00}^{-1}c_{\ell 0})(r, r+m) \\
&= (a_{m-\ell,0}a_{00}^{-1})(r, r+(m-\ell))c_{\ell 0}(r+(m-\ell), r+(m-\ell)+\ell) \\
&= \frac{(-1)^\ell(r+m)!}{(m-\ell)!\ell!}(\beta)_{r+m}
\end{aligned}$$

for $0 \le \ell \le m, 0 \le r \le n-m, 1 \le m \le n$. Now observe that

$$\begin{aligned}\Big(\sum_{\ell=0}^{m} a_{m-\ell,0}a_{00}^{-1}c_{\ell 0}\Big)(r,r+m) &= (r+m)!(\beta)_{m+r}\sum_{\ell=0}^{m}\frac{(-1)^\ell}{(m-\ell)!\ell!}\\ &= \frac{(r+m)!}{m!}(\beta)_{m+r}\sum_{\ell=0}^{m}(-1)^\ell\binom{m}{\ell}\\ &= 0,\end{aligned}$$

which completes the proof of this lemma. □

Lemma 2.1.9. *The matrix entry $\tilde{a}_{k1}(n-k+1,n)$ is a non-zero real number, for $2 \le k \le n+1$, $n \ge 1$. All other entries of $\tilde{a}_{k1}$ are zero.*

Proof. From Lemma 2.1.5 and Lemma 2.1.8, we know that

$$\begin{aligned}\tilde{a}_{k1} &= \sum_{s=1}^{k}\widehat{b}_{s0}\widehat{a}_{k-s,1}+\widehat{a}_{k1}\\ &= \sum_{s=1}^{k}(a_{00}^{-1/2}c_{s0}a_{00}^{-1/2})(a_{00}^{-1/2}a_{k-s,1}a_{00}^{-1/2})+a_{00}^{-1/2}a_{k1}a_{00}^{-1/2}.\end{aligned}$$

Consequently, $a_{00}^{1/2}\tilde{a}_{k1}a_{00}^{1/2} = \sum_{s=1}^{k} c_{s0}a_{00}^{-1}a_{k-s,1}+a_{k1}$ for $1 \le k \le n+1$.

By Lemma 2.1.7 and Lemma 2.1.8, we have

$$\begin{aligned}(c_{s0}a_{00}^{-1})(r,r+s) &= c_{s0}(r,r+s)a_{00}^{-1}(r+s,r+s)\\ &= \frac{(-1)^s(r+s)!}{s!}(\beta)_{r+s}\big((r+s)!(\beta)_{r+s}\big)^{-1}\\ &= \frac{(-1)^s}{s!},\end{aligned}$$

for $0 \le r \le n-s, 0 \le s \le k, 1 \le k \le n+1$.

$$\begin{aligned}&a_{k-s,1}(r,r+(k-s-1))\\ &= \frac{(k+r-s-1)!}{(k-s-1)!}(\beta)_{r+k-s-1}\big(\alpha+(1+\frac{r}{k-s})(\beta+r+k-s-1)\big),\end{aligned}$$

for $k-s-1 \ge 0, 2 \le k \le n+1$. Now,

$$\begin{aligned}&(c_{s0}a_{00}^{-1}a_{k-s,1})(r+s,r+s+(k-s-1))\\ &= (c_{s0}a_{00}^{-1})(r,r+s)a_{k-s,1}(r+s,r+s+(k-s-1))\\ &= \frac{(-1)^s}{s!}\frac{(r+k-1)!}{(k-s-1)!}(\beta)_{r+k-1}\big(\alpha+(1+\frac{r+s}{k-s})(\beta+r+k-1)\big),\end{aligned}$$

for $1 \le s \le k-1, 0 \le r \le n-k+1, 1 \le k \le n+1$. Hence

$$\begin{aligned}&(c_{s0}a_{00}^{-1}a_{k-s,1})(r+s,r+k-1)\\ &= \frac{(-1)^s}{s!}\frac{(r+k-1)!}{(k-s-1)!}(\beta)_{r+k-1}\big(\alpha+\frac{k+r}{k-s}(\beta+r+k-1)\big).\end{aligned}$$

Since $\overline{K(z,w)}^{\rm tr} = K(w,z)$, it follows that $a_{mn} = \overline{a_{nm}}^{\rm tr}$ for $m,n \geq 0$. Thus, by Lemma 2.1.7,

$$a_{01}(r+1,r) = (r+1)!(\beta)_{r+1} \text{ for } 0 \leq r \leq n-1,$$
$$(c_{k0}a_{00}^{-1})(r,r+k) = \frac{(-1)^k}{k!}, \text{ for } 0 \leq r \leq n-k, 1 \leq k \leq n+1$$

and

$$(c_{k0}a_{00}^{-1}a_{01})(r,r+k-1) = (c_{k0}a_{00}^{-1})(r,r+k)a_{01}(r+k,r+k-1) = \frac{(-1)^k}{k!}(r+k)!(\beta)_{r+k},$$

$0 \leq r \leq n-k, 1 \leq k \leq n+1$. Since $c_{00} = a_{00}$, we have for $0 \leq r \leq n-k, 2 \leq k \leq n+1$,

$$\begin{aligned}
(a_{00}^{1/2}\tilde{a}_{k1}a_{00}^{1/2})(r,r+k-1) &= \Big(\sum_{s=1}^{k} c_{s0}a_{00}^{-1}a_{k-s,1} + a_{k1}\Big)(r,r+k-1)\\
&= \Big(\sum_{s=0}^{k-1} c_{s0}a_{00}^{-1}a_{k-s,1} + c_{k0}a_{00}^{-1}a_{01}\Big)(r,r+k-1)\\
&= \sum_{s=0}^{k-1} \tfrac{(-1)^s(k+r-1)!}{s!(k-s-1)!}(\beta)_{r+k-1}\big(\alpha + \tfrac{k+r}{k-s}(\beta+r+k-1)\big) + \frac{(-1)^k(r+k)!}{k!}(\beta)_{r+k}\\
&= \alpha(\beta)_{r+k-1}\tfrac{(k+r-1)!}{(k-1)!}\sum_{s=0}^{k-1}(-1)^s\tbinom{k-1}{s} + (\beta)_{k+r}\big(\sum_{s=0}^{k-1}\tfrac{(-1)^s(k+r)!}{s!(k-s)!} + \tfrac{(-1)^k(k+r)!}{k!}\big)\\
&= \tfrac{(k+r)!}{k!}(\beta)_{k+r}\sum_{s=0}^{k}(-1)^s\binom{k}{s}.
\end{aligned}$$

Therefore $(a_{00}^{1/2}\tilde{a}_{k1}a_{00}^{1/2})(r,r+k-1) = 0$. Now, $c_{00} = a_{00}$ and $(c_{k0}a_{00}^{-1}a_{01})(n-k+1,n) = 0$ for $2 \leq k \leq n+1$. Hence

$$\begin{aligned}
(a_{00}^{1/2}\tilde{a}_{k1}a_{00}^{1/2})(n-k+1,n) &= \Big(\sum_{s=1}^{k} c_{s0}a_{00}^{-1}a_{k-s,1} + a_{k1}\Big)(n-k+1,n)\\
&= \Big(\sum_{s=0}^{k-1} c_{s0}a_{00}^{-1}a_{k-s,1}\Big)(n-k+1,n)\\
&= \sum_{s=0}^{k-1} \tfrac{(-1)^s(k+(n-k+1)-1)!}{s!(k-s-1)!}(\beta)_n\Big(\alpha + \tfrac{k+(n-k+1)}{k-s}(\beta+n)\Big)\\
&= n!(\beta)_n\Big(\alpha\sum_{s=0}^{k-1}\tfrac{(-1)^s}{s!(k-1-s)!} + (n+1)(\beta+n)\sum_{s=0}^{k-1}\tfrac{(-1)^s}{s!(k-s)!}\Big)\\
&= n!(\beta)_n\Big(\tfrac{\alpha}{(k-1)!}\sum_{s=0}^{k-1}(-1)^s\tbinom{k-1}{s} + \tfrac{(n+1)(\beta+n)}{k!}\sum_{s=0}^{k}(-1)^s\tbinom{k}{s} - \tfrac{(-1)^k(n+1)(\beta+n)}{k!}\Big)\\
&= \tfrac{(-1)^{k+1}(n+1)!(\beta)_{n+1}}{k!}, \text{ for } 2 \leq k \leq n+1.
\end{aligned}$$

Since a_{00} is a diagonal matrix with positive diagonal entries, $\tilde{a}_{k1}$ has the form as stated in the lemma, for $2 \leq k \leq n+1, n \geq 1$. □

Here is a simple lemma which will be useful for us in the sequel.

Lemma 2.1.10. *Let $\{A_k\}_{k=0}^{n-1}$ be in $\mathcal{M}_{n+1}$ such that $A_k(k,n) = \lambda_k \neq 0$ for $0 \leq k \leq n-1$, $n \geq 1$. If $AA_k = A_kA$ for some matrix $A = \big(\!\big(A(i,j)\big)\!\big)_{i,j=0}^{n}$ in $\mathcal{M}_{n+1}$ for $0 \leq k \leq n-1$, then A is upper triangular with equal diagonal entries.*

Proof. $AA_k(i,n) = A(i,k)A_k(k,n) = A(i,k)\lambda_k$ and $A_kA(k,j) = A_k(k,n)A(n,j) = \lambda_k A(n,j)$ for $0 \leq i,j \leq n, 0 \leq k \leq n-1$. Putting $i = k$ and $j = n$, we have $AA_k(k,n) = A(k,k)\lambda_k$ and $A_kA(k,n) = \lambda_k A(n,n)$. By hypothesis we have $A(k,k)\lambda_k = \lambda_k A(n,n)$. As $\lambda_k \neq 0$, this implies that $A(k,k) = A(n,n)$ for $0 \leq k \leq n-1$, which is same as saying that A has equal diagonal entries. Now observe that $A_kA(i,j) = 0$ if $i \neq k$ for $0 \leq j \leq n$, which implies that $A_kA(i,n) = 0$ if $i \neq k$. By hypothesis this is same as $AA_k(i,n) = A(i,k)\lambda_k = 0$ if $i \neq k$. This implies $A(i,k) = 0$ if $i \neq k, 0 \leq i \leq n, 0 \leq k \leq n-1$, which is a stronger statement than saying A is upper triangular. ☐

Lemma 2.1.11. *If a matrix A in $\mathcal{M}_{n+1}$ commutes with $\tilde{a}_{k1}$ and $\tilde{a}_{1k}$ for $2 \leq k \leq n+1, n \geq 1$, then A is a scalar.*

Proof. It follows from Lemma 2.1.9 and Lemma 2.1.10 that if A commutes with $\tilde{a}_{k1}$ for $2 \leq k \leq n+1$, then A is upper triangular with equal diagonal entries. As the entries of $\tilde{a}_{k1}$ are real, $\tilde{a}_{1k} = (\tilde{a}_{k1})^{\mathrm{tr}}$. If A commutes with $\tilde{a}_{1k}$ for $2 \leq k \leq n+1$, then by a similar proof as in Lemma 2.1.10, it follows that A is lower triangular with equal diagonal entries. So, A is both upper triangular and lower triangular with equal diagonal entries, hence A is a scalar. ☐

This sequence of Lemmas put together constitutes a proof of Theorem 4.2.

For the operator $M_1^{(\alpha,\beta)^*}$ in the class $\mathrm{B}_2(\mathbb{D})$, we have a proof of irreducibility that avoids the normalization of the kernel. This proof makes use of the fact that if such an operator is reducible then each of the direct summands must belong to the class $\mathrm{B}_1(\mathbb{D})$. We give a precise formulation of this phenomenon along with a proof below. Recall that $B^{(\alpha,\beta)}$ is a positive definite kernel on $\mathbb{D}^2$ and $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ be the corresponding Hilbert space. We know that the pair (M_1, M_2) on $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)$ is in $\mathrm{B}_1(\mathbb{D}^2)$. The operator $M_1^{(\alpha,\beta)^*}$ is the adjoint of the multiplication operator on Hilbert space $\mathbb{A}_{1\ \mathrm{res}}^{(\alpha,\beta)}(\mathbb{D}^2)$ which consists of $\mathbb{C}^2$-valued holomorphic function on $\mathbb{D}$ and possesses the reproducing kernel $B_1^{(\alpha,\beta)}(z,w)$. The operator $M_1^{(\alpha,\beta)^*}$ is in $\mathrm{B}_2(\mathbb{D})$ (cf. [24, Proposition 3.6]).

Proposition 2.1.12. *The operator $M_1^{(\alpha,\beta)^*}$ on Hilbert space $\mathbb{A}_{1\ \mathrm{res}}^{(\alpha,\beta)}(\mathbb{D})$ is irreducible.*

Proof. If possible, let $M_1^{(\alpha,\beta)^*}$ be reducible, that is, $M_1^{(\alpha,\beta)^*} = T_1 \oplus T_2$ for some $T_1, T_2 \in \mathrm{B}_1(\mathbb{D})$. This is the same as saying [18, Proposition 1.18] that the associated bundle $E_{M_1^{(\alpha,\beta)^*}}$ is reducible. A metric on the associated bundle $E_{M_1^{(\alpha,\beta)^*}}$ is given by $h(z) = B_1^{(\alpha,\beta)}(z,z)^{\mathrm{tr}}$. So, there exists a holomorphic change of frame $\psi : \mathbb{D} \longrightarrow GL(2,\mathbb{C})$ such that $\overline{\psi(z)}^{\mathrm{tr}} h(z)\psi(z) = \begin{pmatrix} h_1(z) & 0 \\ 0 & h_2(z) \end{pmatrix}$ for $z \in \mathbb{D}$, where h_1 and h_2 are metrics on the associated line bundles E_{T_1} and E_{T_2} respectively.

So, $\psi(z)^{-1}\mathcal{K}_h(z)\psi(z) = \begin{pmatrix} \mathcal{K}_{h_1}(z) & 0 \\ 0 & \mathcal{K}_{h_2}(z) \end{pmatrix}$, where $\mathcal{K}_h(z) = \frac{\partial}{\partial \bar z}(h^{-1}\frac{\partial}{\partial z}h)(z)$ is the curvature of the bundle $E_{M_1^{(\alpha,\beta)*}}$ with respect to the metric h and $\mathcal{K}_{h_i}(z)$ are the curvatures of the bundles E_{T_i} for $i = 1, 2$ as in [18, pp. 211]. A direct computation shows that

$$\mathcal{K}_h(z) = \begin{pmatrix} \alpha & -2\beta(\beta+1)(1-|z|^2)^{-1}\bar z \\ 0 & \alpha+2\beta+2 \end{pmatrix}(1-|z|^2)^{-2}.$$

Thus the matrix $\psi(z)$ diagonalizes $\mathcal{K}_h(z)$ for $z \in \mathbb{D}$. It follows that $\psi(z)$ is determined, that is, the columns of $\psi(z)$ are eigenvectors of $\mathcal{K}_h(z)$ for $z \in \mathbb{D}$. These are uniquely determined up to multiplication by non-vanishing scalar valued functions f_1 and f_2 on $\mathbb{D}$. Now one set of eigenvectors of $\mathcal{K}_h(z)$ is given by $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\beta\bar z \\ 1-|z|^2 \end{pmatrix}\}$ and it is clear that there does not exist any non-vanishing scalar valued function f_2 on $\mathbb{D}$ such that $f_2(z)\begin{pmatrix} -\beta\bar z \\ 1-|z|^2 \end{pmatrix}$ is an eigenvector for $\mathcal{K}_h(z)$ whose entries are holomorphic functions on $\mathbb{D}$. Hence there does not exist any holomorphic change of frame $\psi : \mathbb{D} \longrightarrow GL(2,\mathbb{C})$ such that $\overline{\psi}^{\mathrm{tr}} h\psi = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$ on $\mathbb{D}$. Hence $M_1^{(\alpha,\beta)^*}$ is irreducible. □

Although, the unitary equivalence class of the curvature $\mathcal{K}_T$ of an operator T does not determine the unitary equivalence class of an operator T in $\mathrm{B}_n(\mathbb{D})$ for $n > 1$, here we show that for the homogeneous operators $M_n^{(\alpha,\beta)}$, the eigenvalues of the curvature $\mathcal{K}_{M_n^{(\alpha,\beta)*}}$ determines the unitary equivalence class of these operators in $\mathcal{W}_n$. Let T and $\tilde T$ denote the operators $M_n^{(\alpha,\beta)}$ and $M_n^{(\tilde\alpha,\tilde\beta)}$ respectively.

Theorem 2.1.13. *The operators T and $\tilde T$ are unitarily equivalent if and only if $\alpha = \tilde\alpha$ and $\beta = \tilde\beta$.*

If $\alpha = \tilde\alpha$ and $\beta = \tilde\beta$ then clearly T and $\tilde T$ are unitarily equivalent. To prove the other implication, recall that [24, Proposition 3.6] $T, \tilde T \in \mathrm{B}_{n+1}(\mathbb{D})$. It follows from [18] that if $T, \tilde T \in \mathrm{B}_{n+1}(\mathbb{D})$ are unitarily equivalent then the curvatures $\mathcal{K}_T, \mathcal{K}_{\tilde T}$ of the associated bundles E_T and $E_{\tilde T}$ respectively, are unitarily equivalent as matrix-valued real-analytic functions on $\mathbb{D}$. In particular, this implies that $\mathcal{K}_T(0)$ and $\mathcal{K}_{\tilde T}(0)$ are unitarily equivalent . Therefore, we compute $\mathcal{K}_T(0)$ and $\mathcal{K}_{\tilde T}(0)$. Let $\tilde{\mathcal{K}}_T$ denote the curvature of the bundle E_T with respect to the metric $\tilde h(z) := \tilde K(z,z)^{\mathrm{tr}}$.

Lemma 2.1.14. *The curvature $\tilde{\mathcal{K}}_T(0)$ at 0 of the bundle E_T equals the coefficient of $z\bar z$ in $\tilde h$, that is, $\tilde{\mathcal{K}}_T(0) = \tilde a_{11}^{\mathrm{tr}}$.*

Proof. The curvature of the bundle E_T with respect to the metric $\tilde h(z) = \tilde K(z,z)^{\mathrm{tr}}$ is $\tilde{\mathcal{K}}_T(z) = \frac{\partial}{\partial \bar z}(\tilde h^{-1}\frac{\partial}{\partial z}\tilde h)(z)$. If $\tilde h(z) = \sum_{m,n\geq 0}\tilde h_{mn} z^m \bar z^n$, then $\tilde h_{mn} = \tilde a_{mn}^{\mathrm{tr}}$ for $m, n \geq 0$. So, $\tilde h_{00} = I$ and $\tilde h_{m0} = \tilde h_{0n} = 0$ for $m, n \geq 1$. Hence

$$\tilde{\mathcal{K}}_T(0) = \bar\partial\tilde h^{-1}(0)\partial\tilde h(0) + \tilde h^{-1}(0)\bar\partial\partial\tilde h(0) = (\bar\partial\tilde h^{-1}(0))\tilde h_{10} + \tilde h_{00}^{-1}\tilde h_{11} = \tilde h_{11} = \tilde a_{11}^{\mathrm{tr}}.$$

□

Lemma 2.1.15. $\big(\tilde{\mathcal{K}}_T(0)\big)(i,i) = \alpha$, *for* $i = 0, \ldots, n-1$ *and* $\big(\tilde{\mathcal{K}}_T(0)\big)(n,n) = \alpha + (n+1)(\beta + n)$ *for* $n \geq 1$.

Proof. From Lemma 2.1.14 and Lemma 2.1.5, we know that

$$\tilde{\mathcal{K}}_T(0) = \tilde{a}_{11}^{\rm tr} = \big(\widehat{a}_{11} + \widehat{b}_{10}\widehat{a}_{01}\big)^{\rm tr}.$$

Thus $\tilde{\mathcal{K}}_T(0)$ is the transpose of $a_{00}^{-1/2}(a_{11} + c_{10}a_{00}^{-1}a_{01})a_{00}^{-1/2}$ by Lemma 2.1.8. Now, by Lemma 2.1.7 and Lemma 2.1.8,

$$\begin{aligned} & c_{10}(r, r+1) = -(r+1)!(\beta)_{r+1} \text{ for } 0 \leq r \leq n-1, \\ & a_{00}(r,r) = r!(\beta)_r, a_{11}(r,r) = r!(\beta)_r\big(\alpha + (r+1)(\beta + r)\big) \text{ for } 0 \leq r \leq n \\ \text{and} \qquad & \big(a_{01}\big)_{r+1,r} = (r+1)!(\beta)_{r+1} \text{ for } 0 \leq r \leq n-1. \end{aligned}$$

Therefore, $\big(c_{10}a_{00}^{-1}a_{01}\big)(r,r) = -(r+1)!(\beta)_{r+1}$ for $0 \leq r \leq n-1$. Also,

$$\begin{aligned} & \big(a_{11} + c_{10}a_{00}^{-1}a_{01}\big)(r,r) = \alpha r!(\beta)_{r+1} \text{ for } 0 \leq r \leq n-1, \\ \text{and} \qquad & \big(a_{11} + c_{10}a_{00}^{-1}a_{01}\big)(n,n) = n!(\beta)_n(\alpha + (n+1)(\beta + n)). \end{aligned}$$

Finally, $\tilde{\mathcal{K}}_T(0) = \tilde{a}_{11}^{\rm tr} = \tilde{a}_{11}$, as $\tilde{a}_{11}$ is a diagonal matrix with real entries. In fact, $\big(\tilde{\mathcal{K}}_T(0)\big)(i,i) = \alpha$, for $i = 0, \ldots, n-1$ and $\big(\tilde{\mathcal{K}}_T(0)\big)(n,n) = \alpha + (n+1)(\beta + n)$. □

We now see that T and $\tilde{T}$ are unitarily equivalent implies that $\alpha = \tilde{\alpha}$ and $\alpha + (n+1)(\beta + n) = \tilde{\alpha} + (n+1)(\tilde{\beta} + n)$, that is, $\alpha = \tilde{\alpha}$ and $\beta = \tilde{\beta}$. This proves Theorem 2.1.13.

2.2 Homogeneity of the operator $M_n^{(\alpha,\beta)}$

Theorem 2.2.1. *The multiplication operator* $M := M_n^{(\alpha,\beta)}$ *on the Hilbert space whose reproducing kernel is* $B_n^{(\alpha,\beta)}$ *is homogeneous.*

This theorem is a particular case of the Lemma 2.1.1. A proof first appeared in [9, Theorem 5.2.]. We give an alternative proof of this Theorem by showing that that the kernel is quasi-invariant, that is,

$$K(z,w) = J_{\varphi^{-1}}(z)K\big(\varphi^{-1}(z), \varphi^{-1}(w)\big)\overline{J_{\varphi^{-1}}(w)}^{\rm tr}$$

for some cocycle

$$J : \text{Möb} \times \mathbb{D} \longrightarrow \mathbb{C}^{(n+1)\times(n+1)},\ \varphi \in \text{Möb},\ z,\ w \in \mathbb{D}.$$

First we prove that $K(z,z) = J_{\varphi^{-1}}(z)K\big(\varphi^{-1}(z), \varphi^{-1}(z)\big)\overline{J_{\varphi^{-1}}(z)}^{\rm tr}$ and then polarize to obtain the final result. We begin with a series of lemmas.

Lemma 2.2.2. *Suppose that $J : \text{Möb} \times \mathbb{D} \longrightarrow \mathbb{C}^{(n+1)\times(n+1)}$ is a cocycle. Then the following are equivalent*

1. $K(z,z) = J_{\varphi^{-1}}(z) K\big(\varphi^{-1}(z), \varphi^{-1}(z)\big) \overline{J_{\varphi^{-1}}(z)}^{\text{tr}}$ *for all* $\varphi \in \text{Möb}$ *and* $z \in \mathbb{D}$;
2. $K(0,0) = J_{\varphi^{-1}}(0) K\big(\varphi^{-1}(0), \varphi^{-1}(0)\big) \overline{J_{\varphi^{-1}}(0)}^{\text{tr}}$ *for all* $\varphi \in \text{Möb}$.

Proof. One of the implications is trivial. To prove the other implication, note that

$$\begin{aligned} J_{\varphi_1^{-1}}(0) K\big(\varphi_1^{-1}(0), \varphi_1^{-1}(0)\big) \overline{J_{\varphi_1^{-1}}(0)}^{\text{tr}} &= K(0,0) \\ &= J_{\varphi_2^{-1}}(0) K\big(\varphi_2^{-1}(0), \varphi_2^{-1}(0)\big) \overline{J_{\varphi_2^{-1}}(0)}^{\text{tr}} \end{aligned}$$

for any $\varphi_1, \varphi_2 \in \text{Möb}$ and $z \in \mathbb{D}$. Now pick $\psi \in \text{Möb}$ such that $\psi^{-1}(0) = z$ and taking $\varphi_1 = \psi, \varphi_2 = \psi\varphi$ in the previous identity we see that

$$\begin{aligned} & J_{\psi^{-1}}(0) K\big(\psi^{-1}(0), \psi^{-1}(0)\big) \overline{J_{\psi^{-1}}(0)}^{\text{tr}} \\ = \; & J_{\varphi^{-1}\psi^{-1}}(0) K\big(\varphi^{-1}\psi^{-1}(0), \varphi^{-1}\psi^{-1}(0)\big) \overline{J_{\varphi^{-1}\psi^{-1}}(0)}^{\text{tr}} \\ = \; & J_{\psi^{-1}}(0) J_{\varphi^{-1}}(\psi^{-1}(0)) K\big(\varphi^{-1}\psi^{-1}(0), \varphi^{-1}\psi^{-1}(0)\big) \overline{J_{\varphi^{-1}}(\psi^{-1}(0))}^{\text{tr}} \overline{J_{\psi^{-1}}(0)}^{\text{tr}} \end{aligned}$$

for $\varphi \in \text{Möb}, z \in \mathbb{D}$. Since $J_{\psi^{-1}}(0)$ is invertible, it follows from the equality of first and third expressions that

$$K\big(\psi^{-1}(0), \psi^{-1}(0)\big) = J_{\varphi^{-1}}(\psi^{-1}(0)) K\big(\varphi^{-1}\psi^{-1}(0), \varphi^{-1}\psi^{-1}(0)\big) \overline{J_{\varphi^{-1}}(\psi^{-1}(0))}^{\text{tr}}.$$

This is the same as $K(z,z) = J_{\varphi^{-1}}(z) K\big(\varphi^{-1}(z), \varphi^{-1}(z)\big) \overline{J_{\varphi^{-1}}(z)}^{\text{tr}}$ by the choice of ψ. The proof of this lemma is therefore complete. □

Let $\mathcal{J}_{\varphi^{-1}}(z) = (J_{\varphi^{-1}}(z)^{\text{tr}})^{-1}, \varphi \in$Möb, $z \in \mathbb{D}$, where X^{tr} denotes the transpose of the matrix X. Clearly, $(J_{\varphi^{-1}}(z)^{\text{tr}})^{-1}$ satisfies the cocycle property if and only if $\mathcal{J}_{\varphi^{-1}}(z)$ does and they uniquely determine each other. It is easy to see that the condition

$$K(0,0) = J_{\varphi^{-1}}(0) K\big(\varphi^{-1}(0), \varphi^{-1}(0)\big) \overline{J_{\varphi^{-1}}(0)}^{\text{tr}}$$

is equivalent to

$$h\big(\varphi^{-1}(0)\big) = \overline{\mathcal{J}_{\varphi^{-1}}(0)}^{\text{tr}} h(0) \mathcal{J}_{\varphi^{-1}}(0), \tag{2.2.1}$$

where $h(z)$ is the transpose of $K(z,z)$ as before. It will be useful to define the two functions:

Notation 2.2.3. *We set*

(i) $c : \text{Möb} \times \mathbb{D} \longrightarrow \mathbb{C}$ *with* $c(\varphi^{-1}, z) = (\varphi^{-1})'(z)$ *and*

(ii) $p : \text{Möb} \times \mathbb{D} \longrightarrow \mathbb{C}$ *with* $p(\varphi^{-1}, z) = \frac{\bar{t}a}{1+\bar{t}az}$

for $\varphi_{t,a} \in \text{Möb}$, $t \in \mathbb{T}$, $a \in \mathbb{D}$. *We point out that the function c is the well-known cocycle for the group* Möb .

Lemma 2.2.4. *With notation as above, we have*

(a) $\varphi_{t,a}^{-1} = \varphi_{\bar{t},-ta}$

(b) $\varphi_{s,b}\varphi_{t,a} = \varphi_{\frac{s(t+\bar{a}b)}{1+ta\bar{b}},\frac{a+\bar{t}b}{1+\bar{t}\bar{a}b}}$

(c) $c(\varphi^{-1},\psi^{-1}(z))c(\psi^{-1}(z)) = c(\varphi^{-1}\psi^{-1},z)$ *for* $\varphi,\psi \in$Möb, $z \in \mathbb{D}$

(d) $p(\varphi^{-1},\psi^{-1}(z))c(\psi^{-1},z) + p(\psi^{-1},z) = p(\varphi^{-1}\psi^{-1},z)$ for $\varphi,\psi \in$Möb, $z \in \mathbb{D}$.

Proof. The proof of (a) is a mere verification. We note that

$$\varphi_{s,b}(\varphi_{t,a}(z)) = s\frac{t\frac{z-a}{1-\bar{a}z} - b}{1 - \bar{b}t\frac{z-a}{1-\bar{a}z}} = s\frac{tz - ta - b + \bar{a}bz}{1 - \bar{a}z - t\bar{b}z + ta\bar{b}} = \frac{s(t+\bar{a}b)}{1+ta\bar{b}}\frac{z - \frac{ta+b}{t+\bar{a}b}}{1 - \frac{\bar{a}+t\bar{b}}{1+ta\bar{b}}z},$$

which is (b). The chain rule gives (c). To prove (d), we first note that for $\varphi = \varphi_{t,a}$ and $\psi = \varphi_{s,b}$, if $\psi^{-1}\varphi^{-1} = \varphi_{t',a'}$ for some $(t',a') \in \mathbb{T}\times\mathbb{D}$ then

$$\overline{t'a'} = \frac{\bar{s}(\bar{t}+a\bar{b})}{1+\overline{ta}b}\frac{\bar{a}+t\bar{b}}{1+ta\bar{b}} = \frac{\bar{s}(\bar{b}+\overline{ta})}{1+\overline{ta}b}.$$

It is now easy to verify that

$$\begin{aligned} p(\varphi^{-1},\psi^{-1}(z))c(\psi^{-1},z) + p(\psi^{-1},z) &= \frac{\overline{ta}}{1+\overline{ta}\psi_{s,b}^{-1}(z)}\frac{\bar{s}(1-|b|^2)}{(1+\overline{sb}z)^2} + \frac{\overline{sb}}{1+\overline{sb}z} \\ &= \frac{\bar{s}(\bar{b}+\overline{ta})}{1+\overline{ta}b+\bar{s}(\bar{b}+\overline{ta})z} \\ &= \left(\frac{\bar{s}(\bar{b}+\overline{ta})}{1+\overline{ta}b}\right)\left(1+\frac{\bar{s}(\bar{b}+\overline{ta})}{1+\overline{ta}b}z\right)^{-1} \\ &= p(\varphi^{-1}\psi^{-1},z). \end{aligned}$$

□

Let

$$\big(\mathcal{J}_{\varphi^{-1}}(z)\big)_{ij} = c(\varphi^{-1},z)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i}\binom{j}{i}c(\varphi^{-1},z)^{n-j}p(\varphi^{-1},z)^{j-i} \tag{2.2.2}$$

for $0 \le i \le j \le n$.

Lemma 2.2.5. $J_{\varphi^{-1}}(z)$ *defines a cocycle for the group* Möb.

Proof. To say that $J_{\varphi^{-1}}(z)$ satisfies the cocycle property is the same as saying $\mathcal{J}_{\varphi^{-1}}(z)$ satisfies the cocycle property, which is what we will verify. Thus we want to show that

$$\big(\mathcal{J}_{\psi^{-1}}(z)\mathcal{J}_{\varphi^{-1}}(\psi^{-1}(z))\big)(i,j) = \big(\mathcal{J}_{\varphi^{-1}\psi^{-1}}(z)\big)(i,j) \text{ for } 0 \le i,j \le n.$$

We note that $\mathcal{J}_{\varphi^{-1}}(z)$ is upper triangular, as the product of two upper triangular matrices is again upper triangular, it suffices to prove this equality for $0 \le i \le j \le n$. Clearly, we have

$$
\begin{aligned}
\big(\mathcal{J}_{\psi^{-1}}(z)\mathcal{J}_{\varphi^{-1}}(\psi^{-1}(z))\big)(i,j) &= \sum_{k=i}^{j} \big(\mathcal{J}_{\psi^{-1}}(z)\big)(i,k)\big(\mathcal{J}_{\varphi^{-1}}(\psi^{-1}(z))\big)(k,j) \\
&= c(\psi^{-1},z)^{-\frac{\alpha+\beta}{2}-n} c(\varphi^{-1},\psi^{-1}(z))^{-\frac{\alpha+\beta}{2}-n} \sum_{k=i}^{j} \Big(\frac{(\beta)_k}{(\beta)_i}\binom{k}{i} c(\psi^{-1},z)^{n-k} \\
&\quad p(\psi^{-1},z)^{k-i}\frac{(\beta)_j}{(\beta)_k}\binom{j}{k} c(\varphi^{-1},\psi^{-1}(z))^{n-j} p(\varphi^{-1},\psi^{-1}(z))^{j-k}\Big) \\
&= c(\varphi^{-1}\psi^{-1},z)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i} c(\psi^{-1},z)^{n-j} c(\varphi^{-1},\psi^{-1}(z))^{n-j} \\
&\quad \sum_{k=i}^{j} \frac{j!}{i!(k-i)!(j-k)!} c(\psi^{-1},z)^{j-k} p(\varphi^{-1},\psi^{-1}(z))^{j-k} p(\psi^{-1},z)^{k-i} \\
&= c(\varphi^{-1}\psi^{-1},z)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i}\binom{j}{i} c(\varphi^{-1}\psi^{-1},z)^{n-j} \\
&\quad \sum_{k=i}^{j} \binom{j-i}{k-i} c(\psi^{-1},z)^{j-k} p(\varphi^{-1},\psi^{-1}(z))^{j-k} p(\psi^{-1},z)^{k-i} \\
&= c(\varphi^{-1}\psi^{-1},z)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i}\binom{j}{i} c(\varphi^{-1}\psi^{-1},z)^{n-j} \\
&\quad \sum_{k=0}^{j-i} \binom{j-i}{k} c(\psi^{-1},z)^{(j-i)-k} p(\varphi^{-1},\psi^{-1}(z))^{(j-i)-k} p(\psi^{-1},z)^{k} \\
&= c(\varphi^{-1}\psi^{-1},z)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i}\binom{j}{i} c(\varphi^{-1}\psi^{-1},z)^{n-j} \\
&\quad \Big(c(\psi^{-1},z)p(\varphi^{-1},\psi^{-1}(z)) + p(\psi^{-1},z)\Big)^{j-i} \\
&= c(\varphi^{-1}\psi^{-1},z)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i}\binom{j}{i} c(\varphi^{-1}\psi^{-1},z)^{n-j} p(\varphi^{-1}\psi^{-1},z)^{j-i} \\
&= \big(\mathcal{J}_{\varphi^{-1}\psi^{-1}}(z)\big)(i,j)
\end{aligned}
$$

for $0 \le i \le j \le n$. The penultimate equality follows from Lemma 2.2.4. □

We need the following beautiful identity to prove Lemma (2.2.7). We provide two proofs, the first one is due to C. Varughese and the second is due to B. Bagchi.

Lemma 2.2.6. *For nonnegative integers $j \ge i$ and $0 \le k \le i$, we have*

$$\sum_{\ell=0}^{i-k}(-1)^{\ell}(\ell+k)!\binom{i}{\ell+k}\binom{j}{\ell+k}\binom{\ell+k}{\ell}(a+j)_{i-\ell-k} = k!\binom{i}{k}\binom{j}{k}(a+k)_{i-k},$$

for all $a \in \mathbb{C}$.

Proof. Here is the first proof due to C. Varughese: For any integer $i \geq 1$ and $a \in \mathbb{C} \setminus \mathbb{Z}$, we have

$$\begin{aligned}
&\sum_{\ell=0}^{i-k}(-1)^\ell(\ell+k)!\binom{i}{\ell+k}\binom{j}{\ell+k}\binom{\ell+k}{\ell}(a+j)_{i-\ell-k}\\
=\;& \frac{i!j!}{k!\Gamma(a+j)}\sum_{\ell=0}^{i-k}\frac{(-1)^\ell}{\ell!(i-k-\ell)!}\frac{\Gamma(a+j+i-\ell-k)}{\Gamma(j-\ell-k+1)}\\
=\;& \frac{i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}\sum_{\ell=0}^{i-k}(-1)^\ell\tbinom{i-k}{\ell}B(a+j+i-k-\ell,1-a-i)\\
=\;& \frac{i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}\sum_{\ell=0}^{i-k}(-1)^\ell\tbinom{i-k}{\ell}\int_0^1 t^{a+j+i-k-\ell-1}(1-t)^{-a-i}dt\\
=\;& \frac{i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}\int_0^1\sum_{\ell=0}^{i-k}(-1)^\ell\tbinom{i-k}{\ell}t^{a+j+i-k-\ell-1}(1-t)^{-a-i}dt\\
=\;& \frac{i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}\int_0^1(1-t)^{-a-i}t^{a+j-1}\Big(\sum_{\ell=0}^{i-k}(-1)^\ell\tbinom{i-k}{\ell}t^{i-k-\ell}\Big)dt\\
=\;& \frac{i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}\int_0^1(1-t)^{-a-i}t^{a+j-1}(t-1)^{i-k}dt\\
=\;& \frac{(-1)^{i-k}i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}B(a+j,1-a-k)\\
=\;& \frac{(-1)^{i-k}i!j!}{k!(i-k)!\Gamma(a+j)\Gamma(1-a-i)}\frac{\Gamma(a+j)\Gamma(1-a-k)}{\Gamma(1+j-k)}\\
=\;& \frac{(-1)^{i-k}i!j!}{k!(i-k)!\Gamma(1-a-i)}\frac{\Gamma(1-a-k)}{(j-k)!}\\
=\;& (-1)^{i-k}k!\binom{i}{k}\binom{j}{k}\frac{\Gamma(1-a-k)}{\Gamma(1-a-i)}\\
=\;& k!\binom{i}{k}\binom{j}{k}\frac{\Gamma(1-a)}{(-1)^k\Gamma(1-a-i)}\frac{(-1)^i\Gamma(1-a-k)}{\Gamma(1-a)}\\
=\;& k!\binom{i}{k}\binom{j}{k}\frac{\Gamma(a+i)\sin(a+i)\pi}{\pi\cos k\pi}\frac{\pi\cos i\pi}{\sin(a+k)\pi\Gamma(a+k)}\\
=\;& k!\binom{i}{k}\binom{j}{k}\frac{\Gamma(a+i)}{\Gamma(a+k)}\\
=\;& k!\binom{i}{k}\binom{j}{k}(a+k)_{i-k}.
\end{aligned}$$

Since we have an equality involving a polynomial of degree $i-k$ for all a in $\mathbb{C} \setminus \mathbb{Z}$, it follows that the equality holds for all $a \in \mathbb{C}$.

Here is another proof due to B. Bagchi: Since $\binom{-x}{n} = \frac{-x(-x-1)\cdots(-x-n+1)}{n!} = (-1)^n\binom{x+n-1}{n}$ and

$(x)_n = x(x+1)\cdots(x+n-1) = n!\binom{x+n-1}{n}$, it follows that

$$\begin{aligned}
&\sum_{l=0}^{i-k}(-1)^{\ell}(\ell+k)!\binom{i}{\ell+k}\binom{j}{\ell+k}\binom{\ell+k}{\ell}(a+j)_{i-\ell-k}\\
&= \frac{i!j!}{k!}\sum_{\ell=0}^{i-k}\frac{(-1)^{\ell}}{\ell!(i-k-\ell)!(j-k-\ell)!}(i-k-\ell)!\binom{a+j+i-k-\ell-1}{i-k-\ell}\\
&= \frac{i!j!}{k!(j-k)!}\sum_{\ell=0}^{i-k}\frac{(-1)^{\ell}(j-k)!}{\ell!(j-k-\ell)!}(-1)^{i-k-\ell}\binom{-a-j}{i-k-\ell}\\
&= i!\binom{j}{k}(-1)^{i-k}\sum_{\ell=0}^{i-k}\binom{j-k}{\ell}\binom{-a-j}{i-k-\ell}\\
&= i!\binom{j}{k}(-1)^{i-k}\binom{-a-k}{i-k}\\
&= i!\binom{j}{k}(-1)^{i-k}(-1)^{i-k}\binom{a+i-1}{i-k}\\
&= k!\binom{i}{k}\binom{j}{k}(a+k)_{i-k},
\end{aligned}$$

where the equality after the last summation symbol follows from Vandermonde's identity which says that for $s,t\in\mathbb{C}$ and $n\geq 0$, one has $\sum_{k=0}^{n}\binom{s}{k}\binom{t}{n-k} = \binom{s+t}{n}$. □

Lemma 2.2.7. *For $\varphi \in$ Möb and $\mathcal{J}_{\varphi^{-1}}(z)$ as in (2.2.2), we have*

$$h\big(\varphi^{-1}(0)\big) = \overline{\mathcal{J}_{\varphi^{-1}}(0)}^{\mathrm{tr}} h(0)\mathcal{J}_{\varphi^{-1}}(0).$$

Proof. Since $\overline{h(z)}^{\mathrm{tr}} = h(z)$, it is enough to show that

$$h\big(\varphi^{-1}(0)\big)(i,j) = \big(\overline{\mathcal{J}_{\varphi^{-1}}(0)}^{\mathrm{tr}} h(0)\mathcal{J}_{\varphi^{-1}}(0)\big)(i,j),\ \text{for } 0\leq i\leq j\leq n.$$

Let $\varphi=\varphi_{t,z}$, $t\in\mathbb{T}$ and $z\in\mathbb{D}$. Since $\big(h\big(\varphi^{-1}(0)\big)\big)(i,j) = \big(h(z)\big)(i,j)$, it follows that

$$\begin{aligned}
\big(h\big(\varphi^{-1}(0)\big)\big)(i,j) &= \bar\partial_2^i\partial_2^j\big(\mathbb{S}(z_1)^{\alpha}\mathbb{S}(z_2)^{\beta}\big)|_{\triangle}\\
&= (\beta)_j\mathbb{S}(z_1)^{\alpha}\bar\partial_2^i\big(\mathbb{S}(z_2)^{\beta+j}\bar z_2^j\big)|_{\triangle}\\
&= (\beta)_j\mathbb{S}(z_1)^{\alpha}\sum_{r=0}^{i}\binom{i}{r}\bar\partial_2^{(i-r)}\big(\mathbb{S}(z_2)^{\beta+j}\big)\bar\partial_2^r(\bar z_2^j)|_{\triangle}\\
&= (\beta)_j\mathbb{S}(z_1)^{\alpha}\sum_{r=0}^{i}\binom{i}{r}(\beta+j)_{i-r}\mathbb{S}(z_2)^{\beta+j+(i-r)}z_2^{i-r}r!\binom{j}{r}\bar z_2^{j-r}|_{\triangle}\\
&= (\beta)_j\mathbb{S}(z)^{\alpha+\beta+i+j}\bar z^{j-i}\sum_{r=0}^{i}r!\binom{i}{r}\binom{j}{r}(\beta+j)_{i-r}\mathbb{S}(z)^{-r}|z|^{2(i-r)},
\end{aligned}$$

for $i\leq j$.

Clearly, $\big(\mathcal{J}_{\varphi^{-1}}(0)\big)(i,j) = c(\varphi^{-1},0)^{-\frac{\alpha+\beta}{2}-n}\frac{(\beta)_j}{(\beta)_i}\binom{j}{i}c(\varphi^{-1},0)^{n-j}p(\varphi^{-1},0)^{j-i}$ and $h(0)(i,i) = i!(\beta)_i$, $0\le i\le j\le n$. We have

$$\begin{aligned}
\big(\overline{\mathcal{J}_{\varphi^{-1}}(0)}^{\rm tr}h(0)\mathcal{J}_{\varphi^{-1}}(0)\big)(i,j) &= \sum_{k=0}^{j}\big(\overline{\mathcal{J}_{\varphi^{-1}}(0)}^{\rm tr}h(0)\big)(i,k)\big(\mathcal{J}_{\varphi^{-1}}(0)\big)(k,j)\\
&= \sum_{k=0}^{i}\sum_{k=0}^{j}\big(\overline{\mathcal{J}_{\varphi^{-1}}(0)}^{\rm tr}\big)(i,k)\big(h(0)\big)(k,k)\big(\mathcal{J}_{\varphi^{-1}}(0)\big)(k,j)\\
&= \sum_{k=0}^{\min(i,j)}\big(\overline{\mathcal{J}_{\varphi^{-1}}(0)}^{\rm tr}\big)(i,k)\big(h(0)\big)(k,k)\big(\mathcal{J}_{\varphi^{-1}}(0)\big)(k,j).
\end{aligned}$$

Now, for $0\le i\le j\le n$,

$$\begin{aligned}
\sum_{k=0}^{\min(i,j)}\big(\overline{\mathcal{J}_{\varphi^{-1}}(0)}^{\rm tr}\big)_{ik}\big(h(0)\big)_{kk}\big(\mathcal{J}_{\varphi^{-1}}(0)\big)_{kj} &= |c(\varphi^{-1},0)|^{-\alpha-\beta-2n}\\
&\quad\sum_{k=0}^{i}\Big(\frac{(\beta)_i}{(\beta)_k}\binom{i}{k}\overline{c(\varphi^{-1},0)}^{n-i}\overline{p(\varphi^{-1},0)}^{i-k}k!(\beta)_k\frac{(\beta)_j}{(\beta)_k}\\
&\qquad\binom{j}{k}c(\varphi^{-1},0)^{n-j}p(\varphi^{-1},0)^{j-k}\Big)\\
&= \mathbb{S}(z)^{\alpha+\beta+2n}\sum_{k=0}^{i}\frac{k!(\beta)_i(\beta)_j}{(\beta)_k}\binom{i}{k}\binom{j}{k}\\
&\qquad\big(t\mathbb{S}(z)\big)^{-n+i}(tz)^{i-k}\big(\bar{t}\mathbb{S}(z)\big)^{-n+j}(\overline{tz})^{j-k}\\
&= (\beta)_j\mathbb{S}(z)^{\alpha+\beta+i+j}\sum_{k=0}^{i}k!\binom{i}{k}\binom{j}{k}\frac{(\beta)_i}{(\beta)_k}z^{i-k}\bar{z}^{j-k}\\
&= (\beta)_j\mathbb{S}(z)^{\alpha+\beta+i+j}\bar{z}^{j-i}\sum_{k=0}^{i}k!\binom{i}{k}\binom{j}{k}\frac{(\beta)_i}{(\beta)_k}|z|^{2(i-k)}.
\end{aligned}$$

Clearly, to prove the desired equality we have to show that

$$\sum_{r=0}^{i}r!\binom{i}{r}\binom{j}{r}(\beta+j)_{i-r}\mathbb{S}(z)^{-r}|z|^{2(i-r)} = \sum_{k=0}^{i}k!\binom{i}{k}\binom{j}{k}\frac{(\beta)_i}{(\beta)_k}|z|^{2(i-k)} \tag{2.2.3}$$

for $0\le i\le j\le n$. But

$$\begin{aligned}
&\sum_{r=0}^{i}r!\binom{i}{r}\binom{j}{r}(\beta+j)_{i-r}(1-|z|^2)^r|z|^{2(i-r)}\\
&= \sum_{r=0}^{i}r!\binom{i}{r}\binom{j}{r}(\beta+j)_{i-r}\sum_{\ell=0}^{r}(-1)^\ell\binom{r}{\ell}|z|^{2\ell}|z|^{2(i-r)}\\
&= \sum_{\ell=0}^{i}\sum_{r=\ell}^{i}(-1)^\ell r!\binom{i}{r}\binom{j}{r}\binom{r}{\ell}(\beta+j)_{i-r}|z|^{2(i-(r-\ell))}\\
&= \sum_{\ell=0}^{i}\sum_{r=0}^{i-\ell}(-1)^\ell(r+\ell)!\binom{i}{r+\ell}\binom{j}{r+\ell}\binom{r+\ell}{\ell}(\beta+j)_{i-r-\ell}|z|^{2(i-r)}.
\end{aligned}$$

For $0 \le k \le i-\ell$, the coefficient of $|z|^{2(i-k)}$ in the left hand side of (2.2.3) is

$$\sum_{\ell=0}^{i}(-1)^\ell (k+\ell)!\binom{i}{k+\ell}\binom{j}{k+\ell}\binom{k+\ell}{\ell}(\beta+j)_{i-k-\ell},$$

which is the same as

$$\sum_{\ell=0}^{i-k}(-1)^\ell (k+\ell)!\binom{i}{k+\ell}\binom{j}{k+\ell}\binom{k+\ell}{\ell}(\beta+j)_{i-k-\ell},$$

for $0 \le \ell \le i-k \le i$. So, to complete the proof we have to show that

$$\sum_{\ell=0}^{i-k}(-1)^\ell (k+\ell)!\binom{i}{k+\ell}\binom{j}{k+\ell}\binom{k+\ell}{\ell}(\beta+j)_{i-k-\ell} = k!\binom{i}{k}\binom{j}{k}\frac{(\beta)_i}{(\beta)_k},$$

for $0 \le k \le i, i \le j$. But this follows from Lemma 2.2.6. □

2.3 The case of the tri-disc $\mathbb{D}^3$

Let $\mathcal{M}$ be a Hilbert space of holomorphic functions on $\mathbb{D}^3$ considered as a Hilbert module over the function algebra $\mathcal{A}(\mathbb{D}^3)$. Assume that $\mathcal{M}$ possesses a reproducing kernel $K : \mathbb{D}^3 \times \mathbb{D}^3 \longrightarrow \mathbb{C}$. For $k \ge 1$ let

$$\mathcal{I}_k := \{I = (i_1, i_2) \in (\mathbb{Z}^+)^2 : |I| = i_1 + i_2 \le k\}$$

and $\triangle := \{(z,z,z) : z \in \mathbb{D}\}$ be the diagonal set in $\mathbb{D}^3$. We consider $\mathcal{I}_k^0 \subseteq \mathcal{I}_k$ such that (i) there is at least one $I \in \mathcal{I}_k^0, |I| = k$ and (ii) that the set

$$\mathcal{M}_{\mathcal{I}_k^0} := \{f \in \mathcal{M} : \partial^I f_{|\triangle} = 0 \text{ for } I \in \mathcal{I}_k^0\}$$

of functions vanishing to order $(k+1)$ on the diagonal is a *submodule of* $\mathcal{M}$. Clearly, $\mathcal{M}_{\mathcal{I}_k^0}$ is a submodule of $\mathcal{M}$ if $\mathcal{I}_k^0 = \mathcal{I}_k$. As we shall see in the second example $\mathcal{M}_{\mathcal{I}_k^0}$ can be a submodule of $\mathcal{M}$ even if $\mathcal{I}_k^0 \subsetneq \mathcal{I}_k$.

Following [24], it is not hard to see that jet construction of that paper applies to this case as well. Consequently, as in that paper, it is possible to describe the quotient module explicitly as a reproducing kernel Hilbert space consisting of $\mathbb{C}^{|\mathcal{I}_k^0|}$- valued holomorphic functions on which the algebra $\mathcal{A}(\mathbb{D}^3)$ acts by pointwise multiplication, where $|\mathcal{I}_k^0|$ denotes the cardinality of $\mathcal{I}_k^0$.

Throughout this section, we take $\mathcal{M} = \mathbb{A}^{(\alpha)}(\mathbb{D}) \otimes \mathbb{A}^{(\beta)}(\mathbb{D}) \otimes \mathbb{A}^{(\gamma)}(\mathbb{D})$ and $K = B^{(\alpha,\beta,\gamma)}$, where

$$B^{(\alpha,\beta,\gamma)}(\boldsymbol{z},\boldsymbol{w}) = (1-z_1\bar{w}_1)^{-\alpha}(1-z_2\bar{w}_2)^{-\beta}(1-z_3\bar{w}_3)^{-\gamma}$$

for $\boldsymbol{z} = (z_1,z_2,z_3), \boldsymbol{w} = (w_1,w_2,w_3) \in \mathbb{D}^3$, $\alpha,\beta,\gamma > 0$.

Example 2.3.1. In particular, take $k = 1$ and $\mathcal{I}_1^0 = \mathcal{I}_1 = \{(0,0),(1,0),(0,1)\}$ and let $B_1^{(\alpha,\beta,\gamma)}$ denote the reproducing kernel for the quotient Hilbert module $\mathcal{M} \ominus \mathcal{M}_{\mathcal{I}_1}$. We have:

$$B_1^{(\alpha,\beta,\gamma)}(z,w) = \begin{pmatrix} K(\boldsymbol{z},\boldsymbol{w}) & \partial_2 K(\boldsymbol{z},\boldsymbol{w}) & \partial_3 K(\boldsymbol{z},\boldsymbol{w}) \\ \bar{\partial}_2 K(\boldsymbol{z},\boldsymbol{w}) & \partial_2\bar{\partial}_2 K(\boldsymbol{z},\boldsymbol{w}) & \bar{\partial}_2\partial_3 K(\boldsymbol{z},\boldsymbol{w}) \\ \bar{\partial}_3 K(\boldsymbol{z},\boldsymbol{w}) & \partial_2\bar{\partial}_3 K(\boldsymbol{z},\boldsymbol{w}) & \bar{\partial}_3\partial_3 K(\boldsymbol{z},\boldsymbol{w}) \end{pmatrix}_{|\mathrm{res}\triangle\times\triangle}, \quad z,\, w \in \mathbb{D}.$$

As in section 2.1, we replace $B_1^{(\alpha,\beta,\gamma)}(z,w)$ by its transpose to retain the usual meaning of ∂ and $\bar\partial$. Let $M_1^{(\alpha,\beta,\gamma)}$ denotes the multiplication operator on the quotient $\mathcal{M}\ominus\mathcal{M}_{\mathcal{I}_1}$. For simplicity of notation, we let $H(z,w) := B_1^{(\alpha,\beta,\gamma)^{\rm tr}}$. In this notation, recalling that the kernel function K on $\mathbb{D}^3$ is

$$K(\boldsymbol{z},\boldsymbol{w}) = (1-z_1\bar w_1)^{-\alpha}(1-z_2\bar w_2)^{-\beta}(1-z_3\bar w_3)^{-\gamma},$$

we have

$$H(z,w) = \begin{pmatrix} (1-z\bar w)^2 & \beta z(1-z\bar w) & \gamma z(1-z\bar w) \\ \beta\bar w(1-z\bar w) & \beta(1+\beta z\bar w) & \beta\gamma z\bar w \\ \gamma\bar w(1-z\bar w) & \beta\gamma z\bar w & \gamma(1+\gamma z\bar w) \end{pmatrix}(1-z\bar w)^{-\alpha-\beta-\gamma-2},$$

for $z,w\in\mathbb{D},\quad \alpha,\beta,\gamma>0$.

Theorem 2.3.2. *The adjoint of the multiplication operator $M_1^{(\alpha,\beta,\gamma)^*}$ on the Hilbert space of $\mathbb{C}^3$ valued holomorphic functions on $\mathbb{D}$ with reproducing kernel $B_1^{(\alpha,\beta,\gamma)}$ is in $\mathrm{B}_3(\mathbb{D})$. It is homogeneous and reducible. Moreover, $M_1^{(\alpha,\beta,\gamma)^*}$ is unitarily equivalent to $M_1^*\oplus M_2^*$ for a pair of irreducible homogeneous operators M_1^* and M_2^* from $\mathrm{B}_1(\mathbb{D})$ and $\mathrm{B}_2(\mathbb{D})$ respectively.*

Proof. Although homogeneity of $M_1^{(\alpha,\beta,\gamma)^*}$ follows along the same line as in [9, Theorem 5.2.], we give an independent proof using the ideas we have developed in this chapter. Recalling the notation $B_1^{(\alpha,\beta,\gamma)} = H$ let

$$\tilde H(z,w) = H(0,0)^{1/2}H(z,0)^{-1}H(z,w)H(0,w)^{-1}H(0,0)^{1/2}.$$

Evidently, $\tilde H(z,0)=I$, that is, $\tilde H$ is a normalized kernel at 0. The form of $\tilde H(z,w)$ for $z,w\in\mathbb{D}$ is

$$\tilde H(z,w) = \begin{pmatrix} \begin{matrix}(1-z\bar w)^2-(\beta+\gamma)(1-z\bar w)z\bar w\\ +(\beta+\gamma)(1+\beta+\gamma)z^2\bar w^2\end{matrix} & -\sqrt{\beta}(1+\beta+\gamma)z^2\bar w & -\sqrt{\gamma}(1+\beta+\gamma)z^2\bar w \\ -\sqrt{\beta}(1+\beta+\gamma)z\bar w^2 & 1+\beta z\bar w & \sqrt{\beta\gamma}z\bar w \\ -\sqrt{\gamma}(1+\beta+\gamma)z\bar w^2 & \sqrt{\beta\gamma}z\bar w & 1+\gamma z\bar w \end{pmatrix}(1-z\bar w)^{-\alpha-\beta-\gamma-2}.$$

Let $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{\beta}{\beta+\gamma}} & \sqrt{\frac{\gamma}{\beta+\gamma}} \\ 0 & -\sqrt{\frac{\gamma}{\beta+\gamma}} & \sqrt{\frac{\beta}{\beta+\gamma}} \end{pmatrix}$ which is unitary on $\mathbb{C}^3$. By a direct computation, we see that the equivalent normalized kernel $U\tilde H(z,w)\overline{U}^{\rm tr}$ is equal to the direct sum $H_2(z,w)\oplus H_1(z,w)$, where $H_1(z,w) = (1-z\bar w)^{-\alpha-\beta-\gamma-2}$ and

$$H_2(z,w) = \begin{pmatrix} \begin{matrix}(1-z\bar w)^2-(\beta+\gamma)(1-z\bar w)z\bar w\\ +(\beta+\gamma)(1+\beta+\gamma)z^2\bar w^2\end{matrix} & -\sqrt{\beta+\gamma}(1+\beta+\gamma)z^2\bar w \\ -\sqrt{\beta+\gamma}(1+\beta+\gamma)z\bar w^2 & 1+(\beta+\gamma)z\bar w \end{pmatrix}(1-z\bar w)^{-\alpha-\beta-\gamma-2}.$$

It follows that $M_1^{(\alpha,\beta,\gamma)^*}$ is unitarily equivalent to a reducible operator by an application of Theorem 2.1.3, that is, $M_1^{(\alpha,\beta,\gamma)^*}$ is reducible. If we replace β by $\beta+\gamma$ in Theorem 2.1.2 and take $n=1$, then

$$B_1^{(\alpha,\beta+\gamma)}(z,w) = \begin{pmatrix} (1-z\bar w)^2 & (\beta+\gamma)z(1-z\bar w) \\ (\beta+\gamma)\bar w(1-z\bar w) & (\beta+\gamma)(1+(\beta+\gamma)z\bar w) \end{pmatrix}(1-z\bar w)^{-\alpha-\beta-\gamma-2},$$

for $z, w \in \mathbb{D}$. We observe that

$$H_2(z,w) = \tilde{B}_1^{(\alpha,\beta+\gamma)}(z,w) \text{ for } z,w \in \mathbb{D},$$

where $\tilde{B}_1^{(\alpha,\beta+\gamma)}$ is the normalization of $B_1^{(\alpha,\beta+\gamma)}$ at 0. The multiplication operator corresponding to the reproducing kernel H_2, which we denote by M_2, is unitarily equivalent to $M_1^{(\alpha,\beta+\gamma)}$ by Theorem 2.1.3. Hence it is in $\mathrm{B}_2(\mathbb{D})$ by [24, Proposition 3.6]. Since both homogeneity and irreducibility are invariant under unitary equivalence, it follows by an easy application of Theorem 2.1.3, Theorem 2.1.2 and Theorem 2.2.1 that M_2^* is a irreducible homogeneous operator in $\mathrm{B}_2(\mathbb{D})$. Irreducibility of M_2^* also follows from Proposition 2.1.12. Let M_1 be the multiplication operator on the Hilbert space of scalar valued holomorphic functions with reproducing kernel H_1. Again, M_1^* is in $\mathrm{B}_1(\mathbb{D})$. The operator M_1 is irreducible by [18, corollary 1.19]. Homogeneity of M_1^* was first established in [32], see also [51]. An alternative proof is obtained when we observe that $\Gamma : \text{Möb} \times \mathbb{D} \longrightarrow \mathbb{C}$, where $\Gamma_{\varphi^{-1}}(z) = \big((\varphi^{-1})'(z)\big)^{\frac{\alpha+\beta+\gamma}{2}+1}$ is a cocycle such that $H_1(z,w) = \Gamma_{\varphi^{-1}}(z) H_1\big(\varphi^{-1}(z), \varphi^{-1}(w)\big)\overline{\Gamma_{\varphi^{-1}}(w)}$ for $z,w \in \mathbb{D}, \varphi \in \text{Möb}$. Now, we conclude that $M_1^{(\alpha,\beta,\gamma)^*}$ is homogeneous as it is unitarily equivalent to the direct sum of two homogeneous operators. Also, $M_1^{(\alpha,\beta,\gamma)^*}$ is in $\mathrm{B}_3(\mathbb{D})$ being the direct sum of two operators from the Cowen-Douglas class. □

Example 2.3.3. Now, let us take $k = 2$ and $\mathcal{I}_2^0 = \{(0,0),(1,0),(0,1),(1,1)\}$. This example enables us to produce an *irreducible homogeneous* operator in $\mathrm{B}_4(\mathbb{D})$ whose associated representation is *not* multiplicity–free. We denote the reproducing kernel for the quotient module $\mathcal{M} \ominus \mathcal{M}_{\mathcal{I}_2^0}$ by $B_2^{(\alpha,\beta,\gamma)}$.Let $M_2^{(\alpha,\beta,\gamma)}$ denotes the multiplication operator on the quotient $\mathcal{M} \ominus \mathcal{M}_{\mathcal{I}_2^0}$. As in Example 2.3.1, we have:

$$B_2^{(\alpha,\beta,\gamma)}(z,w) = \begin{pmatrix} (1-z\bar{w})^4 & \beta(1-z\bar{w})^3 z & \gamma(1-z\bar{w})^3 z & \beta\gamma(1-z\bar{w})^2 z^2 \\ \beta(1-z\bar{w})^3\bar{w} & \beta(1+\beta z\bar{w})(1-z\bar{w})^2 & \beta\gamma z\bar{w}(1-z\bar{w})^2 & \beta\gamma(1+\beta z\bar{w})(1-z\bar{w})z \\ \gamma(1-z\bar{w})^3\bar{w} & \beta\gamma z\bar{w}(1-z\bar{w})^2 & \gamma(1+\gamma z\bar{w})(1-z\bar{w})^2 & \beta\gamma(1+\gamma z\bar{w})(1-z\bar{w})z \\ \beta\gamma(1-z\bar{w})^2\bar{w}^2 & \beta\gamma(1+\beta z\bar{w})(1-z\bar{w})\bar{w} & \beta\gamma(1+\gamma z\bar{w})(1-z\bar{w})\bar{w} & \beta\gamma(1+\beta z\bar{w})(1+\gamma z\bar{w}) \end{pmatrix}(1-z\bar{w})^{-\alpha-\beta-\gamma-4}$$

for $z, w \in \mathbb{D}$.

Theorem 2.3.4. *The multiplication operator $M_2^{(\alpha,\beta,\gamma)}$ on the Hilbert space whose reproducing kernel is $B_2^{(\alpha,\beta,\gamma)}$ is irreducible for $\beta \neq \gamma$.*

The proof will consist of a sequence of lemmas. Before going into the proof let us recall:

Notation 2.3.5. *For any reproducing kernel K on $\mathbb{D}$, the normalized kernel $\tilde{K}(z,w)$ at 0 is defined to be the kernel $K(0,0)^{1/2}K(z,0)^{-1}K(z,w)K(0,w)^{-1}K(0,0)^{1/2}$. This kernel is characterized by the property $\tilde{K}(z,0) = I$ and is therefore uniquely determined up to a conjugation by a constant unitary matrix. Let $K(z,w) = \sum_{k,\ell\geq 0} a_{k\ell} z^k \bar{w}^\ell$ and $\tilde{K}(z,w) = \sum_{k,\ell\geq 0} \tilde{a}_{k\ell} z^k \bar{w}^\ell$, where $a_{k\ell}$ and $\tilde{a}_{k\ell}$ are determined by the real analytic functions K and $\tilde{K}$ respectively, $a_{k\ell}$ and $\tilde{a}_{k\ell}$ are in $\mathcal{M}_n$, for $k, \ell \geq 0$. Since $\tilde{K}(z,w)$ is a normalized kernel, it follows that $\tilde{a}_{00} = I$ and $\tilde{a}_{k0} = \tilde{a}_{0\ell} = 0$ for $k, \ell \geq$*

1. *Let* $K(z,w)^{-1} = \sum_{k,\ell\geq 0} b_{k\ell} z^k \bar{w}^\ell$, *where* $b_{k\ell}$ *is in* $\mathcal{M}_n$ *for* $k,\ell \geq 0$. *Clearly,* $K(z,w)^* = K(w,z)$ *for any reproducing kernel* K *and* $z,w \in \mathbb{D}$. *Therefore,* $a^*_{k\ell} = a_{\ell k}$, $\tilde{a}^*_{k\ell} = \tilde{a}_{\ell k}$ *and* ${b_{k\ell}}^* = b_{\ell k}$ *for* $k,\ell \geq 0$, *where* X^* *denotes the conjugate transpose of the matrix* X.

The following lemma is from [21, Theorem 3.7, Remark 3.8 and Lemma 3.9].

Following Lemma 2.1.4, we will prove irreducibility of $M_2^{(\alpha,\beta,\gamma)}$ for $\beta \neq \gamma$ by showing that only operators on $\mathbb{C}^4$ which commutes with all the coefficients of $\tilde{B}_2^{(\alpha,\beta,\gamma)}(z,w)$ for $\beta \neq \gamma$ are scalars.

Lemma 2.3.6. *The coefficient of* $z^k\bar{w}$ *is* $\tilde{a}_{k1} = a_{00}^{1/2}\big(\sum_{s=1}^k b_{s0}a_{k-s,1}b_{00}\big)a_{00}^{1/2} + a_{00}^{-1/2}a_{k1}a_{00}^{-1/2}$ *for* $1 \leq k \leq 3$.

Proof. Let us denote the coefficient of $z^k\bar{w}^\ell$ in the power series expansion of $\tilde{K}(z,w)$ is $\tilde{a}_{k\ell}$ for $k,\ell \geq 0$. We see that

$$\begin{aligned}
\tilde{a}_{k\ell} &= a_{00}^{1/2}\big(\sum_{s=0}^{k}\sum_{t=0}^{\ell} b_{s0}a_{k-s,\ell-t}b_{0t}\big)a_{00}^{1/2}\\
&= a_{00}^{1/2}\big(\sum_{s=1}^{k}\sum_{t=1}^{\ell}\ a_{s0}a_{k-s,\ell-t}b_{0t} + \sum_{s=1}^{k} b_{s0}a_{k-s,\ell}b_{00} + \sum_{t=1}^{\ell} b_{00}a_{k,\ell-t}b_{0t} + b_{00}a_{k\ell}b_{00}\big)a_{00}^{1/2}
\end{aligned}$$

Also,

$$\begin{aligned}
\tilde{a}_{k1} &= a_{00}^{1/2}\big(\sum_{s=1}^{k} b_{s0}a_{k-s,0}b_{01} + \sum_{s=1}^{k} b_{s0}a_{k-s,1}b_{00} + b_{00}a_{k0}b_{01} + b_{00}a_{k1}b_{00}\big)a_{00}^{1/2}\\
&= a_{00}^{1/2}\big(\big(\sum_{s=0}^{k} b_{s0}a_{k-s,0}\big)b_{01} + \sum_{s=1}^{k} b_{s0}a_{k-s,1}b_{00}\big)a_{00}^{1/2} + a_{00}^{-1/2}a_{k1}a_{00}^{-1/2}\\
&= a_{00}^{1/2}\big(\sum_{s=1}^{k} b_{s0}a_{k-s,1}b_{00}\big)a_{00}^{1/2} + a_{00}^{-1/2}a_{k1}a_{00}^{-1/2}
\end{aligned}$$

as $b_{00} = a_{00}^{-1}$ and coefficient of z^k in $K(z,w)^{-1}K(z,w) = \sum_{s=0}^k b_{s0}a_{k-s,0} = 0$ for $k \geq 1$. ◻

Lemma 2.3.7. *For the reproducing kernel* $B_2^{(\alpha,\beta,\gamma)}$,

$$\tilde{a}_{11} = \begin{pmatrix} \alpha & 0 & 0 & 0\\ 0 & \alpha+2\beta+2 & 0 & 0\\ 0 & 0 & \alpha+2\gamma+2 & 0\\ 0 & 0 & 0 & \alpha+2(\beta+\gamma)+4 \end{pmatrix} \text{ and } \tilde{a}_{21} = \begin{pmatrix} 0 & -\sqrt{\beta}(\beta+1) & -\sqrt{\gamma}(\gamma+1) & 0\\ 0 & 0 & 0 & -\sqrt{\gamma}(\gamma+1)\\ 0 & 0 & 0 & -\sqrt{\beta}(\beta+1)\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. For any reproducing kernel K with

$$K(z,w) = \sum_{m,n\geq 0} a_{mn}z^m\bar{w}^n \text{ and } K(z,w)^{-1} = \sum_{m,n\geq 0} b_{mn}z^m\bar{w}^n$$

the identity $K(z,w)^{-1}K(z,w) = I$ implies that $b_{00} = a_{00}^{-1}$ and $\sum_{\ell=0}^{k} b_{k-\ell,0}a_{\ell 0} = 0$ for $k \geq 1$. For $k = 1$ we have $b_{10} = -a_{00}^{-1}a_{10}a_{00}^{-1}$. We have from Lemma 2.3.6,

$$\begin{aligned}
\tilde{a}_{11} &= a_{00}^{1/2}\big(b_{10}a_{00}b_{00}\big)a_{00}^{1/2} + a_{00}^{-1/2}a_{11}a_{00}^{-1/2}\\
&= a_{00}^{-1/2}\big(a_{11} - a_{10}a_{00}^{-1}a_{01}\big)a_{00}^{-1/2}.
\end{aligned} \tag{2.3.4}$$

For $k = 2$ we have $b_{20} = -\big(b_{10}a_{10} + b_{00}a_{20}\big)a_{00}^{-1} = a_{00}^{-1}\big(a_{10}a_{00}^{-1}a_{10} - a_{20}\big)a_{00}^{-1}$. We get from Lemma 2.3.6

$$\begin{aligned}\tilde{a}_{21} &= a_{00}^{1/2}\big(b_{10}a_{11}b_{00} + b_{20}a_{01}b_{00}\big)a_{00}^{1/2} + a_{00}^{-1/2}a_{21}a_{00}^{-1/2}\\ &= a_{00}^{-1/2}\big(a_{21} - a_{10}a_{00}^{-1}(a_{11} - a_{10}a_{00}^{-1}a_{01}) - a_{20}a_{00}^{-1}a_{01}\big)a_{00}^{-1/2}.\end{aligned} \tag{2.3.5}$$

From the reproducing kernel $B_2^{(\alpha,\beta,\gamma)}$ we see that

$$\begin{aligned} a_{00} = \begin{pmatrix} 1&0&0&0\\ 0&\beta&0&0\\ 0&0&\gamma&0\\ 0&0&0&\beta\gamma\end{pmatrix}, a_{10} &= \begin{pmatrix} 0&\beta&\gamma&0\\ 0&0&0&\beta\gamma\\ 0&0&0&\beta\gamma\\ 0&0&0&0\end{pmatrix}, a_{20} = \begin{pmatrix} 0&0&0&\beta\gamma\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix},\\ a_{11} &= \begin{pmatrix} \alpha+\beta+\gamma&0&0&0\\ 0&\beta(\alpha+2\beta+\gamma+2)&\beta\gamma&0\\ 0&\beta\gamma&\gamma(\alpha+\beta+2\gamma+2)&0\\ 0&0&0&\beta\gamma(\alpha+2(\beta+\gamma)+4)\end{pmatrix},\\ a_{21} &= \begin{pmatrix} 0&\beta(\alpha+\beta+\gamma+1)&\gamma(\alpha+\beta+\gamma+1)&0\\ 0&0&0&\beta\gamma(\alpha+2\beta+\gamma+3)\\ 0&0&0&\beta\gamma(\alpha+\beta+2\gamma+3)\\ 0&0&0&0\end{pmatrix}.\end{aligned}$$

Therefore, $a_{11}-a_{10}a_{00}^{-1}a_{01} = \begin{pmatrix} \alpha&0&0&0\\ 0&\beta(\alpha+2\beta+2)&0&0\\ 0&0&\gamma(\alpha+2\gamma+2)&0\\ 0&0&0&\beta\gamma(\alpha+2(\beta+\gamma)+4)\end{pmatrix}$, hence from Equation (2.3.4), we have $\tilde{a}_{11} = \begin{pmatrix} \alpha&0&0&0\\ 0&\alpha+2\beta+2&0&0\\ 0&0&\alpha+2\gamma+2&0\\ 0&0&0&\alpha+2(\beta+\gamma)+4\end{pmatrix}$. Now, from Equation (2.3.5), we obtain by a routine calculation $\tilde{a}_{21} = \begin{pmatrix} 0&-\sqrt{\beta}(\beta+1)&-\sqrt{\gamma}(\gamma+1)&0\\ 0&0&0&-\sqrt{\gamma}(\gamma+1)\\ 0&0&0&-\sqrt{\beta}(\beta+1)\\ 0&0&0&0\end{pmatrix}$. □

Lemma 2.3.8. *If $P \in \mathcal{M}_4$ commutes with $\tilde{a}_{11}$ and $\tilde{a}_{21}$ for $\beta \neq \gamma$, then P is a scalar matrix.*

Proof. We see from Lemma 2.3.7 that if $\beta \neq \gamma$ then $\tilde{a}_{11}$ is a matrix with distinct diagonal entries. Now, if $P\tilde{a}_{11} = \tilde{a}_{11}P$ then P is a diagonal matrix. If a diagonal matrix P commutes with $\tilde{a}_{21}$ then by direct computation it is easy to see that P has to be a scalar matrix. □

Combining all the lemmas above we have a proof of Theorem 2.3.4.

Theorem 2.3.9. *The multiplication operator $M_2^{(\alpha,\beta,\gamma)}$ on the Hilbert space whose reproducing kernel is $B_2^{(\alpha,\beta,\gamma)}$ is homogeneous .*

We write K for $B_2^{(\alpha,\beta,\gamma)}$ for simplicity of notation. We give a proof of this Theorem by showing that that the kernel is quasi-invariant, that is,

$$K(z,w) = J_{\varphi^{-1}}(z)K\big(\varphi^{-1}(z),\varphi^{-1}(w)\big)\overline{J_{\varphi^{-1}}(w)}^{\mathrm{tr}}$$

for some cocycle

$$J : \text{Möb} \times \mathbb{D} \longrightarrow \mathbb{C}^{4\times 4},\ \varphi \in \text{Möb},\ z,\ w \in \mathbb{D}.$$

First we prove that $K(z,z) = J_{\varphi^{-1}}(z)K\big(\varphi^{-1}(z),\varphi^{-1}(z)\big)\overline{J_{\varphi^{-1}}(z)}^{\mathrm{tr}}$ and then polarize to obtain the final result. It follows from Lemma 2.2.2 that the above equality is same as showing

$$K(0,0) = J_{\varphi^{-1}}(0)K\big(\varphi^{-1}(0),\varphi^{-1}(0)\big)\overline{J_{\varphi^{-1}}(0)}^{\mathrm{tr}} \tag{2.3.6}$$

Let $\mathcal{J}_{\varphi^{-1}}(z) = (J_{\varphi^{-1}}(z)^{\rm tr})^{-1}, \varphi \in$ Möb, $z \in \mathbb{D}$, where $X^{\rm tr}$ denotes the transpose of the matrix X. Clearly, $(J_{\varphi^{-1}}(z)^{\rm tr})^{-1}$ satisfies the cocycle property if and only if $\mathcal{J}_{\varphi^{-1}}(z)$ does and they uniquely determine each other. It is easy to see that the condition (2.3.6) is equivalent to

$$h\big(\varphi^{-1}(0)\big) = \overline{\mathcal{J}_{\varphi^{-1}}(0)}^{\rm tr} h(0)\mathcal{J}_{\varphi^{-1}}(0), \tag{2.3.7}$$

where $h(z)$ is the transpose of $K(z,z)$.

Recalling Notaton 2.2.3 let

$$\mathcal{J}_{\varphi^{-1}}(z) = \begin{pmatrix} c(\varphi^{-1},z)^2 & \beta c(\varphi^{-1},z)p(\varphi^{-1},z) & \gamma c(\varphi,z)p(\varphi^{-1},z) & \beta\gamma p(\varphi^{-1},z) \\ 0 & c(\varphi^{-1},z) & 0 & \gamma p(\varphi^{-1},z) \\ 0 & 0 & c(\varphi^{-1},z) & \beta p(\varphi^{-1},z) \\ 0 & 0 & 0 & 1 \end{pmatrix} c(\varphi^{-1},z)^{-\frac{\alpha+\beta+\gamma}{2}-2} \tag{2.3.8}$$

Lemma 2.3.10. *$J_{\varphi^{-1}}(z)$ defines a cocycle for the group* Möb.

Proof. To say that $J_{\varphi^{-1}}(z)$ satisfies the cocycle property is the same as saying $\mathcal{J}_{\varphi^{-1}}(z)$ satisfies the cocycle property, which we will verify. Thus we want to show that $\mathcal{J}_{\psi^{-1}}(z)\mathcal{J}_{\varphi^{-1}}(\psi^{-1}(z)) = \mathcal{J}_{\varphi^{-1}\psi^{-1}}(z)$. This follows from direct computation and Lemma 2.2.4(d). □

Lemma 2.3.11. *For $\varphi \in$ Möb and $\mathcal{J}_{\varphi^{-1}}(z)$ as in* (2.3.8)*,*

$$h\big(\varphi^{-1}(0)\big) = \overline{\mathcal{J}_{\varphi^{-1}}(0)}^{\rm tr} h(0)\mathcal{J}_{\varphi^{-1}}(0).$$

Proof. Taking $\varphi = \varphi_{t,z}$, $t \in \mathbb{T}$ and $z \in \mathbb{D}$, we get the result by an easy direct computation. □

Thus we have a proof of Theorem 2.3.9.

We briefly describe the class of homogeneous operators which appear in [31].

Notation 2.3.12. *Let λ be a real number and m be a positive integer such that $2\lambda - m > 0$. For brevity, we will write $2\lambda_j = 2\lambda - m + 2j$, $0 \le j \le m$. Let*

$$L(\lambda)(\ell,j) = \begin{cases} \binom{\ell}{j}^2 \frac{(\ell-j)!}{(2\lambda_j)_{\ell-j}} & \text{for } 0 \le j \le \ell \le m; \\ 0 & \textit{otherwise.} \end{cases}$$

and $\mathrm{B} = \mathrm{diag}\ (d_0, d_1, \ldots, d_m)$*. Now, for $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_m)^{\rm tr}$ with $\mu_0 = 1$ and $\mu_\ell > 0$ for $\ell = 1, \ldots, m$, let*

$$\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(z,w) = (1 - z\bar{w})^{-2\lambda-m} D(z\bar{w}) \exp(\bar{w}\mathbb{S}_m)\mathrm{B} \exp(z\mathbb{S}_m^*) D(z\bar{w}), \tag{2.3.9}$$

where B *is a positive diagonal matrix with* $\mathrm{B}(\ell,\ell) = d_\ell = \sum_{j=0}^{\ell} \binom{\ell}{j}^2 \frac{(\ell-j)!}{(2\lambda_j)_{\ell-j}} \mu_j^2$ *for $0 \le \ell \le m$, $D(z\bar{w}) = \mathrm{diag}((1-z\bar{w})^{m-\ell})_{\ell=0}^m)$ and $\mathbb{S}_m$ is the forward shift on $\mathbb{C}^{m+1}$ with weight sequence $(1, \ldots, m)$. Thus, $L(\lambda)\boldsymbol{\mu}^2 = \boldsymbol{d}$ for $\boldsymbol{\mu}^2 := (\mu_0^2, \mu_1^2, \ldots, \mu_m^2)^{\rm tr}$ and $\boldsymbol{d} = (d_0, d_1, \ldots, d_m)^{\rm tr}$.*

The kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ is positive definite. Indeed, it is the reproducing kernel for the Hilbert space $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$ of $\mathbb{C}^{m+1}$ - valued holomorphic functions on $\mathbb{D}$ described in [31]. Let $M^{(\lambda,\boldsymbol{\mu})}$ denote the multiplication operator on the Hilbert space $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$. The Hermitian holomorphic vector bundle associated with $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ is denoted by $E^{(\lambda,\boldsymbol{\mu})}$. In [31], it is shown that $M^{(\lambda,\boldsymbol{\mu})}$ is an irreducible homogeneous operator and $M^{(\lambda,\boldsymbol{\mu})*}$ is in $\mathrm{B}_{m+1}(\mathbb{D})$.

It will be convenient to let $\mathcal{K}^{(\lambda,\boldsymbol{\mu})}$ denote the curvature $\mathcal{K}_{h'}(z) = \frac{\partial}{\partial \bar{z}}\big(h'^{-1}\frac{\partial}{\partial z}h'\big)(z)$, where $h'(z) = \mathbf{B}^{(\lambda,\boldsymbol{\mu})}(z,z)^{\mathrm{tr}}$ for z in $\mathbb{D}$.

Theorem 2.3.13. *The operator $M_2^{(\alpha,\beta,\gamma)}$ described here does not belong to the class discussed in [31].*

The proof of this Theorem will be completed after proving a sequence of Lemmas.

Lemma 2.3.14. trace $\mathcal{K}^{(\lambda,\boldsymbol{\mu})}(0) > m(m+1)$.

Proof. We know that the curvature of the determinant bundle is same as the trace of the curvature of the given bundle. So, $\mathcal{K}_{\det\, h'}(z) = \text{trace } \mathcal{K}^{(\lambda,\boldsymbol{\mu})}(z)$ for $h'(z) = \mathbf{B}^{(\lambda,\boldsymbol{\mu})}(z,z)^{\mathrm{tr}}$, $z \in \mathbb{D}$. Now,

$$\det h'(z) = (1-|z|^2)^{(-2\lambda-m)(m+1)+2\times\frac{m(m+1)}{2}} \det \mathrm{B} = \det \mathrm{B} \times (1-|z|^2)^{-2(m+1)\lambda}.$$

Therefore, $\mathcal{K}_{\det\, h'}(z) = 2(m+1)\lambda(1-|z|^2)^{-2}$, so the trace $\mathcal{K}^{(\lambda,\boldsymbol{\mu})}(0) = 2(m+1)\lambda > m(m+1)$, as $2\lambda > m$ by construction. □

Let $\mathcal{K}^{(\alpha,\beta,\gamma)}$ denote the curvature $\mathcal{K}_h(z) = \frac{\partial}{\partial \bar{z}}\big(h^{-1}\frac{\partial}{\partial z}h\big)(z)$, where $h(z) = B_2^{(\alpha,\beta,\gamma)}(z,z)^{\mathrm{tr}}$ for $z \in \mathbb{D}$.

Lemma 2.3.15. trace $\mathcal{K}^{(\alpha,\beta,\gamma)}(0) = 4(\alpha+\beta+\gamma+2)$.

Proof. From Lemma 2.3.11, it follows that $h\big(\varphi_{t,z}^{-1}(0)\big) = \overline{\mathcal{J}_{\varphi_{t,z}^{-1}}(0)}^{\mathrm{tr}} h(0)\mathcal{J}_{\varphi_{t,z}^{-1}}(0)$ for $z \in \mathbb{D}$ and $t \in \mathbb{T}$. From 2.2.2, we get $\det h(z) = (1-|z|^2)^{8\times(-\frac{\alpha+\beta+\gamma}{2}-2)+8} \det h(0) = \beta^2\gamma^2(1-|z|^2)^{-4(\alpha+\beta+\gamma+2)}$. So, $\mathcal{K}^{(\alpha,\beta,\gamma)}(z) = 4(\alpha+\beta+\gamma+2)(1-|z|^2)^{-2}$. Hence, trace $\mathcal{K}^{(\alpha,\beta,\gamma)}(0) = 4(\alpha+\beta+\gamma+2)$. □

For $m = 3$ in Lemma 2.3.14, we have $\mathcal{K}^{(\lambda,\boldsymbol{\mu})}(0) > 12$. Whereas we see that from Lemma 2.3.15 that one can choose $\alpha,\beta,\gamma > 0$ such that trace $\mathcal{K}^{(\alpha,\beta,\gamma)}(0) \leq 12$. Hence, we have proved Theorem 2.3.13.

Theorem 2.3.16. *The multiplication operator on the Hilbert space whose reproducing kernel is $B_2^{(\alpha,\beta,\gamma)}$ is reducible for $\beta = \gamma$. That is, $M_2^{(\alpha,\beta,\gamma)}$ is unitarily equivalent to $M_1 \oplus M_2$, where M_1 is unitarily equivalent to $M^{(\lambda,\boldsymbol{\mu})}$ for $2\lambda = \alpha + 2\beta + 2$ and $\boldsymbol{\mu} = (1,\mu_1,\mu_2)^{tr}$, $\mu_1,\mu_2 > 0$ and M_2 is a homogeneous operator in* $\mathrm{B}_1(\mathbb{D})$.

Proof. One observes that

$$B_2^{(\alpha,\beta,\gamma)}(z,w) = (1-z\bar{w})^{-\alpha-\beta-\gamma-4} D_1(z\bar{w}) \exp(\bar{w}S_{\beta,\gamma}) B^{(\alpha,\beta,\gamma)}(0,0) \exp(zS_{\beta,\gamma}^*) D_1(z\bar{w})$$

for $z, w \in \mathbb{D}$, where

$$D_1(z\bar{w}) = \begin{pmatrix} (1-z\bar{w})^2 & 0 & 0 & 0 \\ 0 & 1-z\bar{w} & 0 & 0 \\ 0 & 0 & 1-z\bar{w} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } S_{\beta,\gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 \\ 0 & \gamma & \beta & 0 \end{pmatrix}.$$

Let

$$\Psi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, D'(z\bar{w}) = \begin{pmatrix} (1-z\bar{w})^2 & 0 & 0 & 0 \\ 0 & 1-z\bar{w} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-z\bar{w} \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2\beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and observe that $\det \Psi = -2 \neq 0$. Clearly, $\Psi D_1(z\bar{w}) = D'(z\bar{w})\Psi$. For $\beta = \gamma$, one has

$$\Psi S_{\beta,\beta} = S\Psi \text{ and } \Psi B_2^{(\alpha,\beta,\beta)}(0,0)\Psi^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\beta & 0 & 0 \\ 0 & 0 & \beta^2 & 0 \\ 0 & 0 & 0 & 2\beta \end{pmatrix}.$$

Observing that $\Psi S_{\beta,\beta} = S\Psi$ implies $\Psi \exp(\bar{w}S_{\beta,\beta}) = \exp(\bar{w}S)\Psi$ and $\exp(zS^*_{\beta,\beta})\Psi^* = \Psi^* \exp(zS^*)$ we get

$$\Psi B_2^{(\alpha,\beta,\beta)}(z,w)\Psi^* = K_1(z,w) \oplus K_2(z,w)$$

where

$$K_1(z,w) = (1-z\bar{w})^{-\alpha-2\beta-4} D(z\bar{w}) \exp(\bar{w}\tilde{S})\tilde{D}\exp(z\tilde{S}^*)D(z\bar{w}) \text{ for } \tilde{S} = \begin{pmatrix} 0 & 0 & 0 \\ 2\beta & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix}, \tilde{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2\beta & 0 \\ 0 & 0 & \beta^2 \end{pmatrix}$$

and

$$K_2(z,w) = 2\beta(1-z\bar{w})^{-\alpha-2\beta-2}.$$

Taking $\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2\beta} & 0 \\ 0 & 0 & \frac{1}{\beta^2} \end{pmatrix}$, noting that $\Phi\tilde{S} = \mathbb{S}_2\Phi$, $\det \Phi \neq 0$ and arguing as before we get

$$\Phi K_1(z,w)\Phi^* = (1-z\bar{w})^{-\alpha-2\beta-4} D(z\bar{w}) \exp(\bar{w}\mathbb{S}_2)\mathrm{B}\exp(z{\mathbb{S}_2}^*)D(z\bar{w})$$

where $\mathrm{B} = \Phi$.

For $2\lambda = \alpha + 2\beta + 2$ we note that the vector

$$\boldsymbol{\xi} = L(\lambda)^{-1}\boldsymbol{d} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{\alpha+2\beta} & 1 & 0 \\ \frac{2}{(\alpha+2\beta+1)(\alpha+2\beta+2)} & -\frac{4}{\alpha+2\beta+2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2\beta \\ 1/\beta^2 \end{pmatrix}$$

is of the form $(1, \xi_1, \xi_2)^{\mathrm{tr}}$ for $\xi_1, \xi_2 > 0$. Therefore, for $\lambda = \frac{\alpha}{2} + \beta + 1$ and $\boldsymbol{\mu}^2 = \boldsymbol{\xi}$ we have $\Phi K_1(z,w)\Phi^* = \mathbf{B}^{(\lambda,\boldsymbol{\mu})}$. This completes the proof. ∎

3. CONSTRUCTION OF NEW HOMOGENEOUS OPERATORS SIMILAR TO THE GENERALIZED WILKINS' OPERATORS

3.1 The generalized Wilkins' operators

Although, it is not clear at the outset that there exists (α, β) and $(\lambda, \boldsymbol{\mu})$ such that the two homogeneous operators $M_m^{(\alpha,\beta)}$ and $M^{(\lambda,\boldsymbol{\mu})}$ are unitarily equivalent. We calculate those λ and $\boldsymbol{\mu}$ (for a fixed m) as a function of α, β explicitly for which $M_m^{(\alpha,\beta)}$ is unitarily equivalent to $M^{(\lambda,\boldsymbol{\mu})}$. We show in this chapter that the set of homogeneous operators that appear from the first jet construction, is a small subset of those appearing in the second one. The multiplication operators constructed via the first jet construction are known as the "Generalized Wilkins' operators"[9]. However, there is an easy modification of the first jet construction that allows us to construct the entire family of homogeneous operators which were first exhibited in [31].

Let us consider the function $G : \mathbb{D} \times \mathbb{D} \longrightarrow \mathcal{M}_{m+1}$ defined by

$$G(z, w) = (1 - z\bar{w})^{-\alpha-\beta-2m} D(z\bar{w}) \exp(\bar{w} S_\beta) \mathrm{A} \exp(z S_\beta^*) D(z\bar{w})$$

for $S_\beta = S((r(\beta + r - 1)_{r=1}^m)$, $\mathrm{A} = \operatorname{diag}((r!(\beta)_r)_{r=0}^m)$ and $D(z\bar{w}) = \operatorname{diag}((1 - z\bar{w})^{m-\ell})_{\ell=0}^m)$.

Proposition 3.1.1. *In the above notations one has*

$$B_m^{(\alpha,\beta)} = G \text{ on } \mathbb{D} \times \mathbb{D}.$$

The proof of this Proposition will be facilitated by a sequence of lemmas. Since $\overline{B_m^{(\alpha,\beta)}(z, w)}^{\mathrm{tr}} = B_m^{(\alpha,\beta)}(w, z)$ and $\overline{G(z, w)}^{\mathrm{tr}} = G(w, z)$ for $(z, w) \in \mathbb{D} \times \mathbb{D}$ it suffices to show that

$$(B_m^{(\alpha,\beta)}(z, w))(i, j) = (G(z, w))(i, j) \text{ for } 0 \le i \le j \le m.$$

Lemma 3.1.2. *In the above notations we have*

$$G(z, w)(i, j) = (\beta)_j (1 - z\bar{w})^{-\alpha-\beta-i-j} z^{j-i} \sum_{k=0}^{i} \binom{i}{k} \binom{j}{k} (\beta + k)_{i-k} (z\bar{w})^{i-k}$$

for $0 \le i \le j \le m$.

Proof. Only the nonzero entries of the matrices are mentioned throughout this proof except for the last computation. One can easily see that $S_\beta^k(i, i-k) = (i-k+1)_k (\beta + i - k)_k$ for $0 \le i \le m$,

$k \geq 1$. Clearly, $S_\beta^{m+1} = 0$. So, $\exp(\bar{w}S_\beta) = \sum_{k=0}^{m} \frac{\bar{w}^k S_\beta^k}{k!}$, that is,

$$(\exp(\bar{w}S_\beta))(i, i-k) = \frac{(i-k+1)_k}{k!}(\beta+i-k)_k \bar{w}^k = \binom{i}{k}(\beta+i-k)_k \bar{w}^k \text{ for } 0 \leq k \leq i \leq m.$$

So, we have $(\exp(\bar{w}S_\beta))(i,j) = \binom{i}{j}(\beta+j)_{i-j}\bar{w}^{i-j}$ and $(\exp(zS_\beta^*))(j,i) = \binom{i}{j}(\beta+j)_{i-j}z^{i-j}$ for $0 \leq j \leq i \leq m$. Now,

$$\begin{aligned}(\exp(\bar{w}S_\beta)\mathrm{A}\ \exp(zS_\beta^*))(i,j) &= \sum_{k=0}^{\min(i,j)} (\exp(\bar{w}S_\beta)\mathrm{A})(i,k)(\ \exp(zS_\beta^*)(k,j) \\ &= \sum_{k=0}^{i} \binom{i}{k}(\beta+k)_{i-k} k!(\beta)_k(\beta+k)_{j-k} z^{j-i}\bar{w}^{i-k} \\ &= (\beta)_j z^{j-i} \sum_{k=0}^{i} k! \binom{i}{k}\binom{j}{k}(\beta+k)_{i-k}(z\bar{w})^{i-k}\end{aligned}$$

for $0 \leq i \leq j \leq m$. Since $D(z\bar{w})$ is a diagonal matrix one can easily see that $G(z,w)$ has the desired expression. □

Lemma 3.1.3. *We have*

$$B_m^{(\alpha,\beta)}(z,w)(i,j) = (\beta)_j(1-z\bar{w})^{-\alpha-\beta-i-j}z^{j-i}\sum_{k=0}^{i}\binom{i}{k}\binom{j}{k}(\beta+k)_{i-k}(z\bar{w})^{i-k},$$

for $0 \leq j \leq i \leq m$.

Proof.

$$\begin{aligned}\big(B_m^{(\alpha,\beta)}(z,w)\big)(i,j) &= \partial_{z_2}^i \partial_{\bar{w}_2}^j \big((1-z_1\bar{w}_1)^{-\alpha}(1-z_2\bar{w}_2)^{-\beta}\big)|_{\triangle\times\triangle} \\ &= (\beta)_j(1-z_1\bar{w}_1)^{-\alpha}\partial_{z_2}^i\big((1-z_2\bar{w}_2)^{-\beta-j}z_2^j\big)|_{\triangle\times\triangle} \\ &= (\beta)_j(1-z_1\bar{w}_1)^{-\alpha}\sum_{r=0}^{i}\binom{i}{r}\partial_{z_2}^{i-r}\big((1-z_2\bar{w}_2)^{-\beta-j}\big)\partial_{z_2}^r(z_2^j)|_{\triangle\times\triangle} \\ &= (\beta)_j(1-z_1\bar{w}_1)^{-\alpha}\sum_{r=0}^{i}\binom{i}{r}(\beta+j)_{i-r}(1-z_2\bar{w}_2)^{-\beta-j-(i-r)}\bar{w}_2^{i-r}r!\binom{j}{r}z_2^{j-r}|_{\triangle\times\triangle} \\ &= (\beta)_j(1-z\bar{w})^{-\alpha-\beta-i-j}z^{j-i}\sum_{r=0}^{i}r!\binom{i}{r}\binom{j}{r}(\beta+j)_{i-r}(1-z\bar{w})^r(z\bar{w})^{i-r},\end{aligned}$$

for $i \leq j$.

Clearly, to prove the desired equality we have to show that

$$\sum_{r=0}^{i} r!\binom{i}{r}\binom{j}{r}(\beta+j)_{i-r}(1-z\bar{w})^r(z\bar{w})^{i-r} = \sum_{k=0}^{i} k!\binom{i}{k}\binom{j}{k}(\beta+k)_{i-k}(z\bar{w})^{i-k} \qquad (3.1.1)$$

for $0 \leq i \leq j \leq n$. Now

$$\begin{aligned}
&\sum_{r=0}^{i} r!\binom{i}{r}\binom{j}{r}(\beta+j)_{i-r}(1-z\bar{w})^r(z\bar{w})^{i-r} \\
&= \sum_{r=0}^{i} r!\binom{i}{r}\binom{j}{r}(\beta+j)_{i-r}\sum_{\ell=0}^{r}(-1)^\ell\binom{r}{\ell}(z\bar{w})^\ell(z\bar{w})^{i-r} \\
&= \sum_{\ell=0}^{i}\sum_{r=\ell}^{i}(-1)^\ell r!\binom{i}{r}\binom{j}{r}\binom{r}{\ell}(\beta+j)_{i-r}(z\bar{w})^{i-(r-\ell)} \\
&= \sum_{\ell=0}^{i}\sum_{r=0}^{i-\ell}(-1)^\ell(r+\ell)!\binom{i}{r+\ell}\binom{j}{r+\ell}\binom{r+\ell}{\ell}(\beta+j)_{i-r-\ell}(z\bar{w})^{i-r}.
\end{aligned}$$

For $0 \leq k \leq i-\ell$, the coefficient of $(z\bar{w})^{i-k}$ in the left hand side of (3.1.1) is

$$\sum_{\ell=0}^{i}(-1)^\ell(k+\ell)!\binom{i}{k+\ell}\binom{j}{k+\ell}\binom{k+\ell}{\ell}(\beta+j)_{i-k-\ell},$$

which is the same as

$$\sum_{\ell=0}^{i-k}(-1)^\ell(k+\ell)!\binom{i}{k+\ell}\binom{j}{k+\ell}\binom{k+\ell}{\ell}(\beta+j)_{i-k-\ell},$$

for $0 \leq \ell \leq k \leq i$. So, to complete the proof we have to show that

$$\sum_{\ell=0}^{i-k}(-1)^\ell(k+\ell)!\binom{i}{k+\ell}\binom{j}{k+\ell}\binom{k+\ell}{\ell}(\beta+j)_{i-k-\ell} = k!\binom{i}{k}\binom{j}{k}(\beta+k)_{i-k},$$

for $0 \leq k \leq i \leq j$. But this follows from Lemma 2.2.6. Hence the proof is complete. □

Proof of Proposition 3.1.1: Combining Lemma 3.1.2 and Lemma 3.1.3, we have a proof of Proposition 3.1.1. □

Recall Notation 2.3.12: For $2\lambda = \alpha+\beta+m$, $\alpha,\beta > 0, m \geq 1$, one has

$$L(\lambda)(i,j) = \begin{cases} \binom{i}{j}^2 \frac{(i-j)!}{(\alpha+\beta+2j)_{i-j}} & \text{for } 0 \leq j \leq i \leq m; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.1.4. *In the above notation one has*

$$L(\lambda)^{-1}(i,j) = \begin{cases} (-1)^{i+j}\binom{i}{j}^2 \frac{(i-j)!}{(\alpha+\beta+i+j-1)_{i-j}} & \text{for } 0 \leq j \leq i \leq m; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Since $L(\lambda)$ is a lower-triangular matrix with 1 as diagonal entries, it is enough to verify that $\sum_{k=j}^{i} L(\lambda)(i,k)L(\lambda)^{-1}(k,j) = 0$ for $0 \leq j < i \leq m$.

So writing $\alpha+\beta=a$ we have

$$
\begin{aligned}
&\sum_{k=j}^{i} L(\lambda)(i,k)L(\lambda)^{-1}(k,j)\\
= &\sum_{k=0}^{i-j}(-1)^{k+j}\binom{i}{k}^2\binom{k}{j}^2\frac{(i-k)!(k-j)!}{(a+2k)_{i-k}(a+k+j-1)_{k-j}}\\
= &\sum_{k=j}^{i}(-1)^{k+j}\frac{(i!)^2}{(j!)^2(i-k)!(k-j)!}\frac{1}{(a+2k)_{i-k}(a+k+j-1)_{k-j}}\\
= &\frac{(i!)^2}{(j!)^2(i-j)!}\sum_{k=0}^{i-j}(-1)^{k+2j}\frac{(i-j)!}{(i-j-k)!k!}\frac{1}{(a+2j+2k)_{i-j-k}(a+2j+k-1)_k}\\
= &\frac{(i!)^2}{(j!)^2(i-j)!(a+2j)_{2(i-j)}}\sum_{k=0}^{i-j}(-1)^k\binom{i-j}{k}\frac{(a+2j)_{2(i-j)}}{(a+2j+2k)_{i-j-k}(a+2j+k-1)_k}
\end{aligned}
$$

Now, noting that $(x)_n=(-1)^n n!\binom{-x}{n}=n!\binom{x+n-1}{n}$ we get

$$
\begin{aligned}
&\sum_{k=0}^{i-j}(-1)^k\binom{i-j}{k}\frac{(a+2j)_{2(i-j)}}{(a+2j+2k)_{i-j-k}(a+2j+k-1)_k}\\
= &\sum_{k=1}^{i-j-1}(-1)^k\binom{i-j}{k}(a+2j)_{k-1}(a+2j+2k-1)(a+i+j+k)_{i-j-k}\\
&\qquad +(a+i+j)_{i-j}+(-1)^{i-j}(a+2j)_{i-j-1}(a+2i-1)\\
= &\sum_{k=1}^{i-j-1}(-1)^k\binom{i-j}{k}(a+2j)_k(a+i+j+k)_{i-j-k}\\
&\qquad +\sum_{k=1}^{i-j-1}(-1)^k k\binom{i-j}{k}(a+2j)_{k-1}(a+i+j+k)_{i-j-k}\\
&\qquad +(a+i+j)_{i-j}+(-1)^{i-j}(a+2j)_{i-j}+(-1)^{i-j}(i-j)(a+2i)_{i-j-1}\\
= &\sum_{k=0}^{i-j}(-1)^k\binom{i-j}{k}(a+2j)_k(a+i+j+k)_{i-j-k}\\
&\qquad +\sum_{k=1}^{i-j}(-1)^k k\binom{i-j}{k}(a+2j)_{k-1}(a+i+j+k)_{i-j-k}\\
= &(i-j)!\sum_{k=0}^{i-j}\binom{-(a+2j)}{k}\binom{a+2i-1}{i-j-k}-(i-j)!\sum_{k=0}^{i-j-1}\binom{-(a+2j)}{k}\binom{a+2i-1}{i-j-1-k}\\
= &(i-j)!\binom{2(i-j)-1}{i-j}-(i-j)!\binom{2(i-j)-1}{i-j-1}\\
= &0.
\end{aligned}
$$

The last equality follows from the Vandermonde's identity and the conclusion follows from the fact that $\binom{n}{r}=\binom{n}{n-r}$ for $0\le r\le n$. Hence the verification is complete. $\square$

Lemma 3.1.5. *For $\alpha, \beta > 0$ and $n \geq 0$, we have*

$$(-1)^n n! \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(\alpha+\beta+n+k-1)_{n-k}(\beta)_k} = \frac{n!(\alpha)_n}{(\alpha+\beta+n-1)_n(\beta)_n}$$

Proof. Since $(x)_n = (-1)^n n! \binom{-x}{n} = n! \binom{x+n-1}{n}$ we have

$$\begin{aligned}
& (-1)^n n! \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(\alpha+\beta+n+k-1)_{n-k}(\beta)_k} \\
= & \frac{(-1)^n n!}{(\alpha+\beta+n-1)_n(\beta)_n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(\alpha+\beta+n-1)_n(\beta)_n}{(\alpha+\beta+n+k-1)_{n-k}(\beta)_k} \\
= & \frac{(-1)^n n!}{(\alpha+\beta+n-1)_n(\beta)_n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\alpha+\beta+n-1)_k(\beta+k)_{n-k} \\
= & \frac{(-1)^n n!}{(\alpha+\beta+n-1)_n(\beta)_n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (-1)^k k! \binom{-(\alpha+\beta+n-1)}{k} (n-k)! \binom{\beta+n-1}{n-k} \\
= & \frac{(-1)^n (n!)^2}{(\alpha+\beta+n-1)_n(\beta)_n} \sum_{k=0}^{n} \binom{-(\alpha+\beta+n-1)}{k} \binom{\beta+n-1}{n-k} \\
= & \frac{(-1)^n (n!)^2}{(\alpha+\beta+n-1)_n(\beta)_n} \binom{-\alpha}{n} \\
= & \frac{n!(\alpha)_n}{(\alpha+\beta+n-1)_n(\beta)_n}
\end{aligned}$$

The penultimate equality follows from the Vandermonde's identity. ☐

Corollary 3.1.6. *For $2\lambda = \alpha + \beta + m$ and $\boldsymbol{d}(r) = \frac{r!}{(\beta)_r}$ for $0 \leq r \leq m$, $\alpha, \beta > 0$, $m \geq 1$, one has*

$$(L(\lambda)^{-1}\boldsymbol{d})(r) = \frac{n!(\alpha)_n}{(\alpha+\beta+n-1)_n(\beta)_n} \text{ for } 0 \leq r \leq m.$$

Proof. Follows easily from Lemma 3.1.4 and Lemma 3.1.5. ☐

Proposition 3.1.7. *For*

$$2\lambda = \alpha + \beta + m \text{ and } \boldsymbol{\mu}_0(n) = \sqrt{\frac{n!(\alpha)_n}{(\alpha+\beta+n-1)_n(\beta)_n}},$$

$0 \leq n \leq m$, $\alpha, \beta > 0, m \geq 1$, we have $\Phi B_m^{(\alpha,\beta)} \Phi^* = \mathbf{B}^{(\lambda,\boldsymbol{\mu}_0)}$ on $\mathbb{D} \times \mathbb{D}$ for $\Phi = \operatorname{diag}((\frac{1}{(\beta)_r})_{r=0}^m)$.

Proof. It follows from Corollary 3.1.6 that $\mathbf{B}^{(\lambda,\boldsymbol{\mu}_0)}(0,0) = \mathrm{B} = \operatorname{diag}((\frac{r!}{(\beta)_r})_{r=0}^m)$. One observes that $\Phi S_\beta = \mathbb{S}_m \Phi = S((\frac{r}{(\beta)_{r-1}})_{r=1}^m)$ and for $\mathrm{A} = \operatorname{diag}((r!(\beta)_r)_{r=0}^m)$, $\Phi \mathrm{A} \Phi^* = \operatorname{diag}((\frac{r!}{(\beta)_r})_{r=0}^m) = \mathrm{B}$. It follows that $\Phi \exp(\bar{w} S_\beta) = \exp(\bar{w}\mathbb{S}_m)\Phi$ and $\exp(z S_\beta^*)\Phi^* = \Phi^* \exp(z\mathbb{S}_m^*)$. Hence $\Phi G \Phi^* = \mathbf{B}^{(\lambda,\boldsymbol{\mu}_0)}$ on $\mathbb{D} \times \mathbb{D}$. Since $B_m^{(\alpha,\beta)} = G$ on $\mathbb{D} \times \mathbb{D}$ one has the desired conclusion. ☐

The proof of the following Theorem is a consequence of the Proposition 3.1.7 and Theorem 2.1.3.

Theorem 3.1.8. *The operator* $M_m^{(\alpha,\beta)}$ *is unitarily equivalent to* $M^{(\lambda,\boldsymbol{\mu}_0)}$ *if* $2\lambda = \alpha+\beta+m$ *and* $\boldsymbol{\mu}_0(n) = \sqrt{\frac{n!(\alpha)_n}{(\alpha+\beta+n-1)_n(\beta)_n}}$ *for* $0 \le n \le m$, $\alpha, \beta > 0, m \ge 1$.

Remark 3.1.9. *Recall that* $\mathcal{W}_m = \{M_m^{(\alpha,\beta)} : \alpha,\beta > 0\}$ *for* $m \ge 1$. *The inclusion*

$$\mathcal{W}_m \subseteq \{M^{(\lambda,\boldsymbol{\mu})} : 2\lambda > m, \boldsymbol{\mu} = (1,\mu_1,\dots,\mu_m) > 0\}$$

is proper unless $m = 1$. *If* $m = 1$, *then the two sets coincide up to unitary equivalence. Moreover,* $\mathcal{W}_1$ *is same as the complete list of irreducible homogeneous operators in* $\mathrm{B}_2(\mathbb{D})$ *first discovered by Wilkins [51].*

Remark 3.1.10. *One knows from Theorem 3.1.8 that the adjoint of* $M_m^{(\alpha,\beta)}$ *is a member of the class of homogeneous operators described in [31] for* $2\lambda = \alpha+\beta+m$ *and* $\mu_n = \sqrt{\frac{n!(\alpha)_n}{(\alpha+\beta+n-1)_n(\beta)_n}}$, $0 \le n \le m$. *Putting* $2\lambda = \alpha+\beta+m$ *one gets from [31] that* $M_m^{(\alpha,\beta)}$ *acts on a Hilbert space which is isomorphic to* $\oplus_{j=0}^m \mathbb{A}^{(\alpha+\beta+2j)}(\mathbb{D})$. *Consequently, the representation* U_φ *associated with* $M_m^{(\alpha,\beta)}$ *acts on* $\oplus_{j=0}^m \mathbb{A}^{(\alpha+\beta+2j)}(\mathbb{D})$.

3.2 The relationship between the two jet constructions

Let $\mathcal{H}$ be a Hilbert space and $\{e_i\}$ be an orthonormal basis for $\mathcal{H}$. We will let $c\mathcal{H}$ denote the Hilbert space whose orthonormal basis is $\{ce_i\}$ for $c > 0$. This is same as saying $\langle f,g\rangle_{\mathcal{H}} = c^{-2}\langle f,g\rangle_{c\mathcal{H}}$ for $f,g \in \mathcal{H}$. The linear map $\iota : c\mathcal{H} \to \mathcal{H}$, $\iota : f \mapsto f$ has the matrix representation cI with respect to the orthonormal bases $\{ce_i\}$ and $\{e_i\}$ respectively.

Now, let $\mathcal{H}$ be the Hilbert space $\oplus_{j=0}^m \mathcal{H}_j$, an orthogonal direct sum of the Hilbert spaces $\mathcal{H}_j$ having reproducing kernel K_j, $0 \le j \le m$. Let $\mathcal{H}_{\boldsymbol{\eta}}$ be the Hilbert space $\oplus_{j=0}^m \eta_j \mathcal{H}_j$. The inner product $\langle , \rangle_{\boldsymbol{\eta}}$ of the Hilbert space $\mathcal{H}_{\boldsymbol{\eta}}$ is given by

$$\langle \sum_{j=0}^m f_j, \sum_{j=0}^m g_j \rangle_{\boldsymbol{\eta}} = \sum_{j=0}^m \eta_j^{-2} \langle f_j, g_j\rangle_j \text{ for } f_j, g_j \in \mathcal{H}_j,\ \eta_j > 0,$$

where $\langle , \rangle_j$ is the inner product for the Hilbert space $\mathcal{H}_j$, $0 \le j \le m$. Clearly, the reproducing kernels for $\mathcal{H}$ and $\mathcal{H}_{\boldsymbol{\eta}}$ are $\sum_{j=0}^m K_j$ and $\sum_{j=0}^m \eta_j^2 K_j$ respectively.

The next Proposition is now immediate.

Proposition 3.2.1. *The multiplication operators on the Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{H}_{\boldsymbol{\eta}}$ *are similar via the map* $\iota : \mathcal{H}_{\boldsymbol{\eta}} \to \mathcal{H}$.

We know from Theorem [31] that the operator $M := M^{(\lambda,\boldsymbol{\mu})}$ is homogeneous and irreducible. It is natural to ask if there exists an invertible operator L such that $L^{-1}ML$ is homogeneous. The irreducibility of the operator M ensures (cf. [10, Theorem 2.2]) the existence of a unique projective unitary representation U_φ of Möb such that $\varphi(M) = U_\varphi^{-1} M U_\varphi$ for all $\varphi \in$ Möb. Clearly,

$$\varphi(L^{-1}ML) = L^{-1}\varphi(M)L = L^{-1}U_\varphi^{-1}MU_\varphi L,\ \varphi \in \text{Möb}\ .$$

It follows that $L^{-1}ML$ is homogeneous if $LU_\varphi = U_\varphi L$. Since U_φ is a multiplicity free representation acting on the direct sum $\oplus_{j=0}^m \mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})$ of Hilbert spaces it follows by a simple application of Schur's Lemma that $L = \oplus_{j=0}^m c_j I_j$, where $c_j \in \mathbb{C}$ and I_j is the identity on the Hilbert space $\mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})$, $0 \le j \le m$. It is clear that L can be thought of as the linear map $f \mapsto f$ from $\oplus_{j=0}^m c_j\mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})$ to $\oplus_{j=0}^m \mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})$.

Let us recall the following from [31]. Let $\mathrm{Hol}(\mathbb{D}, \mathbb{C}^k)$ be the space of all holomorphic functions taking values in $\mathbb{C}^k$, $k \in \mathbb{N}$. Let λ be a real number and m be a positive integer satisfying $2\lambda - m > 0$. For brevity, we will write $2\lambda_j = 2\lambda - m + 2j$, $0 \le j \le m$.

For each j, $0 \le j \le m$, define the operator $\Gamma_j : \mathbb{A}^{(2\lambda_j)}(\mathbb{D}) \longrightarrow \mathrm{Hol}(\mathbb{D}, \mathbb{C}^{m+1})$ by the formula

$$(\Gamma_j f)(\ell) = \begin{cases} \binom{\ell}{j} \frac{1}{(2\lambda_j)_{\ell-j}} f^{(\ell-j)} & \text{if } \ell \ge j \\ 0 & \text{if } 0 \le \ell < j, \end{cases}$$

for $f \in \mathbb{A}^{(2\lambda_j)}(\mathbb{D})$, $0 \le j \le m$, where $(x)_n := x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol. Here $(\Gamma_j f)(\ell)$ denotes the ℓ-th component of the function $\Gamma_j f$ and $f^{(\ell-j)}$ denotes the $(\ell - j)$-th derivative of the holomorphic function f.

We denote the range of Γ_j by $\mathbf{A}^{(2\lambda_j)}(\mathbb{D})$ and transfer to it the inner product of $\mathbb{A}^{(2\lambda_j)}(\mathbb{D})$, that is, one sets $\langle \Gamma_j f, \Gamma_j g\rangle = \langle f, g\rangle$ for $f, g \in \mathbb{A}^{(2\lambda_j)}(\mathbb{D})$. The Hilbert space $\mathbf{A}^{(2\lambda_j)}(\mathbb{D})$ is a reproducing kernel space because the point evaluation $f \mapsto (\Gamma_j f)(w)$ are continuous for each $w \in \mathbb{D}$. Let $\mathbf{B}^{(2\lambda_j)}$ denote the reproducing kernel for the Hilbert space $\mathbf{A}^{(2\lambda_j)}(\mathbb{D})$. Let

$$\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D}) := \oplus_{j=0}^m \mu_j \mathbf{A}^{(2\lambda_j)}(\mathbb{D}),\ 1 = \mu_0, \mu_1, \ldots, \mu_m > 0.$$

The Hilbert space $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$ has the reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})} = \sum_{j=0}^m \mu_j^2 \mathbf{B}^{(2\lambda_j)}$. The operator $M^{(\lambda,\boldsymbol{\mu})}$ is the multiplication operator on the Hilbert space $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})$. The following Theorem is then obvious.

Theorem 3.2.2. *Then operator $L^{-1}M^{(\lambda,\boldsymbol{\mu})}L$ acting on $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}(\mathbb{D})_{\boldsymbol{\eta}}$ is homogeneous , where $L = \oplus_{j=0}^m \eta_j I_j$. Moreover, the representation associated with $M^{(\lambda,\boldsymbol{\mu})}$ and $L^{-1}M^{(\lambda,\boldsymbol{\mu})}L$ is the same.*

Corollary 3.2.3. *For $2\lambda = \alpha + \beta + m$ and $\boldsymbol{\mu} = (\mu_j)_{j=0}^m$, $1 = \mu_0, \mu_1, \ldots, \mu_m > 0$ arbitrary, the operator $M^{(\lambda,\boldsymbol{\mu})}$ is similar to $M_m^{(\alpha,\beta)}$.*

Proof. From Theorem 3.1.8 and Remark 3.1.10, one gets the desired conclusion by taking

$$\eta_j = \mu_j \sqrt{\frac{(\alpha+\beta+j-1)_j(\beta)_j}{j!(\alpha)_j}} \quad \text{for } 0 \le j \le m \text{ and } L = \oplus_{j=0}^m \eta_j I_j$$

from the discussion that precedes Proposition 3.2.1. That is, for $2\lambda = \alpha + \beta + m$ we have $L^{-1}M_m^{(\alpha,\beta)}L = M^{(\lambda,\boldsymbol{\mu})}$. □

Corollary 3.2.4. *The operators $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\lambda',\boldsymbol{\mu}')}$ are similar if and only if $\lambda = \lambda'$.*

Proof. "if" part: If $\lambda = \lambda'$ then taking $\eta_j = \frac{\mu'_j}{\mu_j}$ for $0 \leq j \leq m$ and $L = \oplus_{j=0}^m \eta_j I_j$, one has $L^{-1} M^{(\lambda,\boldsymbol{\mu})} L = M^{(\lambda',\boldsymbol{\mu}')}$ as in Corollary 3.2.3.

"only if" part: One knows from Theorem 3.1.8 and Remark 3.1.10 that $M^{(\lambda,\boldsymbol{\mu}_0)}$ is unitarily equivalent to $M_m^{(\alpha,\beta)}$ if $2\lambda = \alpha+\beta+m$ and $\boldsymbol{\mu}_0(j) = \sqrt{\frac{j!(\alpha)_j}{(\alpha+\beta+j-1)_j(\beta)_j}}$ for $0 \leq j \leq m$. Also, for $2\lambda = \alpha+\beta+m$ and $1 = \mu_0, \mu_1, \ldots, \mu_m > 0$ arbitrary, $M^{(\lambda,\boldsymbol{\mu})}$ is similar to $M^{(\lambda,\boldsymbol{\mu}_0)}$ by Corollary 3.2.3. Hence $M_m^{(\alpha,\beta)}$ is similar to $M^{(\lambda,\boldsymbol{\mu})}$ if $2\lambda = \alpha+\beta+m$. Similarly, $M_m^{(\alpha',\beta')}$ is similar to $M^{(\lambda',\boldsymbol{\mu}')}$ if $2\lambda' = \alpha'+\beta'+m$. So, to arrive at the desired conclusion it is enough to show the following: The similarity of $M_m^{(\alpha,\beta)}$ and $M_m^{(\alpha',\beta')}$ implies that $\alpha+\beta = \alpha'+\beta'$.

Let T be a bounded invertible such that $T^{-1} M_m^{(\alpha,\beta)} T = M_m^{(\alpha',\beta')}$. We know that $M_m^{(\alpha,\beta)}$ is the compression of $M_1 \otimes I$ to $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) \ominus \mathbb{A}_m^{(\alpha,\beta)}(\mathbb{D}^2) = \oplus_{i=0}^m \mathcal{M}_i$, where M_1 denotes multiplication by z_1 and $\mathcal{M}_i = \mathbb{A}_{i-1}^{(\alpha,\beta)}(\mathbb{D}^2) \ominus \mathbb{A}_i^{(\alpha,\beta)}(\mathbb{D}^2)$ for $1 \leq i \leq m$, $\mathcal{M}_0 = \mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) \ominus \mathbb{A}_0^{(\alpha,\beta)}(\mathbb{D}^2)$. We observe that ${M_m^{(\alpha,\beta)}}^* \mathcal{M}_i \subseteq \mathcal{M}_i$ for $0 \leq i \leq m$. Similarly, ${M_m^{(\alpha',\beta')}}^* \mathcal{M}'_i \subseteq \mathcal{M}'_i$ for $0 \leq i \leq m$. Moreover, $T\mathcal{M}_i \subseteq \mathcal{M}'_i$ for $0 \leq i \leq m$. So, we have that the operators ${M_m^{(\alpha,\beta)}}^*_{|\text{res}\,\mathcal{M}_0}$ and ${M_m^{(\alpha',\beta')}}^*_{|\text{res}\,\mathcal{M}_0}$ are similar. Therefore, the multiplication operators on the Hilbert spaces having reproducing kernels $\big((1-z_1\bar{w}_1)^{-\alpha}(1-z_2\bar{w}_2)^{-\beta}\big)_{|z_1=z_2, w_1=w_2}$ and $\big((1-z_1\bar{w}_1)^{-\alpha'}(1-z_2\bar{w}_2)^{-\beta'}\big)_{|z_1=z_2, w_1=w_2}$ are similar. These are weighted shift operators which are similar if and only if $\alpha+\beta = \alpha'+\beta'$.

□

It is clear from Corollary 3.2.3 that $M_m^{(\alpha,\alpha)}$ is similar to $M^{(\lambda,\boldsymbol{\mu})}$ if $2\lambda = 2\alpha+m$. The family $\{M^{(\lambda,\boldsymbol{\mu})} : 2\lambda = 2\alpha+m, \alpha > 0 \text{ and } 1 = \mu_0, \mu_1, \ldots, \mu_m > 0\}$ is clearly seen to be the same as the family $\{M^{(\lambda,\boldsymbol{\mu})} : 2\lambda > m \text{ and } 1 = \mu_0, \mu_1, \ldots, \mu_m > 0\}$.

So, we have the following Theorem.

Theorem 3.2.5. *We have* $M^{(\lambda,\boldsymbol{\mu})} = L^{-1} M_m^{(\alpha,\alpha)} L$ *if* $2\lambda = 2\alpha+m$, *where* $L = \oplus_{j=0}^m \eta_j I_j$ *for* $\eta_j = \mu_j \sqrt{\frac{(2\alpha+j-1)_j(\beta)_j}{j!(\alpha)_j}}$ *for* $0 \leq j \leq m$.

Remark 3.2.6. Recall that $M_m^{(\alpha,\beta)}$ is the multiplication operator on the Hilbert space whose reproducing kernel is $B_m^{(\alpha,\beta)}$ and $\mathbf{B}^{(2\lambda_j)}$ is the reproducing kernel for the Hilbert space $\mathbf{A}^{(2\lambda_j)}(\mathbb{D}) = \Gamma_j(\mathbb{A}^{(2\lambda_j)}(\mathbb{D}))$ for $2\lambda_j = 2\lambda - m + 2j, 0 \leq j \leq m$. Putting together Proposition 3.1.7 and Theorem 3.1.8 we see that

$$\Phi B_m^{(\alpha,\beta)} \Phi^* = \sum_{j=0}^m \boldsymbol{\mu}_0(j)^2 \mathbf{B}^{(\alpha+\beta+2j)}, \text{ where } \boldsymbol{\mu}_0(j) = \sqrt{\frac{j!(\alpha)_j}{(\alpha+\beta+j-1)_j(\beta)_j}},\ 0 \leq j \leq m. \quad (3.2.2)$$

To prove Theorem 3.2.5 one may put $\alpha = \beta$ and replace the specific $\boldsymbol{\mu}_0$ of Theorem 3.1.8 by an arbitrary $\boldsymbol{\mu}$. By Corollary 3.2.3, we see that a simple similarity will do it. This produces all the $M^{(\lambda,\boldsymbol{\mu})}$ of [31].

Remark 3.2.7. The homogeneous operator $M_m^{(\alpha,\beta)}$ acts on the Hilbert space $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) \ominus \mathbb{A}_m^{(\alpha,\beta)}(\mathbb{D}^2)$ with associated representation $D_\alpha^+ \otimes D^+_{\beta\,|\text{res}(\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) \ominus \mathbb{A}_m^{(\alpha,\beta)}(\mathbb{D}^2))}$ [9]. The operator $M^{(\lambda,\boldsymbol{\mu}_0)}$ acts on the Hilbert space $\oplus_{j=0}^{m}\mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})$ with associated representation $\oplus_{j=0}^{m}D^{+}_{2\lambda-m+2j}$. From Theorem 3.1.8, one knows that $M_m^{(\alpha,\beta)}$ is unitarily equivalent to $M^{(\lambda,\boldsymbol{\mu}_0)}$ if $2\lambda=\alpha+\beta+m$ and $\boldsymbol{\mu}_0(n)=\sqrt{\frac{n!(\alpha)_n}{(\alpha+\beta+n-1)_n(\beta)_n}}$, $0\leq n\leq m$. Since the homogeneous operators $M_m^{(\alpha,\beta)}$ and $M^{(\lambda,\boldsymbol{\mu}_0)}$ are both irreducible the representations associated with them are unique up to equivalence. Moreover, $M_m^{(\alpha,\beta)}$ and $M^{(\lambda,\boldsymbol{\mu}_0)}$ are unitarily equivalent for proper choice of $\lambda,\boldsymbol{\mu}_0$ as functions of α and β. Hence it follows that the representations associated with them are same up to equivalence. Therefore, one has $D^{+}_{\alpha}\otimes D^{+}_{\beta}{}_{|\mathrm{res}(\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)\ominus\mathbb{A}_m^{(\alpha,\beta)}(\mathbb{D}^2))}\backsimeq\oplus_{j=0}^{m}D^{+}_{\alpha+\beta+2j}$. Now, the subspaces $\mathbb{A}_m^{(\alpha,\beta)}(\mathbb{D}^2)$ decrease to $\{0\}$ as $m\to\infty$. Therefore, $D^{+}_{\alpha}\otimes D^{+}_{\beta}\backsimeq\oplus_{j=0}^{\infty}D^{+}_{\alpha+\beta+2j}$. This is the Clebsh-Gordan formula. The identification of $\mathbb{A}^{(\alpha,\beta)}\ominus\mathbb{A}_m^{(\alpha,\beta)}(\mathbb{D}^2)$ with the direct sum $\oplus_{j=0}^{m}\mathbb{A}^{(2\lambda-m+2j)}(\mathbb{D})$ is a special case of [46, Theorem 3.3, page 179], although we arrive at the same conclusion via a different route.

4. ON A QUESTION OF COWEN AND DOUGLAS

For an operator T in the class $\mathrm{B}_n(\Omega)$, introduced by Cowen and Douglas in [18], the simultaneous unitary equivalence class of the curvature and the covariant derivatives up to a certain order of the corresponding bundle E_T determine the unitary equivalence class of the operator T. In a subsequent paper [20], the authors ask if the simultaneous unitary equivalence class of the curvature and these covariant derivatives are necessary to determine the unitary equivalence class of the operator $T \in \mathrm{B}_n(\Omega)$. Although, they have shown in [20] that the curvature and all its covariant derivatives in the list of [18, 20] are necessary to determine the equivalence class of a Hermitian holomorphic vector bundle bundle E but those examples do not necessarily correspond an operator $T \in \mathrm{B}_n(\Omega)$ such that $E = E_T$. Here we show that some of the covariant derivatives are necessary to determine the unitary equivalence class of the operator $T \in \mathrm{B}_n(\Omega)$. Our examples consist of homogeneous operators in $\mathrm{B}_n(\mathbb{D})$. For homogeneous operators, the simultaneous unitary equivalence class of the curvature and all its covariant derivatives at any point w in the unit disc $\mathbb{D}$ are determined from the simultaneous unitary equivalence class at 0. This shows that it is enough to calculate all the invariants and compare them at just one point, say 0. These calculations are then carried out in number of examples. One of our main results is that the curvature along with its covariant derivative of order $(0,1)$ at 0 determines the equivalence class of generic homogeneous Hermitian holomorphic vector bundles associated with the homogeneous operators described in [31]. This result is true for all (generic or not) rank 3 bundles associated with the homogeneous operators discussed in [31].

4.1 Examples from the Jet Construction

Let $S(z,w) = (1 - z\bar{w})^{-1}$ be the Sžego kernel on the unit disc, the Hilbert space corresponding to the non-negative definite kernel $S^\alpha(z,w) = (1 - z\bar{w})^{-\alpha}$ be $\mathbb{A}^{(\alpha)}(\mathbb{D})$ for $\alpha > 0$. We let $M^{(\alpha)} : \mathbb{A}^{(\alpha)}(\mathbb{D}) \longrightarrow \mathbb{A}^{(\alpha)}(\mathbb{D})$ denote the multiplication operator, that is, $(M^{(\alpha)}f)(z) = zf(z)$, $f \in \mathbb{A}^{(\alpha)}(\mathbb{D}), z \in \mathbb{D}$. Following the jet construction of [24] (see also [46]), we construct a Hilbert space $J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\,\triangle}$ $(\alpha,\, \beta > 0,\, k \in \mathbb{N})$ starting from the kernel Hilbert space $\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2) = \mathbb{A}^{(\alpha)}(\mathbb{D}) \otimes \mathbb{A}^{(\beta)}(\mathbb{D})$ with reproducing kernel $B^{(\alpha,\beta)}(\boldsymbol{z}, \boldsymbol{w}) = S^\alpha(z_1, w_1)S^\beta(z_2, w_2)$, $\boldsymbol{z} = (z_1, z_2), \boldsymbol{w} = (w_1, w_2) \in \mathbb{D}^2$. The Hilbert space $J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\,\triangle}$ consists of $\mathbb{C}^{k+1}$-valued holomorphic functions defined on the open unit disc $\mathbb{D}$. It turns out that the reproducing kernel $B_k^{(\alpha,\beta)}$ for $J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\text{res}\triangle}$ is

$$B_k^{(\alpha,\beta)}(z,w) = \left(\!\!\left(\partial_{z_2}^i \partial_{\bar{w}_2}^j B^{(\alpha,\beta)}(\boldsymbol{z},\boldsymbol{w})\right)\!\!\right)_{0\le i,j\le k|\text{res }\triangle\times\triangle}, \tag{4.1.1}$$

where $\triangle := \{(z,z) : z\in\mathbb{D}\} \subseteq \mathbb{D}^2$. The multiplication operator on $J^{(k)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\text{res }\triangle}$ is denoted by $M_k^{(\alpha,\beta)}$.

Example 4.1.1. Consider the operators $M := M^{(\lambda)} \oplus M^{(\mu)}$ and $M' := M_1^{(\alpha,\beta)}$ for $\lambda,\mu,\alpha,\beta > 0$. Wilkins [51] has shown that the operator M'^* is in $\mathrm{B}_2(\mathbb{D})$ and that it is irreducible. This operator is also *homogeneous* , that is, $\varphi(M')$ is unitarily equivalent to M' for all bi-holomorphic automorphisms φ of the open unit disc $\mathbb{D}$ (cf. [9]). It is easy to see that the operators $M^{(\lambda)}$ and $M^{(\mu)}$ are both homogeneous and the adjoint of these operators are in the class $\mathrm{B}_1(\mathbb{D})$. Consequently, the direct sum, namely, M^* is homogeneous and lies in the class $\mathrm{B}_2(\mathbb{D})$. Let

1. $h(z) = \begin{pmatrix} S^\lambda(z,z) & 0 \\ 0 & S^\mu(z,z)\end{pmatrix}$, $\lambda,\ \mu > 0$,

2. $h'(z) = B_1^{(\alpha,\beta)}(z,z)^{\text{tr}} = \begin{pmatrix} (1-|z|^2)^2 & \beta\bar{z}(1-|z|^2) \\ \beta z(1-|z|^2) & \beta(1+\beta|z|^2)\end{pmatrix}(1-|z|^2)^{-\alpha-\beta-2}$, $\alpha,\beta > 0$, for $z\in\mathbb{D}$,

where X^{tr} denotes the transpose of the matrix X.

The bundles (E,h) and (E',h') correspond to the the operators M^* and M'^* respectively. We denote the curvature and the covariant derivative of the curvature of order $(0,1)$ for the bundles (E,h) and (E',h') by $\mathcal{K}$, $\mathcal{K}_{\bar{z}}$ and $\mathcal{K}'$, $\mathcal{K}'_{\bar{z}}$ respectively. By direct computation we have

$$\mathcal{K}(z) = \begin{pmatrix} \frac{\lambda}{(1-|z|^2)^2} & 0 \\ 0 & \frac{\mu}{(1-|z|^2)^2}\end{pmatrix}, \qquad \mathcal{K}_{\bar{z}}(z) = 2\begin{pmatrix} \frac{\lambda z}{(1-|z|^2)^3} & 0 \\ 0 & \frac{\mu z}{(1-|z|^2)^3}\end{pmatrix};$$

$$\mathcal{K}'(z) = \begin{pmatrix} \frac{\alpha}{(1-|z|^2)^2} & \frac{-2\beta(\beta+1)\bar{z}}{(1-|z|^2)^3} \\ 0 & \frac{\alpha+2\beta+2}{(1-|z|^2)^2}\end{pmatrix}, \qquad \mathcal{K}'_{\bar{z}}(z) = 2\begin{pmatrix} \frac{\alpha z}{(1-|z|^2)^3} & \frac{-\beta(\beta+1)(1+2|z|^2)}{(1-|z|^2)^4} \\ 0 & \frac{(\alpha+2\beta+2)z}{(1-|z|^2)^3}\end{pmatrix}.$$

Choose $\lambda,\ \mu > 0$ with $\mu-\lambda > 2$ and set $\alpha = \lambda$ and $\beta = \frac{1}{2}(\mu-\lambda-2)$. Since curvature is self-adjoint the set of eigenvalues is a complete set of unitary invariants for the curvature. The eigenvalues for $\mathcal{K}(z)$ and $\mathcal{K}'(z)$, $z\in\mathbb{D}$, are clearly the same by the choice of λ, μ, α and β. So, these matrices are pointwise unitarily equivalent. Now, we observe that $\mathcal{K}_{\bar{z}}(0) = 0$ and $\mathcal{K}'_{\bar{z}}(0) \neq 0$. Hence they cannot be unitarily equivalent . It follows that the eigenvalues of the curvature alone cannot determine the unitary equivalence class of the bundle. However, in this example, the covariant derivative of order $(0,1)$ suffices to distinguish the equivalence class of the operators M^* and M'^*.

Before we construct the next example, let us recall that for any reproducing kernel K on $\mathbb{D}$, the normalized kernel $\tilde{K}(z,w)$ at 0 (in the sense of Curto-Salinas [21, Remark 4.7 (b)]) is defined to be the kernel $K(0,0)^{1/2}K(z,0)^{-1}K(z,w)K(0,w)^{-1}K(0,0)^{1/2}$, see Notation 2.3.5.

Let $\mathcal{H}$ be a Hilbert space of holomorphic functions on $\mathbb{D}$ possessing the reproducing kernel K. To emphasize the role of the reproducing kernel, we sometimes write $(\mathcal{H}, K)$ for this Hilbert space. If we assume that the adjoint M^* of the multiplication operator M on the Hilbert space $(\mathcal{H}, K)$ is in $\mathrm{B}_k(\mathbb{D})$, then it follows from [21, Lemma 4.8, page. 474] that the operator $\tilde{M}^*$ on the Hilbert space $\tilde{\mathcal{H}}$ determined by the normalized kernel $\tilde{K}$ is unitarily equivalent to M^* on the Hilbert space $(\mathcal{H}, K)$. Hence the adjoint of the multiplication operator $\tilde{M}$ on $(\tilde{\mathcal{H}}, \tilde{K})$ lies in $\mathrm{B}_k(\mathbb{D})$ as well. Let $(\tilde{E}, \tilde{h})$ be the corresponding bundle, where $\tilde{h}(z) = \tilde{K}(z, z)^{\mathrm{tr}}$, $z \in \mathbb{D}$. The curvature of this bundle is $\tilde{\mathcal{K}}(z) = \frac{\partial}{\partial \bar{z}}(\tilde{h}^{-1}\frac{\partial}{\partial z}\tilde{h})(z)$ for $z \in \mathbb{D}$.

Lemma 4.1.2. *Let $\tilde{h}(z)^{\mathrm{tr}} = \tilde{K}(z, z) = \sum_{k,\ell \geq 0} a_{k\ell} z^k \bar{z}^\ell$. In this notation, we have*

(a) $\partial^m \tilde{h}(0) = \bar{\partial}^n \tilde{h}(0) = 0 = \partial^m \tilde{h}^{-1}(0) = \bar{\partial}^n \tilde{h}^{-1}(0)$ for $m, n \geq 1$ and

(b) $\bar{\partial}\partial\tilde{h}(0) = \tilde{a}_{11}^{\mathrm{tr}}$, $\bar{\partial}\partial\tilde{h}^{-1}(0) = -\tilde{a}_{11}^{\mathrm{tr}}$, $\bar{\partial}^2\partial^2\tilde{h}(0) = 4\tilde{a}_{22}^{\mathrm{tr}}$.

Proof. Since $\tilde{K}(z, w)$ is a real analytic function with $\tilde{K}(z, 0) = I$ for $z \in \mathbb{D}$ and $\tilde{h}(z) = \tilde{K}(z, z)^{\mathrm{tr}}$, it follows that $\partial^m \tilde{h}(0) = m! a_{m0}^{\mathrm{tr}} = 0$ and $\bar{\partial}^n \tilde{h}(0) = n! a_{0n}^{\mathrm{tr}} = 0$. By the same token, for $\tilde{h}^{-1}(z) = \tilde{K}^{-1}(z, z)$, we have $\partial^m \tilde{h}^{-1}(0) = 0$ and $\bar{\partial}^n \tilde{h}^{-1}(0) = 0$ since $\tilde{K}^{-1}(z, 0) = I$ as well for all $z \in \mathbb{D}$. This completes the proof of part (a). To prove part (b), we note that $\bar{\partial}\partial\tilde{h}(0) = \tilde{a}_{11}^{\mathrm{tr}}$ and $\bar{\partial}^2\partial^2\tilde{h}(0) = 4\tilde{a}_{22}^{\mathrm{tr}}$. Also, $\tilde{h}(z)\tilde{h}^{-1}(z) = I$ implies $\bar{\partial}\partial\tilde{h}^{-1}(0) = -\tilde{a}_{11}^{\mathrm{tr}}$. □

Lemma 4.1.3. *The curvature $\tilde{\mathcal{K}}$ and the covariant derivative of the curvature $\tilde{\mathcal{K}}_{\bar{z}^n}$ at 0 are given by the formulae:*

$$\tilde{\mathcal{K}}(0) = \tilde{a}_{11}^{\mathrm{tr}} \text{ and } \tilde{\mathcal{K}}_{\bar{z}^n}(0) = (n+1)!\tilde{a}_{1,n+1}^{\mathrm{tr}}.$$

Proof. Since $\tilde{\mathcal{K}}(z) = \frac{\partial}{\partial \bar{z}}(\tilde{h}^{-1}\frac{\partial}{\partial z}\tilde{h})(z)$, it follows that $\tilde{\mathcal{K}}(0) = \bar{\partial}\tilde{h}^{-1}(0)\partial\tilde{h}(0) + \tilde{h}^{-1}(0)\bar{\partial}\partial\tilde{h}(0) = \tilde{a}_{11}^{\mathrm{tr}}$, by the previous Lemma. Also, $\tilde{\mathcal{K}}_{\bar{z}^n}(0) = \bar{\partial}^n\tilde{\mathcal{K}}(0) = \bar{\partial}^{n+1}(\tilde{h}^{-1}\partial\tilde{h})(0)$ (see [18, Proposition 2.17, page 211]). From Lemma 4.1.2, we have $\bar{\partial}^\ell \tilde{h}^{-1}(0) = 0$ for $\ell \geq 1$. Therefore, using the Leibnitz rule,

$$\tilde{\mathcal{K}}_{\bar{z}^n}(0) = \sum_{k=0}^{n+1} \binom{n+1}{k} \bar{\partial}^{n+1-k}\tilde{h}^{-1}(0)\bar{\partial}^k\partial\tilde{h}(0) = \bar{\partial}^{n+1}\partial\tilde{h}(0) = (n+1)!\tilde{a}_{1,n+1}^{\mathrm{tr}}.$$

This proves the second assertion. □

Lemma 4.1.4. *If $\tilde{\mathcal{K}}$ is the curvature of the bundle $(\tilde{E}, \tilde{h})$, then $\tilde{\mathcal{K}}_{z\bar{z}}(0) = 2(2\tilde{a}_{22} - \tilde{a}_{11}^2)^{\mathrm{tr}}$.*

Proof. We know from [18, Proposition 2.17, page 211] that for a bundle map Θ of a Hermitian holomorphic vector bundle $(\tilde{E}, \tilde{h})$, the covariant derivatives Θ_z and $\Theta_{\bar{z}}$ with respect to a holomorphic frame f are given by $\Theta_z(f) = \partial\Theta(f) + [\tilde{h}^{-1}\partial\tilde{h}, \Theta(f)]$ and $\Theta_{\bar{z}}(f) = \bar{\partial}\Theta(f)$. Since the curvature $\tilde{\mathcal{K}}$ is a bundle map, it follows that

$$\begin{aligned}
\tilde{\mathcal{K}}_{z\bar{z}}(z) &= \bar{\partial}\big(\partial\tilde{\mathcal{K}}(z) + [\tilde{h}^{-1}\partial\tilde{h}, \tilde{\mathcal{K}}](z)\big) \\
&= \bar{\partial}\partial\tilde{\mathcal{K}}(z) + [\bar{\partial}(\tilde{h}^{-1}\partial\tilde{h}), \tilde{\mathcal{K}}](z) + [\tilde{h}^{-1}\partial\tilde{h}, \bar{\partial}\tilde{\mathcal{K}}](z) \\
&= \bar{\partial}\partial\tilde{\mathcal{K}}(z) + [\tilde{h}^{-1}\partial\tilde{h}, \bar{\partial}\tilde{\mathcal{K}}](z).
\end{aligned}$$

Since $\partial\tilde{h}(0) = 0$ by Lemma 4.1.2, we have $\tilde{\mathcal{K}}_{z\bar{z}}(0) = \bar{\partial}\partial\tilde{\mathcal{K}}(0)$. Consequently, $\tilde{\mathcal{K}}_{z\bar{z}}(z)|_{z=0} = \bar{\partial}\partial(\bar{\partial}\tilde{h}^{-1}\partial\tilde{h})(z)|_{z=0}$. This simplifies considerably since $\partial\tilde{h}(0) = \partial^2\tilde{h}(0) = \partial\tilde{h}^{-1}(0) = 0$, again by Lemma 4.1.2. Thus we obtain

$$\tilde{\mathcal{K}}_{z\bar{z}}(0) = 2\bar{\partial}\partial\tilde{h}^{-1}(0)\bar{\partial}\partial\tilde{h}(0) + \bar{\partial}^2\partial^2\tilde{h}(0) = -2\tilde{a}_{11}^{\mathrm{tr}}\tilde{a}_{11}^{\mathrm{tr}} + 4a_{22}^{\mathrm{tr}} = 2(2\tilde{a}_{22} - \tilde{a}_{11}^2)^{\mathrm{tr}}.$$

□

Lemma 4.1.5. *The coefficient of $z^{k+1}\bar{w}^{\ell+1}$ in the power series expansion of $\tilde{K}(z,w)$ is*

$$\begin{aligned}\tilde{a}_{k+1,\ell+1} = a_{00}^{1/2}\Big(&\sum_{s=1}^{k}\sum_{t=1}^{\ell} b_{s0}a_{k+1-s,\ell+1-t}b_{0t}+\\ &\sum_{s=1}^{k} b_{s0}a_{k+1-s,\ell+1}b_{00} + \sum_{t=1}^{\ell} b_{00}a_{k+1,\ell+1-t}b_{0t} + b_{00}a_{k+1,\ell+1}b_{00} - b_{k+1,0}a_{00}b_{0,\ell+1}\Big)a_{00}^{1/2}\end{aligned}$$

for $k,\ell \geq 0$.

Proof. From the definition of $\tilde{K}(z,w)$ we see that for $k,\ell \geq 0$

$$\begin{aligned}\tilde{a}_{k+1,\ell+1} &= a_{00}^{1/2}\Big(\sum_{s=0}^{k+1}\sum_{t=0}^{\ell+1} b_{s0}a_{k+1-s,\ell+1-t}b_{0t}\Big)a_{00}^{1/2}\\
&= a_{00}^{1/2}\Big(\sum_{s=1}^{k+1}\sum_{t=1}^{\ell+1} b_{s0}a_{k+1-s,\ell+1-t}b_{0t} + \sum_{s=1}^{k+1} b_{s0}a_{k+1-s,\ell+1}b_{00}\\
&\qquad + \sum_{t=1}^{\ell+1} b_{00}a_{k+1,\ell+1-t}b_{0t} + b_{00}a_{k+1,\ell+1}b_{00}\Big)a_{00}^{1/2}\\
&= a_{00}^{1/2}\Big(\sum_{s=1}^{k}\sum_{t=1}^{\ell} b_{s0}a_{k+1-s,\ell+1-t}b_{0t} + \sum_{s=1}^{k+1} b_{s0}a_{k+1-s,0}b_{0,\ell+1} + \sum_{t=1}^{\ell} b_{k+1,0}a_{0,\ell+1-t}b_{0t}\\
&\qquad + \sum_{s=1}^{k+1} b_{s0}a_{k+1-s,\ell+1}b_{00} + \sum_{t=1}^{\ell+1} b_{00}a_{k+1,\ell+1-t}b_{0t} + b_{00}a_{k+1,\ell+1}b_{00}\Big)a_{00}^{1/2}\\
&= a_{00}^{1/2}\Big(\sum_{s=1}^{k}\sum_{t=1}^{\ell} b_{s0}a_{k+1-s,\ell+1-t}b_{0t} + \big(\sum_{s=0}^{k+1} b_{s0}a_{k+1-s,0}\big)b_{0,\ell+1} + b_{k+1,0}\big(\sum_{t=0}^{\ell+1} a_{0,\ell+1-t}b_{0t}\big)\\
&\qquad + \sum_{s=1}^{k} b_{s0}a_{k+1-s,\ell+1}b_{00} + \sum_{t=1}^{\ell} b_{00}a_{k+1,\ell+1-t}b_{0t} + b_{00}a_{k+1,\ell+1}b_{00} - b_{k+1,0}a_{00}b_{0,\ell+1}\Big)a_{00}^{1/2}\\
&= a_{00}^{1/2}\Big(\sum_{s=1}^{k}\sum_{t=1}^{\ell} b_{s0}a_{k+1-s,\ell+1-t}b_{0t} + \sum_{s=1}^{k} b_{s0}a_{k+1-s,\ell+1}b_{00} + \sum_{t=1}^{\ell} b_{00}a_{k+1,\ell+1-t}b_{0t}\\
&\qquad + b_{00}a_{k+1,\ell+1}b_{00} - b_{k+1,0}a_{00}b_{0,\ell+1}\Big)a_{00}^{1/2},\end{aligned}$$

as the coefficient of z^{k+1} in $K(z,w)^{-1}K(z,w) = \sum_{s=0}^{k+1} b_{s0}a_{k+1-s,0} = 0$ and the coefficient of $\bar{w}^{\ell+1}$ in $K(z,w)K(z,w)^{-1} = \sum_{t=0}^{\ell+1} a_{0,\ell+1-t}b_{0t} = 0$ for $k,\ell \geq 0$. □

The following Theorem will be useful in the sequel. For T in $\mathrm{B}_n(\Omega)$, recall that $\mathcal{K}_T$ denotes the curvature of the bundle E_T corresponding to T.

Theorem 4.1.6. *Suppose that T_1 and T_2 are homogeneous operators in $\mathrm{B}_n(\mathbb{D})$. Then $\mathcal{K}_{T_1}(0)$ and $(\mathcal{K}_{T_1})_{\bar z}(0)$ are simultaneously unitarily equivalent to $\mathcal{K}_{T_2}(0)$ and $(\mathcal{K}_{T_2})_{\bar z}(0)$ respectively if and only if $\mathcal{K}_{T_1}(z)$ and $(\mathcal{K}_{T_1})_{\bar z}(z)$ are simultaneously unitarily equivalent to $\mathcal{K}_{T_2}(z)$ and $(\mathcal{K}_{T_2})_{\bar z}(z)$ respectively for all z in $\mathbb{D}$.*

Recall that $c : \text{Möb}\times\mathbb{D} \longrightarrow \mathbb{C}$ is defined by the formula

$$c(\varphi^{-1}, z) := (\varphi^{-1})'(z),$$

where the prime stands for differentiation with respect to z. See Notation 2.2.3. The cocycle property of c was verified in Lemma 2.2.4.

The following Lemma will be useful to prove Theorem 4.1.6.

Lemma 4.1.7. *Suppose that T in $\mathrm{B}_n(\mathbb{D})$ is homogeneous . Then*

(a) $\mathcal{K}_T(\varphi^{-1}(0)) = |c(\varphi^{-1},0)|^{-2} U_\varphi^{-1}\mathcal{K}_T(0)U_\varphi$ *and*

(b) $(\mathcal{K}_T)_{\bar z}(\varphi^{-1}(0)) = |c(\varphi^{-1},0)|^{-2}\overline{c(\varphi^{-1},0)^{-1}}U_\varphi^{-1}\Big((\mathcal{K}_T)_{\bar z}(0) - \overline{c(\varphi^{-1},0)^{-1}(\varphi^{-1})^{(2)}(0)}\mathcal{K}_T(0)\Big)U_\varphi$

for some unitary operator U_φ, $\varphi \in$ Möb.

Proof. Following [18], using the homogeneity of the operator T, we find that there is a unitary operator $U_{\varphi,z}$ such that

$$\mathcal{K}_{\varphi(T)}(z) = U_{\varphi,z}^{-1}\mathcal{K}_T(z)U_{\varphi,z},\ \varphi \in \text{Möb and } z \in \mathbb{D}. \tag{4.1.2}$$

On the other hand, an application of the chain rule gives the formula

$$\mathcal{K}_{\varphi(T)}(z) = |(\varphi^{-1})'(z)|^2\mathcal{K}_T((\varphi^{-1})(z)),\ \text{for } \varphi \in \text{Möb and } z \in \mathbb{D}. \tag{4.1.3}$$

Putting both of these together, we clearly have

$$U_{\varphi,z}^{-1}\mathcal{K}_T(z)U_{\varphi,z} = |c(\varphi^{-1},z)|^2\mathcal{K}_T((\varphi^{-1})(z)).$$

In particular, if $z = 0$, then

$$U_{\varphi,0}^{-1}\mathcal{K}_T(\varphi^{-1}(0))U_{\varphi,0} = |c(\varphi^{-1},0)|^2\mathcal{K}_T((\varphi^{-1})(0)).$$

Set $U_{\varphi,0} := U_\varphi$. Then

$$\mathcal{K}_T(\varphi^{-1}(0)) = |c(\varphi^{-1},0)|^{-2}U_\varphi^{-1}\mathcal{K}_T(0)U_\varphi$$

for $\varphi \in$ Möb, $z \in \mathbb{D}$. This proves part (a).

To prove part (b), we differentiate $\mathcal{K}_{\varphi(T)}$ respect to $\bar z$ using (4.1.3) to see that

$$\begin{aligned}\bar\partial\mathcal{K}_{\varphi(T)}(z) &= (\varphi^{-1})'(z)\overline{(\varphi^{-1})^{(2)}(z)}\mathcal{K}_T(\varphi^{-1}(z)) + |(\varphi^{-1})'(z)|^2\overline{(\varphi^{-1})'(z)}\bar\partial\mathcal{K}_T(\varphi^{-1}(z))\\ &= c(\varphi^{-1},z)\overline{(\varphi^{-1})^{(2)}(z)}\mathcal{K}_T(\varphi^{-1}(z)) + |c(\varphi^{-1},z)|^2\overline{c(\varphi^{-1},z)}\bar\partial\mathcal{K}_T(\varphi^{-1}(z)).\end{aligned} \tag{4.1.4}$$

Using (4.1.4) and (a), putting $z = 0$ and $U_{\varphi,0} = U_\varphi$, we see that

$$\begin{aligned} U_\varphi^{-1}\bar{\partial}\mathcal{K}_T(0)U_\varphi &= c(\varphi^{-1},0)\overline{(\varphi^{-1})^{(2)}(0)}\mathcal{K}_T(\varphi^{-1}(0)) + |c(\varphi^{-1},0)|^2\overline{c(\varphi^{-1},0)}\bar{\partial}\mathcal{K}_T(\varphi^{-1}(0)) \\ &= c(\varphi^{-1},0)\overline{(\varphi^{-1})^{(2)}(0)}|c(\varphi^{-1},0)|^{-2}U_\varphi^{-1}\mathcal{K}_T(0)U_\varphi + |c(\varphi^{-1},0)|^2\overline{c(\varphi^{-1},0)}\bar{\partial}\mathcal{K}_T(\varphi^{-1}(0)) \\ &= \overline{c(\varphi^{-1},0)^{-1}(\varphi^{-1})^{(2)}(0)}U_\varphi^{-1}\mathcal{K}_T(0)U_\varphi + |c(\varphi^{-1},0)|^2\overline{c(\varphi^{-1},0)}\bar{\partial}\mathcal{K}_T(\varphi^{-1}(0)). \end{aligned} \tag{4.1.5}$$

So,

$$\bar{\partial}\mathcal{K}_T(\varphi^{-1}(0)) = |c(\varphi^{-1},0)|^{-2}\overline{c(\varphi^{-1},0)^{-1}}U_\varphi^{-1}\Big(\bar{\partial}\mathcal{K}_T(0) - \overline{c(\varphi^{-1},0)^{-1}(\varphi^{-1})^{(2)}(0)}\mathcal{K}_T(0)\Big)U_\varphi.$$

The proof of part (b) is complete since $(\mathcal{K}_T)_{\bar{z}} = \bar{\partial}\mathcal{K}_T$ (cf. [18]). □

Corollary 4.1.8. *Suppose that T_1, T_2 are homogeneous operators in* $\mathrm{B}_n(\mathbb{D})$*. Then*

$$(1)\ \ U^{-1}\mathcal{K}_{T_2}(0)U = \mathcal{K}_{T_1}(0), \qquad (2)\ \ U^{-1}(\mathcal{K}_{T_2})_{\bar{z}}(0)U = (\mathcal{K}_{T_1})_{\bar{z}}(0)$$

for some unitary operator U if and only if

$$(\mathrm{i})\ \ V_\varphi^{-1}\mathcal{K}_{T_2}(z)V_\varphi = \mathcal{K}_{T_1}(z), \qquad (\mathrm{ii})\ \ V_\varphi^{-1}(\mathcal{K}_{T_2})_{\bar{z}}(z)V_\varphi = (\mathcal{K}_{T_1})_{\bar{z}}(z)$$

for some unitary operator V_φ, φ in Möb and $z \in \mathbb{D}$.

Proof. The "if" part is obvious. To prove the "only if" part, take $\varphi = \varphi_{t,z}$, where $\varphi_{t,a}(w) = t\frac{w-a}{1-\bar{a}w}$, for $a, w \in \mathbb{D}$ and $t \in \mathbb{T}$. Pick a unitary operator such that (a) and (b) of Lemma 4.1.7 are satisfied. We get from (1) and Lemma 4.1.7(a) that

$$\begin{aligned} \mathcal{K}_{T_1}(z) &= |c(\varphi^{-1},0)|^{-2}U_\varphi^{-1}\mathcal{K}_{T_1}(0)U_\varphi \\ &= |c(\varphi^{-1},0)|^{-2}U_\varphi^{-1}U^{-1}\mathcal{K}_{T_2}(0)UU_\varphi \\ &= |c(\varphi^{-1},0)|^{-2}U_\varphi^{-1}U^{-1}|c(\varphi^{-1},0)|^2U_\varphi\mathcal{K}_{T_2}(z)U_\varphi^{-1}UU_\varphi \\ &= U_\varphi^{-1}U^{-1}U_\varphi\mathcal{K}_{T_2}(z)U_\varphi^{-1}UU_\varphi. \end{aligned}$$

Since $V_\varphi := U_\varphi^{-1}UU_\varphi$ is unitary, the proof of (i) is complete.

From (1), (2) and Lemma 4.1.7(b)

$$\begin{aligned} (\mathcal{K}_{T_1})_{\bar{z}}(z) &= |c(\varphi^{-1},0)|^{-2}\overline{c(\varphi^{-1},0)^{-1}}U_\varphi^{-1}\Big((\mathcal{K}_{T_1})_{\bar{z}}(0) - \overline{c(\varphi^{-1},0)^{-1}(\varphi^{-1})^{(2)}(0)}\mathcal{K}_{T_1}(0)\Big)U_\varphi \\ &= |c(\varphi^{-1},0)|^{-2}\overline{c(\varphi^{-1},0)^{-1}}U_\varphi^{-1}U^{-1}\Big((\mathcal{K}_{T_2})_{\bar{z}}(0) - \overline{c(\varphi^{-1},0)^{-1}(\varphi^{-1})^{(2)}(0)}\mathcal{K}_{T_2}(0)\Big)UU_\varphi \\ &= |c(\varphi^{-1},0)|^{-2}\overline{c(\varphi^{-1},0)^{-1}}U_\varphi^{-1}U^{-1}\Big(\overline{(\varphi^{-1})^{(2)}(0)c(\varphi^{-1},0)^{-1}}\mathcal{K}_{T_2}(0) \\ &\quad + |c(\varphi^{-1},0)|^2\overline{c(\varphi^{-1},0)}U_\varphi(\mathcal{K}_{T_2})_{\bar{z}}(z)U_\varphi^{-1} - \overline{(\varphi^{-1})^{(2)}(0)c(\varphi^{-1},0)^{-1}}\mathcal{K}_{T_2}(0)\Big)UU_\varphi \\ &= U_\varphi^{-1}U^{-1}U_\varphi(\mathcal{K}_{T_2})_{\bar{z}}(z)U_\varphi^{-1}UU_\varphi. \end{aligned} \tag{4.1.6}$$

Taking $V_\varphi = U_\varphi^{-1}UU_\varphi$ as before, we have (ii). □

Proof of Theorem 4.1.6. Combining Lemma 4.1.7 and Corollary 4.1.8, we have a proof of the Theorem 4.1.6. □

Corollary 4.1.9. *Suppose that $T \in \mathrm{B}_n(\mathbb{D})$ is homogeneous . Then the eigenvalues of $\mathcal{K}_T(a)$ are $\{\lambda_i(1-|a|^2)^{-2}\}_{i=1}^k$, $a \in \mathbb{D}$, $\lambda_i > 0$; $\lambda_i(1-|a|^2)^{-2}$ has multiplicity m_i, $1 \le i \le k$.*

Proof. We have from Lemma 4.1.7(a) that

$$\mathcal{K}_T(a) = (1-|a|^2)^{-2} U_\varphi^{-1} \mathcal{K}_T(0) U_\varphi \tag{4.1.7}$$

for some unitary operator U_φ, $\varphi \in$ Möb, where $\varphi = \varphi_{t,a}$ for $(t,a) \in \mathbb{T} \times \mathbb{D}$, $\varphi_{t,a}(z) = t\frac{z-a}{1-\bar{a}z}$. Without loss of generality [18, Proposition 2.20] one can assume that $\mathcal{K}_T(0)$ is diagonal. Let $\{\lambda_i\}_{i=0}^k$ be the distinct diagonal entries of $\mathcal{K}_T(0)$ with multiplicity m_i, $1 \le i \le k$. We know from [18, Proposition 2.20] that $\lambda_i > 0$, $1 \le i \le k$. Without loss of generality we assume that $\lambda_{i+1} > \lambda_i$ for $1 \le i \le k-1$. Let $\{\Lambda_i(a)\}_{i=1}^n$ be the eigenvalues of $\mathcal{K}_T(a)$. Since $\mathcal{K}_T(0)$ has distinct eigenvalues $\{\lambda_i\}_{i=1}^k$ with λ_i having multiplicity m_i for $1 \le i \le k$, $\{\Lambda_i(0)\}_{i=1}^k$ are the distinct eigenvalues of $\mathcal{K}_T(0)$ with $\Lambda_i(0)$ having multiplicity m_i and $\Lambda_i(0) = \lambda_i$ for $1 \le i \le k$. Now, connectedness of $\mathbb{D}$ implies that m_i is a constant function for $1 \le i \le k$. So, Λ_i has multiplicity m_i on $\mathbb{D}$. By real-analyticity of the function $\mathcal{K}_T$ on $\mathbb{D}$ it follows that the function Λ_i is also real-analytic on $\mathbb{D}$ for $1 \le i \le k$. Since $\lambda_{i+1} > \lambda_i$ for $1 \le i \le k-1$, by continuity of Λ_i's there exist a neighborhood W of 0 such that $\Lambda_{i+1}(a) > \Lambda_i(a)$ for $a \in W$, $1 \le i \le k-1$. Therefore, we have from (4.1.7), $\Lambda_i(a) = \lambda_i(1-|a|^2)^{-2}$ for $a \in W$, $1 \le i \le k$. Now, we have the desired conclusion by real-analiticity of Λ_i's. □

Corollary 4.1.10. *Suppose that $T \in \mathrm{B}_2(\mathbb{D})$ is homogeneous and $T = T_1 \oplus T_2$ for $T_1, T_2 \in \mathrm{B}_1(\mathbb{D})$. Then the operators T_1 and T_2 are homogeneous.*

Proof. Let E_T be Hermitian holomorphic vector bundle the associated with $T \in \mathrm{B}_2(\mathbb{D})$. By [18, Proposition 1.18] the hypothesis is equivalent to $E_T = E_{T_1} \oplus E_{T_2}$, E_{T_i} being the bundle associated with $T_i \in \mathrm{B}_1(\mathbb{D})$ for $i = 1, 2$. So, the metric h for the bundle E_T is of the form $h = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$, where h_i is the metric for $E_{T_i}, i = 1, 2$. Now, it follows from Lemma 4.1.7(a) that

$$\mathcal{K}_T(a) = \begin{pmatrix} \mathcal{K}_{h_1}(a) & 0 \\ 0 & \mathcal{K}_{h_2}(a) \end{pmatrix} = (1-|a|^2)^{-2} U_\varphi^{-1} \mathcal{K}_T(0) U_\varphi \tag{4.1.8}$$

for some unitary operator U_φ, $\varphi \in$ Möb, where $\varphi = \varphi_{t,a}$ for $(t,a) \in \mathbb{T} \times \mathbb{D}$, $\varphi_{t,a}(z) = t\frac{z-a}{1-\bar{a}z}$ and $\mathcal{K}_{h_i}$ is the curvature of the line bundle E_{T_i} with respect to the metric h_i for $i = 1, 2$.

Without loss of generality [18, Proposition 2.20] one can assume that $\mathcal{K}_T(0)$ is diagonal, that is, $\mathcal{K}_T(0) = \mathrm{diag}(\lambda_1, \lambda_2)$. We know from [18, Proposition 2.20] that $\lambda_1, \lambda_2 > 0$. The case $\lambda_1 = \lambda_2$ is trivial, so we may assume that $\lambda_1 > \lambda_2$. Since $\mathcal{K}_T(a)$ is similar with $(1-|a|^2)^{-2} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ putting $a = 0$ we see that $\mathcal{K}_T(0) = \begin{pmatrix} \mathcal{K}_{h_1}(0) & 0 \\ 0 & \mathcal{K}_{h_2}(0) \end{pmatrix}$ is similar with $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Without loss of generality one can assume that $\mathcal{K}_{h_i}(0) = \lambda_i$ for $i = 1, 2$. By continuity of $\mathcal{K}_{h_i}$ there exist neighborhoods V of 0 such that $\mathcal{K}_{h_1} > \mathcal{K}_{h_2}$ on V as $\lambda_1 > \lambda_2$. It follows that $\mathcal{K}_{h_i}(a) = \lambda_i(1-|a|^2)^{-2}$ for $a \in V, i = 1, 2$. Hence by real analyticity of $\mathcal{K}_{h_i}$ one concludes that $\mathcal{K}_{h_i}(a) = \lambda_i(1-|a|^2)^{-2}$ for $a \in \mathbb{D}, i = 1, 2$. This is same as saying that the operators T_1 and T_2 are homogeneous [32, 51]. □

Remark 4.1.11. It is pointed out that atomic homogeneous need not always be irreducible. Multiplication operators by the respective co-ordinate functions on the Hilbert spaces $L^2(\mathbb{T})$ and $L^2(\mathbb{D})$ are examples of atomic homogeneous operators which are not irreducible. The Corollary 4.1.10 shows that the atomic homogeneous operators in $\mathrm{B}_2(\mathbb{D})$ are irreducible.

Moreover, one can show the following: If $T \in \mathrm{B}_n(\mathbb{D})$ is homogeneous and $T = \oplus_{i=1}^n T_i$ for $T_i \in \mathrm{B}_1(\mathbb{D}), i = 1, \ldots, n$; then T_i is homogeneous for $i = 1, \ldots, n$.

The proof of this fact is similar to the proof of Corollary 4.1.10.

Notation 4.1.12. *For a positive integer m, let $S(c_1, \ldots, c_m)$ denote the forward shift on $\mathbb{C}^{m+1}$ with weight sequence $(c_1, \ldots, c_m)$, $c_i \in \mathbb{C}$, that is,*

$$S(c_1, \ldots, c_m)(\ell, p) = c_\ell \delta_{p+1,\ell} \quad \textit{for} \quad 0 \le p, \ell \le m.$$

We set $\mathbb{S}_m := S(1, \ldots, m)$. For A in $\mathcal{M}_{p,q}$, we let $A(i,j)$ denote the (i,j)-th entry of the matrix A for $1 \le i \le p$, $1 \le j \le q$. For a vector $\boldsymbol{v}$ in $\mathbb{C}^k$, let $\boldsymbol{v}(i)$ denote the i-th component of the vector $\boldsymbol{v}$, $1 \le i \le k$.

Example 4.1.13. From (4.1.1), we get

$$B_1^{(\alpha,\beta')}(z,w) = \begin{pmatrix} (1-z\bar{w})^2 & \beta' z(1-z\bar{w}) \\ \beta'\bar{w}(1-z\bar{w}) & \beta'(1+\beta' z\bar{w}) \end{pmatrix} (1-z\bar{w})^{-\alpha-\beta'-2}$$

and

$$B_2^{(\alpha,\beta)}(z,w) = \begin{pmatrix} (1-z\bar{w})^4 & \beta(1-z\bar{w})^3 z & \beta(\beta+1)(1-z\bar{w})^2 z^2 \\ \beta(1-z\bar{w})^3\bar{w} & \beta(1+\beta z\bar{w})(1-z\bar{w})^2 & \beta(\beta+1)(2+\beta z\bar{w})(1-z\bar{w})z \\ \beta(\beta+1)(1-z\bar{w})^2\bar{w}^2 & \beta(\beta+1)(2+\beta z\bar{w})(1-z\bar{w})\bar{w} & \beta(\beta+1)(2+(\beta+1)(4+\beta z\bar{w})z\bar{w}) \end{pmatrix} (1-z\bar{w})^{-\alpha-\beta-4}$$

for $\alpha, \beta, \beta' > 0$ and $(z,w) \in \mathbb{D} \times \mathbb{D}$. Let

$$K_1(z,w) := (1-z\bar{w})^{-\alpha} \oplus B_1^{(\alpha,\beta')}(z,w) \text{ and } K_2(z,w) := B_2^{(\alpha,\beta)}(z,w) \text{ for } (z,w) \in \mathbb{D} \times \mathbb{D}.$$

Let M_1 and M_2 be the multiplication operators on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with reproducing kernels K_1 and K_2 respectively. Clearly, M_1 is the direct sum $M^{(\alpha)} \oplus M_1^{(\alpha,\beta')}$ acting on the Hilbert space $\mathbb{A}^{(\alpha)}(\mathbb{D}) \oplus J^{(1)}\mathbb{A}^{(\alpha,\beta')}(\mathbb{D}^2)_{|\mathrm{res}\,\triangle}$ and M_2 is the multiplication operator on the Hilbert space $J^{(2)}\mathbb{A}^{(\alpha,\beta)}(\mathbb{D}^2)_{|\mathrm{res}\,\triangle}$. Wilkins [51] has shown that the adjoint of the operator $M_1^{(\alpha,\beta')}$ on $J^{(1)}\mathbb{A}^{(\alpha,\beta')}(\mathbb{D}^2)_{|\mathrm{res}\,\triangle}$ is in $\mathrm{B}_2(\mathbb{D})$. This operator is also homogeneous . It is easy to see that the operator $M^{(\alpha)}$ is homogeneous and its adjoint is in the class $\mathrm{B}_1(\mathbb{D})$. Consequently, the direct sum, namely, M_1^* is homogeneous and lies in the class $\mathrm{B}_3(\mathbb{D})$. The operator M_2^* is in $\mathrm{B}_3(\mathbb{D})$ by [24, Proposition 3.6] and is homogeneous by [9, Page. 428] and [37, Theorem 5.1]. Let

$$h_1(z) = K_1(z,z)^{\mathrm{tr}} \text{ and } h_2(z) = K_2(z,z)^{\mathrm{tr}}. \tag{4.1.9}$$

Thus h_1 and h_2 are the metrics for the bundles E_1 and E_2 corresponding to the operators M_1^* and M_2^* respectively.

Lemma 4.1.14. *The curvature at zero and the covariant derivatives of curvature at zero of order* $(0,1)$ *and* $(1,1)$ *for the bundles* E_1 *and* E_2 *are*

(a) $\tilde{\mathcal{K}}_1(0) = \operatorname{diag}(\alpha, \alpha, \alpha+2\beta'+2)$, $(\tilde{\mathcal{K}}_1)_{\bar z}(0) = S(0, -2\sqrt{\beta'}(\beta'+1))^{\mathrm{tr}}$ *and* $(\tilde{\mathcal{K}}_1)_{z\bar z}(0) = 2\operatorname{diag}(\alpha, \alpha+\beta'(\beta'+1), \alpha+\beta'(-\beta'+1)+2)$;

(b) $\tilde{\mathcal{K}}_2(0) = \operatorname{diag}\ (\alpha, \alpha, \alpha+3\beta+6)$, $(\tilde{\mathcal{K}}_2)_{\bar z}(0) = S\big(0, -3\sqrt{2(\beta+1)}(\beta+2)\big)^{\mathrm{tr}}$ *and* $(\tilde{\mathcal{K}}_2)_{z\bar z}(0) = \operatorname{diag}\big(\alpha, \alpha+3(\beta+1)(\beta+2), \alpha-3\beta(\beta+2)\big)$,

respectively. Here $\tilde{\mathcal{K}}_i$, $(\tilde{\mathcal{K}}_i)_{\bar z}$ *and* $(\tilde{\mathcal{K}}_i)_{z\bar z}$ *are computed with respect to the metrics* $\tilde h_i$ *for* $i = 1, 2$ *obtained from the corresponding reproducing kernels normalized at* 0.

(If $\tilde h$ is a metric corresponding to a normalized reproducing kernel at 0, then $\tilde h(0) = I$, that is, the basis for the fibre at 0 with respect to which $\tilde h(0)$ is computed is orthonormal.)

Proof. For any reproducing kernel K with

$$K(z,w) = \sum_{m,n\geq 0} a_{mn} z^m \bar w^n \quad \text{and} \quad K(z,w)^{-1} = \sum_{m,n\geq 0} b_{mn} z^m \bar w^n,$$

the identity $K(z,w)^{-1}K(z,w) = I$ implies that

$$b_{00} = a_{00}^{-1} \text{ and } \sum_{\ell=0}^{k} b_{0,k-\ell}a_{0\ell} = 0,\ k \geq 1.$$

For $k=1$, we have $b_{10} = -a_{00}^{-1}a_{10}a_{00}^{-1}$, $b_{01} = (b_{10})^*$. Also, by Lemma 4.1.5, we have

$$\tilde a_{11} = a_{00}^{1/2}\big(b_{00}a_{11}b_{00} - b_{10}a_{00}b_{01}\big)a_{00}^{1/2} = a_{00}^{-1/2}\big(a_{11} - a_{10}a_{00}^{-1}a_{01}\big)a_{00}^{-1/2}. \tag{4.1.10}$$

For $k=2$, we have $b_{02} = -\big(b_{01}a_{01} + b_{00}a_{02}\big)a_{00}^{-1} = a_{00}^{-1}\big(a_{01}a_{00}^{-1}a_{01} - a_{02}\big)a_{00}^{-1}$. Now, Lemma 4.1.5 gives

$$\begin{aligned}\tilde a_{12} &= a_{00}^{1/2}\big(b_{00}a_{11}b_{01} + b_{00}a_{12}b_{00} - b_{10}a_{00}b_{02}\big)a_{00}^{1/2}\\ &= a_{00}^{-1/2}\big(a_{12} - (a_{11} - a_{10}a_{00}^{-1}a_{01})a_{00}^{-1}a_{01} - a_{10}a_{00}^{-1}a_{02}\big)a_{00}^{-1/2}.\end{aligned} \tag{4.1.11}$$

Observing that $b_{20} = {b_{02}}^* = a_{00}^{-1}\big(a_{10}a_{00}^{-1}a_{10} - a_{20}\big)a_{00}^{-1}$, from Lemma 4.1.5, we have

$$\begin{aligned}\tilde a_{22} &= a_{00}^{1/2}\big(b_{10}a_{11}b_{01} + b_{10}a_{12}b_{00} + b_{00}a_{21}b_{01} + b_{00}a_{22}b_{00} - b_{20}a_{00}b_{02}\big)a_{00}^{1/2}\\ &= a_{00}^{-1/2}\big(a_{10}a_{00}^{-1}a_{11}a_{00}^{-1}a_{01} - a_{10}a_{00}^{-1}a_{12} - a_{21}a_{00}^{-1}a_{01} + a_{22}\\ &\qquad -(a_{10}a_{00}^{-1}a_{10} - a_{20})a_{00}^{-1}(a_{01}a_{00}^{-1}a_{01} - a_{02})\big)a_{00}^{-1/2}\\ &= a_{00}^{-1/2}\big(a_{22} + (a_{20}a_{00}^{-1}a_{01} - a_{21})a_{00}^{-1}a_{01} - a_{20}a_{00}^{-1}a_{02}\\ &\qquad -a_{10}a_{00}^{-1}(a_{12} - (a_{11} - a_{10}a_{00}^{-1}a_{01})a_{00}^{-1}a_{01} - a_{10}a_{00}^{-1}a_{02})\big)a_{00}^{-1/2}.\end{aligned} \tag{4.1.12}$$

In particular, choosing $K = K_1$, we have

$$a_{00} = \operatorname{diag}(1,1,\beta'), \qquad a_{01} = S\big(0,\beta'\big);$$
$$a_{11} = \operatorname{diag}(\alpha,\alpha+\beta',\beta'(\alpha+2\beta'+2)), \qquad a_{12} = S\big(0,\beta'(\alpha+\beta'+1)\big);$$

$$a_{22} = \operatorname{diag}\big(\frac{\alpha(\alpha+1)}{2},\frac{(\alpha+\beta')(\alpha+\beta'+1)}{2},\frac{\beta'(\alpha+\beta'+2)(\alpha+3\beta'+3)}{2}\big), \quad \text{and} \quad a_{20}=0.$$

Thus, $a_{11} - a_{10}a_{00}^{-1}a_{01} = \operatorname{diag}(\alpha,\alpha,\beta'(\alpha+2\beta'+2))$. Hence from Lemma 4.1.3 and Equation (4.1.10), we have $\tilde{\mathcal{K}}_1(0) = \tilde{a}_{11}^{\rm tr} = \operatorname{diag}(\alpha,\alpha,\alpha+2\beta'+2)$.

From Equation (4.1.11), we get $\tilde{a}_{12} = S\big(0,-\sqrt{\beta'}(\beta'+1)\big)$. So, from Lemma 4.1.3, we have $(\tilde{\mathcal{K}}_1)_{\bar z}(0) = 2\tilde{a}_{12}^{\rm tr} = S\big(0,-2\sqrt{\beta'}(\beta'+1)\big)^{\rm tr}$.

Similarly, from Equation (4.1.12), $\tilde{a}_{22} = \operatorname{diag}\big(\frac{\alpha(\alpha+1)}{2},\frac{\alpha(\alpha+1)+\beta'(\beta'+1)}{2},\frac{(\alpha+\beta'+2)(\alpha+3\beta'+3)}{2}\big)$. Hence

$$(\tilde{\mathcal{K}}_1)_{z\bar z}(0) = 2(2\tilde{a}_{22}-\tilde{a}_{11}^2)^{\rm tr} = 2\ \operatorname{diag}(\alpha,\alpha+\beta'(\beta'+1),\alpha+\beta'(-\beta'+1)+2)$$

from Lemma 4.1.4. This completes the proof of (a).

To prove (b), choose $K = K_2$ and observe that

$$a_{00} = \operatorname{diag}(1,\beta,2\beta(\beta+1)), \qquad a_{10} = S\big(\beta,2\beta(\beta+1)\big)^{\rm tr},$$

$$a_{12} = S\big(\beta(\alpha+\beta+1),\beta(\beta+1)(2\alpha+3\beta+6)\big), \qquad \big(a_{02}\big)(i,j) = \begin{cases} \beta(\beta+1) & \text{for } i=3, j=1;\\ 0 & \text{otherwise,}\end{cases}$$

$$a_{11} = \operatorname{diag}\ \big(\alpha+\beta,\beta(\alpha+2\beta+2),2\beta(\beta+1)(\alpha+3\beta+6)\big)$$

and

$$\begin{aligned} a_{22} \ &= \ \operatorname{diag}\big(\frac{(\alpha+\beta)(\alpha+\beta+1)}{2},\frac{\beta(\alpha+\beta+2)(\alpha+3\beta+3)}{2},\\ &\qquad \beta(\beta+1)((\alpha+\beta+4)(\alpha+\beta+5)+4(\beta+1)(\alpha+\beta+4)+\beta(\beta+1))\big).\end{aligned}$$

Therefore, $a_{11} - a_{10}a_{00}^{-1}a_{01} = \operatorname{diag}\big(\alpha,\alpha\beta,2\beta(\beta+1)(\alpha+3\beta+6)\big)$. Hence from Lemma 4.1.3 and Equation (4.1.10), we have

$$\tilde{\mathcal{K}}_2(0) = \tilde{a}_{11}^{\rm tr} = \operatorname{diag}(\alpha,\alpha,\alpha+3\beta+6).$$

Also, from Equation (4.1.11), we have

$$\tilde{a}_{12} = S\big(0,-\frac{3}{\sqrt{2}}\sqrt{\beta+1}(\beta+2)\big)$$

and from Lemma 4.1.3, we have

$$(\tilde{\mathcal{K}}_2)_{\bar z}(0) = 2\tilde{a}_{12}^{\rm tr} = S\big(0,-3\sqrt{2(\beta+1)}(\beta+2)\big)^{\rm tr}.$$

Since

$$\tilde{a}_{22} = \operatorname{diag}\big(\frac{\alpha(\alpha+1)}{2}, \frac{\alpha(\alpha+1)+3(\beta+1)(\beta+2)}{2}, \frac{\alpha(\alpha+1)}{2} + 3(\beta+2)(\alpha+\beta+3)\big)$$

from Equation (4.1.12), using Lemma 4.1.4, we get

$$(\tilde{\mathcal{K}}_2)_{z\bar{z}}(0) = 2(2\tilde{a}_{22} - \tilde{a}_{11}^2)^{\mathrm{tr}} = 2\ \operatorname{diag}\big(\alpha, \alpha+3(\beta+1)(\beta+2), \alpha - 3\beta(\beta+2)\big).$$

□

By means of a sequence of lemmas proved below, we construct a unitary operator between the vector spaces $\big((E_1)_0, h_1(0)\big)$ and $\big((E_2)_0, h_2(0)\big)$ which intertwines $\tilde{\mathcal{K}}_1(0)$, $\tilde{\mathcal{K}}_2(0)$ and $(\tilde{\mathcal{K}}_1)_{\bar{z}}(0)$, $(\tilde{\mathcal{K}}_2)_{\bar{z}}(0)$. Here $(E_1)_0$ and $(E_2)_0$ are the fibres over 0 of the corresponding bundles E_1 and E_2 respectively.

Lemma 4.1.15. *A linear transformation $U_0 : (\mathbb{C}^3, h_2(0)) \longrightarrow (\mathbb{C}^3, h_1(0))$ is diagonal and unitary with $U_0 = \operatorname{diag}(u_1, u_2, u_3)$, $u_i \in \mathbb{C}$ for $i = 1, 2, 3$, if and only if $|u_1|^2 = 1, |u_2|^2 = \beta, |u_3|^2 = \frac{2\beta(\beta+1)}{\beta'}$.*

Proof. "only if" part: Since U_0 is a unitary operator we have $U_0^* = U_0^{-1}$, where $*$ denotes the adjoint of U_0. Now, from [24, p. 395]

$$\begin{aligned} U_0^* &= h_2(0)^{-1}\overline{U}_0^{\mathrm{tr}} h_1(0) \\ &= \operatorname{diag}\big(1, \beta^{-1}, (2\beta(\beta+1))^{-1}\big)\operatorname{diag}(\bar{u}_1, \bar{u}_2, \bar{u}_3)\operatorname{diag}(1, 1, \beta') \\ &= \operatorname{diag}\big(\bar{u}_1, \frac{\bar{u}_2}{\beta}, \frac{\bar{u}_3\beta'}{2\beta(\beta+1)}\big) \\ &= \operatorname{diag}(u_1^{-1}, u_2^{-1}, u_3^{-1}) \end{aligned}$$

This implies the desired equalities.

"if" part: Taking $u_1 = 1, u_2 = \sqrt{\beta}, u_3 = \sqrt{\frac{2\beta(\beta+1)}{\beta'}}$, we see that $U_0 = \operatorname{diag}(u_1, u_2, u_3)$ is a unitary operator between the two given vector spaces. □

The proof of the next lemma is just a routine verification.

Lemma 4.1.16. *Suppose that T and $\tilde{T}$ are in $\mathcal{M}_3$ such that*

$$T(i,j) = \begin{cases} \eta & \text{for } i=2, j=3; \\ 0 & \text{otherwise.} \end{cases} \quad \textit{and} \quad \tilde{T}(i,j) = \begin{cases} \tilde{\eta} & \text{for } i=2, j=3; \\ 0 & \text{otherwise.} \end{cases}$$

Then $AT = \tilde{T}A$ for some invertible diagonal matrix $A = \operatorname{diag}(a_1, a_2, a_3)$ if and only if $\frac{\tilde{\eta}}{\eta} = \frac{a_2}{a_3}$.

Lemma 4.1.17. *If $\beta' = \frac{3}{2}\beta + 2$, then $U_0^{-1}\tilde{\mathcal{K}}_1(0)U_0 = \tilde{\mathcal{K}}_2(0)$ and $U_0^{-1}(\tilde{\mathcal{K}}_1)_{\bar{z}}(0)U_0 = (\tilde{\mathcal{K}}_2)_{\bar{z}}(0)$, where $U_0 : \big(\mathbb{C}^3, h_2(0)\big) \longrightarrow \big(\mathbb{C}^3, h_1(0)\big)$, is a diagonal unitary with $U_0 = \operatorname{diag}(u_1, u_2, u_3)$, $u_i \in \mathbb{C}$ for $i = 1, 2, 3$.*

Proof. Our the choice of β' together with Lemma 4.1.14 ensures that $\tilde{\mathcal{K}}_1(0) = \tilde{\mathcal{K}}_2(0)$. The first equality is therefore evident.

Clearly, $(\tilde{\mathcal{K}}_2)_{\bar{z}}(0)$ and $(\tilde{\mathcal{K}}_1)_{\bar{z}}(0)$ are of the form T and $\tilde{T}$ of the previous Lemma. Choose

$$u_1 = 1, u_2 = \sqrt{\beta}, u_3 = \sqrt{\frac{2\beta(\beta+1)}{\beta'}}, \quad \text{with} \quad \beta' = \frac{3}{2}\beta + 2.$$

To complete the proof of the second equality, by Lemma 4.1.16, we only have to verify $\frac{\tilde{\eta}}{\eta} = \frac{u_2}{u_3}$, where $\eta = -3\sqrt{2(\beta+1)}(\beta+2), \tilde{\eta} = -2\sqrt{\beta'}(\beta'+1)$. Now,

$$\frac{u_2}{u_3} = \sqrt{\frac{\beta\beta'}{2\beta(\beta+1)}} = \sqrt{\frac{\frac{3}{2}\beta+2}{2(\beta+1)}} = \frac{1}{2}\sqrt{\frac{3\beta+4}{\beta+1}}$$

and

$$\frac{\tilde{\eta}}{\eta} = \frac{-2\sqrt{\beta'}(\beta'+1)}{-3\sqrt{2(\beta+1)}(\beta+2)} = \frac{2\sqrt{\frac{3}{2}\beta+2}(\frac{3}{2}\beta+2+1)}{3\sqrt{2(\beta+1)}(\beta+2)} = \frac{3\sqrt{3\beta+4}(\beta+2)}{2(3\sqrt{\beta+1})(\beta+2)} = \frac{1}{2}\sqrt{\frac{3\beta+4}{\beta+1}}.$$

□

Since the operators M_1 and M_2 are homogeneous , combining Lemma 4.1.17 with Theorem 4.1.6, we have the following Corollary.

Corollary 4.1.18. *For φ in* Möb*, there is a unitary operator U_φ such that ${U_\varphi}^{-1}\tilde{\mathcal{K}}_1(z)U_\varphi = \tilde{\mathcal{K}}_2(z)$ and ${U_\varphi}^{-1}(\tilde{\mathcal{K}}_1)_{\bar{z}}(z)U_\varphi = (\tilde{\mathcal{K}}_2)_{\bar{z}}(z)$.*

Lemma 4.1.19. *If $\beta' = \frac{3}{2}\beta + 2$ then $(\tilde{\mathcal{K}}_1)_{z\bar{z}}(0)$ and $(\tilde{\mathcal{K}}_2)_{z\bar{z}}(0)$ are not unitarily equivalent .*

Proof. By Lemma 4.1.14, $(\tilde{\mathcal{K}}_i)_{z\bar{z}}(0) = \mathrm{diag}(p_i, q_i, r_i)$ for $i = 1, 2$, where $p_1 = \alpha$, $q_1 = \alpha + \beta'(\beta'+1)$, $r_1 = \alpha + \beta'(-\beta'+1)$ and $p_2 = \alpha$, $q_2 = \alpha + 3(\beta+1)(\beta+2)$, $r_2 = \alpha - 3\beta(\beta+2)$. Clearly,

$$p_1 = p_2, q_1 > r_1 \text{ and } q_2 > r_2. \tag{4.1.13}$$

If the diagonal matrices $(\tilde{\mathcal{K}}_1)_{z\bar{z}}(0)$ and $(\tilde{\mathcal{K}}_2)_{z\bar{z}}(0)$ are unitarily equivalent then $\{p_1, q_1, r_1\} = \{p_2, q_2, r_2\}$, as sets. From (4.1.13), we see that this can happen only if $p_1 = p_2$, $q_1 = q_2$ and $r_1 = r_2$. Since $\beta' = \frac{3}{2}\beta + 2$, $q_1 = \alpha + \frac{3}{4}(\beta+2)(3\beta+4)$. We see that $q_1 \neq q_2$ as $\beta \neq 0$. Hence $(\tilde{\mathcal{K}}_1)_{z\bar{z}}(0)$ and $(\tilde{\mathcal{K}}_2)_{z\bar{z}}(0)$ are not unitarily equivalent . □

The following Theorem is now obvious.

Theorem 4.1.20. *The simultaneous unitary equivalence class of the curvatures and the covariant derivatives of the curvatures of order $(0,1)$ for the operators M_1^* and M_2^* are the same for $\beta' = \frac{3}{2}\beta + 2$. However, the covariant derivatives of the curvatures of order $(1,1)$ are not unitarily equivalent .*

4.2 Irreducible Examples and Permutation of Curvature Eigenvalues

In the Example 4.1.1, one of the two homogeneous operators M^* is reducible while the other M'^* is irreducible. Similarly in the Example 4.1.13, one of the two operators M_1^* is reducible whereas the other M_2^* is irreducible. Irreducibility of M'^* and M_2^* follows from [37]. We are interested in constructing such examples within the class of irreducible operators in $\mathrm{B}_n(\mathbb{D})$. The class of irreducible homogeneous operators in $\mathrm{B}_2(\mathbb{D})$ cannot possibly possess such examples, since the eigenvalues of the curvature at 0 is a complete invariant for these operators (cf. [51]). Therefore, we consider a class of homogeneous operators in $\mathrm{B}_3(\mathbb{D})$ mentioned in Notation 2.3.12 and discussed in [31]. However, we first show that for generic bundles $E^{(\lambda,\boldsymbol{\mu})}$ the simultaneous equivalence class of the curvature and the covariant derivative of the curvature of order $(0,1)$ determine the equivalence class of the homogeneous Hermitian holomorphic vector bundle $E^{(\lambda,\boldsymbol{\mu})}$.

Recalling Notation 2.3.12, we have

Lemma 4.2.1. *For the reproducing kernel* $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$*, we have*

(a) $\tilde{a}_{11} = [\mathrm{B}^{-1}\mathbb{S}_m\mathrm{B}, \mathbb{S}_m^*] + (2\lambda + m)I_{m+1} - 2D_m$,

(b) $\tilde{a}_{12} = \mathrm{B}^{1/2}\big(\frac{1}{2}(\mathrm{B}^{-1}\mathbb{S}_m^2\mathrm{B}\mathbb{S}_m^*\mathrm{B}^{-1} + \mathbb{S}_m^*\mathrm{B}^{-1}\mathbb{S}_m{}^2) + \mathrm{B}^{-1}[D_m, \mathbb{S}_m] - \mathrm{B}^{-1}\mathbb{S}_m\mathrm{B}\mathbb{S}_m^*\mathrm{B}^{-1}\mathbb{S}_m\big)\mathrm{B}^{1/2}$,

where I_k *denotes the identity matrix of order* k *and* $D_m = \mathrm{diag}\ (m, \ldots, 1, 0)$.

Proof. From Equation (4.1.10) in Lemma 4.1.14, we get $\tilde{a}_{11} = a_{00}^{-1/2}\big(a_{11} - a_{10}a_{00}^{-1}a_{01}\big)a_{00}^{-1/2}$. Form the expansion of the reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ we see that

$$a_{00} = \mathrm{B}, a_{10} = \mathrm{B}\mathbb{S}_m^*, a_{01} = \mathbb{S}_m\mathrm{B}, a_{11} = \mathbb{S}_m\mathrm{B}\mathbb{S}_m^* + (2\lambda + m)\mathrm{B} - 2D_m\mathrm{B}.$$

So, $a_{11} - a_{10}a_{00}^{-1}a_{01} = \mathbb{S}_m\mathrm{B}\mathbb{S}_m^* + (2\lambda + m)\mathrm{B} - 2D_m\mathrm{B} - \mathrm{B}\mathbb{S}_m^*\mathrm{B}^{-1}\mathbb{S}_m\mathrm{B}$. The proof of (a) is now complete since the matrices $\mathbb{S}_m\mathrm{B}\mathbb{S}_m^*$, $\mathbb{S}_m\mathrm{B}^{-1}\mathbb{S}_m^*$, B, $\mathrm{B}^{1/2}$, $\mathrm{B}^{-1/2}$ are diagonal.

From Lemma 4.1.5, we have $\tilde{a}_{12} = a_{00}^{1/2}\big(b_{00}a_{11}b_{01} + b_{00}a_{12}b_{00} - b_{10}a_{00}b_{02}\big)a_{00}^{1/2}$. Again, from the expansion of the reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ it is easy to see that

$$a_{12} = \frac{1}{2}\mathbb{S}_m^2\mathrm{B}\mathbb{S}_m^* + (2\lambda + m)\mathbb{S}_m\mathrm{B} - D_m\mathbb{S}_m\mathrm{B} - \mathbb{S}_m\mathrm{B}D_m, b_{00} = \mathrm{B}^{-1}, b_{10} = -\mathbb{S}_m^*\mathrm{B}^{-1}, b_{02} = \frac{1}{2}\mathrm{B}^{-1}\mathbb{S}_m{}^2.$$

The proof of (b) is now complete since the two diagonal matrices B and D_m commute. □

Let $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}$ denote the curvature of the bundle $E^{(\lambda,\boldsymbol{\mu})}$, that is, $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(z) = \frac{\partial}{\partial\bar{z}}\big(\tilde{h}^{-1}\frac{\partial}{\partial z}\tilde{h}\big)(z)$, where $\tilde{h}(z) = \tilde{\mathbf{B}}^{(\lambda,\boldsymbol{\mu})}(z,z)^{\mathrm{tr}}$ for z in $\mathbb{D}$. Recall that $\tilde{\mathbf{B}}^{(\lambda,\boldsymbol{\mu})}$ is the normalized reproducing kernel obtained from the reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$.

Lemma 4.2.2. *The curvature at zero* $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$ *and the covariant derivative of curvature of order* $(0,1)$ *at zero* $(\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})})_{\bar{z}}(0)$ *are given by the formulae:*

(a) $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}\ \big((2\lambda_r + \alpha_r - \alpha_{r+1})_{r=0}^{m}\big)$,

(b) $(\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})})_{\bar{z}}(0) = 2S\big((-\sqrt{\alpha_r}(1+\alpha_r - \frac{1}{2}(\alpha_{r-1}+\alpha_{r+1}))_{r=1}^m\big)^{\mathrm{tr}}$, *where* $\alpha_r = r^2 d_{r-1} d_r^{-1}$ *for* $0 \leq r \leq m$ *with* $\alpha_0 = \alpha_{m+1} = 0$.

Proof. We only write the nonzero entries of the matrices involved. Notice that

$$\mathbb{S}_m \mathrm{B} \mathbb{S}_m^*(r,r) = r^2 d_{r-1} \text{ for } 1 \leq r \leq m,$$
$$\mathrm{B}^{-1}\mathbb{S}_m \mathrm{B} \mathbb{S}_m^*(r,r) = r^2 d_{r-1} d_r^{-1} \text{ for } 1 \leq r \leq m,$$
$$\mathbb{S}_m^* \mathrm{B}^{-1}\mathbb{S}_m(r,r) = (r+1)^2 d_{r+1}^{-1} \text{ for } 0 \leq r \leq m-1$$

and

$$\mathbb{S}_m^* \mathrm{B}^{-1}\mathbb{S}_m \mathrm{B}(r,r) = (r+1)^2 d_r d_{r+1}^{-1} \text{ for } 0 \leq r \leq m-1.$$

Therefore, by Lemma 4.2.1(a), we see that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \tilde{a}_{11}^{\mathrm{tr}} = \operatorname{diag}\left((2\lambda_r + \alpha_r - \alpha_{r+1})_{r=0}^m\right)$. This proves part (a).

To prove part (b), we observe that

$$\begin{aligned}
\mathrm{B}\mathbb{S}_m(r+1,r) &= (r+1)d_{r+1}^{-1} \quad \text{for} \quad 0 \leq r \leq m-1,\\
\mathbb{S}_m \mathrm{B} \mathbb{S}_m^*(r,r) &= r^2 d_{r-1} \quad \text{for} \quad 1 \leq r \leq m,\\
\mathrm{B}^{-1}\mathbb{S}_m^2 \mathrm{B}\mathbb{S}_m^* \mathrm{B}^{-1}(r+1,r) &= r^2(r+1)d_{r-1}d_r^{-1}d_{r+1}^{-1} \quad \text{for} \quad 1 \leq r \leq m-1.
\end{aligned}$$

Equivalently,

$$\mathrm{B}^{-1}\mathbb{S}_m^2 \mathrm{B}\mathbb{S}_m^* B^{-1}(r,r-1) = r(r-1)^2 d_{r-2} d_{r-1}^{-1} d_r^{-1} \quad \text{for} \quad 2 \leq r \leq m.$$

Since

$$\begin{aligned}
\mathbb{S}_m^* \mathrm{B}^{-1}\mathbb{S}_m^2(r,r-1) &= r(r+1)^2 d_{r+1}^{-1} \quad \text{for} \quad 1 \leq r \leq m-1,\\
D_m \mathbb{S}_m(r,r-1) &= (m-r)r \quad \text{for} \quad 1 \leq r \leq m,\\
\mathbb{S}_m D_m(r,r-1) &= r(m-r+1) \quad \text{for} \quad 1 \leq r \leq m,
\end{aligned}$$

it follows that

$$[D_m, \mathbb{S}_m](r,r-1) = -r, \quad \text{that is,} \quad [D_m, \mathbb{S}_m] = -\mathbb{S}_m.$$

Hence $(\mathrm{B}^{-1}[D_m,\mathbb{S}_m])(r,r-1) = -rd_r^{-1}$ for $1 \leq r \leq m$. Now,

$$\begin{aligned}
(\mathrm{B}^{-1}[D_m,\mathbb{S}_m] - \mathrm{B}^{-1}\mathbb{S}_m\mathrm{B}\mathbb{S}_m^*\mathrm{B}^{-1}\mathbb{S}_m)(r,r-1) &= -rd_r^{-1} - r^3 d_{r-1}d_r^{-2}\\
&= -rd_r^{-1}(1 + r^2 d_{r-1}d_r^{-1})\\
&= -rd_r^{-1}(1+\alpha_r) \text{ for } 1 \leq r \leq m.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{1}{2}(\mathrm{B}^{-1}\mathbb{S}_m^2\mathrm{B}\mathbb{S}_m^*\mathrm{B}^{-1} + \mathbb{S}_m^*\mathrm{B}^{-1}\mathbb{S}_m^2)(r,r-1) &= \frac{r}{2}((r-1)^2 d_{i-2}d_{r-1}^{-1}d_r^{-1} + (r+1)^2 d_{r+1}^{-1})\\
&= \frac{r}{2}(\alpha_{r-1}d_r^{-1} + (r+1)^2 d_{r+1}^{-1})
\end{aligned}$$

for $1 \leq r \leq m$ with $\alpha_0 = 0 = d_{m+1}^{-1}$. From Lemma 4.2.1 (b), using $d_{-1} = 0 = d_{m+1}^{-1}$ we get

$$\begin{aligned}\tilde{a}_{12}(r, r-1) &= \frac{r}{2}(d_{r-1}d_r)^{1/2}(\alpha_{r-1}d_r^{-1} + (r+1)^2 d_{r+1}^{-1}) - r(d_{r-1}d_r)^{1/2}d_r^{-1}(1+\alpha_r)\\ &= \frac{r}{2}(d_{r-1}d_r^{-1})^{1/2}(\alpha_{r-1} + (r+1)^2 d_r d_{r+1}^{-1}) - r(d_{r-1}d_r^{-1})^{1/2}(1+\alpha_r)\\ &= -\sqrt{\alpha_r}(1 + \alpha_r - \frac{1}{2}(\alpha_{r-1} + \alpha_{r+1}))\end{aligned}$$

for $1 \leq r \leq m$. This proves part (b). □

The following Corollary follows immediately from Corollary 3.2.4.

Corollary 4.2.3. *The operators $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\lambda',\boldsymbol{\mu}')}$ in $\mathrm{B}_{m+1}(\mathbb{D})$ are similar if and only if* tr $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) =$ tr $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(0)$.

Notation 4.2.4. *Let $\delta_{r+1} = 2\lambda_r + \alpha_r - \alpha_{r+1}$ for $0 \leq r \leq m$ and $\theta_\ell = -\sqrt{\alpha_\ell}(1 + \alpha_\ell - \frac{1}{2}(\alpha_{\ell-1} + \alpha_{\ell+1}))$ for $1 \leq \ell \leq m$. In this notation,*

$$\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}\ ((\delta_{r+1})_{r=0}^m) \text{ and } (\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})})_{\bar{z}}(0) = 2S((\theta_\ell)_{\ell=1}^m)^{\mathrm{tr}}.$$

As in the previous Lemma, we will let $\alpha_r = r^2 d_{r-1}d_r^{-1}$ for $0 \leq r \leq m$ with $\alpha_0 = \alpha_{m+1} = 0$.

Remark 4.2.5. We emphasize that the reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ is computed from a ordered basis, that is, $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(w,w) = \big(\!\!\big(\langle\gamma_i(w), \gamma_j(w)\rangle\big)\!\!\big)_{i,j=1}^{m+1}$, where $\{\gamma_i(w)\}_{i=1}^{m+1}$ is an ordered basis. Consequently, the eigenvalues of $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$, which is diagonal, appear in a fixed order. If one considers $\{\gamma_{\sigma(i)}(w)\}_{i=1}^{m+1}$, it will give rise to a different reproducing kernel $P_\sigma \mathbf{B}^{(\lambda,\boldsymbol{\mu})} P_\sigma^*$, say $\mathbf{B}_\sigma^{(\lambda,\boldsymbol{\mu})}$, where $\sigma \in S_{m+1}$, S_{m+1} denotes the symmetric group of degree $(m+1)$ and $P_\sigma(i,j) = \delta_{\sigma(i),j}$. Hence $\tilde{\mathcal{K}}_\sigma^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}\big((\delta_{\sigma(r+1)})_{r=0}^m\big)$, where $\tilde{\mathcal{K}}_\sigma^{(\lambda,\boldsymbol{\mu})}$ is the curvature with respect to the metric $\tilde{h}_\sigma(z) = \tilde{\mathbf{B}}_\sigma^{(\lambda,\boldsymbol{\mu})}(z,z)^{\mathrm{tr}}$. It follows that the curvature of the corresponding bundle as a matrix depends on the choice of the particular ordered basis. The set of eigenvalues of curvature at 0, which is diagonal in our case, will be thought of as an ordered tuple, namely, the ordered set of diagonal elements of $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$.

Definition 4.2.6. *[18, Def. 3.18, pp. 226] A C^∞ vector bundle E over an open subset Ω of $\mathbb{C}$ with metric-preserving connection D is said to be generic if $\mathcal{K}$ has distinct eigenvalues of multiplicity one at each point of Ω.*

From Lemma 4.1.7 (a) and Lemma 4.2.2 (a), we note that $E^{(\lambda,\boldsymbol{\mu})}$ is generic if and only if δ_{r+1} are all distinct for $0 \leq r \leq m$. Thus, using Corollary 4.1.9 the proof of the following Corollary is complete.

Corollary 4.2.7. *We have $\delta_r = \delta_{r+1}$ if and only if $\theta_r = 0$ for $1 \leq r \leq m$ with $\alpha_0 = \alpha_{m+1} = 0$. In particular, if $E^{(\lambda,\boldsymbol{\mu})}$ is generic then $\theta_r \neq 0$ for $1 \leq r \leq m$.*

Lemma 4.2.8. *If $(\delta_{r+1})_{r=0}^m$ is an ordered tuple of positive numbers such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \operatorname{diag}((\delta_{r+1})_{r=0}^m)$, then*

(i) $\sum_{k=0}^{m} \delta_{k+1} > m(m+1)$

(ii) $\frac{r}{m+1}\sum_{k=0}^{m} \delta_{k+1} - \sum_{k=0}^{r-1} \delta_{k+1} > r(m+1-r)$ *for* $1 \le r \le m$.

Proof. By Lemma 4.2.2, Remark 4.2.5 and the hypothesis of the Lemma, we have

$$2\lambda_r + \alpha_r - \alpha_{r+1} = \delta_{r+1}$$

for $0 \le r \le m$. This is same as $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A(i,j) = \begin{cases} -1, & j = i+1, \\ 1, & j = 0 \text{ or } i = j, \\ 0, & \text{otherwise;} \end{cases}$$

for $0 \le i, j \le m$; $\boldsymbol{x}(0) = 2\lambda_0$, $\boldsymbol{x}(i) = \alpha_i$ for $1 \le i \le m$ and $\boldsymbol{b}(r) = \delta_{r+1} - 2r$ for $0 \le r \le m$.

We observe that $\det A = \left(\begin{smallmatrix} m+1 & \mathbf{0} \\ B & A' \end{smallmatrix}\right) = m+1$, where B is a column vector with $B(i) = 1$ for $1 \le i \le m$ and A' is an upper-triangular matrix of size m with 1 as its diagonal entries. So, the system $A\boldsymbol{x} = \boldsymbol{b}$ of linear equations admits a unique solution. One verifies that

$$2\lambda_0 = \frac{1}{m+1}\sum_{k=0}^{m} \delta_{k+1} - m \quad \text{and} \quad \alpha_r = \frac{r}{m+1}\sum_{k=0}^{m} \delta_{k+1} - \sum_{k=0}^{r-1} \delta_{k+1} - r(m+1-r) \quad \text{for} \quad 1 \le r \le m.$$

Recall that $2\lambda_0$ and $\alpha_r = r^2 d_{r-1} d_r^{-1}$ (for $1 \le r \le m$) are all positive. Therefore, a set of necessary conditions for existence of the positive numbers $\{\delta_{r+1}\}_{r=0}^m$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \operatorname{diag}((\delta_{r+1})_{r=0}^m)$ are the inequalities in the statement of the Lemma. □

As described in Notation 2.3.12, let $\boldsymbol{\mu} = (1, \mu_1, \ldots, \mu_m)^{\text{tr}}$ and $\boldsymbol{\mu}' = (1, \mu_1', \ldots, \mu_m')^{\text{tr}}$ with $\mu_\ell, \mu_\ell' > 0$ for $0 \le \ell \le m$; $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ and $\boldsymbol{\alpha}' = (\alpha_1', \ldots, \alpha_m')$. For $0 \le j \le m$, set $2\gamma_j = 2\gamma - m + 2j$, where $\gamma = \lambda$ or $\gamma = \lambda'$. Set $\boldsymbol{d} = L(\lambda)\boldsymbol{\mu}^2$, $\boldsymbol{d}' = L(\lambda')\boldsymbol{\mu}'^2$, where $\boldsymbol{\mu}^2$ and $\boldsymbol{\mu}'^2$ denote the componentwise square of $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$. Let $2\lambda_0 = 2\lambda - m$, $\alpha_i = i^2 d_{i-1} d_i^{-1}$; $2\lambda_0' = 2\lambda' - m$, $\alpha_i' = i^2 d_{i-1}' d_i'^{-1}$ for $0 \le i \le m$. In this notation, we have:

Lemma 4.2.9. $\binom{\lambda}{\boldsymbol{\mu}} = \binom{\lambda'}{\boldsymbol{\mu}'}$ *if and only if* $(2\lambda_0, \boldsymbol{\alpha}) = (2\lambda_0', \boldsymbol{\alpha}')$.

Proof. We prove the "only if" part. Assuming $(2\lambda_0, \boldsymbol{\alpha}) = (2\lambda_0', \boldsymbol{\alpha}')$ we have $\lambda = \lambda'$ and $\alpha_i = \alpha_i'$ for $1 \le i \le m$. Thus $\boldsymbol{d} = \boldsymbol{d}'$. Now invertibility of $L(\lambda)$ implies that $\boldsymbol{\mu}^2 = \boldsymbol{\mu}'^2$, that is, $\boldsymbol{\mu} = \boldsymbol{\mu}'$. □

Corollary 4.2.10. *Suppose* $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ *and* $\mathbf{B}^{(\lambda',\boldsymbol{\mu}')}$ *are such that* $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(0)$. *Then* $\binom{\lambda}{\boldsymbol{\mu}} = \binom{\lambda'}{\boldsymbol{\mu}'}$.

Proof. Let $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(0) = \operatorname{diag}\big((\delta_{r+1})_{r=0}^m\big)$. Consider the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ and $A\boldsymbol{x'} = \boldsymbol{b}$, where A, $\boldsymbol{x}$, $\boldsymbol{b}$ are as in Lemma 4.2.9 and $\boldsymbol{x}'(0) = 2\lambda_0'$, $\boldsymbol{x}'(r) = \alpha_r'$ for $1 \leq r \leq m$. Since $\det A = m+1$, A is invertible. Hence $\boldsymbol{x} = \boldsymbol{x}'$ that is, $(2\lambda_0, \boldsymbol{\alpha}) = (2\lambda_0', \boldsymbol{\alpha}')$, where $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}'$ as in Lemma 4.2.9. Now by Lemma 4.2.9, we have $\binom{\lambda}{\boldsymbol{\mu}} = \binom{\lambda'}{\boldsymbol{\mu}'}$. □

Recall that $M^{(\lambda,\boldsymbol{\mu})}$ is the multiplication operator on the Hilbert space whose reproducing kernel is $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $E^{(\lambda,\boldsymbol{\mu})}$ denotes Hermitian holomorphic vector bundle associated with the operator $M^{(\lambda,\boldsymbol{\mu})^*}$. We recall a theorem from [31].

Theorem 4.2.11. *[31, Theorem 6.2] The reproducing kernels $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbf{B}^{(\lambda',\boldsymbol{\mu}')}$ are equivalent , that is, the multiplication operators $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\lambda',\boldsymbol{\mu}')}$ are unitarily equivalent if and only if $\binom{\lambda}{\boldsymbol{\mu}} = \binom{\lambda'}{\boldsymbol{\mu}'}$.*

The following Corollary is an easy consequence of Corollary 4.2.10 and Theorem 4.2.11.

Corollary 4.2.12. *Suppose $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbf{B}^{(\lambda',\boldsymbol{\mu}')}$ are such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(0)$. Then the multiplication operators $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\lambda',\boldsymbol{\mu}')}$ are unitarily equivalent .*

Now we state the main theorem of this section.

Theorem 4.2.13. *Suppose that the Hermitian holomorphic vector bundles $E^{(\lambda,\boldsymbol{\mu})}$ and $E^{(\lambda',\boldsymbol{\eta})}$ are generic. Then the multiplication operators $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\lambda',\boldsymbol{\eta})}$ are unitarily equivalent if $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$ and $(\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})})_{\bar{z}}(0)$ are simultaneously unitarily equivalent to $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})}(0)$ and $(\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})})_{\bar{z}}(0)$ respectively.*

The proof of this Theorem will be completed after proving a sequence of Lemmas. We omit the easy proof of the first of these lemmas.

Lemma 4.2.14. *Suppose that $\Delta = \big(\!\!\big(k_i\delta_{ij}\big)\!\!\big)_{i,j=1}^n$, $\Delta_\sigma = \big(\!\!\big(k_{\sigma(i)}\delta_{ij}\big)\!\!\big)_{i,j=1}^n$, $k_i \neq k_j$ if $i \neq j$ and C in $\mathcal{M}_n$ is such that $C\Delta = \Delta_\sigma C$. Then $C = \big(\!\!\big(C_{ij}\delta_{\sigma(i),j}\big)\!\!\big)_{i,j=1}^n$ for $C_{ij} \in \mathbb{C}$ and $i, j = 1, \ldots, n$, where σ is in S_n, S_n denotes the permutation group of degree n.*

Lemma 4.2.15. *Suppose that B in $\mathcal{M}_{n+1}$ is such that $BS\big((\beta_k)_{k=1}^n\big)^{\rm tr} = S\big((\beta_k)_{k=1}^n\big)^{\rm tr}B$ for $\beta_k \neq 0$, $1 \leq k \leq n$. Then B is upper-triangular.*

Proof. Let $B = \big(\!\!\big(B(i,j)\big)\!\!\big)_{i,j=1}^{n+1}$. The $(i,1)$-th entries of $BS\big((\beta_k)_{k=1}^n\big)^{\rm tr}$ and $S\big((\beta_k)_{k=1}^n\big)^{\rm tr}B$ are 0 and $\beta_i B(i+1,1)$ for $1 \leq i \leq n$, respectively. By hypothesis, $B(i+1,1) = 0$ for $1 \leq i \leq n$. We want to show that $B(i+1,j) = 0$ for $j \leq i \leq n$, $1 \leq j \leq n$. We prove this by induction. We know that the assertion is true for $j = 1$. Assume that $B(i+1,j-1) = 0$ for $j-1 \leq i \leq n$, $2 \leq j \leq n+1$, equivalently, $B(i,j-1) = 0$ for $j \leq i \leq n+1$, $2 \leq j \leq n+1$. Equating (i,j)-th entries from $BS\big((\beta_k)_{k=1}^n\big)^{\rm tr}$ and $S\big((\beta_k)_{k=1}^n\big)^{\rm tr}B$ we have

$$B(i,j-1)\beta_{j-1} = \beta_i B(i+1,j) \qquad \text{for} \qquad 1 \leq i \leq n,\ 2 \leq j \leq n+1.$$

We note that the left hand side of the above equality is zero for $j \leq i \leq n+1$, $2 \leq j \leq n+1$, by induction hypothesis. Hence $B(i+1,j) = 0$ for $j \leq i \leq n+1$, $2 \leq j \leq n+1$ as $\beta_i \neq 0$ for $j \leq i \leq n+1$. □

Lemma 4.2.16. *Suppose that* $C = \big(\!\!\big(C_{ij}\delta_{\sigma(i),j}\big)\!\!\big)_{i,j=1}^{n}$ *for* $C_{ij} \in \mathbb{C}$, $i, j = 1, \ldots, n$ *and* σ *is in* S_n, *where* S_n *denotes the permutation group of degree* n. *Then* $|\det C| = \prod_{i=1}^{n} |C_{i,\ \sigma(i)}|$.

Proof. We observe that the only possible nonzero entries of C are the $(i, \sigma(i))$-th entries for $1 \leq i \leq n$ and $C(i,\ \sigma(i)) = C_{i,\sigma(i)}$. Let $\tilde{C} = \text{diag}\ \big((C_{i,\sigma(i)})_{i=1}^{n}\big)$. It is easy to see that $|\det\ \tilde{C}| = |\det\ C|$, as $\tilde{C}$ can be converted to C by interchanging its rows and columns. This proves the Lemma. ☐

The next corollary is immediate.

Corollary 4.2.17. *If* $C = \big(\!\!\big(C_{ij}\delta_{\sigma(i),j}\big)\!\!\big)_{i,j=1}^{n}$ *then* C *is invertible if and only if* $C_{i,\ \sigma(i)} \neq 0$ *for* $\sigma \in S_n$, $1 \leq i \leq n$, *where* S_n *denotes the permutation group of degree* n.

Lemma 4.2.18. *If* C *is invertible and satisfies the hypothesis of Lemma 4.2.14 for* $\text{id} \neq \sigma \in S_n$ *then* C *cannot be a triangular matrix.*

Proof. From Lemma 4.2.14 and Corollary 4.2.17, it follows that the only nonzero entries of C are the $(i, \sigma(i))$-th entries for $1 \leq i \leq n$ and $C(i,\ \sigma(i)) = C_{i,\ \sigma(i)}$. Therefore, it suffices to show that there is $1 \leq\ i, j \leq n$ with $i \neq j$ such that $i > \sigma(i)$ and $j < \sigma(j)$ for $\text{id} \neq \sigma \in S_n$. Since $\sigma \neq \text{id}$, there is $i,\ 1 \leq i \leq n$ such that $\sigma(i) \neq i$. Without loss of generality assume that $i > \sigma(i)$. Now, if possible, let $r \geq \sigma(r)$ for $1 \leq r \leq n$ with strict inequalities for some r. Since σ is a one-to-one map of the finite set $\{1, \ldots, n\}$ onto itself, this is not possible by the pigeon hole principle. Hence there is $j,\ 1 \leq j \leq n$ such that $j < \sigma(j)$. ☐

Proof of Theorem 4.2.13: By hypothesis there is $L \in GL(m+1, \mathbb{C})$ such that

(i) $L^{-1}\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)L = \tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})}(0)$

(ii) $L^{-1}(\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})})_{\bar{z}}(0)L = (\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})})_{\bar{z}}(0)$.

Clearly, (i) implies that the sets of eigenvalues of $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$ and $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})}(0)$ are the same. Since $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$ and $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})}(0)$ are diagonal matrices it follows that either

(a) $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})}(0)$, or

(b) the set of diagonal entries of $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$ equals the set of diagonal entries of $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})}(0)$ but $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) \neq \tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})}(0)$.

Now, (b) is equivalent to the statement that $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})}(0) = \text{diag}\ \big((\delta_{\sigma(r+1)})_{r=0}^{m}\big)$ for $\text{id} \neq \sigma \in S_{m+1}$, where $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \text{diag}\ \big((\delta_{r+1})_{r=0}^{m})\big)$. This implies by Lemma 4.2.18 that L cannot be a triangular matrix. Whereas (ii) implies by Corollary 4.2.7 and Lemma 4.2.15 that L is a upper-triangular matrix. Hence (b) and (ii) cannot occur simultaneously. Having ruled out the possibility of (b), we conclude that (a) must occur. Therefore, by Corollary 4.2.12, we have $\binom{\lambda}{\boldsymbol{\mu}} = \binom{\lambda'}{\boldsymbol{\eta}}$. ☐

4.3 Homogeneous bundles of rank 3

Now we specialize to the case $m = 2$. In this case, conclusions similar to those of Theorem 4.2.13 are true even if $E^{(\lambda,\boldsymbol{\mu})}$ is not assumed to be generic. Recall that the rank of the bundle $E^{(\lambda,\boldsymbol{\mu})}$ is 3 when $m = 2$.

Theorem 4.3.1. *For $m = 2$, the multiplication operators $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\lambda',\boldsymbol{\eta})}$ are unitarily equivalent if $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$ and $(\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})})_{\bar{z}}(0)$ are simultaneously unitarily equivalent to $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})}(0)$ and $(\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})})_{\bar{z}}(0)$ respectively.*

Proof. By Theorem 4.2.13, we only need to consider the case when one of $E^{(\lambda,\boldsymbol{\mu})}$ and $E^{(\lambda',\boldsymbol{\eta})}$ is not generic.

Let $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})} = \operatorname{diag}(\delta_1, \delta_2, \delta_3)$ and $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})} = \operatorname{diag}(\delta_1', \delta_2', \delta_3')$, where $\delta_{i+1} = 2\lambda_i + \alpha_i - \alpha_{i+1}$, $\delta_{i+1}' = 2\lambda_i' + \alpha_i' - \alpha_{i+1}'$ with $2\lambda_i = 2\lambda - 2 + 2i$, $2\lambda_i' = 2\lambda' - 2 + 2i$, $\alpha_i = i^2 d_{i-1} d_i^{-1}$, $\alpha_i' = i^2 d_{i-1}' d_i'^{-1}$ for $i = 0, 1, 2$; $\alpha_0 = \alpha_3 = \alpha_0' = \alpha_3' = 0$ and $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(0,0) = \operatorname{diag}(d_0, d_1, d_2)$, $d_0 = 1$; $\mathbf{B}^{(\lambda',\boldsymbol{\eta})}(0,0) = \operatorname{diag}(d_0', d_1', d_2')$, $d_0' = 1$. We observe that $\delta_3 - \delta_1 = \alpha_1 + \alpha_2 + 4 > 0$ and $\delta_3' - \delta_1' = \alpha_1' + \alpha_2' + 4 > 0$. Now assume that

(i) $L^{-1}\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)L = \tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})}(0)$ for some $L \in GL(3, \mathbb{C})$.

It follows easily from (i) that if one of the two bundles is not generic then the other cannot be generic. Noting that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$ and $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})}(0)$ are diagonal matrices we have the following possibilities.

$$\text{(a) } \delta_1 = \delta_2 \quad \text{and} \quad \delta_1' = \delta_2' \qquad \text{(b) } \delta_2 = \delta_3 \quad \text{and} \quad \delta_2' = \delta_3'$$
$$\text{(c) } \delta_1 = \delta_2 \quad \text{and} \quad \delta_2' = \delta_3' \qquad \text{(d) } \delta_2 = \delta_3 \quad \text{and} \quad \delta_1' = \delta_2'.$$

From (a) we have $\delta_1 = \delta_2 < \delta_3$ and $\delta_1' = \delta_2' < \delta_3'$. As (i) implies that $\{\delta_1, \delta_2, \delta_3\} = \{\delta_1', \delta_2', \delta_3'\}$, as sets. Comparing order of magnitude we get $\delta_1 = \delta_1'$, $\delta_2 = \delta_2'$ and $\delta_3 = \delta_3'$. It follows that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})} = \tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\eta})}$. Therefore by Corollary 3.1.1, we have $\binom{\lambda}{\boldsymbol{\mu}} = \binom{\lambda'}{\boldsymbol{\eta}}$. So, $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\lambda',\boldsymbol{\eta})}$ are unitarily equivalent .

A similar argument shows that the assumptions in (b) lead to the same conclusion.

From (c), we have $\delta_1 = \delta_2 < \delta_3$ and $\delta_1' < \delta_2' = \delta_3'$. From (i) we have $\{\delta_1, \delta_2, \delta_3\} = \{\delta_1', \delta_2', \delta_3'\}$, as sets. Comparing order of magnitude we get $\delta_1 = \delta_2 = \delta_1'$ and $\delta_3 = \delta_2' = \delta_3'$. Comparing multiplicities of δ_1 and δ_2' we have $\delta_1 = \delta_2'$ and $\delta_3 = \delta_1'$. All the equalities together imply that $\delta_1 = \delta_3$ and $\delta_1' = \delta_3'$, which are impossible. Similarly we see that (d) is also impossible as $\delta_3 > \delta_1$ and $\delta_3' > \delta_1'$. This completes the proof. ☐

If $\delta_1, \delta_2, \delta_3$ are the eigenvalues $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$ then we know from [18, Proposition 2.20] that $\delta_i > 0$ for $i = 1, 2, 3$. Now, suppose $(\delta_1, \delta_2, \delta_3)$ is a fixed ordered triple of positive numbers. Then there exists $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ with $\lambda > 1$ and $\mu_\ell > 0$ $(\ell = 1, 2)$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \operatorname{diag}(\delta_1, \delta_2, \delta_3)$ only if δ_i's satisfy the inequalities of Lemma 4.2.8.

Suppose $(\delta_1, \delta_2, \delta_3)$, $\delta_i > 0$ for $i = 1, 2, 3$ is given satisfying the inequalities of Lemma 4.2.8. Then let us find $\lambda > 1$, $\mu_1, \mu_2 > 0$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \operatorname{diag}(\delta_1, \delta_2, \delta_3)$ with $\boldsymbol{\mu} = (1, \mu_1, \mu_2)^{\mathrm{tr}}$. We have $L(\lambda)\boldsymbol{\mu}^2 = \boldsymbol{d}$, which is the same as

$$\boldsymbol{\mu}^2 = L(\lambda)^{-1}\boldsymbol{d} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2(\lambda-1)} & 1 & 0 \\ \frac{1}{\lambda(2\lambda-1)} & -\frac{2}{\lambda} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 1 \\ d_1 - \frac{1}{2(\lambda-1)} \\ d_2 - \frac{2d_1}{\lambda} + \frac{1}{\lambda(2\lambda-1)} \end{pmatrix}.$$

Thus

$$\mu_1^2 = d_1 - \frac{1}{2(\lambda-1)} = \frac{1}{\alpha_1} - \frac{1}{2(\lambda-1)} = \frac{2(\lambda-1) - \alpha_1}{2\alpha_1(\lambda-1)}.$$

Recall from Lemma 4.2.8 that

$$2\lambda_0 = 2\lambda - 2 = \frac{\delta_1 + \delta_2 + \delta_3}{3} - 2 \quad \text{and} \quad \alpha_1 = \frac{\delta_2 + \delta_3 - 2\delta_1 - 6}{3}.$$

So, we have

$$2(\lambda - 1) - \alpha_1 = \frac{\delta_1 + \delta_2 + \delta_3}{3} - 2 - \frac{\delta_2 + \delta_3 - 2\delta_1 - 6}{3} = \delta_1 > 0.$$

Similarly,

$$\mu_2^2 = d_2 - \frac{2d_1}{\lambda} + \frac{1}{\lambda(2\lambda-1)} = \frac{4}{\alpha_1\alpha_2} - \frac{2}{\alpha_1\lambda} + \frac{1}{\lambda(2\lambda-1)} = \frac{2(2\lambda-\alpha_2)(2\lambda-1) + \alpha_1\alpha_2}{\alpha_1\alpha_2\lambda(2\lambda-1)},$$

where α_1, α_2 are as in Lemma 4.2.2. Consequently, we have the following Theorem by an application of Lemma 4.2.8.

Theorem 4.3.2. *There exists* $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ *such that* $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \operatorname{diag}(\delta_1, \delta_2, \delta_3)$ *for some* $\delta_1, \delta_2, \delta_3 > 0$ *if*

$$\begin{gathered} \delta_1 + \delta_2 + \delta_3 > 6, \\ \delta_2 + \delta_3 - 2\delta_1 > 6, \\ 2\delta_3 - \delta_1 - \delta_2 > 6; \\ 2(2\lambda - \alpha_2)(2\lambda - 1) + \alpha_1\alpha_2 > 0, \end{gathered}$$

where α_1, α_2 *are as in Notation 4.2.4.*

Notation 4.3.3. *From now on, we will adhere to the following notational convention (here,* $(\lambda, \boldsymbol{\mu})$ *is fixed but arbitrary).*

$$\begin{aligned} (\lambda, \boldsymbol{\mu}) \quad &: \quad \mathcal{K}^{(\lambda,\boldsymbol{\mu})}(0) = \operatorname{diag}(\delta_1, \delta_2, \delta_3), \\ (\lambda', \boldsymbol{\mu}') \quad &: \quad \mathcal{K}^{(\lambda',\boldsymbol{\mu}')}(0) = \operatorname{diag}(\delta_2, \delta_1, \delta_3); \\ (\widehat{\lambda}, \widehat{\boldsymbol{\mu}}) \quad &: \quad \mathcal{K}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}(0) = \operatorname{diag}(\delta_1, \delta_3, \delta_2). \end{aligned}$$

Proposition 4.3.4. *Suppose* $\delta_i > 0$ *for* $i = 1, 2, 3$ *are such that* $\delta_1 \neq \delta_2$ *and* $2(\delta_1 + \delta_2) > \delta_3 - 6 > \max\{2\delta_1 - \delta_2, 2\delta_2 - \delta_1\}$. *Then there exists reproducing kernels* $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ *and* $\mathbf{B}^{(\lambda',\boldsymbol{\mu}')}$ *such that* $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \operatorname{diag}(\delta_1, \delta_2, \delta_3)$ *and* $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(0) = \operatorname{diag}(\delta_2, \delta_1, \delta_3)$, *where* $\lambda, \lambda' > 1$, $\boldsymbol{\mu} = (1, \mu_1, \mu_2)^{\mathrm{tr}}$, $\boldsymbol{\mu}' = (1, \mu_1', \mu_2')^{\mathrm{tr}}$, $\mu_\ell, \mu_\ell' > 0$ *for* $\ell = 1, 2$.

Proof. Consider $(\delta_1, \delta_2, \delta_3)$, $\delta_i > 0$ for $i = 1, 2, 3$ such that there exists $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1, \delta_2, \delta_3)$ for some $\lambda > 1$, $\boldsymbol{\mu} = (1, \mu_1, \mu_2)^{\mathrm{tr}}$ with $\mu_1, \mu_2 > 0$. So, $\delta_1, \delta_2, \delta_3$ satisfy the inequalities of Lemma 4.2.8. We now produce $\lambda' > 1$, $\boldsymbol{\mu}' = (1, \mu_1', \mu_2')^{\mathrm{tr}}$ with $\mu_1', \mu_2' > 0$ such that $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(0) = \mathrm{diag}(\delta_2, \delta_1, \delta_3)$. We recall that $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}$ is the curvature of the metric $\tilde{\mathbf{B}}^{(\lambda',\boldsymbol{\mu}')}(z,z)^{\mathrm{tr}}$ and $\tilde{\mathbf{B}}^{(\lambda',\boldsymbol{\mu}')}$ denotes the normalization of the reproducing kernel $\mathbf{B}^{(\lambda',\boldsymbol{\mu}')}$. By Lemma 4.2.2 and Remark 4.2.5, we need to consider the equations

$$\begin{aligned}
2\lambda' - \alpha_1' - 2 &= \delta_2,\\
2\lambda' + \alpha_1' - \alpha_2' &= \delta_1,\\
2\lambda' + \alpha_2' + 2 &= \delta_3,
\end{aligned}$$

where $\alpha_1' = {d_1'}^{-1}, \alpha_2' = 4d_1'{d_2'}^{-1}$. This is same as $A\boldsymbol{x}' = \boldsymbol{b}'$, where

$$A = \begin{pmatrix} 1 & -1 & 0\\ 1 & 1 & -1\\ 1 & 0 & 1\end{pmatrix}, \boldsymbol{x}' = \begin{pmatrix} 2\lambda'\\ \alpha_1'\\ \alpha_2'\end{pmatrix}, \boldsymbol{b}' = \begin{pmatrix} \delta_2+2\\ \delta_1\\ \delta_3-2\end{pmatrix}.$$

This system of linear equations has only one solution, namely, $\boldsymbol{x}' = \frac{1}{3}\begin{pmatrix} \delta_1+\delta_2+\delta_3\\ \delta_1+\delta_3-2\delta_2-6\\ 2\delta_3-\delta_1-\delta_2-6\end{pmatrix}$. We observe from Lemma 4.2.8 that $\lambda = \lambda'$ and $\alpha_2 = \alpha_2'$ but $\alpha_1 \neq \alpha_1'$ if $\delta_1 \neq \delta_2$. From Lemma 4.2.8 and Theorem 4.3.2, we know that there exists $\mathbf{B}^{(\lambda',\boldsymbol{\mu}')}$ such that $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(0) = \mathrm{diag}(\delta_2, \delta_1, \delta_3)$ if

$$\begin{aligned}
\delta_1 + \delta_2 + \delta_3 &> 6,\\
\delta_1 + \delta_3 - 2\delta_2 &> 6,\\
2\delta_3 - \delta_1 - \delta_2 &> 6;\\
2(2\lambda' - \alpha_2')(2\lambda' - 1) + \alpha_1'\alpha_2' &> 0.
\end{aligned}$$

Hence there exists $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbf{B}^{(\lambda',\boldsymbol{\mu}')}$ such that

$$\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1, \delta_2, \delta_3) \text{ and } \tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(0) = \mathrm{diag}(\delta_2, \delta_1, \delta_3)$$

if

$$\begin{aligned}
\delta_1 + \delta_2 + \delta_3 &> 6,\\
\delta_2 + \delta_3 - 2\delta_1 &> 6,\\
\delta_1 + \delta_3 - 2\delta_2 &> 6,\\
2\delta_3 - \delta_1 - \delta_2 &> 6;\\
2(2\lambda - \alpha_2)(2\lambda - 1) + \alpha_1\alpha_2 &> 0,\\
2(2\lambda' - \alpha_2')(2\lambda' - 1) + \alpha_1'\alpha_2' &> 0.
\end{aligned}$$

Suppose $\delta_i > 0$ for $i = 1, 2, 3$ are chosen such that $\delta_1 \neq \delta_2$ and

$$\text{(i)}\ \ 2(\delta_1 + \delta_2) > \delta_3 - 6 > \max\{2\delta_1 - \delta_2, 2\delta_2 - \delta_1\}.$$

Then the last part of the inequality (i) is clearly seen to force the two inequalities $\delta_2 + \delta_3 - 2\delta_1 > 6$ and $\delta_1 + \delta_3 - 2\delta_2 > 6$. Adding these two inequalities, we have $2\delta_3 - \delta_1 - \delta_2 > 12$. This choice of δ_i, $i = 1, 2, 3$, also implies $\delta_3 > 6$. Consequently, the first four of the six inequalities listed above are valid. Since $\lambda = \lambda'$ and $\alpha_2 = \alpha_2'$, $2\lambda' - 1 = 2\lambda - 1 > 0$, it follows from the first part of inequality (i) that $2\lambda' - \alpha_2' = 2\lambda - \alpha_2 = \frac{1}{3}(2(\delta_1+\delta_2) - \delta_3) + 2 > 0$. Thus the last two inequalities of the six inequalities listed above are valid with our choice of the δ_i, $i = 1,2,3$. Hence all the inequalities we need for the existence of $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbf{B}^{(\lambda',\boldsymbol{\mu}')}$ are verified by this choice of $\delta_i > 0$ for $i = 1,2,3$. □

Proposition 4.3.5. *Suppose $\delta_i > 0$ for $i = 1,2,3$ are such that $\delta_3 > \delta_2 > 3 + \frac{\delta_3}{2}$ and $\delta_1 < \min\{2\delta_3 - \delta_2, 2\delta_2 - \delta_3\} - 6$. Then there exists reproducing kernels $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbf{B}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \operatorname{diag}(\delta_1,\delta_2,\delta_3)$ and $\tilde{\mathcal{K}}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}(0) = \operatorname{diag}(\delta_1,\delta_3,\delta_2)$, where $\lambda, \widehat{\lambda} > 1$, $\boldsymbol{\mu} = (1,\mu_1,\mu_2)^{\mathrm{tr}}$, $\widehat{\boldsymbol{\mu}} = (1,\widehat{\mu}_1,\widehat{\mu}_2)^{\mathrm{tr}}$, $\mu_\ell, \widehat{\mu}_\ell > 0$ for $\ell = 1,2$.*

Proof. We construct a reproducing kernel $\mathbf{B}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}$ such that $\tilde{\mathcal{K}}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}(0) = \operatorname{diag}(\delta_1,\delta_3,\delta_2)$ for some $\widehat{\lambda} > 1$, $\widehat{\boldsymbol{\mu}} = (1,\widehat{\mu}_1,\widehat{\mu}_2)^{\mathrm{tr}}$, $\widehat{\mu}_\ell > 0$ for $\ell = 1,2$. By Lemma 4.2.2 and Remark 4.2.5, we obtain $(2\widehat{\lambda}, \widehat{\alpha}_1, \widehat{\alpha}_2)$ from the following set of equations

$$\begin{aligned} 2\widehat{\lambda} - \widehat{\alpha}_1 - 2 &= \delta_1, \\ 2\widehat{\lambda} + \widehat{\alpha}_1 - \widehat{\alpha}_2 &= \delta_3, \\ 2\widehat{\lambda} + \widehat{\alpha}_2 + 2 &= \delta_2, \end{aligned}$$

where $\widehat{\alpha}_1 = \widehat{d}_1^{-1}, \widehat{\alpha}_2 = 4\widehat{d}_1\widehat{d}_2^{-1}$. This is same as $A\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{b}}$, where

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}, \widehat{\boldsymbol{x}} = \begin{pmatrix} 2\widehat{\lambda} \\ \widehat{\alpha}_1 \\ \widehat{\alpha}_2 \end{pmatrix}, \widehat{\boldsymbol{b}} = \begin{pmatrix} \delta_2+2 \\ \delta_1 \\ \delta_3-2 \end{pmatrix}.$$

The vector $\widehat{\boldsymbol{x}} = \frac{1}{3}\begin{pmatrix} \delta_1+\delta_2+\delta_3, \\ \delta_2+\delta_3-2\delta_1-6, \\ 2\delta_2-\delta_1-\delta_3-6 \end{pmatrix}$ is the only solution of this system of equations. From Lemma 4.2.8 and Theorem 4.3.2, we know that there exists $\mathbf{B}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}$ such that $\tilde{\mathcal{K}}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}(0) = \operatorname{diag}(\delta_1,\delta_3,\delta_2)$ if

$$\begin{aligned} \delta_1 + \delta_2 + \delta_3 &> 6, \\ \delta_2 + \delta_3 - 2\delta_1 &> 6, \\ 2\delta_2 - \delta_1 - \delta_3 &> 6; \\ 2(2\widehat{\lambda} - \widehat{\alpha}_2)(2\widehat{\lambda} - 1) + \widehat{\alpha}_1\widehat{\alpha}_2 &> 0. \end{aligned}$$

If $(\delta_1,\delta_2,\delta_3)$, $\delta_i > 0$ for $i = 1,2,3$ are such that there exists $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \operatorname{diag}(\delta_1,\delta_2,\delta_3)$. Then δ_i's for $i = 1,2,3$ satisfies the inequalities of Lemma 4.2.8. So, by Theorem 4.3.2, there exist $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbf{B}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}$ such that

$$\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \operatorname{diag}(\delta_1,\delta_2,\delta_3) \text{ and } \tilde{\mathcal{K}}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}(0) = \operatorname{diag}(\delta_1,\delta_3,\delta_2)$$

if

$$\begin{aligned} \delta_1 + \delta_2 + \delta_3 &> 6, \\ \delta_2 + \delta_3 - 2\delta_1 &> 6, \\ 2\delta_2 - \delta_1 - \delta_3 &> 6, \\ 2\delta_3 - \delta_1 - \delta_2 &> 6; \\ 2(2\lambda - \alpha_2)(2\lambda - 1) + \alpha_1\alpha_2 &> 0, \\ 2(2\widehat{\lambda} - \widehat{\alpha}_2)(2\widehat{\lambda} - 1) + \widehat{\alpha}_1\widehat{\alpha}_2 &> 0. \end{aligned}$$

We observe that $\lambda = \widehat{\lambda}$ and $\alpha_1 = \widehat{\alpha}_1$ but $\alpha_2 \neq \widehat{\alpha}_2$ if $\delta_2 \neq \delta_3$. Suppose $\delta_i > 0$ for $i = 1, 2, 3$ are chosen satisfying

$$\text{(a)} \;\; \delta_3 > \delta_2 > 3 + \frac{\delta_3}{2} \quad \text{and} \quad \text{(b)} \;\; \delta_1 < \min\{2\delta_3 - \delta_2, 2\delta_2 - \delta_3\} - 6.$$

Then the inequality (a) implies that $\delta_3 > 6$, hence the first of the set of six inequalities above holds. The inequality (b) implies that $2\delta_3 - \delta_1 - \delta_2 > 6$ and $2\delta_2 - \delta_1 - \delta_3 > 6$, adding these two inequalities we have $\delta_2 + \delta_3 - 2\delta_1 > 12$. Hence the first four inequalities, from the list of six inequalities given above, are verified. The second, third and the second, fourth from the set of the six inequalities respectively imply that $\delta_2 - \delta_1 > 4$ and $\delta_3 - \delta_1 > 4$. An easy computation involving the expressions for $\lambda, \alpha_1, \alpha_2$ and $\widehat{\lambda}, \widehat{\alpha}_1, \widehat{\alpha}_2$ in terms of δ_i for $i = 1, 2, 3$ shows that $2(2\lambda - \alpha_2)(2\lambda - 1) + \alpha_1\alpha_2 > 0$ and $2(2\widehat{\lambda} - \widehat{\alpha}_2)(2\widehat{\lambda} - 1) + \widehat{\alpha}_1\widehat{\alpha}_2 > 0$ together is equivalent to $(\delta_1 + \delta_2)(2\delta_1 + \delta_2) + \delta_3(\delta_2 - \delta_1) + 6\delta_1 > 0$ and $(\delta_1 + \delta_3)(2\delta_1 + \delta_3) + \delta_2(\delta_3 - \delta_1) + 6\delta_1 > 0$. These are satisfied as $\delta_2 - \delta_1 > 4$ and $\delta_3 - \delta_1 > 4$. Hence all the required inequalities for the existence of $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbf{B}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}$ are met by this choice of $\delta_i > 0$ for $i = 1, 2, 3$. □

Remark 4.3.6. The set $\{\delta_i > 0 : i = 1, 2, 3\}$ satisfying the inequalities of Proposition 4.3.4 is non-empty. For instance, take $\delta_1 = 1$, $\delta_2 = 2$ and any δ_3 in the open interval $(9, 12)$. Then $\{\delta_1, \delta_2, \delta_3\}$ meets the requirement. Similarly, taking any δ_1 in the open interval $(0, 1)$, $\delta_2 = 7.5$ and $\delta_3 = 8$, we find that $\{\delta_1, \delta_2, \delta_3\}$ satisfies the inequalities prescribed in Proposition 4.3.5. Thus, the two sets which are obtained from Propositions 4.3.4 and 4.3.5 are not identical.

Corollary 4.3.7. *In Proposition 4.3.4 and Proposition 4.3.5,* $\binom{\lambda}{\boldsymbol{\mu}} \neq \binom{\lambda'}{\boldsymbol{\mu}'}$ *and* $\binom{\lambda}{\boldsymbol{\mu}} \neq \binom{\widehat{\lambda}}{\widehat{\boldsymbol{\mu}}}$.

Proof. By Lemma 4.2.9, it suffices to show that $(2\lambda, \alpha_1, \alpha_2) \neq (2\lambda', \alpha_1', \alpha_2')$ and $(2\lambda, \alpha_1, \alpha_2) \neq (2\widehat{\lambda}, \widehat{\alpha}_1, \widehat{\alpha}_2)$. However, in Proposition 4.3.4, $\alpha_1 \neq \alpha_1'$ since $\delta_1 \neq \delta_2$. Similarly, in Proposition 4.3.5, $\alpha_2 \neq \widehat{\alpha}_2$ since $\delta_2 \neq \delta_3$. □

Recall that $M^{(\lambda,\boldsymbol{\mu})}$ denotes the multiplication operator on the reproducing kernel Hilbert spaces whose reproducing kernel is $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$.

Corollary 4.3.8. *Suppose that* $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$, $\mathbf{B}^{(\lambda',\boldsymbol{\mu}')}$ *and* $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$, $\mathbf{B}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}$ *are as in Proposition 4.3.4 and Proposition 4.3.5 respectively. Then*

(a) the multiplication operators $M^{(\lambda,\boldsymbol{\mu})}$ *and* $M^{(\lambda',\boldsymbol{\mu}')}$ *are not unitarily equivalent .*

(b) the multiplication operators $M^{(\lambda,\boldsymbol{\mu})}$ *and* $M^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}$ *are not unitarily equivalent .*

Proof. The proof is immediate from Theorem 4.2.11 and Corollary 4.3.7. □

Remark 4.3.9. In Proposition 4.3.4 and Proposition 4.3.5, we have shown the following: Given a reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \operatorname{diag}(\delta_1, \delta_2, \delta_3)$ there exists a reproducing kernel $\mathbf{B}^{(\lambda',\boldsymbol{\mu}')}$ with $\binom{\lambda}{\boldsymbol{\mu}} \neq \binom{\lambda'}{\boldsymbol{\mu}'}$ such that $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(0) = \operatorname{diag}(\delta_{\rho(1)}, \delta_{\rho(2)}, \delta_{\rho(3)})$ and given a reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1,\delta_2,\delta_3)$ there exists a reproducing kernel $\mathbf{B}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}$ with $\begin{pmatrix}\lambda\\ \boldsymbol{\mu}\end{pmatrix} \neq \begin{pmatrix}\widehat{\lambda}\\ \widehat{\boldsymbol{\mu}}\end{pmatrix}$ such that $\tilde{\mathcal{K}}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}(0) = \mathrm{diag}(\delta_{\tau(1)},\delta_{\tau(2)},\delta_{\tau(3)})$, where $\rho,\tau \in S_3$ with $\rho(1)=2,\rho(2)=1,\rho(3)=3$ and $\tau(1)=1,\tau(2)=3,\tau(3)=2$. In the next Proposition we prove that if there exists a reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1,\delta_2,\delta_3)$ there does not exist $\mathbf{B}^{(\vartheta,\boldsymbol{\xi})}$ with $\begin{pmatrix}\lambda\\ \boldsymbol{\mu}\end{pmatrix} \neq \begin{pmatrix}\vartheta\\ \boldsymbol{\xi}\end{pmatrix}$ such that $\tilde{\mathcal{K}}^{(\vartheta,\boldsymbol{\xi})}(0) = \mathrm{diag}(\delta_{\sigma(1)},\delta_{\sigma(2)},\delta_{\sigma(3)})$ unless $\sigma=\rho$ or τ, for $\sigma,\rho,\tau \in S_3$. Obviously, there exists $\mathbf{B}_\sigma^{(\lambda,\boldsymbol{\mu})} := P_\sigma \mathbf{B}^{(\lambda,\boldsymbol{\mu})} P_\sigma^*$ such that $\tilde{\mathcal{K}}_\sigma^{(\lambda,\boldsymbol{\mu})}(0)$ =diag $(\delta_{\sigma(1)},\delta_{\sigma(2)},\delta_{\sigma(3)})$ for all $\sigma \in S_3$, where P_σ is in $\mathcal{M}_3$ such that $P_\sigma(i,j) = \delta_{\sigma(i),j}$ and $\tilde{\mathcal{K}}_\sigma^{(\lambda,\boldsymbol{\mu})}$ is the curvature with respect to the metric $\tilde{h}_\sigma(z) = \tilde{\mathbf{B}}_\sigma^{(\lambda,\boldsymbol{\mu})}(z,z)^{\mathrm{tr}}$. The reproducing kernels $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbf{B}_\sigma^{(\lambda,\boldsymbol{\mu})}$ are equivalent, that is, the multiplication operators on the reproducing kernel Hilbert spaces with reproducing kernels $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbf{B}_\sigma^{(\lambda,\boldsymbol{\mu})}$ are unitarily equivalent . Therefore, we do not distinguish between the two reproducing kernels $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbf{B}_\sigma^{(\lambda,\boldsymbol{\mu})}$.

Notation 4.3.10. *Let $\rho,\tau \in S_3$ such that*

$$\rho(1)=2,\rho(2)=1,\rho(3)=3 \text{ and } \tau(1)=1,\tau(2)=3,\tau(3)=2.$$

Proposition 4.3.11. *Given a reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1,\delta_2,\delta_3)$ there does not exist a reproducing kernel $\mathbf{B}^{(\vartheta,\boldsymbol{\xi})}$ such that $\tilde{\mathcal{K}}^{(\vartheta,\boldsymbol{\xi})}(0) = \mathrm{diag}(\delta_{\sigma(1)},\delta_{\sigma(2)},\delta_{\sigma(3)})$ with $\begin{pmatrix}\lambda\\ \boldsymbol{\mu}\end{pmatrix} \neq \begin{pmatrix}\vartheta\\ \boldsymbol{\xi}\end{pmatrix}$ unless $\sigma=\rho$ or $\sigma=\tau$.*

Proof. Case 1. Pick $\sigma \in S_3$ such that $\sigma(1)=3,\sigma(2)=2,\sigma(3)=1$.

The existence of two reproducing kernels $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbf{B}^{(\vartheta,\boldsymbol{\xi})}$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1,\delta_2,\delta_3)$ and $\tilde{\mathcal{K}}^{(\vartheta,\boldsymbol{\xi})}(0) = \mathrm{diag}(\delta_{\sigma(1)},\delta_{\sigma(2)},\delta_{\sigma(3)})$ would imply, by an application of Lemma 4.2.8 to the ordered triples $(\delta_1,\delta_2,\delta_3)$ and $(\delta_{\sigma(1)},\delta_{\sigma(2)},\delta_{\sigma(3)}) = (\delta_3,\delta_2,\delta_1)$ that

$$\begin{aligned}
\delta_1+\delta_2+\delta_3 &> 6,\\
\delta_2+\delta_3-2\delta_1 &> 6,\\
2\delta_3-\delta_1-\delta_2 &> 6;\\
\delta_{\sigma(1)}+\delta_{\sigma(2)}+\delta_{\sigma(3)} &> 6,\\
\delta_{\sigma(2)}+\delta_{\sigma(3)}-2\delta_{\sigma(1)} &> 6,\\
2\delta_{\sigma(3)}-\delta_{\sigma(1)}-\delta_{\sigma(2)} &> 6.
\end{aligned}$$

This set of inequalities are equivalent to

$$\begin{aligned}
\delta_1+\delta_2+\delta_3 &> 6,\\
\delta_2+\delta_3-2\delta_1 &> 6,\\
2\delta_3-\delta_1-\delta_2 &> 6,\\
\delta_1+\delta_2-2\delta_3 &> 6,\\
2\delta_1-\delta_2-\delta_3 &> 6.
\end{aligned}$$

Adding the third and the fourth from these inequalities gives $0 > 12$.

Case 2. Choose $\sigma \in S_3$ such that $\sigma(1)=2,\sigma(2)=3,\sigma(3)=1$.

As in the first case the existence of two reproducing kernels $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbf{B}^{(\vartheta,\boldsymbol{\xi})}$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1, \delta_2, \delta_3)$ and $\tilde{\mathcal{K}}^{(\vartheta,\boldsymbol{\xi})}(0) = \mathrm{diag}(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)})$ would imply, by an application of Lemma 4.2.8 to the ordered triples $(\delta_1, \delta_2, \delta_3)$ and $(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)}) = (\delta_2, \delta_3, \delta_1)$, that

$$\begin{aligned} \delta_1 + \delta_2 + \delta_3 &> 6, \\ \delta_2 + \delta_3 - 2\delta_1 &> 6, \\ 2\delta_3 - \delta_1 - \delta_2 &> 6, \\ \delta_1 + \delta_3 - 2\delta_2 &> 6, \\ 2\delta_1 - \delta_2 - \delta_3 &> 6. \end{aligned}$$

Adding second and fifth of these inequalities gives $0 > 12$.

Case 3. Take $\sigma \in S_3$ such that $\sigma(1) = 3, \sigma(2) = 1, \sigma(3) = 2$.

Finally, continuing in the same manner in the previous two cases, the existence of two reproducing kernels $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ and $\mathbf{B}^{(\vartheta,\boldsymbol{\xi})}$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1, \delta_2, \delta_3)$ and $\tilde{\mathcal{K}}^{(\vartheta,\boldsymbol{\xi})}(0) = \mathrm{diag}(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)})$ would imply, by an application of Lemma 4.2.8 to the ordered triples $(\delta_1, \delta_2, \delta_3)$ and $(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)}) = (\delta_3, \delta_1, \delta_2)$, that

$$\begin{aligned} \delta_1 + \delta_2 + \delta_3 &> 6, \\ \delta_2 + \delta_3 - 2\delta_1 &> 6, \\ 2\delta_3 - \delta_1 - \delta_2 &> 6, \\ \delta_1 + \delta_2 - 2\delta_3 &> 6, \\ 2\delta_2 - \delta_3 - \delta_1 &> 6. \end{aligned}$$

Adding third and fourth inequalities from this set of inequalities, we have $0 > 12$. ☐

Corollary 4.3.12. *There does not exist any multiplication operator $M^{(\vartheta,\boldsymbol{\xi})}$ other than $M^{(\lambda',\boldsymbol{\mu}')}$ or $M^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}$ such that the sets of eigenvalues of $\tilde{\mathcal{K}}^{(\vartheta,\boldsymbol{\xi})}(0)$ and $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$ are equal but $\tilde{\mathcal{K}}^{(\vartheta,\boldsymbol{\xi})}(0) \neq \tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$, where $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$, $\mathbf{B}^{(\lambda',\boldsymbol{\mu}')}$, $\mathbf{B}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}$ are as in Proposition 4.3.4 and Proposition 4.3.5.*

Proof. Combining Corollary 4.3.8, Corollary 4.2.10, Theorem 4.2.11 and Proposition 4.3.11, we obtain a proof of this corollary. ☐

Remark 4.3.13. We discuss the case $m = 1$. From Lemma 4.2.2, we see that

$$\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(2\lambda - \alpha_1 - 1, 2\lambda + \alpha_1 + 1),$$

where $\lambda > 1/2$, $\boldsymbol{\mu} = (1, \mu_1)$, $\mu_1 > 0$, $\alpha_1 = {d_1}^{-1}$, d_1 is defined as before. If $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1, \delta_2)$, $\delta_i > 0$ for $i = 1, 2$, for some $\lambda > 1/2$ and $\boldsymbol{\mu} = (1, \mu_1)$, $\mu_1 > 0$. Then arguing as in Lemma 4.2.8, one notes that $2\lambda = \frac{\delta_1+\delta_2}{2}$, $\alpha_1 = \frac{\delta_2-\delta_1-2}{2}$. As $2\lambda > 1$ and $\alpha_1 = d^{-1} > 0$ it follows that $\delta_1 + \delta_2 > 2$ and $\delta_2 - \delta_1 > 2$ are necessary conditions for existence of a reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1, \delta_2)$. If $\delta_i > 0$ for $i = 1, 2$, proceeding as in Theorem 4.3.2, one observes that $\delta_2 - \delta_1 > 2$, $\delta_1 + \delta_2 > 2$ and $d_1 > \frac{1}{2\lambda-1} = \frac{2}{\delta_1+\delta_2-2}$ are the sufficient conditions for existence of a reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1, \delta_2)$. Conversely, if $\delta_i > 0$ for $i = 1, 2$ and $\delta_2 - \delta_1 > 2$ then clearly $\delta_1 + \delta_2 > 2$ and $d_1 = \frac{2}{\delta_2-\delta_1-2} > \frac{2}{\delta_1+\delta_2-2}$. So, $\delta_i > 0$ for $i = 1, 2$ and $\delta_2-\delta_1 > 2$ are the necessary and sufficient conditions for the existence of reproducing kernel $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1, \delta_2)$.

Remark 4.3.14. If $\delta_i > 0$ for $i = 1, 2$ such that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1, \delta_2)$ there does not exist a reproducing kernel $\mathbf{B}^{(\vartheta,\boldsymbol{\xi})}$ other than $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$ (up to equivalence as discussed in Remark 4.3.9) such that $\tilde{\mathcal{K}}^{(\vartheta,\boldsymbol{\xi})}(0) = \mathrm{diag}(\delta_2, \delta_1)$. If $\mathbf{B}^{(\vartheta,\boldsymbol{\xi})}$ exists satisfying the above requirements then from Remark 4.3.13, we see that both of $\delta_2 - \delta_1 > 2$ and $\delta_1 - \delta_2 > 2$ have to be simultaneously satisfied. This is impossible. Hence there does not exist inequivalent multiplication operators $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\vartheta,\boldsymbol{\xi})}$ such that the set of eigenvalues of $\tilde{\mathcal{K}}^{(\vartheta,\boldsymbol{\xi})}(0)$ equals those of $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$ but $\tilde{\mathcal{K}}^{(\vartheta,\boldsymbol{\xi})}(0) \neq \tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$.

Theorem 4.3.15. *Suppose that $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$, $\mathbf{B}^{(\lambda',\boldsymbol{\mu}')}$ and $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$, $\mathbf{B}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}$ are as in Proposition 4.3.4 and Proposition 4.3.5 respectively. Then*

(i) the multiplication operators $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\lambda',\boldsymbol{\mu}')}$ are not unitarily equivalent although $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(z)$ and $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(z)$ are unitarily equivalent for z in $\mathbb{D}$.

(ii) the multiplication operators $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}$ are not unitarily equivalent although $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(z)$ and $\tilde{\mathcal{K}}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}(z)$ are unitarily equivalent for z in $\mathbb{D}$.

Proof. From Proposition 4.3.4, we see that the curvatures of the associated bundles have the same set of eigenvalues at zero namely, $\{\delta_1, \delta_2, \delta_3\}$. Since curvature is self-adjoint the set of eigenvalues is the complete set of unitary invariants for the curvature. So, $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0)$ and $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(0)$ are unitarily equivalent. Since the operators $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\lambda',\boldsymbol{\mu}')}$ are homogeneous , by an application of Theorem 4.1.6, we see that $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(z)$ and $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(z)$ are unitarily equivalent for $z \in \mathbb{D}$. Now, (i) follows from part (a) of Corollary 4.3.8. The proof of part (ii) of this theorem is similar. □

The proof of the next Theorem will be completed after proving a sequence of Lemmas.

Theorem 4.3.16. *Suppose that $M^{(\lambda,\boldsymbol{\mu})}$ and $M^{(\vartheta,\boldsymbol{\xi})}$ are not unitarily equivalent and the two curvatures $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(z)$ and $\tilde{\mathcal{K}}^{(\vartheta,\boldsymbol{\xi})}(z)$ are unitarily equivalent for $z \in \mathbb{D}$. Then there does not exist any invertible matrix L in $\mathcal{M}_3$ satisfying $L\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \tilde{\mathcal{K}}^{(\vartheta,\boldsymbol{\xi})}(0)L$ for which $L(\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})})_{\bar{z}}(0) = (\tilde{\mathcal{K}}^{(\vartheta,\boldsymbol{\xi})})_{\bar{z}}(0)L$ also. In other words, the covariant derivative of order $(0, 1)$ detects the inequivalence.*

Lemma 4.3.17. *Suppose that there exists reproducing kernels $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$, $\mathbf{B}^{(\lambda',\boldsymbol{\mu}')}$ with $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1, \delta_2, \delta_3)$, $\tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(0) = \mathrm{diag}(\delta_{\rho(1)}, \delta_{\rho(2)}, \delta_{\rho(3)})$, $\delta_1 \neq \delta_2$ and C in $\mathcal{M}_3$ is such that $C\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \tilde{\mathcal{K}}^{(\lambda',\boldsymbol{\mu}')}(0)C$. Then $C = \left(\!\left(C_{ij}\delta_{\rho(i),j}\right)\!\right)$ for $C_{ij} \in \mathbb{C}$, $i, j = 1, 2, 3$, where $\rho \in S_3$ is given by $\rho(1) = 2, \rho(2) = 1, \rho(3) = 3$.*

Proof. The proof of this Lemma is immediate from Lemma 4.2.14, once we ensure that $\delta_1, \delta_2, \delta_3$ are distinct. Recalling notations from Lemma 4.2.2, we write $\delta_1 = 2\lambda - \alpha_1 - 2$, $\delta_2 = 2\lambda + \alpha_1 - \alpha_2$,

$\delta_3 = 2\lambda + \alpha_2 + 2$. Clearly, $\delta_3 - \delta_1 = \alpha_1 + \alpha_2 + 4 > 0$. Recalling notations from Proposition 4.3.4, one has $\delta_2 = 2\lambda' - \alpha_1' - 2$, $\delta_1 = 2\lambda' + \alpha_1' - \alpha_2'$, $\delta_3 = 2\lambda' + \alpha_2' + 2$. So, $\delta_3 - \delta_2 = \alpha_1' + \alpha_2' + 4 > 0$. We have $\delta_3 > \delta_1$, $\delta_3 > \delta_2$ and $\delta_1 \neq \delta_2$ by hypothesis. Hence the proof is complete. $\square$

The proof of the next Lemma is similar and is therefore omitted.

Lemma 4.3.18. *Suppose that there exists reproducing kernels $\mathbf{B}^{(\lambda,\boldsymbol{\mu})}$, $\mathbf{B}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}$ with $\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \mathrm{diag}(\delta_1, \delta_2, \delta_3)$, $\tilde{\mathcal{K}}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}(0) = \mathrm{diag}(\delta_{\tau(1)}, \delta_{\tau(2)}, \delta_{\tau(3)})$, $\delta_2 \neq \delta_3$ and C in $\mathcal{M}_3$ is such that $C\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})}(0) = \tilde{\mathcal{K}}^{(\widehat{\lambda},\widehat{\boldsymbol{\mu}})}(0)C$. Then $C = \big(\!\big(C_{ij}\delta_{\tau(i),j}\big)\!\big)_{i,j=1}^3$ for $C_{ij} \in \mathbb{C}$, $i,j = 1,2,3$, where $\tau \in S_3$ is given by $\tau(1) = 1, \tau(2) = 3, \tau(3) = 2$.*

Lemma 4.3.19. *Suppose that $C = \big(\!\big(C_{ij}\delta_{\sigma(i),j}\big)\!\big)_{i,j=1}^3$ for $\sigma = \rho$ or τ in S_3. Then C is invertible if and only if $C(i,\sigma(i)) \neq 0$ for $i = 1,2,3$ and $\sigma = \rho$ or τ in S_3.*

Proof. We observe that the only possible nonzero entries of C are the $(i,\sigma(i))$-th entries for $1 \le i \le 3$ and $C(i,\sigma(i)) = C_{i,\sigma(i)}$. Since $|\det C| = |C_{1,\sigma(1)}C_{2,\sigma(2)}C_{3,\sigma(3)}|$, it follows that $\det C \neq 0$ if and only if $C(i,\sigma(i)) \neq 0$ for $i = 1,2,3$ and $\sigma = \rho$ or τ in S_3. The proof is therefore complete. $\square$

The proof of the following Lemma is straight forward. We recall that $S(c_1,\ldots,c_m)(\ell,p) = c_\ell\delta_{p+1,\ell}$, $0 \le p,\ell \le m$.

Lemma 4.3.20. *Suppose that $C = \big(\!\big(C_{ij}\delta_{\sigma(i),j}\big)\!\big)_{i,j=1}^3$, $C_{i,\sigma(i)} \neq 0$ for $i = 1,2,3$ and $\sigma = \rho,\tau$ in S_3 is such that $CS(c_1,c_2)^{\mathrm{tr}} = S(\tilde{c}_1,\tilde{c}_2)^{\mathrm{tr}}C$ for $c_i, \tilde{c}_i$ in $\mathbb{C}$, $i = 1,2$. Then $c_i = \tilde{c}_i = 0$ for $i = 1,2$.*

Lemma 4.3.21. *$(\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})})_{\bar{z}}(0)$ is not the zero matrix.*

Proof. If possible let $(\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})})_{\bar{z}}(0) = 0$. Then it follows from Lemma 4.2.2 that $-\sqrt{\alpha}_1(1+\alpha_1-\frac{\alpha_2}{2}) = -\sqrt{\alpha}_2(1+\alpha_2-\frac{\alpha_1}{2}) = 0$. Equivalently, $1+\alpha_1-\frac{\alpha_2}{2} = 1+\alpha_2-\frac{\alpha_1}{2}$, as α_1 and α_2 are positive. This implies that $\alpha_1 = \alpha_2$. So, $(\tilde{\mathcal{K}}^{(\lambda,\boldsymbol{\mu})})_{\bar{z}}(0) = 0$ implies by an application of Lemma 4.2.2 that $-\sqrt{\alpha}_1(1+\frac{\alpha_1}{2}) = 0$, which is impossible as α_1 is positive. $\square$

Proof of Theorem 4.3.16: We observe by applying Proposition 4.3.4, Proposition 4.3.5 and Proposition 4.3.11 that if $M^{(\vartheta,\boldsymbol{\xi})}$ is a multiplication operator not unitarily equivalent to $M^{(\lambda,\boldsymbol{\mu})}$ then $(\vartheta,\boldsymbol{\xi}) = (\lambda',\boldsymbol{\mu}')$ or $(\widehat{\lambda},\widehat{\boldsymbol{\mu}})$. We arrive at the desired conclusion by an straight forward application of Lemma 4.3.17, Lemma 4.3.18, Lemma 4.3.19, Lemma 4.3.20 and Lemma 4.3.21. $\square$

Remark 4.3.22. The calculations for all the homogeneous operators constructed in [31] are not very different. However, we have not succeeded in completely answering the question raised in [20, page. 39] using these calculations. Indeed, for generic bundles associated with the entire class of operators from [31], we have shown 4.2.13 that the simultaneous unitary equivalence class of the curvature at 0 along with the covariant derivative of curvature at 0 of order $(0,1)$ is a complete set of unitary invariants for these operators.

BIBLIOGRAPHY

[1] J. Arazy, S. Fisher and J. Peetre, *Möbius invariant function spaces.* J. Reine Angew. Math. 363 (1985), 110–145.

[2] J. Arazy, S. Fisher and J. Peetre, *Möbius invariant spaces of analytic functions.* Complex analysis, I (College Park, Md., 1985–86), 10–22, Lecture Notes in Math., 1275, Springer, Berlin, 1987.

[3] J. Arazy and H. Upmeier, *Invariant inner product in spaces of holomorphic functions on bounded symmetric domains.* Doc. Math. 2 (1997), 213–261 (electronic).

[4] J. Arazy and G. Zhang, *Homogeneous multiplication operators on bounded symmetric domains*, J. Func. Anal. **202** (2003), 44 – 66.

[5] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer.Math. Soc. **68** (1950), 337 – 404.

[6] B. Bagchi and G. Misra, *Homogeneous tuples of operators and systems of imprimitivity*, Contemporary Mathematics, **185** (1995), 67 – 76.

[7] ———, *Homogeneous tuples of multiplication operators on twisted Bergman space*, J. Funct. Anal. **136** (1996), 171 – 213.

[8] ———, *Constant characteristic functions and Homogeneous operators*, J. Operator Theory, **37** (1997), 51 – 65.

[9] ———, *Homogeneous operators and the projective representations of the Möbius group: a survey*, Proc. Indian Acad. Sci. Math. Sci **111** (2001), no. 4, 415 – 437.

[10] ———, *The homogeneous shifts*, J. Funct. Anal. **204** (2003), 293 – 319.

[11] F. A. Berezin, *General concept of quantization.* Comm. Math. Phys. 40 (1975), 153–174.

[12] I. Biswas and G. Misra, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-*homogeneous vector bundles*, preprint, 2006.

[13] X. Chen and R. G. Douglas, *Localization of Hilbert modules.* Michigan Math. J. 39 (1992), no. 3, 443–454.

[14] X. Chen and K. Guo, *Analytic Hilbert modules.* Chapman and Hall/CRC Research Notes in Mathematics, 433. Chapman and Hall/CRC, Boca Raton, FL, 2003.

[15] D. N. Clark and G. Misra, *On homogeneous contractions and unitary representations of* SU(1, 1). J. Operator Theory 30 (1993), no. 1, 109–122.

[16] L. A. Coburn, *Deformation estimates for the Berezin-Toeplitz quantization*. Comm. Math. Phys. 149 (1992), no. 2, 415–424.

[17] L. A. Coburn *Berezin-Toeplitz quantization*. Algebraic methods in operator theory, 101–108, Birkhuser Boston, Boston, MA, 1994.

[18] M. J. Cowen and R. G. Douglas, *Complex geometry and Operator theory*, Acta Math. **141** (1978), 187 – 261.

[19] ———, *On operators possessing an open set of eigenvalues*, Memorial Conf. for Féjer-Riesz, Colloq. Math. Soc. J. Bolyai, 1980, pp. 323 – 341.

[20] M. J. Cowen and R. G. Douglas, *Equivalence of connections*, Adv. in Math. 56 (1985), no. 1, 39–91.

[21] R. E. Curto and N. Salinas, *Generalized Bergman kernels and the Cowen-Douglas theory*, Amer. J. Math. **106** (1984), 447 – 488.

[22] K. R. Davidson and R. G. Douglas, *The generalized Berezin transform and commutator ideals*. Pacific J. Math. 222 (2005), no. 1, 29–56.

[23] ———, *Equivalence of quotient Hilbert modules – II*, To appear, Trans. Amer. Math. Soc.

[24] R. G. Douglas, G. Misra and C. Varughese, *On quotient modules- the case of arbitrary multiplicity*, J. Funct. Anal. **174**(2000), 364 – 398.

[25] R. G. Douglas and V. I. Paulsen, *Hilbert modules over function algebras*. Pitman Research Notes in Mathematics Series, 217. Longman Scientific and Technical, Harlow; copublished in the United States with John Wiley and Sons, Inc., New York, 1989.

[26] J. Eschmeier and M. Putinar, *Spectral decompositions and analytic sheaves*. London Mathematical Society Monographs. New Series, 10. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.

[27] S. H. Ferguson and R. Rochberg, *Higher order Hilbert-Schmidt Hankel forms and tensors of analytic kernels*. Math. Scand. 96 (2005), no. 1, 117–146.

[28] S. H. Ferguson and R. Rochberg, *Description of certain quotient Hilbert modules*. Operator theory 20, 93–109, Theta Ser. Adv. Math., 6, Theta, Bucharest, 2006.

[29] H. P. Jakobsen, *Tensor products, reproducing kernels, and power series*, J. Funct. Anal. 31 (1979), no. 3, 293–305.

[30] A. Korányi and G. Misra, *New Constructions of Homogeneous operators*, C. R. Acad. Sci. Paris, ser. I **342** (2006), 933 – 936.

[31] A. Korányi and G. Misra, *Homogeneous operators on Hilbert spaces of holomorphic functions*, Preprint, 2006.

[32] G. Misra, *Curvature and the backward shift operators*, Proc. Amer. Math. Soc. **91** (1984), no. 1, 105 – 107.

[33] G. W. Mackey, *Mathematical foundations of quantum mechanics.* With a foreword by A. S. Wightman. Reprint of the 1963 original. Dover Publications, Inc., Mineola, NY, 2004.

[34] G. Misra, *Curvature and Discrete series representations of* $SL_2(\mathbb{R})$, J. Int. Eqn. Oper. Th. **9** (1986), 452 – 459.

[35] G. Misra and N. S. N. Sastry, *Homogeneous tuples of operators and holomorphic discrete series representation of some classical groups*, J. Operator Theory **24** (1990), 23 – 32.

[36] G. Misra and S. Shyam Roy, *The curvature invariant for a class of homogeneous operators*, preprint, 2006.

[37] G. Misra and S. Shyam Roy, *On Irreducibility of a class of homogeneous operators*, Operator Theory: Advances and Applications, Vol. 176, 165-198, 2007 Birkhauser Verlag Basel/ Switzerland.

[38] G. Ølafsson and B. Ørsted, *The holomorphic discrete series of an affine symmetric space and representations with reproducing kernels.* Trans. Amer. Math. Soc. 326 (1991), no. 1, 385–405.

[39] B. Ørsted, *Composition series for analytic continuations of holomorphic discrete series representations of* $\mathrm{SU}(n, n)$. Trans. Amer. Math. Soc. 260 (1980), no. 2, 563–573.

[40] B. Ørsted and G. Zhang, *Reproducing kernels and composition series for spaces of vector-valued holomorphic functions on tube domains.* J. Funct. Anal. 124 (1994), no. 1, 181–204.

[41] B. Ørsted and G. Zhang, *Reproducing kernels and composition series for spaces of vector-valued holomorphic functions.* Pacific J. Math. 171 (1995), no. 2, 493–510.

[42] B. Ørsted and G. Zhang, *Tensor products of analytic continuations of holomorphic discrete series.* Canad. J. Math. 49 (1997), no. 6, 1224–1241.

[43] J. Peetre, *Invariant function spaces connected with the holomorphic discrete series.* Anniversary volume on approximation theory and functional analysis (Oberwolfach, 1983), 119–134, Internat. Schriftenreihe Numer. Math., 65, Birkhuser, Basel, 1984.

[44] J. Peetre, *Moebius invariant function spaces—the case of hyperbolic space.* Proc. Roy. Irish Acad. Sect. A 92 (1992), no. 2, 243–265.

[45] M. M. Peloso, *Möbius invariant spaces on the unit ball.* Michigan Math. J. 39 (1992), no. 3, 509–536.

[46] L. Peng and G. Zhang, *Tensor products of holomorphic representations and bilinear differential operators*, J. Funct. Anal. **210** (2004), 171 – 192.

[47] T. Steger and M. K. Vemuri, *Inductive algebras for* $\mathrm{SL}(2,\mathbb{R})$. Illinois J. Math. 49 (2005), no. 1, 139–151 (electronic).

[48] H. Upmeier, *Jordan algebras in analysis, operator theory, and quantum mechanics.* CBMS Regional Conference Series in Mathematics, 67. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1987.

[49] H. Upmeier, *Toeplitz operators and index theory in several complex variables.* Operator Theory: Advances and Applications, 81. Birkhuser Verlag, Basel, 1996.

[50] R. O. Wells, *Differential analysis on complex manifolds*, Springer, 1973.

[51] D. R. Wilkins, *Homogeneous vector bundles and Cowen-Douglas operators*, Internat. J. Math. **4** (1993), no. 3, 503 – 520.