GEOMETRIC INVARIANTS FROM THE RESOLUTION FOR THE QUOTIENT MODULE ALONG A MULTI-DIMENSIONAL GRID

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Abstract. We construct a resolution for certain class of quotient modules of a Hilbert module consisting of holomorphic functions on a bounded open connected set $\Omega \subseteq \mathbb{C}^m$. Each one of these quotient modules is obtained from the submodule of all functions vanishing on an analytic variety $\mathcal{Z} \subseteq \Omega$ using a multi-dimensional grid. A construction of the fundamental class of the variety $\mathcal{Z}$ using this resolution is given. The relationship of the resolution for the quotient module with the Koszul complex is also explicitly described.

1. Introduction

Let Ω be a nonempty open set in $\mathbb{C}^m$, and let $\mathcal{A}(\Omega)$ be a function algebra of holomorphic functions defined on Ω. If $\mathcal{M}$ is a Hilbert module over $\mathcal{A}(\Omega)$, and $\mathcal{M}_0$ is the submodule of it consisting of functions in $\mathcal{M}$ that vanish on a subvariety $\mathcal{Z} \subset \Omega$, it is interesting to obtain geometric invariants for the quotient module. A study of such invariants has yielded results of various types (cf. [12], [14]). The final goal is, of course, to obtain invariants that are complete and usable in specific contexts. Since these invariants are intended to characterise the quotient module, it is advantageous if they are described using only data obtained from $\mathcal{M}$ and $\mathcal{M}_0$.

The fundamental class of the variety $\mathcal{Z}$ is one such invariant. It is consequently of interest to characterise this fundamental class in terms of information obtained from $\mathcal{M}$ and $\mathcal{M}_0$. When $\mathcal{Z}$ is a hypersurface and $\mathcal{M}_0$ is the largest collection of functions that vanish on $\mathcal{Z}$, a description of the fundamental class was provided in [12]. The following natural resolution of the quotient module was used there:

$$0 \longleftarrow \mathcal{M}_{\mathrm{q}} \longleftarrow \mathcal{M} \xleftarrow{X} \mathcal{M}_0 \longleftarrow 0 \tag{1.1}$$

Here X is the inclusion map. The fundamental class of $\mathcal{Z}$ was characterised in [12] in terms of $\mathcal{M}, \mathcal{M}_0$ and the inclusion map X.

In what follows, we consider the case where the subvariety $\mathcal{Z}$ has co-dimension greater than one, but with $\mathcal{M}_0$ still being the largest collection of functions in $\mathcal{M}$ that vanish on $\mathcal{Z}$. The approach described here uses a generalised resolution of the quotient module. The resolution is by means of a grid of short exact sequences with the dimension of the grid being the same as the co-dimension of $\mathcal{Z}$ (see also [22]). The information obtained from this grid provides the necessary ingredients

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to determine the fundamental class of $\mathcal{Z}$ using the following theorem which is a generalisation of the Poincaré-Lelong formula (cf. [8]).

THEOREM 1.1. *Let $\varphi = (\varphi_1, \cdots, \varphi_p) : U \to \mathbb{C}^p$ be a holomorphic mapping. If $Y_j = \varphi_j^{-1}\{0\}$ are the loci of zeros of the components of φ, one has*

$$\overline{\partial}\partial \log|\varphi_1|^2 \wedge \cdots \wedge \overline{\partial}\partial \log|\varphi_p|^2 = 2\pi\sqrt{-1}[Y_1] \wedge \cdots \wedge 2\pi\sqrt{-1}[Y_p] = (2\pi\sqrt{-1})^p[Y],$$

where $[Y]$ is the current corresponding to the intersection $Y_1 \cap \cdots \cap Y_p$.

We outline below the general assumptions made on $\mathcal{M}, \mathcal{M}_0$ and $\mathcal{A}(\Omega)$. By a hypersurface $\mathcal{Z} \subseteq \Omega$ we mean a complex submanifold of dimension $m-1$. It follows that, given $z^{(0)} \in \mathcal{Z}$, there is a neighborhood $U \subseteq \Omega$ and a holomorphic map $\varphi : U \to \mathbb{C}$ such that $(\partial\varphi/\partial z_j)(z^{(0)}) \neq 0$ for some j, with $1 \leq j \leq m$ and

$$U \cap \mathcal{Z} = \{z \in U : \varphi(z) = 0\}.$$

In this case, we say that φ is a defining function for the hypersurface $\mathcal{Z}$. Whenever we discuss a zero variety, we will assume that it is the common zero set of holomorphic functions $\varphi_1, \ldots, \varphi_n$ defined on Ω. It must be pointed out that our interest lies in equivalence classes of modules consisting of holomorphic functions defined on Ω. The restriction of such modules to an open subset U of Ω yield equivalent modules as pointed out in [14, p. 370]. Consequently, we can (by going to a smaller open set if necessary) ensure that the zero variety has the description given above. The only assumption we make is that if the zero variety is the intersection of a number of hypersurfaces, then it must be a complete intersection. The assumption of complete intersection allows us to get around a number of technical difficulties - none very serious. We hope to study the general case in the near future.

Let $\Omega \subseteq \mathbb{C}^m$ be a bounded, simply connected domain in $\mathbb{C}^m$. Let $\mathcal{A}(\Omega)$ denote the closure with respect to the supremum norm, on Ω, of functions holomorphic in a neighborhood of $\bar{\Omega}$, the closure of the domain $\Omega \subseteq \mathbb{C}^m$. Then $\mathcal{A}(\Omega)$ is a function algebra and consists of continuous functions on $\bar{\Omega}$ which are holomorphic on Ω. We assume that Ω is polynomially convex which then ensures that $\mathcal{A}(\Omega)$ is the closure of polynomials with respect to the supremum norm on Ω.

Let $\mathcal{M}$ be a Hilbert module over $\mathcal{A}(\Omega)$. We assume that the point evaluation $f \mapsto f(w)$, $f \in \mathcal{M}$ is bounded for each $w \in \Omega$. Consequently, $\mathcal{M}$ admits a reproducing kernel $K : \Omega \times \Omega \to \mathbb{C}$. We assume, in addition, that $\mathcal{M}$ consists of holomorphic functions on Ω and contains all the polynomials, and lies in the Cowen-Douglas class $B_1(\Omega^*)$, that is, $(M_1^*, \ldots, M_m^*)$ is in $B_1(\Omega^*)$. Although it may not be absolutely necessary for what follows, we also assume that $\mathcal{M}$ is of rank 1.

2. The co-dimension 2 case

For the sake of clarity of exposition, we first present the grid construction when the co-dimension is two, and then we proceed to the general case. By the assumptions that we have made, the zero variety $\mathcal{Z}$ is a complete intersection of two hypersurfaces $\mathcal{Z}_1$ and $\mathcal{Z}_2$ with defining function φ_1 and φ_2 respectively. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be the maximal set of functions which vanish on the hypersurfaces $\mathcal{Z}_1$ and $\mathcal{Z}_2$

respectively, and let $\mathcal{I}_{12} = \mathcal{I}_1 \cap \mathcal{I}_2$. We recall that the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & \mathcal{I}_2^1 & \longleftarrow & \mathcal{I}_2 & \longleftarrow & \mathcal{I}_{12} & \longleftarrow & 0 \\ & & \downarrow & & \downarrow X_2 & & \downarrow & & \\ 0 & \longleftarrow & \mathcal{I}^1 & \longleftarrow & \mathcal{M} & \xleftarrow{X_1} & \mathcal{I}_1 & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & \mathcal{I}^{12} & \longleftarrow & \mathcal{I}^2 & \longleftarrow & \mathcal{I}_1^2 & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{2.2}$$

where each connecting map is either an inclusion or a quotient map was considered in [15, page 260]. We require that each row and column be a short exact sequence of Hilbert modules. Hence all the undefined symbols in the above diagram represent quotient modules at intermediate stages. Localisation of the grid above provides the necessary ingredients to obtain the current of integration on the zero variety $\mathcal{Z}_1 \cap \mathcal{Z}_2$, using Theorem 1.1. It is the case $p = 2$ of the theorem which is relevant to the above grid.

Notice that $\mathcal{I}^{12}$ in the grid above is the quotient module we are attempting to describe. The result above is then a natural generalisation of the approach described following the linear resolution (1.1) of the quotient module.

Let $K(w, w)$ be the reproducing kernel for the Hilbert module $\mathcal{M}$. Then as pointed out in [15, Section 2.3, equation 2.4]), the reproducing kernel for the submodule $\mathcal{I}_\ell$, $\ell = 1, 2$, is of the form $|\varphi_\ell(w)|^2 \chi_\ell(w, w)$ for some nonvanishing real analytic function χ. Thus we have

$$X_\ell(w)^* X_\ell(w) = \frac{K(w, w)}{|\varphi_\ell(w)|^2 \chi_\ell(w, w)},\ \ell = 1, 2. \tag{2.3}$$

Consequently, we find that

$$\begin{aligned} &\sum_{i,j=1}^{m} \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log(X_\ell(w)^* X_\ell(w)) dw_i \wedge d\bar{w}_j - \mathcal{K}(w) + \mathcal{K}_\ell(w) \\ &\qquad = \sum_{i,j=1}^{m} \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log |\varphi_\ell(w)|^2 dw_i \wedge d\bar{w}_j,\ \ell = 1, 2, \end{aligned} \tag{2.4}$$

where $\mathcal{K}$ is the curvature of the module $\mathcal{M}$ and $\mathcal{K}_\ell$ is the curvature of the submodule $\mathcal{I}_\ell$, $\ell = 1, 2$. Using Theorem 1.1, we conclude that the product of the two currents described in (2.4) yield the fundamental class of the cycle $\mathcal{Z}$.

3. The general case

If the co-dimension of $\mathcal{Z}$ is k, it follows by our assumption that $\mathcal{Z}$ is the complete intersection of k hypersurfaces $\mathcal{Z}_1, \mathcal{Z}_2, \cdots, \mathcal{Z}_k$. Let $\mathcal{I}_j$, for $1 \le j \le k$, be the largest collection of functions in $\mathcal{M}$ which vanish on the hypersurfaces $\mathcal{Z}_j$, and let

$$\mathcal{I}_{j_1 j_2 \cdots j_l} = \mathcal{I}_{j_1} \cap \mathcal{I}_{j_2} \cdots \cap \mathcal{I}_{j_l}\,.$$

If α is a multi-index, we define a family of modules inductively by the following prescription

$$\mathcal{I}_{\alpha}^{i_1 i_2 \cdots i_l} = \mathcal{I}_{\alpha}^{i_1 i_2 \cdots i_{l-1}} / \mathcal{I}_{i_l \alpha}^{i_1 i_2 \cdots i_{l-1}}, \tag{3.5}$$

where '$i_l \alpha$' has the obvious meaning.

We first note that the quotient module $\mathcal{M}_q$ can be obtained by a sequence of successive quotients. Let $X_p = \mathcal{I}_p^{12\cdots(p-1)}$. Define $\mathcal{Q}_0, \mathcal{Q}_1, \cdots, \mathcal{Q}_k$ (where k is the codimension of $\mathcal{Z}$) inductively as follows:

$$\mathcal{Q}_0 = \mathcal{M},\ \mathcal{Q}_p = \mathcal{Q}_{p-1}/X_p \text{ for } 1 \leq p \leq k$$

Then $\mathcal{Q}_k$ is indeed the quotient module $\mathcal{M}_q$.

We now construct the grid of modules mentioned earlier. The vertices of the grid are the points in $\mathbb{R}^k$ with coordinates $(x_1, x_2, \cdots, x_k)$ with $x_j = 0, +1$ or -1 for $1 \leq j \leq k$. Each such vertex is identified with a module over $\mathcal{A}(\Omega)$ as follows. Consider the vertex $(x_1, x_2, \cdots, x_k)$. Assume that

$$x_{i_1} = x_{i_2} = \cdots = x_{i_p} = +1$$

$$x_{j_1} = x_{j_2} = \cdots = x_{j_q} = -1$$

and all other coordinates are zero. The module associated with this vertex is $\mathcal{I}_{i_1 i_2 \cdots i_p}^{j_1 j_2 \cdots j_q}$. Recall that $\mathcal{I} = \mathcal{M}$ and this module is associated with the vertex $(0, 0, \cdots, 0)$. Also by the earlier discussion, the quotient module is $\mathcal{I}^{12\cdots k}$ and is associated with the vertex $(-1, -1, \cdots, -1)$.

It will be shown later (Proposition 3.1) that any three vertices which lie on a line parallel to any of the coordinate axes determine a short exact sequence (the 0's that terminate the short exact sequence are understood). The grid thus includes $k \cdot 3^{k-1}$ short exact sequences. That these sequences are exact follows from a sequence of lemmas which we state and prove below.

Lemma 3.1. *If I is an ideal in $I + A$ and in $I + B$, then*

$$\left(\frac{I+A}{I}\right) \cap \left(\frac{I+B}{I}\right) = (I + A \cap B)/I.$$

Proof: Consider $x \in \left(\frac{I+A}{I}\right) \cap \left(\frac{I+B}{I}\right)$. Let $x = (i + a) \pmod I = (i' + b) \pmod I$. Since

$$\begin{aligned} (i' + b) \pmod I &= i + (i' - i) + b \pmod I \\ &= i + b \pmod I, \end{aligned}$$

it follows that $a = b (\text{mod } I)$. Hence $x \in (A \cap B) \pmod I$ and consequently $x \in \frac{I + A\cap B}{I}$. Thus

$$\left(\frac{I+A}{I}\right) \cap \left(\frac{I+B}{I}\right) \subseteq (I + A \cap B)/I.$$

The reverse inclusion is easy to prove. □

Lemma 3.2. *The module $\mathcal{I}_{i_1 \cdots i_p}^{j_1 \cdots j_q}$ is symmetric in the indices $i_1 \cdots i_p$.*

Proof: We note that $\mathcal{I}_{i_1 \cdots i_p}^{j_1 \cdots j_q}$ can be written as a continued quotient (using the recursion relation) containing only terms of the form $\mathcal{I}_\alpha$ for α a multi-index. The indices

$i_1 \cdots i_p$ appear in the same combination in the continued quotient as subscripts. The result follows since $\mathcal{I}_\alpha$ is symmetric in the indices that occur in α. □

LEMMA 3.3. *The equality $\mathcal{I}^\beta_{\alpha\gamma} = \mathcal{I}^\beta_\alpha \cap \mathcal{I}^\beta_\gamma$ holds for multi-indices α, β, γ.*

Proof: We use induction on the length of β. For length zero, we have $\mathcal{I}_{\alpha\gamma} = \mathcal{I}_\alpha \cap \mathcal{I}_\gamma$ by definition. Otherwise,

$$\begin{aligned}
\mathcal{I}^{\beta k}_{\alpha\gamma} &= \mathcal{I}^\beta_{\alpha\gamma}/\mathcal{I}^\beta_{k\alpha\gamma} \\
&= \mathcal{I}^\beta_{\alpha\gamma}/(\mathcal{I}^\beta_k \cap \mathcal{I}^\beta_{\alpha\gamma}) \text{ (by induction hypothesis on } \mathcal{I}^\beta_{k\alpha\gamma}) \\
&= (\mathcal{I}^\beta_k + \mathcal{I}^\beta_{\alpha\gamma})/\mathcal{I}^\beta_k \\
&= (\mathcal{I}^\beta_k + (\mathcal{I}^\beta_\alpha \cap \mathcal{I}^\beta_\gamma))/\mathcal{I}^\beta_k \text{ (by induction hypothesis on } \mathcal{I}^\beta_{\alpha\gamma}) \\
&= \left(\frac{\mathcal{I}^\beta_k + \mathcal{I}^\beta_\alpha}{\mathcal{I}^\beta_k}\right) \cap \left(\frac{\mathcal{I}^\beta_k + \mathcal{I}^\beta_\gamma}{\mathcal{I}^\beta_k}\right) \text{ (by Lemma3.1)} \\
&= (\mathcal{I}^\beta_\alpha/(\mathcal{I}^\beta_k \cap \mathcal{I}^\beta_\alpha)) \cap (\mathcal{I}^\beta_\gamma/(\mathcal{I}^\beta_k \cap \mathcal{I}^\beta_\gamma)) \\
&= (\mathcal{I}^\beta_\alpha/\mathcal{I}^\beta_{k\alpha}) \cap (\mathcal{I}^\beta_\gamma/\mathcal{I}^\beta_{k\gamma}) \text{ (by induction hypothesis on } \mathcal{I}^\beta_{k\alpha}, \mathcal{I}^\beta_{k\gamma}) \\
&= \mathcal{I}^{\beta k}_\alpha \cap \mathcal{I}^{\beta k}_\gamma.
\end{aligned}$$

□

LEMMA 3.4. *The module $\mathcal{I}^{j_1 \cdots j_k}_\alpha$ is symmetric in the indices $j_1 \cdots j_k$.*

Proof: It is enough to prove the result for an adjacent flip since any permutation can be written as a sequence of adjacent flips. By definition,

$$\mathcal{I}^{j_1 \cdots j_k}_\alpha = \mathcal{I}^{j_1 \cdots j_{k-1}}_\alpha / \mathcal{I}^{j_1 \cdots j_{k-1}}_{j_k \alpha}. \tag{3.6}$$

We use induction on k the length of the superscript. The result is true for $k = 1$ since the permutation group is trivial. Assume that the result is true for length $(k-1)$ and any subscript α. Now consider an adjacent flip in $j_1 \cdots j_k$. If the flip does not involve j_k, the result is true by the induction hypothesis and (3.6) above. We therefore need to prove that

$$\mathcal{I}^{\beta ij}_\alpha = \mathcal{I}^{\beta ji}_\alpha.$$

Now,

$$\begin{aligned}
\mathcal{I}^{\beta ij}_\alpha &= \mathcal{I}^{\beta i}_\alpha/\mathcal{I}^{\beta i}_{j\alpha} \\
&= (\mathcal{I}^\beta_\alpha/\mathcal{I}^\beta_{i\alpha})/(\mathcal{I}^\beta_{j\alpha}/\mathcal{I}^\beta_{ij\alpha}).
\end{aligned}$$

By Lemma 3.3,

$$\begin{aligned}
\mathcal{I}^\beta_{ij\alpha} &= \mathcal{I}^\beta_i \cap \mathcal{I}^\beta_{j\alpha} \\
&= \mathcal{I}^\beta_i \cap \mathcal{I}^\beta_j \cap \mathcal{I}^\beta_\alpha \\
&= \mathcal{I}^\beta_i \cap \mathcal{I}^\beta_\alpha \cap \mathcal{I}^\beta_j \cap \mathcal{I}^\beta_\alpha \\
&= \mathcal{I}^\beta_{i\alpha} \cap \mathcal{I}^\beta_{j\alpha}.
\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{I}_\alpha^{\beta ij} &= (\mathcal{I}_\alpha^\beta/\mathcal{I}_{i\alpha}^\beta)/(\mathcal{I}_{j\alpha}^\beta/(\mathcal{I}_{i\alpha}^\beta \cap \mathcal{I}_{j\alpha}^\beta))\\ &= (\mathcal{I}_\alpha^\beta/\mathcal{I}_{i\alpha}^\beta)/((\mathcal{I}_{i\alpha}^\beta + \mathcal{I}_{j\alpha}^\beta)/\mathcal{I}_{i\alpha}^\beta)\\ &= \mathcal{I}_\alpha^\beta/(\mathcal{I}_{i\alpha}^\beta + \mathcal{I}_{j\alpha}^\beta),\end{aligned}$$

which is symmetric in i and j. □

We now proceed to prove that the edges of the grid define exact sequences.

Proposition 3.1. *Any three vertices in the grid which lie on a line parallel to any of the coordinate axes determine a short exact sequence (the terminating* 0*'s being understood). There are* $k \cdot 3^{k-1}$ *such exact sequences.*

Proof: Any three vertices which lie on a line parallel to any of the coordinate axes have all their coordinates common except those in one position. Assume this position is the j-th where the three vertices have the coordinates $1, 0, -1$. The modules at these three vertices are of the form $\mathcal{I}_\alpha^{\beta j}, \mathcal{I}_\alpha^\beta$ and $\mathcal{I}_{\alpha j}^\beta$ where α and β are multi-indices. Here we use the symmetry properties proved earlier. Since $\mathcal{I}_\alpha^{\beta j} = \mathcal{I}_\alpha^\beta/\mathcal{I}_{\alpha j}^\beta$, these three vertices determine a short exact sequence.

Since j can be chosen in k ways and there are 3^{k-1} choices of coordinates for the remaining common coordinates, we get $k \cdot 3^{k-1}$ short exact sequences. □

The module $\mathcal{I}_j$ appears at the vertex with coordinate 1 in the j-th place and 0's elsewhere. For each j, the three vertices with $1, 0$ and -1 in the j-th place and zeros everywhere else determine the following short exact sequence:

$$0 \longleftarrow \mathcal{I}^j \longleftarrow \mathcal{I} \longleftarrow \mathcal{I}_j \longleftarrow 0, \tag{3.7}$$

that is,

$$0 \longleftarrow \mathcal{M}/\mathcal{I}_j \longleftarrow \mathcal{M} \xleftarrow{X_j} \mathcal{I}_j \longleftarrow 0. \tag{3.8}$$

This gives us k short exact sequences.

By following the procedure outlined in the co-dimension 2 case, we get the necessary ingredients to determine the fundamental class of $\mathcal{Z}$. Specifically, as before, we have

$$X_\ell(w)^* X_\ell(w) = \frac{K(w,w)}{|\varphi_\ell(w)|^2 \chi_\ell(w,w)},\ \ell = 1, 2, \dots, k. \tag{3.9}$$

Consequently, as in the case of $k = 2$, we have

$$\begin{aligned}\mathrm{AltSum}_\ell(w) &:= \sum_{i,j=1}^m \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log(X_\ell(w)^* X_\ell(w)) dw_i \wedge d\bar{w}_j - \mathcal{K}(w) + \mathcal{K}_\ell(w)\\ &= \sum_{i,j=1}^m \tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log |\varphi_\ell(w)|^2 dw_i \wedge d\bar{w}_j,\ \ell = 1, 2, \dots, k, \end{aligned}\tag{3.10}$$

where $\mathcal{K}$ is the curvature of the module $\mathcal{M}$ and $\mathcal{K}_\ell$ is the curvature of the submodule $\mathcal{I}_\ell$, $\ell = 1, 2, \dots, k$. The following Theorem is now evident by appealing to generalization of the Poincare - Lelong formula given in Theorem 1.1.

Theorem 3.1. *The product* $\bigwedge_{\ell=0}^k \mathrm{AltSum}_\ell(w)$ *is a current which represents the fundamental class* $[\mathcal{Z}]$ *of the cycle* $\mathcal{Z}$.

4. The Koszul complex

The grid of short exact sequence of modules described in Section 3 is not a resolution of the quotient module in the usual sense. We describe below, using the Koszul complex, how to obtain a more conventional resolution for the quotient module. Let $J \subseteq \{1, 2, \cdots, k\}$ be a subset. As before, let

$$\mathcal{I}_J := \bigcap_{j \in J} \mathcal{I}_j$$

be the intersection. Consequently,

$$\mathcal{I}_J \subseteq \mathcal{I}_{J'} \tag{4.11}$$

if $J' \subseteq J$. The cardinality of J will be denoted by $\#J$. For any $l \in [1, k]$, let

$$\mathcal{V}_l := \bigoplus_{\{J \mid \#J = l\}} \mathcal{I}_J$$

be the direct sum. Set $\mathcal{V}_0 := \mathcal{M}$ and $\mathcal{V}_{k+1} = 0$.

We will describe a homomorphism from $\mathcal{V}_l$ to $\mathcal{V}_{l-1}$. Take any J' with $\#J' = l-1$ and set $J = J' \bigcup j$, where $j \notin J'$. There is an obvious inclusion homomorphism of $\mathcal{I}_J$ in $\mathcal{I}_{J'}$ obtained from (4.11). Let $n(J, J') \in \mathbb{Z}$ be the number of elements in J' larger than j. In other words,

$$n(J, J') := \#\{z \in J' \mid z > j\}.$$

Let

$$\Phi_{J,J'} : \mathcal{I}_J \longrightarrow \mathcal{I}_{J'} \tag{4.12}$$

denote the injective homomorphism which is $(-1)^{n(J,J')}$-times the inclusion map defined in (4.11). For any $j \in [1, k]$, set $\Phi_{j,\emptyset}$ to be the inclusion map of $\mathcal{I}_J$ in $\mathcal{M}$.

Finally, for $l \in [1, k]$, define

$$\Phi_l : \mathcal{V}_l \longrightarrow \mathcal{V}_{l-1} \tag{4.13}$$

as the direct sum

$$\Phi_l := \bigoplus_{\{J, J' \mid J' \subset J, \#J' = l-1\}} \Phi_{J,J'},$$

where $\Phi_{J,J'}$ is defined in (4.12) and the sum is taken over all possible J and J' satisfying the above conditions. Note that

$$\Phi_1 = \bigoplus_{j=1}^{k} \Phi_{j,\emptyset}\,.$$

Now it is easy to check that $\Phi_{l-1} \circ \Phi_l = 0$. Furthermore, the following

$$0 \longrightarrow \mathcal{V}_k \xrightarrow{\Phi_k} \mathcal{V}_{k-1} \xrightarrow{\Phi_{k-1}} \mathcal{V}_{k-2} \xrightarrow{\Phi_{k-2}} \cdots \xrightarrow{\Phi_2} \mathcal{V}_1 \xrightarrow{\Phi_1} \mathcal{V}_0 \tag{4.14}$$

is exact, where Φ_l is defined in (4.13). (Recall that $\mathcal{V}_0 := \mathcal{M}$.) The quotient $\mathcal{M}/\Phi_1(\mathcal{V}_1)$ evidently coincides with the earlier defined quotient module $\mathcal{I}^{12\cdots k}$.

It is straight-forward to recover the earlier grid of modules from the sequence (4.14). For example, the grid in (2.2) is obtained from the sequence (4.14) with $k = 2$

$$0 \longrightarrow \mathcal{I}_{12} \longrightarrow \mathcal{I}_1 \oplus \mathcal{I}_2 \longrightarrow \mathcal{M}$$

by setting $\mathcal{I}_2^1$ and $\mathcal{I}_1^2$ to be the quotient of the homomorphisms $\Phi_{12,2}$ and $\Phi_{12,1}$ respectively, defined in (4.12), and by setting $\mathcal{I}^1$ (respectively, $\mathcal{I}^2$) to be the quotient of the homomorphism $\Phi_{1,\emptyset}$ (respectively, $\Phi_{2,\emptyset}$).

It may be noted that any $\mathcal{I}_J$ is canonically identifed, as a $\mathcal{M}$-module, with $\mathcal{M}$. Indeed, the homomorphism

$$\mathcal{M} \longrightarrow \mathcal{I}_J$$

defined by $f \longmapsto f\prod_{j\in J}\varphi_j$ gives the canonical identification, where φ_j as before is a generator of the ideal $\mathcal{I}_j$. The exterior product $\bigwedge^l \mathbb{C}^k$ has a natural basis of the form $\{e_{j_1}\wedge e_{j_1}\wedge\cdots\wedge e_{j_l}\}$ where $1\le j_1<j_2<\cdots<j_l\le k$. Therefore, $\mathcal{V}_l$ gets identified with $(\bigwedge^l \mathbb{C}^k)\bigotimes_{\mathbb{C}}\mathcal{M}$.

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