ON QUOTIENT MODULES

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Abstract. In this paper we obtain an extension of one of the main results in [5] relating the fundamental class of the zero set defining a quotient Hilbert module $\mathcal{M}_{\mathrm{q}}$, the curvatures of the two modules $\mathcal{M}$, $\mathcal{M}_0$ and the map X in a topologically exact sequence

$$0 \longrightarrow \mathcal{M}_0 \xrightarrow{X} \mathcal{M} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Let Ω be a bounded open connected set in $\mathbb{C}^m$. Let $\mathcal{A}(\Omega)$ denote the closure of the algebra of holomorphic functions in some neighborhood of $\bar{\Omega}$ with respect to the supremum norm. Let $\mathcal{M}$ be a Hilbert module, consisting of holomorphic functions on Ω over the function algebra $\mathcal{A}(\Omega)$. In an earlier paper [5], we considered the problem of finding invariants for $\mathcal{Q}$ using the resolution

$$0 \longrightarrow \mathcal{M}_0 \xrightarrow{X} \mathcal{M} \longrightarrow \mathcal{Q} \longrightarrow 0, \tag{1}$$

where $X : \mathcal{M}_0 \to \mathcal{M}$ is the inclusion map and $\mathcal{M}_0$ is the submodule of all functions vanishing on a hypersurface $\mathcal{Z} \subseteq \Omega$. In this paper we reconsider this problem extending our earlier results.

Let $\mathbb{C}_w$ be the one dimensional module over $\mathcal{A}(\Omega)$, where the module map is given by evaluation at $w \in \Omega$, that is, $(f, \lambda) \to f(w)\lambda$, $f \in \mathcal{A}(\Omega)$, $\lambda \in \mathbb{C}_w$. Let

$$X \otimes_{\mathcal{A}(\Omega)} 1 : \mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w \to \mathcal{N} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w$$

be the map obtained by localising a module map $X : \mathcal{M} \to \mathcal{N}$ between any two Hilbert modules $\mathcal{M}$ and $\mathcal{N}$ over a function algebra $\mathcal{A}(\Omega)$ (cf. [6, p. 114 - 115]). We let $X(w)$ denote the map $X \otimes_{\mathcal{A}(\Omega)} 1$. Finally, let

$$\mathcal{K}_X(w) \stackrel{\text{def}}{=} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \|X(w)\|^2 dw_i \wedge d\bar{w}_j. \tag{2}$$

Note that $\|X(w)\|^2$ vanishes on $\mathcal{Z}$ and that the right hand side in the above definition is thought of as a (1,1) form with distributional co-efficients.

Let $\mathcal{K}$ and $\mathcal{K}_0$ be the curvatures associated with the modules $\mathcal{M}$ and the submodule $\mathcal{M}_0$ of functions vanishing on the hypersurface $\mathcal{Z}$ respectively. In the paper [5], we proved that if Ω is a bounded domain in $\mathbb{C}^m$ for which the second Cousin problem is solvable, $X : \mathcal{M} \to \mathcal{M}_0$ is the inclusion map, then

$$\mathcal{K}_X(w) - \mathcal{K}_0(w) + \mathcal{K}(w)$$

represents the fundamental class $[\mathcal{Z}]$ of the hypersurface $\mathcal{Z}$.

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In this note, we show that this alternating sum represents the fundamental class $[\mathcal{Z}]$ not just for the inclusion map but for any injective map $X : \tilde{\mathcal{M}} \to \mathcal{M}$ which has dense range in $\mathcal{M}_0 \subseteq \mathcal{M}$.

We begin by describing the exact hypothesis on the domain Ω and the module $\mathcal{M}$. Let $\mathcal{Z}$ be an irreducible analytic hypersurface in Ω (complex submanifold of dimension $m-1$) in the sense of [8, Definition 8, p. 17]. We assume that the second Cousin problem is solvable on the domain Ω. Consequently, as pointed out in the remark preceding Corollary 3 in [8, p. 34], there exists a global defining function φ for the hypersurface $\mathcal{Z}$. We also assume that Ω is polynomially convex. Then the algebra $\mathcal{A}(\Omega)$ equals the uniform limits of polynomials with respect to the supremum norm on Ω. We assume that $\mathcal{M}$ is a complex separable Hilbert space of holomorphic functions on Ω and that the evaluation functionals on $\mathcal{M}$ are bounded. Consequently, recall from [1] that $\mathcal{M}$ admits a reproducing kernel K. The reproducing kernel $K : \Omega \times \Omega \to \mathbb{C}$ is holomorphic in the first variable and anti-holomorphic in the second variable. Further, $K(\cdot, w) \in \mathcal{M}$ for each fixed $w \in \Omega$ and $K(z,w) = \overline{K(w,z)}$. Finally, K has the reproducing property

$$\langle h, K(\cdot, w)\rangle = h(w) \text{ for } w \in \Omega,\ \ h \in \mathcal{M}.$$

Since $K(w,w) = \Big\langle K(\cdot,w), K(\cdot,w)\Big\rangle$, it follows that $K(w,w) \neq 0$ for $w \in \Omega$ whenever $K(\cdot, w)$ is a nonzero vector in $\mathcal{M}$. Assume that $\mathcal{M}$ is a bounded module over $\mathcal{A}(\Omega)$, in particular, the tuple $\boldsymbol{M}^* \stackrel{\text{def}}{=} (M_1^*, \dots, M_m^*)$ is bounded. Here M_k denotes the multiplication operator on $\mathcal{M}$ defined by $(M_k h)(w) = w_k h(w)$ for $h \in \mathcal{M}$ and $w \in \Omega$. Finally, we assume that the tuple $\boldsymbol{M}^*$ is in the class $B_1(\Omega)$ introduced in [2] and [3]. As shown in [3], in this case, the curvature of the module $\mathcal{M}$

$$\mathcal{K}_{\mathcal{M}}(w) = \sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log K(w,w) dw_i \wedge d\bar{w}_j$$

is a complete unitary invariant.

Lemma 1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two modules satisfying the hypotheses stated in the preceding paragraph. Let $L : \mathcal{M} \to \mathcal{N}$ be a module map with dense range. Then*

$$\mathcal{K}_L(w) = \mathcal{K}_{\mathcal{N}}(w) - \mathcal{K}_{\mathcal{M}}(w).$$

Proof. Assuming that the tuple $\boldsymbol{M}^*$ is in $B_1(\Omega)$ ensures on the one hand, the existence of eigenvectors

$$\{\gamma(w) : M_f^* \gamma(w) = \overline{f(w)}\gamma(w),\ \gamma(w) \in \mathcal{M},\ w \in \Omega\}$$

which span $\mathcal{M}$, and on the other hand, it also ensures that $\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w$ is the one dimensional module spanned by $\gamma(w)$. Similarly, $\mathcal{N} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w$ is spanned by the eigenvector $\tilde{\gamma}(w)$. Furthermore, $\gamma,\ \tilde{\gamma} : \Omega \to \mathcal{M}_k \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w$ may be chosen to be anti-holomorphic. Indeed if the Hilbert module $\mathcal{M}$ consists of holomorphic functions on Ω such that all of them do not vanish at any point in Ω, then $\gamma(w)$ may be taken to be the reproducing kernel at $w \in \Omega$, that is, $\gamma(w) = K(\cdot, w)$. In this case, $K(\cdot, w) \neq 0$ for each $w \in \Omega$. Hence $\|\gamma(w)\|^2 = K(w,w) \neq 0$. If all the functions vanish on a common zero set $\mathcal{Z} \subseteq \Omega$, then it can be shown (cf. [5, p. 91]) that the reproducing kernel factors as $K(z,w) = \varphi(z)\chi(z,w)\overline{\varphi(w)}$, where $\chi(w,w)$ does not vanish for any $w \in \Omega$ and φ is holomorphic. In this case, we may use $\chi(w,w)$ for calculating the curvature $\mathcal{K}_{\mathcal{M}}$. Consequently, without loss of generality, we shall assume that all reproducing kernels are nonvanishing.

Of course, similar considerations applies to the module $\mathcal{N}$ as well.

The fact that L is a module map implies $L \otimes_{\mathcal{A}(\Omega)} 1(\mathcal{M} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w) \subseteq \mathcal{N} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w$. However this map cannot be zero since the range of L is dense. Hence $L \otimes_{\mathcal{A}(\Omega)} 1\gamma(w) = a(w)\tilde{\gamma}(w)$, where $a(w)$ is a non-vanishing anti-holomorphic function. Therefore

$$\|L \otimes_{\mathcal{A}(\Omega)} 1\| = \|\tilde{\gamma}(w)\||a(w)|/\|\gamma(w)\|.$$

Taking logarithm on both sides and differentiating verifies the claim. □

Let $\mathcal{N},\ \mathcal{N}_k,\ k = 1, 2$ be Hilbert modules. We say that

$$\mathcal{N}_1 \xrightarrow{X_1} \mathcal{N} \xrightarrow{X_2} \mathcal{N}_2$$

is *topologically exact* at $\mathcal{N}$ if the clos(ran X_1) = $\ker X_2$. This differs from the usual notion of exactness in that the range of the module map X_1 is not assumed to be closed.

Let $\mathcal{M},\ \tilde{\mathcal{M}}$ and $\mathcal{N},\ \tilde{\mathcal{N}}$, be modules over the function algebra $\mathcal{A}(\Omega)$ satisfying the hypotheses stated in the paragraph preceding Lemma 1. Suppose that $L : \mathcal{M} \to \mathcal{N}$ is a bijective module map. In other words, the two Hilbert spaces $\mathcal{M}$ and $\mathcal{N}$ are isomorphic. In this case, it is easy to see that both $\mathcal{M}$ and $\mathcal{N}$ consist of the same set of functions on Ω. To see this, first note that for each $w \in \Omega$, we must have $LK_{\mathcal{M}}(\cdot, w) = c(w)K_{\mathcal{N}}(\cdot, w)$, for some scalar $c(w) \in \mathbb{C}$. The fact that L is a bounded invertible transform is equivalent to saying that there are positive constants a, b such that

$$aK_{\mathcal{M}}(z, w) \leq c(z)\overline{c(w)}K_{\mathcal{N}}(z, w) \leq bK_{\mathcal{M}}(z, w).$$

As usual, these are to be interpreted as ineqaulities involving the positive matrices $K_{\mathcal{M}}(w_j, w_i),\ 1 \leq i, j \leq n$ (respectively, $K_{\mathcal{N}}(w_j, w_i),\ 1 \leq i, j \leq n$) for all finite subsets $\{w_1, \ldots w_n\} \subseteq \Omega$. From these inequalities, it also follows that $a' \leq |c(z)| \leq b'$ for all $z \in \Omega$ and some positive constants a', b'. This implies, in view of [1, Corollary IV_3, page 383], that the module $\mathcal{M}$ coincides with the module whose reproducing kernel is given by the positive definite kernel $c(z)\overline{c(w)}K_{\mathcal{N}}(z, w)$. Clearly, this latter module coincides with $\mathcal{N}$.

Theorem 1. *Let $L : \mathcal{M} \to \mathcal{N}$ be a bijective module map with $L(\text{ran } X) \subseteq \text{ran } Y$. Assume that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\mathcal{M}} & \xrightarrow{X} & \mathcal{M} & \xrightarrow{p} & \mathcal{Q} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow L & & \| & & \\ 0 & \longrightarrow & \tilde{\mathcal{N}} & \xrightarrow{Y} & \mathcal{N} & \xrightarrow{q} & \mathcal{Q} & \longrightarrow & 0 \end{array}$$

is topologically exact, the range of the module map Y is closed and that $L(\ker p) = \ker q$. Then we have

$$\mathcal{K}_X(w) - \mathcal{K}_{\tilde{\mathcal{M}}}(w) + \mathcal{K}_{\mathcal{M}}(w) = \mathcal{K}_Y(w) - \mathcal{K}_{\tilde{\mathcal{N}}}(w) + \mathcal{K}_{\mathcal{N}}(w).$$

Proof. Notice that the assumption of topological exactness at $\mathcal{M}$ together with the fact that ran Y is closed implies

$$\text{ran } Y = \ker q = L(\ker p) = L(\text{clos}(\text{ran } X)).$$

Furthermore, from the exactness at $\mathcal{Q}$, it follows that $\mathcal{Q}$ may be identified with the quotient module $\mathcal{N}/\text{ran } Y = L(\mathcal{M})/L(\text{ran X})$. Define a map $Z : \tilde{\mathcal{M}} \to \tilde{\mathcal{N}}$ by setting

$$Zh = Y^{-1}LX(h),\ h \in \tilde{\mathcal{M}}.$$

This definition of the map Z makes the square on the left of our diagram commutative. Clearly, the map Z is a module map which has dense range in $\tilde{\mathcal{N}}$. Tensoring the entire diagram given in the statement of the theorem with the module $\mathbb{C}_w$, $w \in \Omega$, we obtain a new diagram of one dimensional Hilbert modules in which the map

$$Z \otimes_{\mathcal{A}(\Omega)} 1 : \tilde{\mathcal{M}} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w \to \tilde{\mathcal{N}} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w$$

is surjective. Also, the square of Hilbert modules remains commutative even after the localisation, that is,

$$Z \otimes_{\mathcal{A}(\Omega)} 1 = (Y \otimes_{\mathcal{A}(\Omega)} 1)^{-1} (L \otimes_{\mathcal{A}(\Omega)} 1)(X \otimes_{\mathcal{A}(\Omega)} 1).$$

However, since all these operators act on one dimensional Hilbert spaces, it follows that $\|Z \otimes_{\mathcal{A}(\Omega)} 1\| = \|X \otimes_{\mathcal{A}(\Omega)} 1\| \|(Y \otimes_{\mathcal{A}(\Omega)} 1)\|^{-1} \|L \otimes_{\mathcal{A}(\Omega)} 1)\|$. Hence in the notation of (2), we have

$$\mathcal{K}_Z(w) = \mathcal{K}_X(w) - \mathcal{K}_Y(w) + \mathcal{K}_L(w). \tag{3}$$

Lemma 1 shows that $\mathcal{K}_Z(w) = \mathcal{K}_{\tilde{\mathcal{N}}}(w) - \mathcal{K}_{\tilde{\mathcal{M}}}(w)$. Similarly, $\mathcal{K}_L(w) = \mathcal{K}_{\mathcal{N}}(w) - \mathcal{K}_{\mathcal{M}}(w)$. Going back to the equation (3), we arrive at the desired conclusion when we substitute the values for $\mathcal{K}_Z(w)$ and $\mathcal{K}_L(w)$. □

Recall that $\mathcal{M}_0$ is the sub-module of functions in $\mathcal{M}$ which vanish on the hypersurface $\mathcal{Z}$. Let $\mathcal{Q}$ be the quotient $\mathcal{M}/\mathcal{M}_0$. Let $0 \to \tilde{\mathcal{M}} \xrightarrow{X} \mathcal{M} \to \mathcal{Q} \to 0$ be a topologically exact resolution. Then the range of X must be dense in $\mathcal{M}_0$ and conversely. If $Y : \tilde{\mathcal{N}} \to \mathcal{N}$ is any injective module map such that the range of Y coincides with $L($ clos(ran $X))$ then we can apply the theorem. Specialise to the case, where $\tilde{\mathcal{N}} = \mathcal{M}_0$, $\mathcal{N} = \mathcal{M}$ and $Y : \mathcal{M}_0 \to \mathcal{M}$ is the inclusion map. Note that in this case, the alternating sum

$$\mathcal{K}_Y(w) - \mathcal{K}_{\tilde{\mathcal{N}}}(w) + \mathcal{K}_{\mathcal{N}}(w) = \sum_{i,j=1}^{m} \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log |\varphi(w)|^2 dw_i \wedge d\bar{w}_j \tag{4}$$

represents the fundamental class of the hypersurface $\mathcal{Z}$ [5, Theorem 1.4]. Having established the equation (4) for a module map with closed range, we can apply the theorem to arrive at the same conclusion for the alternating sum

$$\mathcal{K}_X(w) - \mathcal{K}_{\tilde{\mathcal{M}}}(w) + \mathcal{K}_{\mathcal{M}}(w)$$

for an injective module map $X : \tilde{\mathcal{M}} \to \mathcal{M}$ which has dense but not necessarily closed range in $\mathcal{M}_0 \subseteq \mathcal{M}$. We have therefore proved

Corollary 1. *Let $0 \longrightarrow \tilde{\mathcal{M}} \xrightarrow{X} \mathcal{M} \longrightarrow \mathcal{Q} \longrightarrow 0$ be a short topologically exact sequence of Hilbert modules. Suppose that ran X is dense in $\mathcal{M}_0$. Then the alternating sum*

$$\mathcal{K}_X(w) - \mathcal{K}_{\tilde{\mathcal{M}}}(w) + \mathcal{K}_{\mathcal{M}}(w)$$

represents the fundamental class of the hypersurface $\mathcal{Z}$.

One way to view this result is that it provides an invariant for any topologically exact resolution of quotient modules of the form $\mathcal{M}/\mathcal{M}_0$. Such resolutions give an analogue of the Sz.-Nagy–Foias model for contraction operators to the multivariate case (cf. the Introduction in [5]).

We now give some examples of maps $X : \tilde{\mathcal{M}} \to \mathcal{M}$ which satisfy the hypothesis of the Corollary. Let $\Omega_1 \times \Omega_2 \subseteq \mathbb{C}$ be a product domain containing $(0,0)$. Let $\mathcal{M}$

be a module over the function algebra $\mathcal{A}(\Omega_1 \times \Omega_2)$. Assume that the reproducing kernel K of the module is of the form

$$K(z,w) = K_1(z_1,w_1)K_2(z_2,w_2),\ z_i, w_i \in \Omega_i,\ i = 1,2$$

and that $K_1,\ K_2$ possess diagonal expansion about the origin, that is,

$$K_i(z_i,w_i) = \sum_{n=0}^{\infty} a_n(i) z_i^n \bar{w}_i^n,\ i = 1,2.$$

As shown in [5, Proposition 2.4], the reprducing kernel for the submodule $\mathcal{M}_0$ of functions vanishing on the set $\{(0,z_2) \in \Omega_1 \times \Omega_2\}$ is of the form

$$K_0(z,w) = z_1\bar{w}_1\Big(\sum_{n=1}^{\infty} a_n(1) z_1^{n-1}\bar{w}_1^{n-1}\Big)K_2(z_2,w_2).$$

Consequently, $\mathcal{M}_0 = \{z_1 f : f \in \mathcal{M}\}$. If $X : \mathcal{M} \to \mathcal{M}$ is the multiplication operator defined by $(Xf)(z_1,z_2) = z_1 f(z_1,z_2)$ then we see that the range of this operator coincides with $\mathcal{M}_0$.

We point out that similar considerations as in the paragraph preceding the Corollary apply to the commutative square on the right of the diagram in our theorem. This yields the relation

$$\mathcal{K}_{\mathcal{M}}(w) - \mathcal{K}_{\mathcal{Q}}(w) + \mathcal{K}_p(w) = \mathcal{K}_{\mathcal{N}}(w) - \mathcal{K}_{\mathcal{Q}}(w) + \mathcal{K}_q(w) \tag{5}$$

for $w \in \mathcal{Z}$. This relationship is obtained by restricting p to $\mathcal{M}/(\text{ran } X)$ and then writing the identity map on $\mathcal{Q}$ as qLp^{-1}. In particular, if we take $p : \mathcal{M} \to \mathcal{Q}$ to be the quotient map, it is easy to verify that

$$p \otimes_{\mathcal{A}(\Omega)} 1 : \mathcal{M}/(\text{ran } X) \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w \to \mathcal{Q} \otimes_{\mathcal{A}(\Omega)} \mathbb{C}_w$$

is the constant map identically equal to 1. Now if we further assume that $X(\tilde{\mathcal{M}})$ is dense in $\mathcal{M}_0$, then $\mathcal{Q}$ equals the quotient $\mathcal{M}_0^{\perp}$. It was shown, in this case, in [5] that $\mathcal{K}_{\mathcal{Q}}(w) = \mathcal{K}_{\mathcal{M}}(w)$ for $w \in \mathcal{Z}$. Therefore we obtain the equality $\mathcal{K}_{\mathcal{M}}(w) - \mathcal{K}_{\mathcal{Q}}(w) + \mathcal{K}_p(w) = 0$ for $w \in \Omega$. Hence in view of equation (5), this alternating sum is seen to be 0 even if p is not necessarily the quotient map.

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