Decomposition of the tensor product of Hilbert modules via the jet construction and weakly homogeneous operators

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Soumitra Ghara

Department of Mathematics
Indian Institute of Science
Bangalore - 560012

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Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Prof. Gadadhar Misra at the Department of Mathematics, Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

Soumitra Ghara
S. R. No. 10-06-00-10-31-11-1-08647

Indian Institute of Science,
Bangalore,
July, 2018.

Prof. Gadadhar Misra
(Research advisor)

Dedicated to my Family

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Abstract

Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function. We show that if $\alpha, \beta > 0$ be such that the functions K^α and K^β, defined on $\Omega \times \Omega$, are non-negative definite kernels, then the $\mathscr{M}_m(\mathbb{C})$ valued function

$$\mathbb{K}^{(\alpha,\beta)}(z,w) := K^{\alpha+\beta}(z,w)\Big(\big(\partial_i\bar{\partial}_j \log K\big)(z,w)\Big)_{i,j=1}^m,\ z,w \in \Omega,$$

is also a non-negative definite kernel on $\Omega \times \Omega$. Then a realization of the Hilbert space $(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$ determined by the kernel $\mathbb{K}^{(\alpha,\beta)}$ in terms of the tensor product $(\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta)$ is obtained. For two reproducing kernel Hilbert modules $(\mathscr{H}, K_1)$ and $(\mathscr{H}, K_2)$, let $\mathscr{A}_n$, $n \geq 0$, be the submodule of the Hilbert module $(\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)$ consisting of functions vanishing to order n on the diagonal set $\Delta := \{(z,z) : z \in \Omega\}$. Setting $\mathscr{S}_0 = \mathscr{A}_0^\perp$, $\mathscr{S}_n = \mathscr{A}_{n-1} \ominus \mathscr{A}_n$, $n \geq 1$, leads to a natural decomposition of $(\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)$ into infinite direct sum $\bigoplus_{n=0}^\infty \mathscr{S}_n$. A theorem of Aronszajn shows that the module $\mathscr{S}_0$ is isomorphic to the push-forward of the module $(\mathscr{H}, K_1K_2)$ under the map $\iota : \Omega \to \Omega \times \Omega$, where $\iota(z) = (z,z)$, $z \in \Omega$. We prove that if $K_1 = K^\alpha$ and $K_2 = K^\beta$, then the module $\mathscr{S}_1$ is isomorphic to the push-forward of the module $(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$ under the map ι.

Let Möb denote the group of all biholomorphic automorphisms of the unit disc $\mathbb{D}$. An operator T in $B(\mathscr{H})$ is said to be weakly homogeneous if $\sigma(T) \subseteq \bar{\mathbb{D}}$ and $\varphi(T)$ is similar to T for each φ in Möb. For a sharp non-negative definite kernel $K : \mathbb{D} \times \mathbb{D} \to \mathscr{M}_k(\mathbb{C})$, we show that the multiplication operator M_z on $(\mathscr{H}, K)$ is weakly homogeneous if and only if for each φ in Möb, there exists a $g_\varphi \in \mathrm{Hol}(\mathbb{D}, GL_k(\mathbb{C}))$ such that the weighted composition operator $M_{g_\varphi} C_{\varphi^{-1}}$ is bounded and invertible on $(\mathscr{H}, K)$. We also obtain various examples and nonexamples of weakly homogeneous operators in the class $\mathscr{F}B_2(\mathbb{D})$. Finally, it is shown that there exists a Möbius bounded weakly homogeneous operator which is not similar to any homogeneous operator.

We also show that if K_1 and K_2 are two positive definite kernels on $\mathbb{D} \times \mathbb{D}$ such that the multiplication operators M_z on the corresponding reproducing kernel Hilbert spaces are subnormal, then the multiplication operator M_z on the Hilbert space determined by the sum $K_1 + K_2$ need not be subnormal. This settles a recent conjecture of Gregory T. Adams, Nathan S. Feldman and Paul J. McGuire in the negative.

Contents

NOTATIONS

$\mathbb{D}$	the open unit disc
$\mathbb{B}_m$	the open unit ball $\{z \in \mathbb{C}^m : \sum_{i=1}^m \|z_i\|^2 < 1\}$
Ω^*	$\{z : \bar{z} \in \Omega\}$
$K \succeq 0$	K is non-negative definite
$K \preceq 0$	$-K$ is non-negative definite
$K_1 \succeq K_2$	$K_1 - K_2$ is non-negative definite
$K_1 \preceq K_2$	$K_2 - K_1$ is non-negative definite
$(\mathscr{H}, K)$	the Hilbert space determined by the kernel K
B_Ω	the Bergman kernel of the domain Ω
$\mathbb{S}_\Omega$	the Szegö kernel of the domain Ω
$\mathbb{K}^{(\alpha,\beta)}$	the function $K^{\alpha+\beta}\big(\partial_i\bar{\partial}_j \log K\big)_{i,j=1}^m$
$\mathbb{K}$	the function $K^2\big(\partial_i\bar{\partial}_j \log K\big)_{i,j=1}^m$
$K^{(\lambda)}$	the kernel $\frac{1}{(1-z\bar{w})^\lambda}$ on $\mathbb{D} \times \mathbb{D}$
$K_{(\gamma)}$	the kernel $\sum_{n=0}^\infty (n+1)^\gamma (z\bar{w})^n, z, w \in \mathbb{D}$
$\mathscr{H}^{(\lambda)}$	the Hilbert space determined by $K^{(\lambda)}$
M_f	the operator of multiplication by f
δ_{ij}	the Kronecker delta function
Möb	the group of all biholomorphic automorphisms of $\mathbb{D}$
Aut(Ω)	the group of all biholomorphic automorphisms of Ω
$\varphi_{\theta,a}$	the automorphism of the unit disc $e^{i\theta}\frac{z-a}{1-\bar{a}z}$
$\bar{\partial}_j$	$\frac{\partial}{\partial \bar{w}_j}$
$\partial^{\boldsymbol{i}}, \bar{\partial}^{\boldsymbol{j}}$	$\frac{\partial^{\|\boldsymbol{i}\|}}{\partial z_1^{i_1}\cdots\partial z_m^{i_m}}, \frac{\partial^{\|\boldsymbol{j}\|}}{\partial \bar{w}_1^{j_1}\cdots\partial \bar{w}_m^{j_m}}$, respectively
$\big(\frac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}}$	$\frac{\partial^{\|\boldsymbol{i}\|}}{\partial \zeta_1^{i_1}\cdots\partial \zeta_m^{i_m}}$

$J_k(K_1,K_2)_{\vert \mathrm{res}\,\Delta}$	the kernel $\big(K_1(z,w)\partial^{\boldsymbol{i}}\bar\partial^{\boldsymbol{j}}K_2(z,w)\big)_{\vert\boldsymbol{i}\vert,\vert\boldsymbol{j}\vert=0}^{k}$, $\boldsymbol{i},\boldsymbol{j}\in\mathbb{Z}_+^m$
$M_i^{(\ell)}$, $\ell=1,2$	the operator M_{z_i} on $(\mathscr{H},K_\ell)$, $\ell=1,2$
$M_i^{(\alpha)}$	the operator M_{z_i} on $(\mathscr{H},K^\alpha)$
$\mathbb{M}_i^{(\alpha,\beta)}$	the operator M_{z_i} on $(\mathscr{H},\mathbb{K}^{(\alpha,\beta)})$
J_kM_i	the operator M_{z_i} on $\big(\mathscr{H},J_k(K_1,K_2)_{\vert\mathrm{res}\,\Delta}\big)$
$M^{(\alpha)}$	the operator M_z on $(\mathscr{H},K^\alpha)$
$\mathbb{M}^{(\alpha,\beta)}$	the operator M_z on $(\mathscr{H},\mathbb{K}^{(\alpha,\beta)})$
$\boldsymbol{M}^{(\ell)}$, $\ell=1,2$	the m-tuple $(M_1^{(\ell)},\ldots,M_m^{(\ell)})$, $\ell=1,2$
$\boldsymbol{J}_k\boldsymbol{M}$	the m-tuple $(J_kM_1,\ldots,J_kM_m)$.
$D_{(T_1,\ldots,T_m)}$	the operator $h\mapsto T_1h\oplus\cdots\oplus T_mh$
$D^{(T_1,\ldots,T_m)}$	the operator $(h_1,\ldots,h_m)\mapsto T_1h_1+\cdots+T_mh_m$
$\boldsymbol{T}^{\boldsymbol{i}}$	the operator $T_1^{i_1}\cdots T_m^{i_m}$
Hol(Ω)	the space of all holomorphic functions from Ω to $\mathbb{C}$
Hol(Ω,Ω')	the space of all holomorphic functions from Ω to Ω'
$B(\mathscr{H})$	the space of all bounded linear operators on $\mathscr{H}$
P_M	the orthogonal projection to the closed subspace M
$\iota_\star\mathscr{M}$	the push-forward of the module $\mathscr{M}$ under the map ι
Mult $(\mathscr{H}_1,\mathscr{H}_2)$	$\big\{\psi\in\mathrm{Hol}(\mathbb{D}):\psi f\in\mathscr{H}_2$ whenever $f\in\mathscr{H}_1\big\}$.

Chapter 1

Introduction

(Reproducing kernel Hilbert spaces): Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $\mathscr{H}$ be a Hilbert space consisting of $\mathbb{C}^k$ valued holomorphic functions on Ω. Assume that the evaluation map $E_w : \mathscr{H} \to \mathbb{C}^k$, defined by $E_w(f) = f(w)$, $f \in \mathscr{H}$, is bounded for each $w \in \Omega$. The function $K : \Omega \times \Omega \to \mathscr{M}_k(\mathbb{C})$, defined by

$$K(z,w) := E_z E_w^*, \;\; z, w \in \Omega,$$

is called the reproducing kernel of $\mathscr{H}$, and $\mathscr{H}$ is called the reproducing kernel Hilbert space with the reproducing kernel K. The kernel function K satisfies the following two properties. For all w in Ω and η in $\mathbb{C}^k$,

(i) the function $K(\cdot, w)\eta$ is in $\mathscr{H}$,

(ii) $\langle f, K(\cdot, w)\eta \rangle_{\mathscr{H}} = \langle f(w), \eta \rangle_{\mathbb{C}^k}$ for all f in $\mathscr{H}$.

Every reproducing kernel K is a non-negative definite kernel in the following sense.

A function $K : \Omega \times \Omega \to \mathscr{M}_k(\mathbb{C})$ is said to be a non-negative definite kernel if for any subset $\{w_1, \ldots, w_n\}$ of Ω, the $n \times n$ block matrix $\big(K(w_i, w_j)\big)_{i,j=1}^n$ is non-negative definite, that is,

$$\sum_{i,j=1}^n \langle K(w_i, w_j)\eta_j, \eta_i \rangle \geq 0, \;\; \eta_1, \ldots, \eta_n \in \mathbb{C}^k.$$

Analogously, a function $K : \Omega \times \Omega \to \mathscr{M}_k(\mathbb{C})$ is said to be a positive definite kernel if for any subset $\{w_1, \ldots, w_n\}$ of Ω, the $n \times n$ block matrix $\big(K(w_i, w_j)\big)_{i,j=1}^n$ is positive definite, that is, it is non-negative definite and invertible. We always assume that the kernel $K(z,w)$ is sesqui-analytic, that is, it is holomorphic in z and anti-holomorphic in w. If K is the reproducing kernel of a reproducing kernel Hilbert space $\mathscr{H}$, then for any subset $\{w_1, \ldots, w_n\}$ of Ω and

$\eta_1,\ldots,\eta_n$ in $\mathbb{C}^k$, we have

$$\sum_{i,j=1}^{n}\left\langle K(w_i,w_j)\eta_j,\eta_i\right\rangle = \sum_{i,j=1}^{n}\left\langle K(\cdot,w_j)\eta_j,K(\cdot,w_i)\eta_i\right\rangle$$
$$= \Big\|\sum_{i=1}^{n}K(\cdot,w_i)\eta_i\Big\|^2 \geq 0.$$

Thus every reproducing kernel is a non-negative definite kernel.

Conversely, given any non-negative definite kernel $K:\Omega\times\Omega\to\mathcal{M}_k(\mathbb{C})$, let $\mathcal{H}_0$ be the vector space consisting of functions of the form $\sum_{i=1}^{n}K(\cdot,w_i)\eta_i,\ w_i\in\Omega,\ \eta_i\in\mathbb{C}^k,\ i=1,\ldots n,\ n\geq 1$. Define an inner product on $\mathcal{H}_0$ by the following formula:

$$\left\langle \sum_{i=1}^{n}K(\cdot,w_i)\eta_i,\sum_{j=1}^{n}K(\cdot,w_j')\eta_j'\right\rangle = \sum_{i,j=1}^{n}\left\langle K(w_j',w_i)\eta_i,\eta_j'\right\rangle,\ w_i,w_i'\in\Omega,\ \eta_i,\eta_i'\in\mathbb{C}^k. \tag{1.1}$$

Using the non-negative definiteness of K, it is easily verified that (1.1) indeed defines an inner product on $\mathcal{H}_0$. Let $\mathcal{H}$ be the completion of $\mathcal{H}_0$. Then $\mathcal{H}$ is a Hilbert space consisting of functions on Ω and K is the reproducing kernel of $\mathcal{H}$. Note that if the kernel K is sesqui-analytic, then the Hilbert space $\mathcal{H}$ consists of holomorphic functions on Ω taking values in $\mathbb{C}^k$. This completes the proof of the Theorem due to Moore stated below.

Theorem 1.1.1 (Moore)**.** *If $K:\Omega\times\Omega\to\mathcal{M}_k(\mathbb{C})$ is a sesqui-analytic non-negative definite kernel, then there exists a unique Hilbert space $\mathcal{H}$ consisting of $\mathbb{C}^k$ valued holomorphic functions on Ω such that the evaluation map E_w is bounded for each $w\in\Omega$ and K is the reproducing kernel of $\mathcal{H}$.*

We let $(\mathcal{H},K)$ denote the unique reproducing kernel Hilbert space $\mathcal{H}$ determined by the non-negative definite kernel K. Also for $K:\Omega\times\Omega\to\mathcal{M}_k(\mathbb{C})$, we write $K\succeq 0$ to denote that K is non-negative definite. For two functions $K_1,K_2:\Omega\times\Omega\to\mathcal{M}_k(\mathbb{C})$, we write $K_1\succeq K_2$ if $K_1-K_2\succeq 0$. Analogously, we write $K\preceq 0$ if $-K$ is non-negative definite and $K_1\preceq K_2$ if $K_1-K_2\preceq 0$.

We refer to [4] and [47] for the relationship between non-negative definite kernels and Hilbert spaces with the reproducing property as above.

Let $\Omega\subset\mathbb{C}^m$ be a bounded domain and $K:\Omega\times\Omega\to\mathbb{C}$ be a non-zero sesqui-analytic function. Let $t>0$ be any arbitrary positive real number. The function K^t is defined in the usual manner, namely $K^t(z,w)=\exp(t\log K(z,w)),\ z,w\in\Omega$, assuming that a continuous branch of the logarithm of K exists on $\Omega\times\Omega$. Clearly, K^t is also sesqui-analytic. However, if K is non-negative definite, then K^t need not be non-negative definite unless t is a natural number. A direct computation, assuming the existence of a continuous branch of logarithm of K on $\Omega\times\Omega$, shows that for $1\leq i,j\leq m$,

$$\partial_i\bar\partial_j\log K(z,w) = \frac{K(z,w)\partial_i\bar\partial_jK(z,w)-\partial_iK(z,w)\bar\partial_jK(z,w)}{K(z,w)^2},\ \ z,w\in\Omega,$$

where ∂_i and $\bar\partial_j$ denote $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial \bar w_j}$, respectively.

For any *positive integer* $s \geq 2$, define $(K^s \partial_i \bar\partial_j \log K(z,w))_{i,j=1}^m$ to be

$$K(z,w)^{s-2}\big(K(z,w)\partial_i\bar\partial_j K(z,w) - \partial_i K(z,w)\bar\partial_j K(z,w)\big)_{i,j=1}^m,$$

where we have not assumed that a continuous branch of the logarithm of $K(z,w)$ exists on $\Omega\times\Omega$. Also, unless t is a positive integer, we write K^t with the understanding that a continuous branch of logarithm of K exists on $\Omega\times\Omega$. Similarly, with the same hypothesis on K, we write $\big(\partial_i\bar\partial_j \log K(z,w)\big)_{i,j=1}^m$.

For a sesqui-analytic function $K:\Omega\times\Omega\to\mathbb{C}$ satisfying $K(z,z)>0$, an alternative interpretation of $K(z,w)^t$ (resp. $\log K(z,w)$) is possible using the notion of polarization. The real analytic function $K(z,z)^t$ (resp. $\log K(z,z)$) defined on Ω extends to a unique sesqui-analytic function in some neighbourhood U of the diagonal set $\{(z,z): z\in\Omega\}$ in $\Omega\times\Omega$. If the principal branch of logarithm of K exists on $\Omega\times\Omega$, then it is easy to verify that these two definitions of $K(z,w)^t$ (resp. $\log K(z,w)$) agree on the open set U.

(Hilbert Modules): We will find it useful to state many of our results in the language of Hilbert modules. The notion of a Hilbert module was introduced by R. G. Douglas (cf. [29]), which we recall below. We point out that in the original definition, the module multiplication was assumed to be continuous in both the variables. However, for our purposes, it would be convenient to assume that it is continuous only in the second variable.

Definition (Hilbert module)**.** *A Hilbert module $\mathscr{M}$ over a unital, complex algebra $\mathbb{A}$ consists of a complex Hilbert space $\mathscr{M}$ and a map $(a,h)\mapsto a\cdot h$, $a\in\mathbb{A}, h\in\mathscr{M}$, such that*

(i) $1\cdot h = h$

(ii) $(ab)\cdot h = a\cdot(b\cdot h)$

(iii) $(a+b)\cdot h = a\cdot h + b\cdot h$

(iv) *for each a in $\mathbb{A}$, the map $\mathbf{m}_a:\mathscr{M}\to\mathscr{M}$, defined by $\mathbf{m}_a(h) = a\cdot h$, $h\in\mathscr{M}$, is a bounded linear operator on $\mathscr{M}$.*

A closed subspace $\mathscr{S}$ of $\mathscr{M}$ is said to be a submodule of $\mathscr{M}$ if $\mathbf{m}_a h\in\mathscr{S}$ for all $h\in\mathscr{S}$ and $a\in\mathbb{A}$. The quotient module $\mathscr{Q} := \mathscr{H}/\mathscr{S}$ is the Hilbert space $\mathscr{S}^\perp$, where the module multiplication is defined to be the compression of the module multiplication on $\mathscr{H}$ to the subspace $\mathscr{S}^\perp$, that is, the module action on $\mathscr{Q}$ is given by $\mathbf{m}_a(h) = P_{\mathscr{S}^\perp}(\mathbf{m}_a h)$, $h\in\mathscr{S}^\perp$.

Two Hilbert modules $\mathscr{M}_1$ and $\mathscr{M}_2$ over $\mathbb{A}$ are said to be isomorphic if there exists a unitary operator $U:\mathscr{M}_1\to\mathscr{M}_2$ such that $U(a\cdot h) = a\cdot Uh$, $a\in\mathbb{A}$, $h\in\mathscr{M}_1$.

Now, if $\mathscr{M}_0, \mathscr{M}_1$, with $\mathscr{M}_1\subseteq\mathscr{M}_0$, are a pair of nested submodules of $\mathscr{M}$, then the quotient $\mathscr{M}_0/\mathscr{M}_1$ inherits a module multiplication from $\mathscr{M}_0$. Notice that the two projection operators

$P_0 : \mathscr{M}_0 \to \mathscr{M}_0 \ominus \mathscr{M}_1$ and $P : \mathscr{M} \to \mathscr{M}_0 \ominus \mathscr{M}_1$ agree on the subspace $\mathscr{M}_0$. Thus $\mathbf{m}_a(f) = P(ah)$, $h \in \mathscr{M}_0 \ominus \mathscr{M}_1$, $a \in \mathbb{A}$, defines a module multiplication on $\mathscr{M}_0 \ominus \mathscr{M}_1$. Note that, in general, the module $\mathscr{M}_0 \ominus \mathscr{M}_1$ is neither a submodule nor a quotient module of $\mathscr{M}$. Indeed, one may say it is a semi-invariant module of the Hilbert module $\mathscr{M}$.

Let $K : \Omega \times \Omega \to \mathscr{M}_k(\mathbb{C})$ be a non-negative definite kernel. Assume that the multiplication operator M_{z_i} by the ith coordinate function z_i is bounded on $(\mathscr{H}, K)$ for $i = 1, \ldots, m$. Then $(\mathscr{H}, K)$ may be realized as a Hilbert module over the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ with the module action given by the point-wise multiplication:

$$\mathbf{m}_p(h) = ph,\ h \in (\mathscr{H}, K),\ p \in \mathbb{C}[z_1, \ldots, z_m].$$

(Tensor products): Let K_1 and K_1 be two scalar valued non-negative definite kernels defined on $\Omega \times \Omega$. We identify the tensor product $(\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)$ as a Hilbert space of holomorphic functions defined on $\Omega \times \Omega$. Then it is the reproducing kernel Hilbert space with the reproducing kernel $K_1 \otimes K_2$ where $K_1 \otimes K_2 : (\Omega \times \Omega) \times (\Omega \times \Omega) \to \mathbb{C}$ is given by

$$(K_1 \otimes K_2)(z, \zeta; w, \rho) = K_1(z, w) K_2(\zeta, \rho),\ \ z, \zeta, w, \rho \in \Omega.$$

We also make the standing assumption that the multiplication operators M_{z_i}, $i = 1, \ldots, m$, are bounded on $(\mathscr{H}, K_1)$ as well as on $(\mathscr{H}, K_2)$. Thus the map

$$\mathbf{m} : \mathbb{C}[z_1, \ldots, z_{2m}] \times \big((\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)\big) \to (\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)$$

defined by

$$\mathbf{m}_p(h) = ph,\ h \in (\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2),\ p \in \mathbb{C}[z_1, \ldots, z_{2m}],$$

provides a module multiplication on $(\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)$ over the polynomial ring $\mathbb{C}[z_1, \ldots, z_{2m}]$. The module $(\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)$ admits a natural direct sum decomposition as follows. First, we recall some multi-index notations.

Let $\mathbb{Z}_+$ denote the set of all non-negative integers. For $\boldsymbol{i} = (i_1, \ldots, i_m) \in \mathbb{Z}_+^m$, let $|\boldsymbol{i}| = i_1 + \cdots + i_m$. For a holomorphic function $f : \Omega \times \Omega \to \mathbb{C}$, let $\big(\frac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)$ be the function $\frac{\partial^{|\boldsymbol{i}|}}{\partial \zeta_1^{i_1} \cdots \partial \zeta_m^{i_m}} f(z, \zeta)$ and $\big(\big(\frac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)\big)_{|\Delta}$ be the restriction of $\big(\frac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)$ to the set Δ, where Δ is the diagonal set $\{(z, z) : z \in \Omega\}$. Also if $K : \Omega \times \Omega \to \mathbb{C}$ is non-negative definite, then $\bar{\partial}^{\boldsymbol{i}} K(\cdot, w)$ denotes the function $\frac{\partial^{|\boldsymbol{i}|}}{\partial \bar{w}_1^{i_1} \cdots \partial \bar{w}_m^{i_m}} K(\cdot, w)$.

For a non-negative integer k, let $\mathscr{A}_k$ be the subspace of $(\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)$ defined by

$$\mathscr{A}_k := \Big\{ f \in (\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2) : \Big(\big(\tfrac{\partial}{\partial \zeta}\big)^{\boldsymbol{i}} f(z, \zeta)\Big)_{|\Delta} = 0,\ \boldsymbol{i} \in \mathbb{Z}_+^m, |\boldsymbol{i}| \le k \Big\}. \tag{1.2}$$

It is verified that the subspaces $\mathscr{A}_k$, $k \ge 0$, are closed and also invariant under the multiplication by any polynomial in $\mathbb{C}[z_1, \ldots, z_{2m}]$, therefore, they are submodules of $(\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)$.

Setting $\mathscr{S}_0 = \mathscr{A}_0^\perp$ and $\mathscr{S}_k := \mathscr{A}_{k-1} \ominus \mathscr{A}_k$, $k \geq 1$, we obtain a direct sum decomposition of the Hilbert space

$$(\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2) = \bigoplus_{k=0}^{\infty} \mathscr{S}_k.$$

In this decomposition, the subspaces $\mathscr{S}_k \subseteq (\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)$, $k \geq 0$, are not necessarily submodules. As we have already mentioned, these are called semi-invariant modules following the terminology commonly used in operator theory. We aim to study the compression of the module action to these subspaces analogous to the ones studied in operator theory. Also, such a decomposition is similar to the Clebsch-Gordan decomposition, which describes the decomposition of the tensor product of two irreducible representations, say ϱ_1 and ϱ_2 of a group G when restricted to the diagonal subgroup in $G \times G$:

$$\varrho_1(g) \otimes \varrho_2(g) = \bigoplus_k d_k \pi_k(g),$$

where π_k, $k \in \mathbb{Z}_+$, are irreducible representation of the group G and d_k, $k \in \mathbb{Z}_+$, are natural numbers. However, the decomposition of the tensor product of two Hilbert modules cannot be expressed as the direct sum of submodules. Noting that $\mathscr{S}_0$ is a quotient module, describing all the semi-invariant modules $\mathscr{S}_k$, $k \geq 1$, would appear to be a natural question. To describe the equivalence classes of $\mathscr{S}_0, \mathscr{S}_1, \ldots$ etc., it would be useful to recall the notion of the push-forward of a module.

Let $\iota : \Omega \to \Omega \times \Omega$ be the map $\iota(z) = (z, z)$, $z \in \Omega$. Any Hilbert module $\mathscr{M}$ over the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ may be thought of as a module $\iota_\star \mathscr{M}$ over the ring $\mathbb{C}[z_1, \ldots, z_{2m}]$ by re-defining the multiplication: $\mathbf{m}_p(h) = (p \circ \iota)h$, $h \in \mathscr{M}$ and $p \in \mathbb{C}[z_1, \ldots, z_{2m}]$.

Definition (push-forward module under ι)**.** *The module $\iota_\star \mathscr{M}$ over $\mathbb{C}[z_1, \ldots, z_{2m}]$ is defined to be the push-forward of the module $\mathscr{M}$ over $\mathbb{C}[z_1, \ldots, z_m]$ under the inclusion map ι.*

In [4], Aronszajn proved that the Hilbert space $(\mathscr{H}, K_1 K_2)$ corresponding to the point-wise product $K_1 K_2$ of two non-negative definite kernels K_1 and K_2 is obtained by the restriction of the functions in the tensor product $(\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)$ to the diagonal set Δ.

Theorem 1.1.2 (Aronszajn, [4])**.** *Let $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ be two non-negative definite kernels. Then $K_1 K_2$ is a non-negative kernel and the Hilbert space determined by $K_1 K_2$ is given by*

$$(\mathscr{H}, K_1 K_2) = \big\{h_{|\Delta} : h \in (\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)\big\},$$

with

$$\|f\|^2_{(\mathscr{H}, K_1 K_2)} = \min\big\{\|h\|^2 : h \in (\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2) \text{ and } h_{|\Delta} = f\big\}.$$

Building on his work, it was shown in [28] that the restriction map is isometric on the subspace $\mathscr{S}_0$ onto $(\mathscr{H}, K_1 K_2)$ intertwining the module actions on $\iota_\star(\mathscr{H}, K_1 K_2)$ and $\mathscr{S}_0$. However, using the jet construction given below, it is possible to describe the quotient modules $\mathscr{A}_k^\perp$, $k \geq 0$. Here we address the question of describing the semi-invariant modules, namely, $\mathscr{S}_1, \mathscr{S}_2, \ldots$. Unfortunately, we have been able to succeed in describing only $\mathscr{S}_1$ assuming that $K_1 = K^\alpha$ and $K_2 = K^\beta$ for some sesqui-analytic function $K : \Omega\times\Omega \to \mathbb{C}$ and a pair of positive real numbers α, β. In the particular case, when $K_1 = (1 - z\bar{w})^{-\alpha}$ and $K_2 = (1 - z\bar{w})^{-\beta}$, $\alpha, \beta > 0$, the semi-invariant modules $\mathscr{S}_k$, $k \geq 0$, were described by Ferguson and Rochberg.

Theorem 1.1.3 (Ferguson-Rochberg, [31]). *If $K_1(z,w) = \frac{1}{(1-z\bar{w})^\alpha}$ and $K_2(z,w) = \frac{1}{(1-z\bar{w})^\beta}$ on $\mathbb{D}\times\mathbb{D}$ for some $\alpha, \beta > 0$, then the Hilbert modules $\mathscr{S}_n$ and $\iota_\star(\mathscr{H}, (1 - z\bar{w})^{-(\alpha+\beta+2n)})$ are isomorphic.*

(The jet construction): For a bounded domain $\Omega \subset \mathbb{C}^m$, let K_1 and K_2 be two scalar valued non-negative kernels defined on $\Omega\times\Omega$. Assume that the multiplication operators M_{z_i}, $i = 1, \ldots, m$, are bounded on $(\mathscr{H}, K_1)$ as well as on $(\mathscr{H}, K_2)$. For a non-negative integer k, let $\mathscr{A}_k$ be the subspace defined in (1.2).

Let d be the cardinality of the set $\{\boldsymbol{i} \in \mathbb{Z}_+^m, |\boldsymbol{i}| \leq k\}$, which is $\binom{m+k}{m}$. Define the linear map $J_k : (\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2) \to \mathrm{Hol}(\Omega\times\Omega, \mathbb{C}^{\mathrm{d}})$ by

$$(J_k f)(z,\zeta) = \sum_{|\boldsymbol{i}|\leq k} \left(\tfrac{\partial}{\partial\zeta}\right)^{\boldsymbol{i}} f(z,\zeta) \otimes e_{\boldsymbol{i}},\ f \in (\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2), \tag{1.3}$$

where $\{e_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathbb{Z}_+^m, |\boldsymbol{i}|\leq k}$ is the standard orthonormal basis of $\mathbb{C}^d$. Let $R : \mathrm{ran}\ J_k \to \mathrm{Hol}(\Omega, \mathbb{C}^{\mathrm{d}})$ be the restriction map, that is, $R(\mathbf{h}) = \mathbf{h}_{|\Delta}$, $\mathbf{h} \in \mathrm{ran}\ J_k$. Clearly, $\ker RJ_k = \mathscr{A}_k$. Hence the map $RJ_k : \mathscr{A}_k^\perp \to \mathrm{Hol}(\Omega, \mathbb{C}^{\mathrm{d}})$ is one to one. Therefore we can give a natural inner product on ran RJ_k, namely,

$$\langle RJ_k(f), RJ_k(g)\rangle = \langle P_{\mathscr{A}_k^\perp} f, P_{\mathscr{A}_k^\perp} g\rangle,\ f, g \in (\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2).$$

In what follows, we think of ran RJ_k as a Hilbert space equipped with this inner product. The theorem stated below is a straightforward generalization of one of the main results from [28].

Theorem 1.1.4. *(* [28, Proposition 2.3]) Let $K_1, K_2 : \Omega\times\Omega \to \mathbb{C}$ be two non-negative definite kernels. Then ran RJ_k is a reproducing kernel Hilbert space and its reproducing kernel $J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}$ is given by the formula

$$J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}(z,w) := \left(K_1(z,w)\partial^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{j}} K_2(z,w)\right)_{|\boldsymbol{i}|,|\boldsymbol{j}|=0}^{k},\ z, w \in \Omega.$$

Now for any polynomial p in z, ζ, define the operator $\mathscr{T}_p$ on ran RJ_k as

$$(\mathscr{T} p)(RJ_k f) = \sum_{|\boldsymbol{l}|\leq k} \Big(\sum_{\boldsymbol{q}\leq\boldsymbol{l}} \tbinom{\boldsymbol{l}}{\boldsymbol{q}} \Big(\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{q}} p(z,\zeta)\Big)_{|\Delta} \Big(\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{l}-\boldsymbol{q}} f(z,\zeta)\Big)_{|\Delta}\Big) \otimes e_{\boldsymbol{l}}, f \in (\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2),$$

where $\boldsymbol{l} = (l_1, \ldots, l_m), \boldsymbol{q} = (q_1, \ldots, q_m) \in \mathbb{Z}_+^m$, and $\boldsymbol{q} \leq \boldsymbol{l}$ means $q_i \leq l_i,\ i = 1, \ldots, m$ and $\binom{\boldsymbol{l}}{\boldsymbol{q}} = \binom{l_1}{q_1}\cdots\binom{l_m}{q_m}$. The proof of the Proposition below follows from a straightforward computation using the Leibniz rule, see [28].

Proposition 1.1.5. *For any polynomial p in $\mathbb{C}[z_1,\dots,z_{2m}]$, the operator $P_{\mathscr{A}_k^\perp} M_{p|\mathscr{A}_k^\perp}$ is unitarily equivalent to the operator $\mathscr{T}_p$ on* $(\operatorname{ran} RJ_k)$.

(The Cowen-Douglas class): We now discuss an important class of operators introduced by Cowen and Douglas in the very influential paper [21]. The case of 2 variables was discussed in [22], while a detailed study in the general case appeared later in [26]. The definition below is taken from [26]. Let $\boldsymbol{T} := (T_1,\dots,T_m)$ be a m-tuple of commuting bounded linear operators on a separable Hilbert space $\mathscr{H}$. Let $D_{\boldsymbol{T}}:\mathscr{H}\to\mathscr{H}\oplus\cdots\oplus\mathscr{H}$ be the operator defined by $D_{\boldsymbol{T}}(x) = (T_1x,\dots,T_mx),\ x\in\mathscr{H}$.

Definition 1.1.6 (Cowen-Douglas class operator)**.** *Let $\Omega\subset\mathbb{C}^m$ be a bounded domain. The operator $\boldsymbol{T}$ is said to be in the Cowen-Douglas class $B_n(\Omega)$ if $\boldsymbol{T}$ satisfies the following requirements:*

(i) $\dim\ker D_{\boldsymbol{T}-w} = n,\ \ w\in\Omega$

(ii) $\operatorname{ran} D_{\boldsymbol{T}-w}$ is closed for all $w\in\Omega$

(iii) $\overline{\bigvee}\{\ker D_{\boldsymbol{T}-w} : w\in\Omega\} = \mathscr{H}$.

If $\boldsymbol{T}\in B_n(\Omega)$, then for each $w\in\Omega$, there exist functions $\gamma_1,\dots,\gamma_n$ holomorphic in a neighbourhood $\Omega_0\subseteq\Omega$ containing w such that $\ker D_{\boldsymbol{T}-w'} = \bigvee\{\gamma_1(w'),\dots,\gamma_n(w')\}$ for all $w'\in\Omega_0$ (cf. [22]). Consequently, every $\boldsymbol{T}\in B_n(\Omega)$ corresponds to a rank n holomorphic hermitian vector bundle $E_{\boldsymbol{T}}$ defined by

$$E_{\boldsymbol{T}} = \{(w,x)\in\Omega\times\mathscr{H} : x\in\ker D_{\boldsymbol{T}-w}\}$$

and $\pi(w,x) = w$, $(w,x)\in E_{\boldsymbol{T}}$.

For a bounded domain Ω in $\mathbb{C}^m$, let $\Omega^* = \{z : \bar z\in\Omega\}$. It is known that if T is an operator in $B_n(\Omega^*)$, then for each $w\in\Omega$, T is unitarily equivalent to the adjoint of the multiplication tuple $(M_{z_1},\dots,M_{z_m})$ on some reproducing kernel Hilbert space $(\mathscr{H},K)\subseteq\operatorname{Hol}(\Omega_0,\mathbb{C}^n)$ for some open subset $\Omega_0\subseteq\Omega$ containing w. Here the kernel K can be described explicitly as follows. Let $\Gamma = \{\gamma_1,\dots,\gamma_n\}$ be a holomorphic frame of the vector bundle $E_{\boldsymbol{T}}$ on a neighbourhood $\Omega_0^*\subseteq\Omega^*$ containing $\bar w$. Define $K_\Gamma : \Omega_0\times\Omega_0\to\mathscr{M}_n(\mathbb{C})$ by $K_\Gamma(z,w) = \big(\langle\gamma_j(\bar w),\gamma_i(\bar z)\rangle\big)_{i,j=1}^n$, $z,w\in\Omega_0$. Setting $K = K_\Gamma$, one may verify that the operator $\boldsymbol{T}$ is unitarily equivalent to the adjoint of the m-tuple of multiplication operators $(M_{z_1},\dots,M_{z_m})$ on the Hilbert space $(\mathscr{H},K)$.

If $T\in B_1(\Omega^*)$, the curvature matrix $\mathscr{K}_T(\bar w)$ at a fixed but arbitrary point $\bar w\in\Omega^*$ is defined by

$$\mathscr{K}_T(\bar w) = \big(\partial_i\bar\partial_j\log\|\gamma(\bar w)\|^2\big)_{i,j=1}^m,$$

where γ is a holomorphic frame of E_T defined on some open subset $\Omega_0^*\subseteq\Omega^*$ containing $\bar w$. If T is realized as the adjoint of the multiplication tuple $(M_{z_1},\dots,M_{z_m})$ on some reproducing kernel Hilbert space $(\mathscr{H},K)\subseteq\operatorname{Hol}(\Omega_0)$, where $w\in\Omega_0$, the curvature $\mathscr{K}_T(\bar w)$ is then equal to

$$\big(\partial_i\bar\partial_j\log K(w,w)\big)_{i,j=1}^m.$$

The study of operators in the Cowen-Douglass class using the properties of the kernel functions was initiated by Curto and Salinas in [26]. The following definition is taken from [49].

Definition 1.1.7 (Sharp kernel and generalized Bergman kernel). *A positive definite kernel $K:\Omega\times\Omega\to\mathscr{M}_k(\mathbb{C})$ is said to be sharp if*

(i) *the multiplication operator M_{z_i} is bounded on $(\mathscr{H},K)$ for $i=1,\dots,m$,*

(ii) $\ker D_{(\boldsymbol{M}_z-w)^*}=\operatorname{ran} K(\cdot,w)$, $w\in\Omega$,

where $\boldsymbol{M}_z$ denotes the m-tuple $(M_{z_1},M_{z_2},\dots,M_{z_m})$ on $(\mathscr{H},K)$. Moreover, if $\operatorname{ran} D_{(\boldsymbol{M}_z-w)^*}$ *is closed for all $w\in\Omega$, then K is said to be a generalized Bergman kernel.*

Some of the results in this thesis generalize, among other things, one of the main results of [49], which is reproduced below.

Theorem 1.1.8 (Salinas, [49, Theorem 2.6]). *Let $\Omega\subset\mathbb{C}^m$ be a bounded domain. If $K_1,K_2:\Omega\times\Omega\to\mathbb{C}$ are two sharp kernels (resp. generalized Bergman kernels), then $K_1\otimes K_2$ and K_1K_2 are also sharp kernels (resp. generalized Bergman kernels).*

(Homogeneous and weakly homogeneous operators): Let Möb denote the group of all biholomorphic automorphisms of the unit disc $\mathbb{D}$. Recall that an operator T in $B(\mathscr{H})$ is said to be homogeneous if $\sigma(T)\subseteq\overline{\mathbb{D}}$ and $\varphi(T)$ is unitarily equivalent to T for all φ in Möb, where $\varphi(T)$ is defined by using the usual Riesz functional calculus. It follows from the spectral mapping theorem that the spectrum of a homogeneous operator is invariant under the action of Möb and therefore, it is either the unit circle $\mathbb{T}$ or the closed unit disc $\bar{\mathbb{D}}$.

For $\lambda>0$, let $K^{(\lambda)}$ denote the positive definite kernel $(1-z\bar{w})^{-\lambda}$ on $\mathbb{D}\times\mathbb{D}$ and let $\mathscr{H}^{(\lambda)}$ denote the Hilbert space determined by the kernel $K^{(\lambda)}$. It known that the adjoint M_z^* of the multiplication operator by the coordinate function z on $\mathscr{H}^{(\lambda)}$, $\lambda>0$, is homogeneous and upto unitary equivalence, every homogeneous operator in $B_1(\mathbb{D})$ is of this form , see [44].

An operator T in $B(\mathscr{H})$ is said to be weakly homogeneous if $\sigma(T)\subseteq\overline{\mathbb{D}}$ and $\varphi(T)$ is similar to T for all φ in Möb, see [16], [10]. As in the case of homogeneous operators, the spectrum of a weakly homogeneous operator is also $\mathbb{T}$ or $\overline{\mathbb{D}}$. It is easy to verify that every operator T which is similar to a homogeneous operator is weakly homogeneous. But the converse of this is not true. To see this, it would be useful to recall the definition of a Möbius bounded operator.

Möbius bounded operators were introduced in [51] by Shields. An operator T on a Banach space $\mathscr{B}$ is said to be Möbius bounded if $\sigma(T)\subseteq\overline{\mathbb{D}}$ and $\sup_{\varphi\in\text{Möb}}\|\varphi(T)\|<\infty$. We will only discuss Möbius bounded operators on Hilbert spaces. By the von Neumann's inequality, every contraction on a Hilbert space is Möbius bounded. Also, if T is an operator which is similar to a homogeneous operator, then it is easily verified that T is Möbius bounded. In [10], the existence of a weakly homogeneous operator which is not Möbius bounded was given. Hence it cannot be similar to any homogeneous operator. In the same paper, the following question was raised.

Question 1.1.9 (Bagchi-Misra, [10, Question 10])**.** *Is it true that every Möbius bounded weakly homogeneous operator is similar to a homogeneous operator?*

In [51], it was shown that every power bounded operator is Möbius bounded. An example of an operator on a Banach space which is Möbius bounded but not power bounded was also given in that paper. The multiplication operator M_z on the Hilbert space $(\mathscr{H}, K^{(\lambda)})$, $0<\lambda<1$, is homogeneous, therefore, Möbius bounded, however, it is not power bounded. This was noted in [10]. Although a Möbius bounded operator need not be power bounded, Shields proved that if T is a Möbius bounded operator on a Banach space, then $\|T^n\| \leq c(n+1)$, $n \in \mathbb{Z}_+$, for some constant $c>0$. But for operators on Hilbert spaces, he made the following conjecture.

Conjecture 1.1.10 (Shields, [52])**.** *If T is a Möbius bounded operator on a Hilbert space, then $\|T^n\| \leq c(n+1)^{\frac{1}{2}}$, $n \in \mathbb{Z}_+$, for some constant $c>0$.*

This conjecture is verified for the class of quasi-homogeneous operators introduced recently in the paper [38].

(subnormal operators on reproducing kernel Hilbert spaces): Recall that an operator T in $B(\mathscr{H})$ is said to be *subnormal* if there exists a Hilbert space $\mathscr{K} \supset \mathscr{H}$ and a normal operator N in $B(\mathscr{K})$ such that $N(\mathscr{H}) \subset \mathscr{H}$ and $N_{|\mathscr{H}} = T$. For the basic theory of subnormal operators, we refer to [18].

Completely hyperexpansive operators were introduced in [6]. An operator $T \in B(\mathscr{H})$ is said to be completely hyperexpansive if

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} T^{*j} T^j \leq 0 \qquad (n \geq 1).$$

The theory of subnormal and completely hyperexpansive operators is closely related with the theory of *completely monotone* and *completely alternating* sequences (cf. [5], [6]). A sequence $\{a_k\}_{k\in\mathbb{Z}_+}$ of positive real numbers is said to be completely monotone if

$$\sum_{j=0}^{n} (-1)^j \tbinom{n}{j} a_{m+j} \geq 0 \ \ (m, n \geq 0). \tag{1.4}$$

It is known that a sequence $\{a_k\}_{k\in\mathbb{Z}_+}$ is completely monotone if and only if it is a Hausdorff moment sequence, that is, there exists a positive measure ν supported in $[0,1]$ such that $a_k = \int_{[0,1]} x^k d\nu(x)$ for all $k \in \mathbb{Z}_+$ (see [12]). The measure ν is called the representing measure of the sequence $\{a_k\}_{k\in\mathbb{Z}_+}$.

Similarly, a sequence $\{a_k\}_{k\in\mathbb{Z}_+}$ of positive real numbers is said to be completely alternating if

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}a_{m+j}\le 0 \qquad (m\ge 0, n\ge 1). \tag{1.5}$$

Note that $\{a_k\}_{k\in\mathbb{Z}_+}$ is completely alternating if and only if the sequence $\{\Delta a_k\}_{k\in\mathbb{Z}_+}$ is completely monotone, where $\Delta a_k := a_{k+1}-a_k$.

Let $(\mathscr{H},K)$ be a reproducing kernel Hilbert space consisting of holomorphic functions on the unit disc $\mathbb{D}$ where K has the diagonal expansion $\sum_{k\in\mathbb{Z}_+}a_k(z\bar{w})^k$, $a_k>0$. Consider the operator M_z of multiplication by the coordinate function z on $(\mathscr{H},K)$. As is well-known, such a multiplication operator is unitarily equivalent to a weighted shift operator W with the weight sequence $\big\{(\frac{a_k}{a_{k+1}})^{1/2}\big\}_{k\in\mathbb{Z}_+}$. The operator M_z on $(\mathscr{H},K)$ is contractive subnormal if and only if $\{\frac{1}{a_k}\}_{k\in\mathbb{Z}_+}$ is a Hausdorff moment sequence (cf. [18, Theorem 6.10]). We will often call the representing measure of the Hausdorff moment sequence $\{\frac{1}{a_k}\}_{k\in\mathbb{Z}_+}$ as the representing measure of the subnormal operator M_z. On the other hand, the operator M_z on $(\mathscr{H},K)$ is completely hyperexpansive if and only if the sequence $\{\frac{1}{a_k}\}_{k\in\mathbb{Z}_+}$ is completely alternating (see [6, Proposition 3]).

For any two positive definite kernels K_1 and K_2 defined on $\mathbb{D}\times\mathbb{D}$, their sum K_1+K_2 is again a positive definite kernel on $\mathbb{D}\times\mathbb{D}$ and therefore determines a Hilbert space $(\mathscr{H},K_1+K_2)$ of functions on $\mathbb{D}$. It was shown in [4] that

$$(\mathscr{H},K_1+K_2)=\{f=f_1+f_2 : f_1\in(\mathscr{H},K_1), f_2\in(\mathscr{H},K_2)\},$$

and the norm is given by

$$\|f\|^2_{(\mathscr{H},K_1+K_2)}:=\inf\Big\{\|f_1\|^2_{(\mathscr{H},K_1)}+\|f_2\|^2_{(\mathscr{H},K_2)} : f=f_1+f_2,\ f_1\in(\mathscr{H},K_1),\ f_2\in(\mathscr{H},K_2)\Big\}.$$

The sum of two kernel functions is also discussed by Salinas in [49]. He proved that if K_1 and K_2 are generalized Bergman kernels, then so is K_1+K_2. Although not explicitly stated in [4], it is not hard to verify that the multiplication operator M_z on $(\mathscr{H},K_1+K_2)$ is unitarily equivalent to the operator $P_{\mathcal{N}^\perp}(M^{(1)}\bigoplus M^{(2)})_{|\mathcal{N}^\perp}$, where $M^{(i)}$ is the operator of multiplication by the coordinate function z on $(\mathscr{H},K_i)$, $i=1,2$ and

$$\mathcal{N}=\big\{(g,-g)\in(\mathscr{H},K_1)\oplus(\mathscr{H},K_2) : g\in(\mathscr{H},K_1)\cap(\mathscr{H},K_2)\big\}.$$

Evidently, if $M^{(1)}$ and $M^{(2)}$ are subnormal, then so is $M^{(1)}\oplus M^{(2)}$. In chapter 5, we discuss the subnormality of the compression $P_{\mathcal{N}^\perp}(M^{(1)}\oplus M^{(2)})_{|\mathcal{N}^\perp}$ for a class of kernels. In particular, we show that the subnormality of $M^{(1)}$ and $M^{(2)}$ need not imply that $P_{\mathcal{N}^\perp}(M^{(1)}\oplus M^{(2)})_{|\mathcal{N}^\perp}$ is subnormal.

A similar question on subnormality involving the point-wise product of two positive definite kernels was raised in [49]. Recently, a counterexample of the conjecture has been found, see [2, Theorem 1.5]. The conjecture below is similar except that it involves the sum of two kernels.

Conjecture 1.1.11 (Adams-Feldman-McGuire, [1, page 22])**.** *Let* $K_1(z,w)=\sum_{k\in\mathbb{Z}_+} a_k(z\bar{w})^k$ *and* $K_2(z,w)=\sum_{k\in\mathbb{Z}_+} b_k(z\bar{w})^k$ *be any two reproducing kernels satisfying:*

(a) $\lim \frac{a_k}{a_{k+1}} = \lim \frac{b_k}{b_{k+1}} = 1$

(b) $\lim a_k = \lim b_k = \infty$

(c) $\frac{1}{a_k} = \int_{[0,1]} t^k d\nu_1(t)$ *and* $\frac{1}{b_k} = \int_{[0,1]} t^k d\nu_2(t)$ *for all* $k\in\mathbb{Z}_+$*, where* ν_1 *and* ν_2 *are two positive measures supported in* $[0,1]$.

Then the multiplication operator M_z *on* $(\mathscr{H}, K_1+K_2)$ *is a subnormal operator.*

An equivalent formulation, in terms of the moment sequence criterion, of the conjecture is the following. If $\{\frac{1}{a_k}\}_{k\in\mathbb{Z}_+}$ and $\{\frac{1}{b_k}\}_{k\in\mathbb{Z}_+}$ are Hausdorff moment sequences, does it necessarily follow that $\{\frac{1}{a_k+b_k}\}_{k\in\mathbb{Z}_+}$ is also a Hausdorff moment sequence?

1.1.1 Main results of the thesis

In this section, we present the main results of this thesis.

In Chapter 2, a decomposition of the tensor product of Hilbert modules via the jet construction is discussed. First, the following proposition is proved.

Proposition. 2.1.4. *Let* Ω *be a bounded domain in* $\mathbb{C}^m$ *and* $K:\Omega\times\Omega\to\mathbb{C}$ *be a sesqui-analytic function. Suppose that* K^α *and* K^β*, defined on* $\Omega\times\Omega$*, are non-negative definite for some* $\alpha,\beta>0$. *Then the function* $\mathbb{K}^{(\alpha,\beta)}:\Omega\times\Omega\to\mathscr{M}_m(\mathbb{C})$ *defined by*

$$\mathbb{K}^{(\alpha,\beta)}(z,w) = K^{\alpha+\beta}(z,w)\Big(\big(\partial_i\bar{\partial}_j \log K\big)(z,w)\Big)_{i,j=1}^m,\ z,w\in\Omega,$$

is a non-negative definite kernel.

The following corollary is an immediate consequence.

Corollary. 2.1.5. *Let* Ω *be a bounded domain in* $\mathbb{C}^m$. *If* $K:\Omega\times\Omega\to\mathbb{C}$ *is a non-negative definite kernel, then*

$$\mathbb{K}(z,w) := K^2(z,w)\Big(\big(\partial_i\bar{\partial}_j \log K\big)(z,w)\Big)_{i,j=1}^m \tag{1.6}$$

is also a non-negative definite kernel, defined on $\Omega\times\Omega$*, taking values in* $\mathscr{M}_m(\mathbb{C})$.

Next, a realization of the Hilbert space determined by the non-negative definite kernel $\mathbb{K}^{(\alpha,\beta)}$ is obtained. For this, first define a linear map $\mathscr{R}_1 : (\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta) \to \mathrm{Hol}(\Omega, \mathbb{C}^m)$ by

$$\mathscr{R}_1(f) = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \begin{pmatrix} (\beta\partial_1 f - \alpha\partial_{m+1} f)_{|\Delta} \\ \vdots \\ (\beta\partial_m f - \alpha\partial_{2m} f)_{|\Delta} \end{pmatrix}, \; f \in (\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta). \tag{1.7}$$

We have shown that $(\ker \mathscr{R}_1)^\perp = \mathscr{A}_0 \ominus \mathscr{A}_1$, where $\mathscr{A}_0$ and $\mathscr{A}_1$ are defined by (1.2). Therefore, the map $\mathscr{R}_{1|\mathscr{A}_0 \ominus \mathscr{A}_1} \to \mathrm{ran}\, \mathscr{R}_1$ is one-to-one and onto. Require this map to be unitary by defining an appropriate inner-product on $\mathrm{ran}\, \mathscr{R}_1$, that is, define

$$\langle \mathscr{R}_1(f), \mathscr{R}_1(g) \rangle := \langle P_{\mathscr{A}_0 \ominus \mathscr{A}_1} f, P_{\mathscr{A}_0 \ominus \mathscr{A}_1} g \rangle, \; f, g \in (\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta), \tag{1.8}$$

where $P_{\mathscr{A}_0 \ominus \mathscr{A}_1}$ is the orthogonal projection of $(\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta)$ onto the subspace $\mathscr{A}_0 \ominus \mathscr{A}_1$.

Theorem. 2.2.3. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function. Suppose that the functions K^α and K^β, defined on $\Omega \times \Omega$, are non-negative definite for some $\alpha, \beta > 0$. Then the Hilbert space determined by the non-negative definite kernel $\mathbb{K}^{(\alpha,\beta)}$ coincides with the space* $\mathrm{ran}\, \mathscr{R}_1$ *and the inner product given by* (1.8) *on* $\mathrm{ran}\, \mathscr{R}_1$ *agrees with the one induced by the kernel* $\mathbb{K}^{(\alpha,\beta)}$.

Then a description of the Hilbert module $\mathscr{S}_1$ is given using Theorem 2.2.3.

Theorem. 2.2.5. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function. Suppose that the functions K^α and K^β, defined on $\Omega \times \Omega$, are non-negative definite for some $\alpha, \beta > 0$, and the multiplication operators M_{z_i}, $i = 1, 2, \ldots, m$, are bounded on both $(\mathscr{H}, K^\alpha)$ and $(\mathscr{H}, K^\beta)$. Then the Hilbert modules $\mathscr{S}_1$ and $\iota_\star(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$ are isomorphic via the module map $\mathscr{R}_{1|\mathscr{S}_1}$.*

The jet construction gives rise to a family of non-negative definite kernels $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$, $k \geq 0$. In case $k = 0$, it is the point-wise product $K_1 K_2$. The next two results are generalization of Theorem 1.1.8 for all kernels of the form $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$.

Theorem. 2.3.14. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. If $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ are two sharp kernels, then so is the kernel $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$, $k \geq 0$.*

Theorem. 2.3.16. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. If $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ are generalized Bergman kernels, then so is the kernel $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$, $k \geq 0$.*

In Chapter 3, we discuss the generalized Bergman metrics and the generalized Wallach set.

The notion of a generalized Bergman metric was introduced in [26], however, it has a different meaning in what follows. Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and let B_Ω denote the

Bergman kernel of Ω. Assume that a continuous branch of logarithm of B_Ω exists. Then B_Ω^t is well defined for all $t \in \mathbb{R}$. A function of the form $\left(B_\Omega^t(w,w)\partial_i\bar\partial_j \log B_\Omega(w,w)\right)_{i,j=1}^m$, $t \in \mathbb{R}$, $w \in \Omega$, is said to be a generalized Bergman metric. Note that a generalized Bergman metric is non-negative definite at each $w \in \Omega$ and for all $t \in \mathbb{R}$.

The ordinary Wallach set associated with the Bergman kernel of a bounded symmetric domain Ω is the set $\{t > 0 : B_\Omega^t$ is non-negative definite$\}$. It has been determined explicitly in this case, see [30]. Replacing the Bergman kernel in the definition of the Wallach set by an arbitrary non-negative definite kernel, we define the ordinary Wallach set $\mathscr{W}_\Omega(K)$ for K to be the set

$$\{t > 0 : K^t \text{ is non-negative definite}\}.$$

More importantly, we introduce the generalized Wallach set for any kernel K as follows:

$$G\mathscr{W}_\Omega(K) := \left\{t \in \mathbb{R} :\ K^{t-2}\mathbb{K} \text{ is non-negative definite}\right\}, \tag{1.9}$$

where we have assumed that K^t is well defined for all $t \in \mathbb{R}$ and $\mathbb{K}$ is the function defined in (1.6). In the particular case of the Euclidean unit ball in $\mathbb{C}^m$ and the Bergman kernel, the generalized Wallach set $G\mathscr{W}_{\mathbb{B}_m}(B_{\mathbb{B}_m})$, $m > 1$, is shown to be the set $\{t \in \mathbb{R} :\ t \geq 0\}$. If $m = 1$, then it is the set $\{t \in \mathbb{R} :\ t \geq -1\}$.

Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $\mathrm{Aut}(\Omega)$ denote the group of all biholomorphic automorphisms of Ω. Let $J : \mathrm{Aut}(\Omega) \times \Omega \to \mathrm{GL_k}(\mathbb{C})$ be a function such that $J(\varphi, \cdot)$ is holomorphic for each φ in $\mathrm{Aut}(\Omega)$, where $GL_k(\mathbb{C})$ is the set of all invertible matrices in $\mathscr{M}_k(\mathbb{C})$.

A non-negative definite kernel $K : \Omega \times \Omega \to \mathscr{M}_k(\mathbb{C})$ is said to be quasi-invariant with respect to J if K satisfies the following transformation rule:

$$J(\varphi, z)K(\varphi(z), \varphi(w))J(\varphi, w)^* = K(z, w),\ z, w \in \Omega,\ \varphi \in \mathrm{Aut}(\Omega). \tag{1.10}$$

In this chapter, we show that if $K : \Omega \times \Omega \to \mathscr{M}_k(\mathbb{C})$ is a quasi-invariant kernel, then $K^{t-2}\mathbb{K}$ is also a quasi-invariant kernel whenever t is in $G\mathscr{W}_\Omega(K)$.

In Chapter 4, we study weakly homogeneous operators.

It is shown that if $K : \mathbb{D} \times \mathbb{D} \to \mathscr{M}_k(\mathbb{C})$ is a sharp kernel, then the following conditions are equivalent.

(i) The multiplication operator M_z on $(\mathscr{H}, K)$ is weakly homogeneous.

(ii) For each $\varphi \in$ Möb, there exists a $g_\varphi \in \mathrm{Hol}(\mathbb{D}, \mathrm{GL_k}(\mathbb{C}))$ such that the weighted composition operator $M_{g_\varphi}C_{\varphi^{-1}}$ on $(\mathscr{H}, K)$ is bounded and invertible.

Thus, if K is a sharp kernel such that multiplication operator M_z on $(\mathscr{H}, K)$ is not weakly homogeneous, then there exists a $\varphi \in$ Möb such that the weighted composition operator $M_{g_\varphi}C_{\varphi^{-1}}$ on $(\mathscr{H}, K)$ is not simultaneously bounded and invertible for any choice of g_φ in

$\mathrm{Hol}(\mathbb{D}, \mathrm{GL_k}(\mathbb{C}))$. In particular, there must exist a $\varphi \in$ Möb such that the composition operator C_φ is not bounded.

Although there are examples, see [41, Theorem (1.1)′] and [43, Theorem 3.3]), of scalar valued sharp kernels K such that the composition operators C_φ, $\varphi \in$ Möb, are not bounded on $(\mathscr{H}, K)$, it does not necessarily follow that the multiplication operator M_z on $(\mathscr{H}, K)$ fails to be weakly homogeneous. In many other examples excluding the ones in [41] and [43], the operator C_φ is bounded for all φ in Möb showing that the corresponding multiplication operator M_z is weakly homogeneous. While the question of the existence of an operator M_z which is not weakly homogeneous on a Hilbert space $(\mathscr{H}, K)$, where K is a scalar valued sharp kernel, remains unanswered, in this chapter, we find such examples where the kernel K takes values in $\mathscr{M}_2(\mathbb{C})$. Indeed, the theorem given below provides many examples and nonexamples of weakly homogeneous operators in the class $\mathscr{F}B_2(\mathbb{D}) \subset B_2(\mathbb{D})$, see [37].

Theorem. 4.3.16. *Let $1 \le \lambda \le \mu < \lambda + 2$ and ψ be a non-zero function in $C(\bar{\mathbb{D}}) \cap \mathrm{Hol}(\mathbb{D})$. Then the operator $T = \begin{pmatrix} M_z^* & M_\psi^* \\ 0 & M_z^* \end{pmatrix}$ on $\mathscr{H}^{(\lambda)} \oplus \mathscr{H}^{(\mu)}$ is weakly homogeneous if and only if ψ is non-vanishing on $\bar{\mathbb{D}}$.*

In this chapter, we also study Möbius bounded operators. Some necessary conditions for a weighted shift to be Möbius bounded are obtained. As a result, it is shown that the Dirichlet shift, which has the weight sequence $\{(\frac{n+2}{n+1})^{\frac{1}{2}}\}_{n \in \mathbb{Z}_+}$, is not Möbius bounded. In the class of quasi-homogeneous operators, recently introduced by Ji, Jiang and Misra, see [38], the Möbius bounded operators have been identified.

Theorem. 4.4.11. *A quasi-homogeneous operator T is Möbius bounded if and only if $\Lambda(T) \ge 2$.*

As a consequence of this theorem, it is shown that the Shields' conjecture has an affirmative answer for the class of quasi homogeneous operators. Finally, we show that there exists a Möbius bounded weakly homogeneous operator which is not not similar to any homogeneous operator. This answers Question 1.1.9 in the negative.

Theorem. 4.5.3. *Let $K(z,w) = \sum_{n=0}^\infty a_n (z\bar{w})^n$, $z, w \in \mathbb{D}$, be a positive definite kernel such that for each $\gamma \in \mathbb{R}$, $\lim_{|z| \to 1}(1-|z|^2)^\gamma K(z,z)$ is either 0 or ∞. Assume that the adjoint M_z^* of the multiplication operator by the coordinate function z on $(\mathscr{H}, K)$ is in $B_1(\mathbb{D})$ and is weakly homogeneous. Then the multiplication operator M_z on $(\mathscr{H}, KK^{(\lambda)})$, $\lambda > 0$, is a Möbius bounded weakly homogeneous operator which is not similar to any homogeneous operator.*

In Chapter 5, we discuss the subnormality of the multiplication operator on the Hilbert space determined by the sum of two positive definite kernels. It is shown that if K_1 and K_2 are two positive definite kernels on $\mathbb{D} \times \mathbb{D}$ such that the multiplication operators M_z on the corresponding Hilbert spaces are subnormal, then the multiplication operator M_z on the Hilbert space determined by the sum $K_1 + K_2$ need not be subnormal. This settles the Conjecture 1.1.11 of Gregory T. Adams, Nathan S. Feldman and Paul J. McGuire in the negative. We also discuss some cases for which the answer to this conjecture is affirmative.

Chapter 2

Decomposition of the tensor product of two Hilbert modules

Given a pair of positive real numbers α, β and a sesqui-analytic function K on a bounded domain $\Omega \subset \mathbb{C}^m$, in this chapter, we investigate the properties of the sesqui-analytic function $\mathbb{K}^{(\alpha,\beta)} := K^{\alpha+\beta}\big(\partial_i \bar{\partial}_j \log K\big)_{i,j=1}^m$, taking values in $m \times m$ matrices. One of the key findings is that $\mathbb{K}^{(\alpha,\beta)}$ is non-negative definite whenever K^α and K^β are non-negative definite. In this case, a realization of the Hilbert space determined by the kernel $\mathbb{K}^{(\alpha,\beta)}$ is obtained. Let $\mathscr{M}_i$, $i = 1,2$, be two Hilbert modules over the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$. The polynomial ring $\mathbb{C}[z_1, \ldots, z_{2m}]$ then naturally acts on the tensor product $\mathscr{M}_1 \otimes \mathscr{M}_2$. The restriction of this action to the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ obtained using the restriction map $p \mapsto p_{|\Delta}$ leads to a natural decomposition of the tensor product $\mathscr{M}_1 \otimes \mathscr{M}_2$, which is investigated in this chapter. Two of the initial pieces in this decomposition are identified. The first one is the push-forward of the module corresponding to the non-negative definite kernel $K^{\alpha+\beta}$ while the second one is the push-forward of the Hilbert module determined by the kernel $\mathbb{K}^{(\alpha,\beta)}$. In the section 2.3, a class of matrix valued kernels arising out of the jet construction are studied and are shown to be generalized Bergman kernels. Various other properties, which is preserved by the kernel $\mathbb{K}^{(\alpha,\beta)}$, is also discussed.

2.1 A new non-negative definite kernel

The following lemma is undoubtedly well-known, however, we provide the easy proof here.

Lemma 2.1.1. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $\mathscr{H}$ be a Hilbert space. If $\phi_1, \phi_2, \ldots, \phi_k$ are anti-holomorphic functions from Ω into $\mathscr{H}$, then the function $K : \Omega \times \Omega \to \mathscr{M}_k(\mathbb{C})$ defined by $K(z,w) = \big(\langle \phi_j(w), \phi_i(z) \rangle_{\mathscr{H}}\big)_{i,j=1}^k$, $z, w \in \Omega$, is a sesqui-analytic non-negative definite kernel.*

Proof. Let $z_1, z_2, \ldots, z_n$ be n arbitrary points in Ω and $\eta_1, \eta_2, \ldots, \eta_n$ be n arbitrary vectors in $\mathbb{C}^k$, where $\eta_p = (\eta_{p,1}, \eta_{p,2}, \ldots, \eta_{p,k})$, $\eta_{p,j} \in \mathbb{C}$. Then

$$\begin{aligned}\sum_{p,q=1}^{n} \langle K(z_p, z_q)\eta_q, \eta_p\rangle_{\mathbb{C}^k} &= \sum_{p,q=1}^{n} \sum_{i,j=1}^{k} \langle \phi_j(z_q), \phi_i(z_p)\rangle_{\mathscr{H}} \eta_{q,j} \overline{\eta_{p,i}} \\ &= \sum_{p,q=1}^{n} \sum_{i,j=1}^{k} \langle \eta_{q,j}\phi_j(z_q), \eta_{p,i}\phi_i(z_p)\rangle_{\mathscr{H}} \\ &= \Big\| \sum_{p=1}^{n} \sum_{i=1}^{k} \eta_{p,i}\phi_i(z_p) \Big\|_{\mathscr{H}}^2 \geq 0,\end{aligned}$$

proving that K is non-negative definite. Also since $\phi_1, \phi_2, \ldots, \phi_k$ are anti-holomorphic, it follows easily that K is sesqui-analytic. □

Remark 2.1.2. *It is clear from the proof that if the vectors $\phi_1(z), \phi_2(z), \ldots, \phi_k(z)$ are linearly independent in $\mathscr{H}$, then the matrix $K(z,z)$ is positive definite.*

For any reproducing kernel Hilbert space $(\mathscr{H}, K)$ the following proposition from [26, Lemma 4.1] is a basic tool in what follows. While the proof is not difficult, we provide the details for the sake of completeness.

In what follows, the symbol $\frac{\partial}{\partial \bar{w}_j}$ denotes differentiation with respect to the complex conjugate of the variable w_j. We will often write $\bar{\partial}_j$ instead of $\frac{\partial}{\partial \bar{w}_j}$. Also, for any non-negative definite kernel $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ and $\eta \in \mathbb{C}^k$, let $\bar{\partial}^{\boldsymbol{i}} K(\cdot, w)\eta$ denote the function $\big(\frac{\partial}{\partial \bar{w}_1}\big)^{i_1} \cdots \big(\frac{\partial}{\partial \bar{w}_m}\big)^{i_m} K(\cdot, w)\eta$ and $(\partial^{\boldsymbol{i}} f)(z)$ be the function $\big(\frac{\partial}{\partial z_1}\big)^{i_1} \cdots \big(\frac{\partial}{\partial z_m}\big)^{i_m} f(z)$, $\boldsymbol{i} = (i_1, \ldots, i_m) \in \mathbb{Z}_+^m$.

Proposition 2.1.3. *Let $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ be a non-negative definite kernel. For every $\boldsymbol{i} \in \mathbb{Z}_+^m$, $\eta \in \mathbb{C}^k$ and $w \in \Omega$, we have*

(i) *$\bar{\partial}^{\boldsymbol{i}} K(\cdot, w)\eta$ is in $(\mathscr{H}, K)$,*

(ii) *$\langle f, \bar{\partial}^{\boldsymbol{i}} K(\cdot, w)\eta\rangle_{(\mathscr{H},K)} = \langle (\partial^{\boldsymbol{i}} f)(w), \eta\rangle_{\mathbb{C}^k}$, $f \in (\mathscr{H}, K)$.*

Proof. For any $1 \leq j \leq m$, we prove that the function $\bar{\partial}_j K(\cdot, w)\eta$ belongs to $(\mathscr{H}, K)$. Then the proof, by induction, showing that $\bar{\partial}^{\boldsymbol{i}} K(\cdot, w)\eta$ is in $(\mathscr{H}, K)$ for any $\boldsymbol{i} \in \mathbb{Z}_+^m$ is omitted.

First, choose a sequence $\{h_n\}_{n \in \mathbb{Z}_+}$ of complex numbers such that $w + h_n e_j \in \Omega$ and $h_n \to 0$, where e_j is the jth standard basis vector of $\mathbb{C}^m$. Define

$$S(h_n) = \frac{K(\cdot, w + h_n e_j)\eta - K(\cdot, w)\eta}{\bar{h}_n}, n \in \mathbb{Z}_+.$$

Since $S(h_n)$ belongs to $(\mathscr{H}, K)$ and f is holomorphic, it follows that

$$\lim_{n\to\infty} \langle f, S(h_n)\rangle_{(\mathscr{H},K)} = \lim_{n\to\infty} \left\langle \frac{f(w + h_n e_j) - f(w)}{h_n}, \eta \right\rangle_{\mathbb{C}^k} = \langle \partial_j f(w), \eta\rangle_{\mathbb{C}^k}, \tag{2.1}$$

for all f in $(\mathcal{H}, K)$. Therefore the sequence $\{S(h_n)\}_{n\in\mathbb{Z}_+}$ is weakly bounded. Consequently, using the uniform boundedness principle, we conclude that $\{S(h_n)\}_{n\in\mathbb{Z}_+}$ is bounded. Hence

$$|\langle \partial_j f(w), \eta\rangle_{\mathbb{C}^k}| = \lim_{n\to\infty} |\langle f, S(h_n)\rangle_{(\mathcal{H},K)}| \le \sup_{n\in\mathbb{Z}_+} \|S(h_n)\| \|f\|.$$

Thus the linear functional $f \mapsto \langle \partial_j f(w), \eta\rangle$ is bounded on $(\mathcal{H}, K)$. By the Riesz representation theorem, there exists a vector $L_{w,\eta}$ in $(\mathcal{H}, K)$ such that $\langle \partial_j f(w), \eta\rangle_{\mathbb{C}^k} = \langle f, L_{w,\eta}\rangle_{(\mathcal{H},K)}$ for f in $(\mathcal{H}, K)$. From (2.1), we see that $\{S(h_n)\}_{n\in\mathbb{Z}_+}$ converges to $L_{w,\eta}$ weakly. Moreover, since

$$\begin{aligned}\langle L_{w,\eta}(z), \eta'\rangle_{\mathbb{C}^k} &= \langle L_{w,\eta}, K(\cdot, z)\eta'\rangle_{\mathbb{C}^k}\\ &= \lim_{n\to\infty} \langle S(h_n), K(\cdot, z)\eta'\rangle_{(\mathcal{H},K)}\\ &= \lim_{n\to\infty} \langle S(h_n)(z), \eta'\rangle_{\mathbb{C}^k}\\ &= \langle \bar{\partial}_j K(z, w)\eta, \eta'\rangle_{\mathbb{C}^k},\end{aligned}$$

for all $z \in \Omega$ and $\eta' \in \mathbb{C}^k$, it follows that $\bar{\partial}_j K(\cdot, w)\eta = L_{w,\eta}$. Hence $\bar{\partial}_j K(\cdot, w)\eta$ is in $(\mathcal{H}, K)$.

The proof of part (ii) is implicit in the proof of part (i) given above. □

Proposition 2.1.4. *Let Ω be a bounded domain in $\mathbb{C}^m$ and $K : \Omega\times\Omega \to \mathbb{C}$ be a sesqui-analytic function. Suppose that K^α and K^β, defined on $\Omega\times\Omega$, are non-negative definite for some $\alpha, \beta > 0$. Then the function*

$$K^{\alpha+\beta}(z, w)\Big(\big(\partial_i\bar{\partial}_j \log K\big)(z, w)\Big)_{i,j=1}^m, \; z, w \in \Omega,$$

is a non-negative definite kernel on $\Omega\times\Omega$ taking values in $\mathcal{M}_m(\mathbb{C})$.

Proof. For $1 \le i \le m$, set $\phi_i(z) = \beta\bar{\partial}_i K^\alpha(\cdot, z) \otimes K^\beta(\cdot, z) - \alpha K^\alpha(\cdot, z) \otimes \bar{\partial}_i K^\beta(\cdot, z)$. From Proposition 2.1.3, it follows that each ϕ_i is a function from Ω into the Hilbert space $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$. Then we have

$$\begin{aligned}\langle \phi_j(w), \phi_i(z)\rangle &= \beta^2\partial_i\bar{\partial}_j K^\alpha(z, w)K^\beta(z, w) + \alpha^2 K^\alpha(z, w)\partial_i\bar{\partial}_j K^\beta(z, w)\\ &\quad - \alpha\beta\big(\partial_i K^\alpha(z, w)\bar{\partial}_j K^\beta(z, w) + \bar{\partial}_j K^\alpha(z, w)\partial_i K^\beta(z, w)\big)\\ &= \beta^2\big(\alpha(\alpha-1)K^{\alpha+\beta-2}(z, w)\partial_i K(z, w)\bar{\partial}_j K(z, w) + \alpha K^{\alpha+\beta-1}(z, w)\partial_i\bar{\partial}_j K(z, w)\big)\\ &\quad + \alpha^2\big(\beta(\beta-1)K^{\alpha+\beta-2}(z, w)\partial_i K(z, w)\bar{\partial}_j K(z, w) + \beta K^{\alpha+\beta-1}(z, w)\partial_i\bar{\partial}_j K(z, w)\big)\\ &\qquad - 2\alpha^2\beta^2 K^{\alpha+\beta-2}(z, w)\partial_i K(z, w)\bar{\partial}_j K(z, w)\\ &= (\alpha^2\beta + \alpha\beta^2)K^{\alpha+\beta-2}(z, w)\big(K(z, w)\partial_i\bar{\partial}_j K(z, w) - \partial_i K(z, w)\bar{\partial}_j K(z, w)\big)\\ &= \alpha\beta(\alpha+\beta)K^{\alpha+\beta}(z, w)\partial_i\bar{\partial}_j \log K(z, w).\end{aligned}$$

An application of Lemma 2.1.1 now completes the proof. □

The particular case, when $\alpha = 1 = \beta$ occurs repeatedly in the following. We therefore record it separately as a corollary.

Corollary 2.1.5. *Let Ω be a bounded domain in $\mathbb{C}^m$. If $K : \Omega\times\Omega \to \mathbb{C}$ is a non-negative definite kernel, then*

$$K^2(z,w)\big(\big(\partial_i\bar\partial_j \log K\big)(z,w)\big)_{i,j=1}^m$$

is also a non-negative definite kernel, defined on $\Omega\times\Omega$, taking values in $\mathcal{M}_m(\mathbb{C})$.

A more substantial corollary is the following taken from [13]. Here we provide a slightly different proof. Recall that a non-negative definite kernel $K : \Omega\times\Omega \to \mathbb{C}$ is said to be *infinitely divisible* if for all $t > 0$, the function K^t is also non-negative definite.

Corollary 2.1.6. *Let Ω be a bounded domain in $\mathbb{C}^m$. Suppose that $K : \Omega\times\Omega \to \mathbb{C}$ is an infinitely divisible kernel. Then the function $\big(\big(\partial_i\bar\partial_j \log K\big)(z,w)\big)_{i,j=1}^m$ is a non-negative definite kernel taking values in $\mathcal{M}_m(\mathbb{C})$.*

Proof. For $t > 0$, the function $K^t(z,w)$ is non-negative definite by hypothesis. It follows, from Corollary 2.1.5, that $\big(K^{2t}\partial_i\bar\partial_j \log K^t(z,w)\big)_{i,j=1}^m$ is non-negative definite. Hence the function $\big(K^{2t}\partial_i\bar\partial_j \log K(z,w)\big)_{i,j=1}^m$ is non-negative definite for all $t > 0$. Taking the limit as $t \to 0$, we conclude that $\big(\partial_i\bar\partial_j \log K(z,w)\big)_{i,j=1}^m$ is non-negative definite since it is the point-wise limit of the non-negative definite kernels $\big(K^{2t}\partial_i\bar\partial_j \log K(z,w)\big)_{i,j=1}^m$. □

The kernel $K(z,w)^{\alpha+\beta}\big(\partial_i\bar\partial_j \log K(z,w)\big)_{i,j=1}^m$ is going to appear repeatedly in our study of the Hilbert module $\big(\mathcal{H},\big(K(z,w)^{\alpha+\beta}\partial_i\bar\partial_j \log K(z,w)\big)_{i,j=1}^m\big)$ in this chapter. We begin by setting up some helpful notations.

Notation 2.1.7. *Let $\mathbb{K}^{(\alpha,\beta)}$ denote the kernel $K^{\alpha+\beta}(z,w)\Big(\big(\partial_i\bar\partial_j \log K\big)(z,w)\Big)_{i,j=1}^m$. If $\alpha = 1 = \beta$, then we write $\mathbb{K}$ instead of $\mathbb{K}^{(1,1)}$.*

Remark 2.1.8. *It is known that even if K is a positive definite kernel, $\big(\big(\partial_i\bar\partial_j \log K\big)(z,w)\big)_{i,j=1}^m$ need not be a non-negative definite kernel. In fact, $\big(\big(\partial_i\bar\partial_j \log K\big)(z,w)\big)_{i,j=1}^m$ is non-negative definite if and only if K is infinitely divisible (see [13, Theorem 3.3]).*

Let $K : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ be the positive definite kernel given by $K(z,w) = 1+\sum_{i=1}^\infty a_i z^i \bar w^i$, $z, w \in \mathbb{D}$, $a_i > 0$. For any $t > 0$, a direct computation gives

$$\begin{aligned}\big(K^t\partial\bar\partial\log K\big)(z,w) &= \big(1+\sum_{i=1}^\infty a_i z^i\bar w^i\big)^t\,\partial\bar\partial\Big(\sum_{i=1}^\infty a_i z^i \bar w^i - \frac{(\sum_{i=1}^\infty a_i z^i\bar w^i)^2}{2}+\cdots\Big)\\ &= (1+t a_1 z\bar w+\cdots)(a_1+2(2a_2-a_1^2)z\bar w+\cdots)\\ &= a_1+(4a_2+(t-2)a_1^2)z\bar w+\cdots.\end{aligned}$$

Thus, if $t < 2$, one may choose $a_1, a_2 > 0$ such that $4a_2+(t-2)a_1^2 < 0$. Hence $\big(K^t\partial\bar\partial\log K\big)(z,w)$ cannot be a non-negative definite kernel. Therefore, in general, for $\big(\big(K^t\partial_i\bar\partial_j\log K\big)(z,w)\big)_{i,j=1}^m$ to be non-negative definite, it is necessary that $t \geq 2$.

Remark 2.1.9. *Let $\Omega \subset \mathbb{C}$ be open and $\rho : \Omega \to \mathbb{R}_+$ be a C^2-smooth function. The Gaussian curvature of the metric ρ is given by the formula*

$$\mathscr{K}_G(z,\rho) = -\frac{\big(\partial\bar{\partial}\log\rho\big)(z)}{\rho(z)^2}, z \in \Omega. \tag{2.2}$$

If $K : \Omega \times \Omega \to \mathbb{C}$ is a non-negative definite kernel with $K(z,z) > 0$, then the function $\frac{1}{K}$ defines a metric on Ω and its Gaussian curvature is given by the formula

$$\mathscr{K}_G(z,\tfrac{1}{K}) = K(z,z)^2\big(\partial\bar{\partial}\log K\big)(z,z),\ z \in \Omega.$$

2.1.1 Boundedness of the multiplication operator on $\big(\mathscr{H},\mathbb{K}\big)$

For a holomorphic function $f : \Omega \to \mathbb{C}$, the operator M_f of multiplication by f on the linear space $\mathrm{Hol}(\Omega,\mathbb{C}^k)$ is defined by the rule $M_f h = f\,h$, $h \in \mathrm{Hol}(\Omega,\mathbb{C}^k)$, where $(f\,h)(z) = f(z)h(z)$, $z \in \Omega$. The boundedness criterion for the multiplication operator M_f restricted to the Hilbert space $(\mathscr{H},K)$ is well-known for the case of positive definite kernels. In what follows, often we have to work with a kernel which is merely non-negative definite. A verification of the boundedness criterion is therefore given below assuming only that the kernel K is non-negative definite.

Lemma 2.1.10. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathscr{M}_k(\mathbb{C})$ be a non-negative definite kernel. Let $f : \Omega \to \mathbb{C}$ be an arbitrary holomorphic function. Then the operator M_f of multiplication by f is bounded on $(\mathscr{H},K)$ if and only if there exists a constant $c > 0$ such that $\big(c^2 - f(z)\overline{f(w)}\big)K(z,w)$ is non-negative definite on $\Omega \times \Omega$. In case M_f is bounded, $\|M_f\|$ is the infimum of all $c > 0$ such that $\big(c^2 - f(z)\overline{f(w)}\big)K(z,w)$ is non-negative definite.*

Proof. Suppose that the multiplication operator M_f is bounded on $(\mathscr{H},K)$. Then for any $h \in (\mathscr{H},K)$, $w \in \Omega, \eta \in \mathbb{C}^k$, we see that

$$\begin{aligned}\big\langle M_f h, K(\cdot,w)\eta\big\rangle &= \big\langle (M_f h)(w),\eta\big\rangle\\ &= \big\langle f(w)h(w),\eta\big\rangle\\ &= \Big\langle h(w),\overline{f(w)}\eta\Big\rangle\\ &= \Big\langle h,\overline{f(w)}K(\cdot,w)\eta\Big\rangle.\end{aligned}$$

Therefore

$$M_f^* K(\cdot,w)\eta = \overline{f(w)}K(\cdot,w)\eta,\ w \in \Omega, \eta \in \mathbb{C}^k. \tag{2.3}$$

Since M_f is bounded, for any points $z_1,\ldots,z_n$ in Ω and vectors $\eta_1,\ldots,\eta_n$ in $\mathbb{C}^k$, we have

$$\big\|M_f^*\big(\sum_{j=1}^n K(\cdot,z_j)\eta_j\big)\big\|^2 \le \big\|M_f^*\big\|^2\big\|\sum_{j=1}^n K(\cdot,z_j)\eta_j\big\|^2. \tag{2.4}$$

A straightforward computation using (2.3), shows that (2.4) is equivalent to

$$\sum_{j,l=1}^{n} f(z_l)\overline{f(z_j)}\left\langle K(z_l,z_j)\eta_j,\eta_l\right\rangle \le \|M_f\|^2 \sum_{j,l=1}^{n}\left\langle K(z_l,z_j)\eta_j,\eta_l\right\rangle.$$

Therefore we conclude that $\big(c^2 - f(z)\overline{f(w)}\big)K(z,w)$ is non-negative definite on $\Omega\times\Omega$ where $c = \|M_f\|$.

Conversely, suppose that there exists a constant $c>0$ such that $(c^2 - f(z)\overline{f(w)})K(z,w)$ is non-negative definite on $\Omega\times\Omega$. Let $\mathscr{H}_0$ be the linear subspace of $(\mathscr{H},K)$ spanned by the elements $K(\cdot,w)\eta,\ w\in\Omega,\eta\in\mathbb{C}^k$. Define an operator $\mathring{T}_f$ on $\mathscr{H}_0$ by the following formula:

$$\mathring{T}_f\Big(\sum_{i=1}^{n} K(\cdot,w_i)\eta_i\Big) = \sum_{i=1}^{n}\overline{f(w_i)}K(\cdot,w_i)\eta_i,$$

where $w_1,\ldots,w_n\in\Omega,\ \eta_1,\ldots,\eta_n\in\mathbb{C}^k,\ n\ge 1$. First, we show that the operator $\mathring{T}_f$ is well-defined on $\mathscr{H}_0$. From the assumption on K, it follows that

$$\begin{aligned}\Big\|\sum_{i=1}^{n}\overline{f(w_i)}K(\cdot,w_i)\eta_i\Big\|^2 &= \sum_{i,j=1}^{n}\Big\langle f(w_i)\overline{f(w_j)}K(w_i,w_j)\eta_j,\eta_i\Big\rangle\\ &\le c^2\sum_{i,j=1}^{n}\left\langle K(w_i,w_j)\eta_j,\eta_i\right\rangle\\ &= c^2\Big\|\sum_{i=1}^{n}K(\cdot,w_i)\eta_i\Big\|^2.\end{aligned}$$

Therefore, if $\sum_{i=1}^{n}K(\cdot,w_i)\eta_i = 0$ for some points $w_1,\ldots,w_n\in\Omega$ and vectors $\eta_1,\ldots,\eta_n\in\mathbb{C}^k$, then $\sum_{i=1}^{n}\overline{f(w_i)}K(\cdot,w_i)\eta_i$ must be 0. Consequently, the operator $\mathring{T}_f$ is well-defined on $\mathscr{H}_0$. It is also evident from the above computation that

$$\|\mathring{T}_f(h)\| \le c\|h\|,\ h\in\mathscr{H}_0.$$

Since $\mathscr{H}_0$ is a dense subspace of $(\mathscr{H},K)$, it follows that $\mathring{T}_f$ can be extended to a unique bounded linear operator T_f on $(\mathscr{H},K)$ and $\|T_f\|\le c$. Finally, note that for $w\in\Omega,\eta\in\mathbb{C}^k$ and $h\in(\mathscr{H},K)$,

$$\begin{aligned}\Big\langle (T_f^*h)(w),\eta\Big\rangle &= \Big\langle T_f^*h, K(\cdot,w)\eta\Big\rangle\\ &= \left\langle h, T_f(K(\cdot,w)\eta)\right\rangle\\ &= \left\langle f(w)h, K(\cdot,w)\eta\right\rangle\\ &= \left\langle f(w)h(w),\eta\right\rangle.\end{aligned}$$

Therefore, we conclude that $T_f^* = M_f$ and $\|M_f\|\le c$.

From the proof, it is also clear that if the operator M_f is bounded, then $\|M_f\|$ is the infimum of all $c>0$ such that $\big(c^2 - f(z)\overline{f(w)}\big)K(z,w)$ is non-negative definite on $\Omega\times\Omega$. □

The proof of the lemma stated below is not significantly different from the one we have just proved. However, it would be useful for some of our later arguments, exactly in the form given below.

Lemma 2.1.11. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ be a non-negative definite kernel. Then the operator M_{z_i} of multiplication by the ith coordinate function z_i is bounded on $(\mathcal{H}, K)$ for $i = 1, \ldots, m$, if and only if there exists a constant $c > 0$ such that $\big(c^2 - \langle z, w\rangle\big)K(z, w)$ is non-negative definite on $\Omega \times \Omega$.*

Proof. Suppose that the operator M_{z_i} is bounded on $(\mathcal{H}, K)$ for $i = 1, \ldots, m$. Then it follows that the operator $D_{\boldsymbol{M}_z^*} : (\mathcal{H}, K) \to (\mathcal{H}, K) \bigoplus \cdots \bigoplus (\mathcal{H}, K)$ taking h to $(M_{z_1}^* h \oplus \cdots \oplus M_{z_m}^* h)$ is also bounded. Now, set $c = \|D_{\boldsymbol{M}_z^*}\|$. For this c, following the proof of the first half of Lemma 2.1.10, we conclude that $\big(c^2 - \langle z, w\rangle\big)K(z, w)$ is non-negative definite.

Conversely, assume that there exists a constant $c > 0$ such that $(c^2 - \langle z, w\rangle)K(z, w)$ is non-negative definite on $\Omega \times \Omega$. Let $\mathcal{H}_0$ be the subspace of $(\mathcal{H}, K)$ spanned by the vectors $K(\cdot, w)\eta$, $w \in \Omega, \eta \in \mathbb{C}^k$. Let $\mathring{D}_T : \mathcal{H}_0 \to \mathcal{H}_0 \bigoplus \cdots \bigoplus \mathcal{H}_0$ be given by the formula:

$$\mathring{D}_T\Big(\sum_{i=1}^n K(\cdot, w_i)\eta_i\Big) = \sum_{i=1}^n \big(\bar{w}_{i,1}K(\cdot, w_i)\eta_i, \ldots, \bar{w}_{i,m}K(\cdot, w_i)\eta_i\big), \tag{2.5}$$

where $w_i = (w_{i,1}, \ldots, w_{i,m}) \in \Omega$ and $\eta_i \in \mathbb{C}^k$, $i = 1, \ldots, n,\ n \geq 1$. As in Lemma 2.1.10, the following computation

$$\begin{aligned}
&\Big\| \sum_{i=1}^m \big(\bar{w}_{i,1}K(\cdot, w_i)\eta_i, \ldots, \bar{w}_{i,m}K(\cdot, w_i)\eta_i\big)\Big\|^2 \\
&\quad= \sum_{i,j=1}^n \big\langle\big(\bar{w}_{i,1}K(\cdot, w_i)\eta_i, \ldots, \bar{w}_{i,m}K(\cdot, w_i)\eta_i\big), \big(\bar{w}_{j,1}K(\cdot, w_j)\eta_j, \ldots, \bar{w}_{j,m}K(\cdot, w_j)\eta_j\big)\big\rangle \\
&\quad= \sum_{i,j=1}^n \sum_{p=1}^m w_{j,p}\bar{w}_{i,p}\big\langle K(\cdot, w_i)\eta_i, K(\cdot, w_j)\eta_j\big\rangle \\
&\quad= \sum_{i,j=1}^n \big\langle w_j, w_i\big\rangle\big\langle K(w_j, w_i)\eta_i, \eta_j\big\rangle \\
&\quad\leq c^2 \sum_{i,j=1}^n \big\langle K(w_j, w_i)\eta_i, \eta_j\big\rangle \\
&\quad= c^2 \Big\| \sum_{i=1}^n K(\cdot, w_i)\eta_i\Big\|^2,
\end{aligned}$$

shows that $\mathring{D}_T$ is well defined and $\|\mathring{D}_T(h)\| \leq c\|h\|$, $h \in \mathcal{H}_0$. Consequently, for $1 \leq j \leq m$, the operator $\mathring{D}_{T_j}$ is well-defined and $\|\mathring{D}_{T_j}(h)\| \leq c\|h\|$, $h \in \mathcal{H}_0$, where

$$\mathring{D}_{T_j}\Big(\sum_{i=1}^n K(\cdot, w_i)\eta_i\Big) = \sum_{i=1}^n \bar{w}_{i,j}K(\cdot, w_i)\eta_i,\ \ w_i \in \Omega, \eta_i \in \mathbb{C}^k,\ i = 1, \ldots, n,\ n \geq 1.$$

Therefore each $\mathring{D}_{T_j}$ can be extended to a bounded linear operator D_{T_j} on $(\mathscr{H}, K)$ with $\|D_{T_j}\| \leq c$. Now a similar argument used towards the end of the proof of Lemma 2.1.10 shows that ${D_{T_j}}^* = M_{z_j}$ on $(\mathscr{H}, K)$, $j = 1, \ldots, m$, completing the proof. □

As we have pointed out, the distinction between the non-negative definite kernels and the positive definite ones is very significant. Indeed, as shown in [26, Lemma 3.6], it is interesting that if the operator $\boldsymbol{M}_z := (M_{z_1}, \ldots, M_{z_m})$ is bounded on $(\mathscr{H}, K)$ for some non-negative definite kernel K such that $K(z,z)$, $z \in \Omega$, is invertible, then K is positive definite. We give a proof of this statement which is different from the inductive proof of Curto and Salinas. First, let us recall a generalization of the Openheim inequality for the block Hadamard product of two block matrices.

Let $A = \big(A_{ij}\big)_{i,j=1}^n$, $B = \big(B_{ij}\big)_{i,j=1}^n$ be two $n \times n$ block matrices where each block is of size $k \times k$. The block Hadamard product $A \square B$ of A and B is defined by $A \square B = \big(A_{ij}B_{ij}\big)_{i,j=1}^n$, where $A_{ij}B_{ij}$ denotes the usual matrix product. If each block $A_{i,j}$ of A commutes with every block B_{pq} of B, then A and B are said to be block commuting. The statement in the lemma given below combines [34, Corollary 3.3] and [34, Proposition 3.8].

Lemma 2.1.12. *Let $A = \big(A_{ij}\big)_{i,j=1}^n$, $B = \big(B_{ij}\big)_{i,j=1}^n$ be two $n \times n$ block matrices where each block is of size $k \times k$. Suppose that A and B are non-negative definite and block commuting. Then the block Hadamard product $A \square B$ is non-negative definite and*

$$\det(A \square B) \geq \det A \Big(\prod_{i=1}^n \det B_{ii} \Big).$$

The addendum at the end of [26, Lemma 3.6] follows immediately from the following lemma.

Lemma 2.1.13. *Let X be an arbitrary set. Let $k_1 : X \times X \to \mathbb{C}$ be a positive definite kernel and $K_2 : X \times X \to \mathscr{M}_k(\mathbb{C})$ be a non-negative definite kernel. Suppose that $K_2(x,x)$ is invertible for all $x \in X$. Then the product $k_1 K_2$ is positive definite on $X \times X$.*

Proof. Let $x_1, \ldots, x_n$ be a set of n arbitrary points from X and let C be the $n \times n$ block matrix $\big(k_1(x_i,x_j)K_2(x_i,x_j)\big)_{i,j=1}^n$, which is of the form $A \square B$, where

$$A = \Big(k_1(x_i,x_j) I_k\Big)_{i,j=1}^n \quad \text{and} \quad B = \Big(K_2(x_i,x_j)\Big)_{i,j=1}^n.$$

Since K_2 is non-negative definite on $X \times X$, we have that B is non-negative definite. Furthermore, since k_1 is positive definite and

$$\Big(k_1(x_i,x_j) I_k\Big)_{i,j=1}^n = \Big(k_1(x_i,x_j)\Big)_{i,j=1}^n \bigotimes I_k, \tag{2.6}$$

it follows that A is positive definite. It is easily verified that A and B are block commuting. Hence by Lemma 2.1.12, we have that C is non-negative definite and

$$\det C \geq \det A \Big(\prod_{i=1}^{n} \det K_2(x_i, x_i) \Big). \tag{2.7}$$

From (2.6), we see that $\det A = \det\big(k_1(x_i, x_j)\big)_{i,j=1}^{n} > 0$. Also, by hypothesis, $\det K_2(x_i, x_i) > 0$ for $i = 1, \ldots, n$. Hence, from (2.7), it follows that C is positive definite, completing the proof of the lemma. □

Proposition 2.1.14. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ be a non-negative definite kernel. Suppose that $K(z,z)$ is invertible for all $z \in \Omega$ and the multiplication operator M_{z_i} on $(\mathcal{H}, K)$ is bounded for $i = 1, \ldots, m$. Then K is positive definite on $\Omega \times \Omega$.*

Proof. Since the m-tuple of operators $(M_{z_1}, \ldots, M_{z_m})$ on $(\mathcal{H}, K)$ is bounded, by Corollary 2.1.11, there exists a constant $c > 0$ such that $\hat{K}(z, w) := (c^2 - \langle z, w\rangle)K(z, w)$, $z, w \in \Omega$, is non-negative definite on $\Omega \times \Omega$. Therefore, using the positive definiteness of $K(z,z)$, we find that $\|z\|_2 \leq c$, $z \in \Omega$. Since Ω is an open subset of $\mathbb{C}$, it follows that $\|z\|_2 < c$, $z \in \Omega$. Thus $|\langle z, w\rangle| < c^2$ for $z, w \in \Omega$. Consequently, $(c^2 - \langle z, w\rangle)$ is non-vanishing on $\Omega \times \Omega$, and K can be written as the product

$$K(z, w) = \big(c^2 - \langle z, w\rangle\big)^{-1} \hat{K}(z, w), \;\; z, w \in \Omega.$$

Note that

$$(c^2 - \langle z, w\rangle)^{-1} = c^{-2}\left(1 - \left\langle \tfrac{z}{c}, \tfrac{w}{c} \right\rangle\right)^{-1}, \;\; z, w \in \Omega. \tag{2.8}$$

Since $(1 - \langle z, w\rangle)^{-1}$ is a positive definite kernel on $\mathbb{B}_m \times \mathbb{B}_m$ (where $\mathbb{B}_m$ is the Euclidean unit ball in $\mathbb{C}^m$) and $\|z\|_2 < c$ on Ω, by (2.8), we conclude that the function $\big(c^2 - \langle z, w\rangle\big)^{-1}$ is positive definite on $\Omega \times \Omega$. Also, since $K(z,z)$ is invertible, we see that $\hat{K}(z,z)$ is also invertible for all $z \in \Omega$. Hence, by Lemma 2.1.13, it follows that K is positive definite. □

Lemma 2.1.15. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a non-negative definite kernel. Let $f : \Omega \to \mathbb{C}$ be an arbitrary holomorphic function. Suppose that there exists a constant $c > 0$ such that $\big(c^2 - f(z)\overline{f(w)}\big)K(z, w)$ is non-negative definite on $\Omega \times \Omega$. Then the function $\big(c^2 - f(z)\overline{f(w)}\big)^2 \mathbb{K}(z, w)$ is non-negative definite on $\Omega \times \Omega$.*

Proof. Without loss of generality, we assume that f is non-constant and K is non-zero. The function $G(z, w) := \big(c^2 - f(z)\overline{f(w)}\big)K(z, w)$ is non-negative definite on $\Omega \times \Omega$ by hypothesis. We claim that $|f(z)| < c$ for all z in Ω. If not, then by the open mapping theorem, there exists an open set $\Omega_0 \subset \Omega$ such that $|f(z)| > c$, $z \in \Omega_0$. Since $\big(c^2 - |f(z)|^2\big)K(z,z) \geq 0$, it follows that $K(z,z) = 0$ for all $z \in \Omega_0$. Now, let h be an arbitrary vector in $(\mathcal{H}, K)$. Clearly, $|h(z)| = |\langle h, K(\cdot, z)\rangle| \leq \|h\|\|K(\cdot, z)\| = \|h\| K(z,z)^{\frac{1}{2}} = 0$ for all $z \in \Omega_0$. Consequently, $h(z) = 0$ on Ω_0. Since

Ω is connected and h is holomorphic, it follows that $h=0$. This contradicts the assumption that K is non-zero verifying the validity of our claim.

From the claim, we have that the function $c^2 - f(z)\overline{f(w)}$ is non-vanishing on $\Omega\times\Omega$. Therefore, the kernel K can be written as the product

$$K(z,w) = \frac{1}{\left(c^2 - f(z)\overline{f(w)}\right)} G(z,w),\ z,w\in\Omega.$$

Since $|f(z)| < c$ on Ω, the function $\frac{1}{\left(c^2 - f(z)\overline{f(w)}\right)}$ has a convergent power series expansion, namely,

$$\frac{1}{\left(c^2 - f(z)\overline{f(w)}\right)} = \sum_{n=0}^{\infty} \frac{1}{c^{2(n+1)}} f(z)^n \overline{f(w)}^n,\ \ z,w\in\Omega.$$

Therefore it defines a non-negative definite kernel on $\Omega\times\Omega$. Note that

$$\begin{aligned}
&\left(K(z,w)^2 \partial_i\bar\partial_j \log K(z,w)\right)_{i,j=1}^m \\
&= \left(K(z,w)^2 \partial_i\bar\partial_j \log \frac{1}{\left(c^2 - f(z)\overline{f(w)}\right)}\right)_{i,j=1}^m + \left(K(z,w)^2 \partial_i\bar\partial_j \log G(z,w)\right)_{i,j=1}^m \\
&= \frac{1}{\left(c^2 - f(z)\overline{f(w)}\right)^2}\left(K(z,w)^2\left(\partial_i f(z)\overline{\partial_j f(w)}\right)_{i,j=1}^m + G(z,w)^2\left(\partial_i\bar\partial_j \log G(z,w)\right)_{i,j=1}^m\right),
\end{aligned}$$

where for the second equality, we have used that

$$\partial_i\bar\partial_j \log \frac{1}{\left(c^2 - f(z)\overline{f(w)}\right)} = \frac{\partial_i f(z)\overline{\partial_j f(w)}}{\left(c^2 - f(z)\overline{f(w)}\right)^2},\ z,w\in\Omega,\ 1\le i,j\le m.$$

Thus

$$\begin{aligned}
&\left(c^2 - f(z)\overline{f(w)}\right)^2 \mathbb{K}(z,w) \\
&\qquad = K(z,w)^2\left(\partial_i f(z)\overline{\partial_j f(w)}\right)_{i,j=1}^m + \left(G(z,w)^2 \partial_i\bar\partial_j \log G(z,w)\right)_{i,j=1}^m.
\end{aligned} \tag{2.9}$$

By Lemma 2.1.1, the function $\left(\partial_i f(z)\overline{\partial_j f(w)}\right)_{i,j=1}^m$ is non-negative definite on $\Omega\times\Omega$. Thus the product $K(z,w)^2\left(\partial_i f(z)\overline{\partial_j f(w)}\right)_{i,j=1}^m$ is also non-negative definite on $\Omega\times\Omega$. Since G is non-negative definite on $\Omega\times\Omega$, by Corollary 2.1.5, the function $\left(G(z,w)^2\partial_i\bar\partial_j \log G(z,w)\right)_{i,j=1}^m$ is also non-negative definite on $\Omega\times\Omega$. The proof is now complete since the sum of two non-negative definite kernels remains non-negative definite. □

We use the lemma we have just proved in the proof of the following theorem giving a sufficient condition for the boundedness of the multiplication operator on the Hilbert space $(\mathscr{H},\mathbb{K})$.

Theorem 2.1.16. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a non-negative definite kernel. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Suppose that the multiplication operator M_f on $(\mathscr{H}, K)$ is bounded. Then the multiplication operator M_f is also bounded on $(\mathscr{H}, \mathbb{K})$.*

Proof. Since the operator M_f is bounded on $(\mathscr{H}, K)$, by Lemma 2.1.10, we find a constant $c > 0$ such that $\big(c^2 - f(z)\overline{f(w)}\big)K(z, w)$ is non-negative definite on $\Omega \times \Omega$. Then , by Lemma 2.1.15, it follows that $\big(c^2 - f(z)\overline{f(w)}\big)^2\mathbb{K}(z, w)$ is non-negative definite on $\Omega \times \Omega$. Also, from the proof of Lemma 2.1.15, we have that $\big(c^2 - f(z)\overline{f(w)}\big)^{-1}$ is non-negative definite on $\Omega \times \Omega$ (assuming that f is non-constant). Hence $\big(c - f(z)\overline{f(w)}\big)\mathbb{K}(z, w)$, being the product of two non-negative definite kernels, is non-negative definite on $\Omega \times \Omega$. An application of Lemma 2.1.10, a second time, completes the proof. □

The corollary given below provides a sufficient condition for the positive definiteness of the kernel $\mathbb{K}$.

Corollary 2.1.17. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a non-negative definite kernel satisfying $K(w, w) > 0$, $w \in \Omega$. Suppose that the multiplication operator M_{z_i} on $(\mathscr{H}, K)$ is bounded for $i = 1, \ldots, m$. Then the kernel $\mathbb{K}$ is positive definite on $\Omega \times \Omega$.*

Proof. By Corollary 2.1.5, we already have that $\mathbb{K}$ is non-negative definite. Moreover, since M_{z_i} on $(\mathscr{H}, K)$ is bounded for $i = 1, \ldots, m$, it follows from Theorem 2.1.16 that M_{z_i} is bounded on $(\mathscr{H}, \mathbb{K})$ also. Therefore, in view of Proposition 2.1.14, $\mathbb{K}$ is positive definite if $\mathbb{K}(w, w)$ is invertible for all $w \in \Omega$. To verify this, set

$$\phi_i(w) = \bar{\partial}_i K(\cdot, w) \otimes K(\cdot, w) - K(\cdot, w) \otimes \bar{\partial}_i K(\cdot, w),\ 1 \le i \le m.$$

From the proof of Proposition 2.1.4, we see that $\mathbb{K}(w, w) = \frac{1}{2}\big(\langle \phi_j(w), \phi_i(w)\rangle\big)_{i,j=1}^m$. Therefore, in view of Remark 2.1.2, $\mathbb{K}(w, w)$ is invertible if the vectors $\phi_1(w), \ldots, \phi_m(w)$ are linearly independent. Note that for $w = (w_1, \ldots, w_m)$ in Ω and $j = 1, \ldots, m$, we have $(M_{z_j} - w_j)^* K(\cdot, w) = 0$. Differentiating this equation with respect to $\bar{w}_i$, we obtain

$$(M_{z_j} - w_j)^* \bar{\partial}_i K(\cdot, w) = \delta_{ij} K(\cdot, w),\ 1 \le i, j \le m.$$

Thus

$$\big((M_{z_j} - w_j)^* \otimes I\big)\big(\phi_i(w)\big) = \delta_{ij} K(\cdot, w) \otimes K(\cdot, w),\ 1 \le i, j \le m. \tag{2.10}$$

Now assume that $\sum_{i=1}^m c_i \phi_i(w) = 0$ for some scalars $c_1, \ldots, c_m$. Then, for $1 \le j \le m$, we have that $\sum_{i=1}^m \big((M_{z_j} - w_j)^* \otimes I\big)\big(\phi_i(w)\big) = 0$. Thus, using (2.10), we see that $c_j K(\cdot, w) \otimes K(\cdot, w) = 0$. Since $K(w, w) > 0$, we conclude that $c_j = 0$. Hence the vectors $\phi_1(w), \ldots, \phi_m(w)$ are linearly independent. This completes the proof. □

Remark 2.1.18. *Recall that any operator T in $B_1(\mathbb{D})$ is unitarily equivalent to the adjoint M_z^* of the multiplication operator by the coordinate function z on some reproducing kernel Hilbert space $(\mathscr{H}, K) \subseteq \mathrm{Hol}(\mathbb{D})$. In particular, any contraction T in $B_1(\mathbb{D})$, modulo unitary equivalence, is of this form. For such a contractive operator M_z^* in $B_1(\mathbb{D})$, the curvature inequality [45, Corollary 1.2'] takes the form (see [13]):*

$$-\partial\bar{\partial}\log K(z,z) \leq -\partial\bar{\partial}\log \mathbb{S}_{\mathbb{D}}(z,z),\ z \in \mathbb{D}, \tag{2.11}$$

where $\mathbb{S}_{\mathbb{D}}(z,w) = \frac{1}{1-z\bar{w}}$, $z, w \in \mathbb{D}$, is the Szegö kernel of the unit disc $\mathbb{D}$. Since M_z^* on $(\mathscr{H}, K)$ is a contraction, by Lemma 2.1.10*, it follows that the function $G(z,w) := (1-z\bar{w})K(z,w)$ is non-negative definite on $\mathbb{D}\times\mathbb{D}$. Hence, from* (2.9)*, we have that*

$$\begin{aligned} &-G(z,w)^2\partial\bar{\partial}\log G(z,w) \\ &\qquad = (1-z\bar{w})^2K(z,w)^2\big(-\partial\bar{\partial}\log K(z,w) + \partial\bar{\partial}\log \mathbb{S}_{\mathbb{D}}(z,w)\big),\ z,w \in \mathbb{D}. \end{aligned}$$

Therefore, applying Corollary 2.1.5 *for $G(z,w)$, we obtain that*

$$(1-z\bar{w})^2K(z,w)^2\big(-\partial\bar{\partial}\log K(z,w) + \partial\bar{\partial}\log \mathbb{S}_{\mathbb{D}}(z,w)\big) \preceq 0. \tag{2.12}$$

In particular, evaluating (2.12) *at a fixed but arbitrary point, the inequality* (2.11) *is evident. However, for any contraction in $B_1(\mathbb{D})$,* (2.12) *gives a much stronger (curvature) inequality. Conversely, whether it is strong enough to force contractivity of the operator is not clear.*

Let Ω be a finitely connected bounded planar domain and $\mathrm{Rat}(\Omega)$ *be the ring of rational functions with poles off $\bar{\Omega}$. Let T be an operator in $B_1(\Omega)$ with $\sigma(T) = \bar{\Omega}$. Suppose that the homomorphism $q_T : \mathrm{Rat}(\Omega) \to B(\mathscr{H})$ given by*

$$q_T(f) = f(T),\ f \in \mathrm{Rat}(\Omega),$$

is contractive, that is, $\|f(T)\| \leq \|f\|_{\Omega,\infty}$, $f \in \mathrm{Rat}(\Omega)$. Setting $G_f(z,w) = (1-f(z)\overline{f(w)})K(z,w)$ and using (2.9)*, as before, we have*

$$\begin{aligned} 0 \preceq &G_f(z,w)^2\partial\bar{\partial}\log G_f(z,w) \\ = &G_f(z,w)^2\Big(-\tfrac{f'(z)\overline{f'(w)}}{(1-f(z)\overline{f(w)})^2} + \partial\bar{\partial}\log K(z,w)\Big) \\ = &-K(z,w)^2f'(z)\overline{f'(w)} + (1-f(z)\overline{f(w)})^2K(z,w)^2\partial\bar{\partial}\log K(z,w) \end{aligned}$$

for any rational function f with poles off $\bar{\Omega}$ and $|f(z)| \leq 1$, $z \in \Omega$. As in the case of the disc, in particular, evaluating this inequality at a fixed but arbitrary point $z \in \Omega$, we have

$$\partial\bar{\partial}\log K(z,z) \geq \sup\Big\{\tfrac{|f'(z)|^2}{(1-|f(z)|^2)^2} : f \in \mathrm{Rat}(\Omega), \|f\|_{\Omega,\infty} \leq 1\Big\} = \mathbb{S}_{\Omega}(z,z)^2,$$

where $\mathbb{S}_{\Omega}$ is the Szegö kernel of the domain Ω. This is the curvature inequality for contractive homomorphisms (cf. [45, Corollary 1.2']).

2.2 Realization of $\left(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)}\right)$

Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function. Suppose that the functions K^α and K^β are non-negative definite for some $\alpha, \beta > 0$. In this section, we give a description of the Hilbert space $\left(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)}\right)$. As before, we set

$$\phi_i(w) = \beta \bar{\partial}_i K^\alpha(\cdot, w) \otimes K^\beta(\cdot, w) - \alpha K^\alpha(\cdot, w) \otimes \bar{\partial}_i K^\beta(\cdot, w),\ 1 \le i \le m,\ w \in \Omega. \tag{2.13}$$

Let $\mathscr{N}$ be the subspace of $(\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta)$ which is the closed linear span of the vectors

$$\left\{\phi_i(w) : w \in \Omega, 1 \le i \le m\right\}.$$

From the definition of $\mathscr{N}$, it is not easy to determine which vectors are in it. A useful alternative description of the space $\mathscr{N}$ is given below.

Recall that $K^\alpha \otimes K^\beta$ is the reproducing kernel for the Hilbert space $(\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta)$, where the kernel $K^\alpha \otimes K^\beta$ on $(\Omega \times \Omega) \times (\Omega \times \Omega)$ is given by

$$K^\alpha \otimes K^\beta(z, \zeta; z', \zeta') = K^\alpha(z, z') K^\beta(\zeta, \zeta'),$$

$z = (z_1, \ldots, z_m)$, $\zeta = (\zeta_1, \ldots, \zeta_m)$, $z' = (z_{m+1}, \ldots, z_{2m})$, $\zeta' = (\zeta_{m+1}, \ldots, \zeta_{2m})$ are in Ω. We realize the Hilbert space $(\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta)$ as a space consisting of holomorphic functions on $\Omega \times \Omega$. Let $\mathscr{A}_0$ and $\mathscr{A}_1$ be the subspaces defined by

$$\mathscr{A}_0 = \left\{f \in (\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta) : f_{|\Delta} = 0\right\}$$

and

$$\mathscr{A}_1 = \left\{f \in (\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta) : f_{|\Delta} = (\partial_{m+1} f)_{|\Delta} = \cdots = (\partial_{2m} f)_{|\Delta} = 0\right\},$$

where Δ is the diagonal set $\{(z, z) \in \Omega \times \Omega : z \in \Omega\}$, $\partial_i f$ is the derivative of f with respect to the ith variable, and $f_{|\Delta}$, $(\partial_i f)_{|\Delta}$ denote the restrictions to the set Δ of the functions f, $\partial_i f$, respectively. It is easy to see that both $\mathscr{A}_0$ and $\mathscr{A}_1$ are closed subspaces of the Hilbert space $(\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta)$ and $\mathscr{A}_1$ is a closed subspace of $\mathscr{A}_0$.

Now observe that, for $1 \le i \le m$, we have

$$\begin{aligned} \bar{\partial}_i (K^\alpha \otimes K^\beta)(\cdot, (z', \zeta')) &= \bar{\partial}_i K^\alpha(\cdot, z') \otimes K^\beta(\cdot, \zeta'),\ z', \zeta' \in \Omega \\ \bar{\partial}_{m+i} (K^\alpha \otimes K^\beta)(\cdot, (z', \zeta')) &= K^\alpha(\cdot, z') \otimes \bar{\partial}_i K^\beta(\cdot, \zeta'),\ z', \zeta' \in \Omega. \end{aligned} \tag{2.14}$$

Hence, taking $z' = \zeta' = w$, we see that

$$\phi_i(w) = \beta \bar{\partial}_i (K^\alpha \otimes K^\beta)(\cdot, (w, w)) - \alpha \bar{\partial}_{m+i} (K^\alpha \otimes K^\beta)(\cdot, (w, w)). \tag{2.15}$$

We now prove a useful lemma on the Taylor coefficients of an analytic function.

Lemma 2.2.1. *Suppose that* $f : \Omega \times \Omega \to \mathbb{C}$ *is a holomorphic function satisfying* $f_{|\Delta} = 0$. *Then*

$$(\partial_i f)_{|\Delta} + (\partial_{m+i} f)_{|\Delta} = 0, \quad 1 \le i \le m.$$

Proof. Recall that the map $\iota : \Omega \to \Omega \times \Omega$ is defined by $\iota(z) = (z, z)$, $z \in \Omega$. Let $g(z) = (f \circ \iota)(z)$, $z \in \Omega$. Clearly, the condition $f_{|\Delta} = 0$ is equivalent to saying that g is identically zero on Ω. Thus, if $f_{|\Delta} = 0$, then it follows that $\partial_i g(z) = 0$ on Ω, $1 \le i \le m$. Setting $\iota_j(z) = \iota_{m+j}(z) = z_j$, $z \in \Omega$, $1 \le j \le m$, and applying the chain rule (cf. [48, page 8]), we obtain

$$\begin{aligned}\partial_i g(z) &= \sum_{j=1}^{2m} (\partial_j f)(\iota(z)) \partial_i \iota_j(z) + \sum_{j=1}^{2m} (\bar{\partial}_j f)(\iota(z)) \partial_i \bar{\iota}_j(z)\\ &= \sum_{j=1}^{2m} (\partial_j f)(\iota(z)) \partial_i \iota_j(z)\\ &= (\partial_i f)(z, z) + (\partial_{m+i} f)(z, z), \;\; z \in \Omega.\end{aligned}$$

This completes the proof. □

An alternative description of the subspace $\mathcal{N}$ of $(\mathcal{H}, K^\alpha) \otimes (\mathcal{H}, K^\beta)$ is provided below.

Proposition 2.2.2. $\mathcal{N} = \mathcal{A}_0 \ominus \mathcal{A}_1$.

Proof. For all $z \in \Omega$, we see that

$$\phi_i(w)(z, z) = \alpha\beta K^{\alpha+\beta-1}(z, w) \bar{\partial}_i K(z, w) - \alpha\beta K^{\alpha+\beta-1}(z, w) \bar{\partial}_i K(z, w) = 0.$$

Hence each $\phi_i(w)$, $w \in \Omega, 1 \le i \le m$, belongs to $\mathcal{A}_0$ and consequently, $\mathcal{N} \subset \mathcal{A}_0$. Therefore, to complete the proof of the proposition, it is enough to show that $\mathcal{A}_0 \ominus \mathcal{N} = \mathcal{A}_1$.

To verify this, note that $f \in \mathcal{N}^\perp$ if and only if $\langle f, \phi_i(w) \rangle = 0$, $1 \le i \le m$, $w \in \Omega$. Now, in view of (2.15) and Proposition 2.1.3, we have that

$$\begin{aligned}\langle f, \phi_i(w) \rangle &= \left\langle f, \beta \bar{\partial}_i (K^\alpha \otimes K^\beta)(\cdot, (w, w)) - \alpha \bar{\partial}_{m+i} (K^\alpha \otimes K^\beta)(\cdot, (w, w)) \right\rangle\\ &= \beta (\partial_i f)(w, w) - \alpha (\partial_{m+i} f)(w, w), \; 1 \le i \le m, \; w \in \Omega.\end{aligned} \tag{2.16}$$

Thus $f \in \mathcal{N}^\perp$ if and only if the function $\beta\, (\partial_i f)_{|\Delta} - \alpha\, (\partial_{m+i} f)_{|\Delta} = 0$, $1 \le i \le m$. Combining this with Lemma 2.2.1, we see that any $f \in \mathcal{A}_0 \ominus \mathcal{N}$, satisfies

$$\begin{aligned}&\beta (\partial_i f)_{|\Delta} - \alpha (\partial_{m+i} f)_{|\Delta} = 0,\\ &(\partial_i f)_{|\Delta} + (\partial_{m+i} f)_{|\Delta} = 0,\end{aligned}$$

for $1 \le i \le m$. Therefore, we have $(\partial_i f)_{|\Delta} = (\partial_{m+i} f)_{|\Delta} = 0$, $1 \le i \le m$. Hence f belongs to $\mathcal{A}_1$.

Conversely, let $f \in \mathscr{A}_1$. In particular, $f \in \mathscr{A}_0$. Hence invoking Lemma 2.2.1 once again, we see that

$$(\partial_i f)_{|\Delta} + (\partial_{m+i} f)_{|\Delta} = 0,\ 1 \le i \le m.$$

Since f is in $\mathscr{A}_1$, $(\partial_{m+i} f)_{|\Delta} = 0$, $1 \le i \le m$, by definition. Therefore, $(\partial_i f)_{|\Delta} = (\partial_{m+i} f)_{|\Delta} = 0,\ 1 \le i \le m$, which implies

$$\beta(\partial_i f)_{|\Delta} - \alpha(\partial_{m+i} f)_{|\Delta} = 0,\ 1 \le i \le m.$$

Hence $f \in \mathscr{A}_0 \ominus \mathscr{N}$, completing the proof. □

We now give a description of the Hilbert space $\left(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)}\right)$. For this, first define a linear map $\mathscr{R}_1 : (\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta) \to \mathrm{Hol}(\Omega, \mathbb{C}^m)$ by

$$\mathscr{R}_1(f) = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \begin{pmatrix} (\beta\partial_1 f - \alpha\partial_{m+1} f)_{|\Delta} \\ \vdots \\ (\beta\partial_m f - \alpha\partial_{2m} f)_{|\Delta} \end{pmatrix},\ f \in (\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta). \tag{2.17}$$

Note that

$$\mathscr{R}_1(f)(w) = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \begin{pmatrix} \left\langle f, \phi_1(w) \right\rangle \\ \vdots \\ \left\langle f, \phi_m(w) \right\rangle \end{pmatrix},\ w \in \Omega,\ f \in (\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta). \tag{2.18}$$

From the above equality, it is easy to see that $\ker \mathscr{R}_1 = \mathscr{N}^\perp$. Since $\mathscr{N} = \mathscr{A}_0 \ominus \mathscr{A}_1$ from Proposition 2.2.2, it follows that $\ker \mathscr{R}_1^\perp = \mathscr{A}_0 \ominus \mathscr{A}_1$. Therefore, the map $\mathscr{R}_{1|\mathscr{A}_0 \ominus \mathscr{A}_1} \to \mathrm{ran}\ \mathscr{R}_1$ is one-to-one and onto. Require this map to be a unitary by defining an appropriate inner product on ran $\mathscr{R}_1$, that is, define

$$\left\langle \mathscr{R}_1(f), \mathscr{R}_1(g) \right\rangle := \left\langle P_{\mathscr{A}_0 \ominus \mathscr{A}_1} f, P_{\mathscr{A}_0 \ominus \mathscr{A}_1} g \right\rangle,\ f, g \in (\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta), \tag{2.19}$$

where $P_{\mathscr{A}_0 \ominus \mathscr{A}_1}$ is the orthogonal projection of $(\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta)$ onto the subspace $\mathscr{A}_0 \ominus \mathscr{A}_1$.

Theorem 2.2.3. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function. Suppose that the functions K^α and K^β are non-negative definite for some $\alpha, \beta > 0$. Let $\mathscr{R}_1$ be the map defined by* (2.17)*. Then the Hilbert space determined by the non-negative definite kernel $\mathbb{K}^{(\alpha,\beta)}$ coincides with the space* ran $\mathscr{R}_1$ *and the inner product given by* (2.19) *on* ran $\mathscr{R}_1$ *agrees with the one induced by the kernel $\mathbb{K}^{(\alpha,\beta)}$.*

Proof. Let $\{e_1, \ldots, e_m\}$ be the standard orthonormal basis of $\mathbb{C}^m$. For $1 \le i, j \le m$, from the proof of Proposition 2.1.4, we have

$$\left\langle \phi_j(w), \phi_i(z) \right\rangle = \alpha\beta(\alpha+\beta) K^{\alpha+\beta}(z,w) \partial_i \bar{\partial}_j \log K(z,w) \tag{2.20}$$

$$= \alpha\beta(\alpha+\beta) \left\langle \mathbb{K}^{(\alpha,\beta)}(z,w) e_j, e_i \right\rangle_{\mathbb{C}^m},\ z, w \in \Omega. \tag{2.21}$$

Therefore, from (2.18), it follows that for all $w \in \Omega$ and $1 \le j \le m$,

$$\mathscr{R}_1(\phi_j(w)) = \sqrt{\alpha\beta(\alpha+\beta)}\mathbb{K}^{(\alpha,\beta)}(\cdot, w)e_j.$$

Hence, for all $w \in \Omega$ and $\eta \in \mathbb{C}^m$, $\mathbb{K}^{(\alpha,\beta)}(\cdot, w)\eta$ belongs to ran $\mathscr{R}_1$. Let $\mathscr{R}_1(f)$ be an arbitrary element in ran $\mathscr{R}_1$ where $f \in \mathscr{A}_0 \ominus \mathscr{A}_1$. Then

$$\begin{aligned}\left\langle \mathscr{R}_1(f), \mathbb{K}^{(\alpha,\beta)}(\cdot, w)e_j \right\rangle &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \left\langle \mathscr{R}_1(f), \mathscr{R}_1(\phi_j(w)) \right\rangle \\ &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \left\langle f, \phi_j(w) \right\rangle \\ &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} (\beta\partial_j f(w,w) - \alpha\partial_{m+j} f(w,w)) \\ &= \left\langle \mathscr{R}_1(f)(w), e_j \right\rangle_{\mathbb{C}^m},\end{aligned}$$

where the second equality follows since both f and $\phi_j(w)$ belong to $\mathscr{A}_0 \ominus \mathscr{A}_1$.

This completes the proof. □

Let $\boldsymbol{z} = (z_1, \ldots, z_m)$ and let $\mathbb{C}[\boldsymbol{z}] := \mathbb{C}[z_1, \ldots, z_m]$ denote the ring of polynomials in m-variables. The following proposition gives a sufficient condition for density of $\mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}^m$ in the Hilbert space $\left(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)}\right)$.

Proposition 2.2.4. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are non-negative definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. Suppose that both the Hilbert spaces $(\mathscr{H}, K^\alpha)$ and $(\mathscr{H}, K^\beta)$ contain the polynomial ring $\mathbb{C}[\boldsymbol{z}]$ as a dense subset. Then the Hilbert space $\left(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)}\right)$ contains the ring $\mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}^m$ as a dense subset.*

Proof. Since $C[\boldsymbol{z}]$ is dense in both the Hilbert spaces $(\mathscr{H}, K^\alpha)$ and $(\mathscr{H}, K^\beta)$, it follows that $C[\boldsymbol{z}] \otimes C[\boldsymbol{z}]$, which is $\mathbb{C}[z_1, \ldots, z_{2m}]$, is contained in the Hilbert space $(\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta)$ and is dense in it. Since $\mathscr{R}_1$ maps $(\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta)$ onto $\left(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)}\right)$, to complete the proof, it suffices to show that $\mathscr{R}_1(\mathbb{C}[z_1, \ldots, z_{2m}]) = \mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}^m$. It is easy to see that $\mathscr{R}_1(\mathbb{C}[z_1, \ldots, z_{2m}]) \subseteq \mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}^m$. Conversely, if $\sum_{i=1}^m p_i(z_1, \ldots, z_m) \otimes e_i$ is an arbitrary element of $\mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}^m$, then it is easily verified that the function $p(z_1, \ldots, z_{2m}) := \sqrt{\frac{\alpha\beta}{\alpha+\beta}} \sum_{i=1}^m (z_i - z_{m+i}) p_i(z_1, \ldots, z_m)$ belongs to $\mathbb{C}[z_1, \ldots, z_{2m}]$ and $\mathscr{R}_1(p) = \sum_{i=1}^m p_i(z_1, \ldots, z_m) \otimes e_i$. Therefore $\mathscr{R}_1(\mathbb{C}[z_1, \ldots, z_{2m}]) = \mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}^m$, completing the proof. □

2.2.1 Hilbert modules

In this section, we study certain decomposition of the tensor product of two Hilbert modules. For the basic definitions and properties related to Hilbert module, the reader is referred to chapter 1.

Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. Let K_1 and K_2 are two scalar valued non-negative definite kernels defined on $\Omega\times\Omega$. We identify the tensor product $(\mathscr{H},K_1)\otimes(\mathscr{H},K_2)$ as a spaces of holomorphic functions defined on $\Omega\times\Omega$. We assume that the multiplication operators M_{z_i}, $i=1,\ldots,m$, are bounded on $(\mathscr{H},K_1)$ as well as on $(\mathscr{H},K_2)$. Thus the map

$$\mathbf{m}:\mathbb{C}[z_1,\ldots,z_{2m}]\times\big((\mathscr{H},K_1)\otimes(\mathscr{H},K_2)\big)\to(\mathscr{H},K_1)\otimes(\mathscr{H},K_2)$$

defined by

$$\mathbf{m}_p(h)=ph,\ h\in(\mathscr{H},K_1)\otimes(\mathscr{H},K_2),\ p\in\mathbb{C}[z_1,\ldots,z_{2m}],$$

provides a module multiplication on $(\mathscr{H},K_1)\otimes(\mathscr{H},K_2)$ over the polynomial ring $\mathbb{C}[z_1,\ldots,z_{2m}]$. The Hilbert space $(\mathscr{H},K_1)\otimes(\mathscr{H},K_2)$ admits a natural direct sum decomposition as follows.

For a non-negative integer k, let $\mathscr{A}_k$ be the subspace of $(\mathscr{H},K_1)\otimes(\mathscr{H},K_2)$ defined by

$$\mathscr{A}_k:=\Big\{f\in(\mathscr{H},K_1)\otimes(\mathscr{H},K_2):\big(\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}}f(z,\zeta)\big)_{|\Delta}=0,\ \boldsymbol{i}\in\mathbb{Z}_+^m,\ |\boldsymbol{i}|\le k\Big\}. \tag{2.22}$$

By Proposition 2.1.3, the vector $K_1(\cdot,w)\otimes\bar\partial^{\boldsymbol{i}}K_2(\cdot,w)$, $\boldsymbol{i}\in\mathbb{Z}_+^m$, belongs to $(\mathscr{H},K_1)\otimes(\mathscr{H},K_2)$ and

$$\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}}f(z,\zeta)_{|z=\zeta=w}=\big\langle f,K_1(\cdot,w)\otimes\bar\partial^{\boldsymbol{i}}K_2(\cdot,w)\big\rangle,\ \ f\in(\mathscr{H},K_1)\otimes(\mathscr{H},K_2). \tag{2.23}$$

Thus

$$\mathscr{A}_k^{\perp}=\bigvee\big\{K_1(\cdot,w)\otimes\bar\partial^{\boldsymbol{i}}K_2(\cdot,w):w\in\Omega,\ \boldsymbol{i}\in\mathbb{Z}_+^m,\ |\boldsymbol{i}|\le k\big\}. \tag{2.24}$$

From (2.23), it also follows that these subspaces $\mathscr{A}_k$, $k\ge0$, are closed. Moreover, using the Leibniz rule, it is verified that the closed subspaces $\mathscr{A}_k$, $k\ge0$, are invariant under the multiplication by any polynomial p in $\mathbb{C}[z_1,\ldots,z_{2m}]$ and therefore they are sub-modules of $(\mathscr{H},K_1)\otimes(\mathscr{H},K_2)$.

Setting $\mathscr{S}_0=\mathscr{A}_0^{\perp}$, $\mathscr{S}_k:=\mathscr{A}_{k-1}\ominus\mathscr{A}_k$, $k=1,2,\ldots$, we obtain the direct sum decomposition of the Hilbert module

$$(\mathscr{H},K_1)\otimes(\mathscr{H},K_2)=\bigoplus_{k=0}^{\infty}\mathscr{S}_k.$$

As we have discussed in chapter 1, each $\mathscr{S}_k$, $k\ge0$, is a semi-invariant module of $(\mathscr{H},K_1)\otimes(\mathscr{H},K_2)$ with the module multiplication given by $\mathbf{m}_p(f)=P_{\mathscr{S}_k}(pf)$, $p\in\mathbb{C}[z_1,\ldots,z_{2m}]$, $f\in\mathscr{S}_k$. The following theorem gives a description of the Hilbert module $\mathscr{S}_1$ in the particular case when $K_1=K^\alpha$ and $K_2=K^\beta$ for some sesqui-analytic function K defined on $\Omega\times\Omega$ and a pair of positive real numbers α,β.

Recall that the map $\iota:\Omega\to\Omega\times\Omega$ is defined by $\iota(z)=(z,z)$, $z\in\Omega$. For the definition of push-forward of a module over $\mathbb{C}[z_1,\ldots,z_m]$, the reader is referred to chapter 1.

Theorem 2.2.5. *Let $K:\Omega\times\Omega\to\mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β, defined on $\Omega\times\Omega$, are non-negative definite for some $\alpha,\beta>0$. Suppose that the multiplication operators M_{z_i}, $i=1,2,\ldots,m$, are bounded on both $(\mathscr{H},K^\alpha)$ and $(\mathscr{H},K^\beta)$. Then the Hilbert module $\mathscr{S}_1$ is isomorphic to the push-forward module $\iota_\star\big(\mathscr{H},\mathbb{K}^{(\alpha,\beta)}\big)$ via the module map $\mathscr{R}_{1|\mathscr{S}_1}$.*

Proof. From Theorem 2.2.3, it follows that the map $\mathscr{R}_1$ defined in (2.17) is a unitary map from $\mathscr{S}_1$ onto $(\mathscr{H},\mathbb{K}^{(\alpha,\beta)})$. Now we will show that $\mathscr{R}_1 P_{\mathscr{S}_1}(ph) = (p\circ\iota)\mathscr{R}_1 h,\ h\in\mathscr{S}_1, p\in\mathbb{C}[z_1,\ldots,z_{2m}]$. Let h be an arbitrary element of $\mathscr{S}_1$. Since $\ker\mathscr{R}_1 = \mathscr{S}_1^{\perp}$ (see the discussion before Theorem 2.2.3), it follows that $\mathscr{R}_1 P_{\mathscr{S}_1}(ph) = \mathscr{R}_1(ph),\ p\in\mathbb{C}[z_1,\ldots,z_{2m}]$. Hence

$$\begin{aligned}
\mathscr{R}_1 P_{\mathscr{S}_1}(ph) &= \mathscr{R}_1(ph)\\
&= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}\sum_{j=1}^{m}(\beta\partial_j(ph)-\alpha\partial_{m+j}(ph))_{|\Delta}\otimes e_j\\
&= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}\sum_{j=1}^{m}p_{|\Delta}(\beta\partial_j h-\alpha\partial_{m+j}h)_{|\Delta}\otimes e_j+\sum_{j=1}^{m}h_{|\Delta}(\beta\partial_j p-\alpha\partial_{m+j}p)_{|\Delta}\otimes e_j\\
&= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}\sum_{j=1}^{m}p_{|\Delta}(\beta\partial_j h-\alpha\partial_{m+j}h)_{|\Delta}\otimes e_j\ \ (\text{since } h\in\mathscr{S}_1)\\
&= (p\circ\iota)\mathscr{R}_1 h,
\end{aligned}$$

completing the proof. □

Notation 2.2.6. *For $1\le i\le m$, let $M_i^{(1)}$ and $M_i^{(2)}$ denote the operators of multiplication by the coordinate function z_i on the Hilbert spaces $(\mathscr{H},K_1)$ and $(\mathscr{H},K_2)$, respectively. If $m=1$, we let $M^{(1)}$ and $M^{(2)}$ denote the operators $M_1^{(1)}$ and $M_1^{(2)}$, respectively.*

In case $K_1=K^\alpha$ and $K_2=K^\beta$, let $M_i^{(\alpha)}$, $M_i^{(\beta)}$ and $M_i^{(\alpha+\beta)}$ denote the operators of multiplication by the coordinate function z_i on the Hilbert spaces $(\mathscr{H},K^\alpha)$, $(\mathscr{H},K^\beta)$ and $(\mathscr{H},K^{\alpha+\beta})$, respectively. If $m=1$, we write $M^{(\alpha)}$, $M^{(\beta)}$ and $M^{(\alpha+\beta)}$ instead of $M_1^{(\alpha)}$, $M_1^{(\beta)}$ and $M_1^{(\alpha+\beta)}$, respectively.

Finally, let $\mathbb{M}_i^{(\alpha,\beta)}$ denote the operator of multiplication by the coordinate function z_i on $(\mathscr{H},\mathbb{K}^{(\alpha,\beta)})$. Also let $\mathbb{M}^{(\alpha,\beta)}$ denote the operator $\mathbb{M}_1^{(\alpha,\beta)}$ whenever $m=1$.

Remark 2.2.7. *It is verified that $(M_i^{(\alpha)}\otimes I)^*(\phi_j(w)) = \bar{w}_i\phi_j(w)+\beta\delta_{ij}K^\alpha(\cdot,w)\otimes K^\beta(\cdot,w)$ and $(I\otimes M_i^{(\beta)})^*(\phi_j(w)) = \bar{w}_i\phi_j(w)-\alpha\delta_{ij}K^\alpha(\cdot,w)\otimes K^\beta(\cdot,w)$, $1\le i,j\le m, w\in\Omega$. Therefore,*

$$P_{\mathscr{S}_1}(M_i^{(\alpha)}\otimes I)_{|\mathscr{S}_1} = P_{\mathscr{S}_1}(I\otimes M_i^{(\beta)})_{|\mathscr{S}_1},\ i=1,2,\ldots,m.$$

Corollary 2.2.8. *The m-tuple of operators $\big(P_{\mathscr{S}_1}(M_1^{(\alpha)}\otimes I)_{|\mathscr{S}_1},\ldots,P_{\mathscr{S}_1}(M_m^{(\alpha)}\otimes I)_{|\mathscr{S}_1}\big)$ is unitarily equivalent to the m-tuple of operators $(\mathbb{M}_1^{(\alpha,\beta)},\ldots,\mathbb{M}_m^{(\alpha,\beta)})$ on $\big(\mathscr{H},\mathbb{K}^{(\alpha,\beta)}\big)$.*

In particular, if either the m-tuple of operators $(M_1^{(\alpha)},\ldots,M_m^{(\alpha)})$ on $(\mathscr{H},K^\alpha)$ or the m-tuple of operators $(M_{(1)}^{(\beta)},\ldots,M_m^{(\beta)})$ on $(\mathscr{H},K^\beta)$ is bounded, then the m-tuple $(\mathbb{M}_1^{(\alpha,\beta)},\ldots,\mathbb{M}_m^{(\alpha,\beta)})$ is also bounded on $\big(\mathscr{H},\mathbb{K}^{(\alpha,\beta)}\big)$.

Proof. The proof of the first statement follows from Theorem 2.2.5 and the proof of the second statement follows from the first together with Remark 2.2.7. □

2.2.2 Description of the quotient module $\mathscr{A}_1^{\perp}$

In this subsection, we give a description of the quotient module $\mathscr{A}_1^{\perp}$. Let $(\mathscr{H}, K^{\alpha+\beta}) \widehat{\oplus} (\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$ be the Hilbert module, which is the Hilbert space $(\mathscr{H}, K^{\alpha+\beta}) \oplus (\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$, equipped with the multiplication (distinct from the natural multiplication on it induced by the direct sum of the multiplication operators on $(\mathscr{H}, K^{\alpha+\beta})$ and $(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$) over the polynomial ring $\mathbb{C}[z_1, \ldots, z_{2m}]$ induced by the $2m$-tuple of operators $(T_1, \ldots, T_m, T_{m+1}, \ldots, T_{2m})$ described below. First, for any polynomial $p \in \mathbb{C}[z_1, \ldots, z_{2m}]$, let $p^*(z) := (p \circ \iota)(z) = p(z, z)$, $z \in \Omega$ and let $S_p : (\mathscr{H}, K^{\alpha+\beta}) \to (\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$ be the operator given by

$$S_p(f_0) = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \sum_{j=1}^{m} (\beta(\partial_j p)^* - \alpha(\partial_{m+j} p)^*) f_0 \otimes e_j, f_0 \in (\mathscr{H}, K^{\alpha+\beta}).$$

On the Hilbert space $(\mathscr{H}, K^{\alpha+\beta}) \oplus (\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$, let $T_i = \left(\begin{smallmatrix} M_{z_i} & 0 \\ S_{z_i} & M_{z_i} \end{smallmatrix}\right)$, and $T_{m+i} = \left(\begin{smallmatrix} M_{z_i} & 0 \\ S_{z_{m+i}} & M_{z_i} \end{smallmatrix}\right)$, $1 \leq i \leq m$. Now, a straightforward verification shows that the module multiplication induced by these $2m$-tuple of operators is given by the formula:

$$\mathbf{m}_p(f_0 \oplus f_1) = \begin{pmatrix} M_{p^*} f_0 & 0 \\ S_p f_0 & M_{p^*} f_1 \end{pmatrix}, \; f_0 \oplus f_1 \in (\mathscr{H}, K^{\alpha+\beta}) \oplus (\mathscr{H}, \mathbb{K}^{(\alpha,\beta)}). \tag{2.25}$$

Theorem 2.2.9. *Let $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β, defined on $\Omega \times \Omega$, are non-negative definite for some $\alpha, \beta > 0$. Suppose that the multiplication operators M_{z_i}, $i = 1, 2, \ldots, m$, are bounded on both $(\mathscr{H}, K^\alpha)$ and $(\mathscr{H}, K^\beta)$. Then the quotient module $\mathscr{A}_1^{\perp}$ and the Hilbert module $(\mathscr{H}, K^{\alpha+\beta}) \widehat{\oplus} (\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$ are isomorphic.*

Proof. The proof is accomplished by showing that the compression operator $P_{\mathscr{A}_1^{\perp}} M_{p|\mathscr{A}_1^{\perp}}$ is unitarily equivalent to the operator $\left(\begin{smallmatrix} M_{p^*} & 0 \\ S_p & M_{p^*} \end{smallmatrix}\right)$ on $\left(\mathscr{H}, K^{\alpha+\beta}\right) \bigoplus \left(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)}\right)$ for any polynomial p in $\mathbb{C}[z_1, \ldots, z_{2m}]$.

We recall that the map $\mathscr{R}_0 : (\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta) \to (\mathscr{H}, K^{\alpha+\beta})$ given by $\mathscr{R}_0(f) = f_{|\Delta}$, f in $(\mathscr{H}, K^\alpha) \otimes (\mathscr{H}, K^\beta)$ defines a unitary map from $\mathscr{S}_0$ onto $(\mathscr{H}, K^{\alpha+\beta})$, and it intertwines the operators $P_{\mathscr{S}_0} M_{p|\mathscr{S}_0}$ on $\mathscr{S}_0$ and M_{p^*} on $(\mathscr{H}, K^{\alpha+\beta})$, that is, $M_{p^*}\mathscr{R}_{0|\mathscr{S}_0} = \mathscr{R}_{0|\mathscr{S}_0} P_{\mathscr{S}_0} M_{p|\mathscr{S}_0}$. Combining this with Theorem 2.2.3, we conclude that the map $\mathscr{R} = \left(\begin{smallmatrix} \mathscr{R}_{0|\mathscr{S}_0} & 0 \\ 0 & \mathscr{R}_{1|\mathscr{S}_1} \end{smallmatrix}\right)$ is unitary from $\mathscr{S}_0 \bigoplus \mathscr{S}_1$ (which is $\mathscr{A}_1^{\perp}$) to $(\mathscr{H}, K^{\alpha+\beta}) \bigoplus (\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$. Since $\mathscr{S}_0$ is invariant under M_p^*, it follows that $P_{\mathscr{S}_1} M^*_{p|\mathscr{S}_0} = 0$. Hence

$$\mathscr{R} P_{\mathscr{A}_1^{\perp}} M^*_{p|\mathscr{A}_1^{\perp}} \mathscr{R}^* = \begin{pmatrix} \mathscr{R}_0 P_{\mathscr{S}_0} M^*_{p|\mathscr{S}_0} \mathscr{R}_0^* & \mathscr{R}_0 P_{\mathscr{S}_0} M^*_{p|\mathscr{S}_1} \mathscr{R}_1^* \\ 0 & \mathscr{R}_1 P_{\mathscr{S}_1} M^*_{p|\mathscr{S}_1} \mathscr{R}_1^* \end{pmatrix}$$

on $\mathscr{S}_0 \bigoplus \mathscr{S}_1$. We have $\mathscr{R}_0 P_{\mathscr{S}_0} M^*_{p|\mathscr{S}_0} \mathscr{R}_0^* = (M_{p^*})^*$, already, on $(\mathscr{H}, K^{\alpha+\beta})$. From Theorem 2.2.5, we see that $\mathscr{R}_1 P_{\mathscr{S}_1} M^*_{p|\mathscr{S}_1} \mathscr{R}_1^* = (M_{p^*})^*$ on $(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$. Thus we will be done if we can show that $\mathscr{R}_0 P_{\mathscr{S}_0} M^*_{p|\mathscr{S}_1} \mathscr{R}_1^* = S_p^*$.

To verify this, we claim that

$$M_p^*(\phi_j(w)) = \overline{p(w,w)}\phi_j(w) + (\overline{\beta(\partial_j p)(w,w) - \alpha(\partial_{m+j} p)(w,w)})\, K^\alpha(\cdot,w)\otimes K^\beta(\cdot,w),$$

where $\phi_j(w)$ is defined in (2.13), $1 \le j \le m,\ w \in \Omega$. If h is an arbitrary element of $(\mathscr{H}, K^\alpha)\otimes(\mathscr{H}, K^\beta)$, then

$$\begin{aligned}
\left\langle h, M_p^*(\phi_j(w))\right\rangle &= \langle M_p h, \phi_j(w)\rangle \\
&= \langle ph, \phi_j(w)\rangle \\
&= \beta\big(\partial_j(ph)\big)(w,w) - \alpha\big(\partial_{m+j}(ph)\big)(w,w) \qquad (\text{ by } (2.16)) \\
&= \beta\big((\partial_j p)(w,w)h(w,w) + p(w,w)(\partial_j h)(w,w)\big) \\
&\qquad - \alpha\big((\partial_{m+j} p)(w,w)h(w,w) + p(w,w)(\partial_{m+j} h)(w,w)\big) \\
&= \big(\beta(\partial_j p)(w,w) - \alpha(\partial_{m+j} p)(w,w)\big)h(w,w) \\
&\qquad + \big(\beta(\partial_j h)(w,w) - \alpha(\partial_{m+j} h)(w,w)\big)p(w,w) \\
&= \left\langle h, \overline{(\beta(\partial_j p)(w,w) - \alpha(\partial_{m+j} p)(w,w))} K^\alpha(\cdot,w)\otimes K^\beta(\cdot,w)\right\rangle \\
&\qquad + \left\langle h, \overline{p(w,w)}\phi_j(w)\right\rangle.
\end{aligned}$$

Hence our claim is verified. Since $\phi_j(w) \in \mathscr{A}_0$ (which is $\mathscr{S}_0^\perp$), in the computation below, the third equality follows:

$$\begin{aligned}
&\mathscr{R}_0 P_{\mathscr{S}_0} M^*_{p_{|\mathscr{S}_1}} \mathscr{R}_1^*(\mathbb{K}^{(\alpha,\beta)}(\cdot,w)e_j) \\
&= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \mathscr{R}_0 P_{\mathscr{S}_0} M^*_{p_{|\mathscr{S}_1}}(\phi_j(w)) \\
&= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \mathscr{R}_0 P_{\mathscr{S}_0}\Big(\overline{p(w,w)}\phi_j(w) + \big(\overline{\beta(\partial_j p)(w,w) - \alpha(\partial_{m+j} p)(w,w)}\big) K^\alpha(\cdot,w)\otimes K^\beta(\cdot,w)\Big) \\
&= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \mathscr{R}_0 P_{\mathscr{S}_0}\Big(\big(\overline{\beta(\partial_j p)(w,w) - \alpha(\partial_{m+j} p)(w,w)}\big) K^\alpha(\cdot,w)\otimes K^\beta(\cdot,w)\Big) \\
&= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} (\overline{\beta(\partial_j p)(w,w) - \alpha(\partial_{m+j} p)(w,w)}) \mathscr{R}_0 (K^\alpha(\cdot,w)\otimes K^\beta(\cdot,w)) \\
&= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} (\overline{\beta(\partial_j p)(w,w) - \alpha(\partial_{m+j} p)(w,w)}) K^{\alpha+\beta}(\cdot,w).
\end{aligned}$$

Set $S_p^\sharp = \mathscr{R}_1 P_{\mathscr{S}_1} M_{p_{|\mathscr{S}_0}} \mathscr{R}_0^*$. Then the above computation gives

$$(S_p^\sharp)^*(\mathbb{K}^{(\alpha,\beta)}(\cdot,w)e_j) = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} (\overline{\beta(\partial_j p)(w,w) - \alpha(\partial_{m+j} p)(w,w)}) K^{\alpha+\beta}(\cdot,w),$$

for $1 \leq j \leq m$ and $w \in \Omega$. If f is an arbitrary element in $(\mathscr{H}, K^{\alpha+\beta})$, then we see that

$$\begin{aligned}\langle S_p^\sharp f(z), e_j\rangle &= \langle S_p^\sharp f, \mathbb{K}^{(\alpha,\beta)}(\cdot, z)e_j\rangle \\ &= \langle f, (S_p^\sharp)^*(\mathbb{K}^{(\alpha,\beta)}(\cdot, z)e_j)\rangle \\ &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}(\beta(\partial_j p)(z,z) - \alpha(\partial_{m+j}p)(z,z))\langle f, K^{\alpha+\beta}(\cdot, z)\,\rangle \\ &= \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}}\big(\beta(\partial_j p)(z,z) - \alpha(\partial_{m+j}p)(z,z)\big)f(z).\end{aligned}$$

Hence $S_p^\sharp = S_p$, completing the proof of the theorem. □

Corollary 2.2.10. *Let $\Omega \subset \mathbb{C}$ be a bounded domain. The operator $P_{\mathscr{A}_1^\perp}(M^{(\alpha)} \otimes I)_{|\mathscr{A}_1^\perp}$ is unitarily equivalent to the operator $\begin{pmatrix} M^{(\alpha+\beta)} & 0 \\ \delta\,\mathsf{inc} & \mathbb{M}^{(\alpha,\beta)} \end{pmatrix}$ on $(\mathscr{H}, K^{\alpha+\beta}) \bigoplus (\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$, where $\delta = \frac{\beta}{\sqrt{\alpha\beta(\alpha+\beta)}}$ and* inc *is the inclusion operator from $(\mathscr{H}, K^{\alpha+\beta})$ into $(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$.*

2.3 Generalized Bergman Kernels

In this section, we study the generalized Bergman kernels introduced by Curto and Salinas [26]. We refer the reader to chapter 1 for the definitions and motivation related to generalized Bergman kernels. We start with the following lemma (cf. [27, page 285]) which provides a sufficient condition for the sharpness of a non-negative definite kernel K.

Lemma 2.3.1. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathscr{M}_k(\mathbb{C})$ be a non-negative definite kernel. Assume that the multiplication operator M_{z_i} on $(\mathscr{H}, K)$ is bounded for $1 \leq i \leq m$. If the vector valued polynomial ring $\mathbb{C}[z_1, \ldots, z_m] \otimes \mathbb{C}^k$ is contained in $(\mathscr{H}, K)$ as a dense subset, then K is a sharp kernel.*

Corollary 2.3.2. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are non-negative definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. Suppose that either the m-tuple of operators $(M_1^{(\alpha)}, \ldots, M_m^{(\alpha)})$ on $(\mathscr{H}, K^\alpha)$ or the m-tuple of operators $(M_1^{(\beta)}, \ldots, M_m^{(\beta)})$ on $(\mathscr{H}, K^\beta)$ is bounded. If both the Hilbert spaces $(\mathscr{H}, K^\alpha)$ and $(\mathscr{H}, K^\beta)$ contain the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ as a dense subset, then the kernel $\mathbb{K}^{(\alpha,\beta)}$ is sharp.*

Proof. By Corollary 2.2.8, we have that the m-tuple of operators $(\mathbb{M}_1^{(\alpha,\beta)}, \ldots, \mathbb{M}_m^{(\alpha,\beta)})$ is bounded on $\big(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)}\big)$. If both the Hilbert spaces $(\mathscr{H}, K^\alpha)$ and $(\mathscr{H}, K^\beta)$ contain the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$ as a dense subset, then by Proposition 2.2.4, we see that the ring $\mathbb{C}[z_1, \ldots, z_m] \otimes \mathbb{C}^m$ is contained in $(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$ and is dense in it. An application of Lemma 2.3.1 now completes the proof. □

2.3.1 Jet Construction

For two scalar valued non-negative definite kernels K_1 and K_2, defined on $\Omega \times \Omega$, the jet construction (Theorem 1.1.4) gives rise to a family of non-negative kernels $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$, $k \geq 0$, where

$$J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}(z,w) := \big(K_1(z,w)\partial^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{j}}K_2(z,w)\big)_{|\boldsymbol{i}|,|\boldsymbol{j}|=0}^{k},\ \ z,w\in\Omega.$$

In the particular case when $k = 0$, it coincides with the point-wise product K_1K_2. In this section, we generalize Theorem 1.1.8 for all kernels of the form $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$. First, we discuss two important corollaries of the jet construction which will be used later in this chapter.

For $1 \leq i \leq m$, let J_kM_i denote the operator of multiplication by the ith coordinate function z_i on the Hilbert space $\big(\mathscr{H}, J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}\big)$. In case $m = 1$, we write J_kM instead of J_kM_1.

Taking $p(z,\zeta)$ to be the ith coordinate function z_i in Proposition 1.1.5, we obtain the following corollary.

Corollary 2.3.3. *Let $K_1, K_2 : \Omega\times\Omega \to \mathbb{C}$ be two non-negative definite kernels. Then the m-tuple of operators $\big(P_{\mathscr{A}_k^\perp}(M_1^{(1)}\otimes I)_{|\mathscr{A}_k^\perp}, \ldots, P_{\mathscr{A}_k^\perp}(M_m^{(1)}\otimes I)_{|\mathscr{A}_k^\perp}\big)$ is unitarily equivalent to the m-tuple $(J_kM_1,\ldots,J_kM_m)$ on the Hilbert space $\big(\mathscr{H}, J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}\big)$.*

Combining this with Corollary 2.2.10 we obtain the following result.

Corollary 2.3.4. *Let $\Omega\subset\mathbb{C}$ be a bounded domain and $K:\Omega\times\Omega\to\mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are non-negative definite on $\Omega\times\Omega$ for some $\alpha,\beta>0$. The following operators are unitarily equivalent:*

(i) *the operator $P_{\mathscr{A}_1^\perp}(M^{(\alpha)}\otimes I)_{|\mathscr{A}_1^\perp}$*

(ii) *the multiplication operator J_1M on $\big(\mathscr{H}, J_1(K^\alpha,K^\beta)_{|\mathrm{res}\,\Delta}\big)$*

(iii) *the operator $\begin{pmatrix} M^{(\alpha+\beta)} & 0 \\ \delta\,\mathsf{inc} & \mathbb{M}^{(\alpha,\beta)}\end{pmatrix}$ on $(\mathscr{H},K^{\alpha+\beta})\bigoplus(\mathscr{H},\mathbb{K}^{(\alpha,\beta)})$ where $\delta = \frac{\beta}{\sqrt{\alpha\beta(\alpha+\beta)}}$ and inc is the inclusion operator from $(\mathscr{H},K^{\alpha+\beta})$ into $(\mathscr{H},\mathbb{K}^{(\alpha,\beta)})$.*

We need the following lemmas for the generalization of Theorem 1.1.8.

Lemma 2.3.5. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces and T be a bounded linear operator on $\mathscr{H}_1$. Then*

$$\ker(T\otimes I_{\mathscr{H}_2}) = \ker T\otimes\mathscr{H}_2.$$

Proof. It is easily seen that $\ker T\otimes\mathscr{H}_2\subset\ker(T\otimes I_{\mathscr{H}_2})$. To establish the opposite inclusion, let x be an arbitrary element in $\ker(T\otimes I_{\mathscr{H}_2})$. Fix an orthonormal basis $\{f_i\}$ of $\mathscr{H}_2$. Note that x is of the form $\sum v_i \otimes f_i$ for some v_i's in $\mathscr{H}_1$. Since $x \in \ker(T \otimes I_{\mathscr{H}_2})$, we have $\sum T v_i \otimes f_i = 0$. Moreover, since $\{f_i\}$ is an orthonormal basis of $\mathscr{H}_2$, it follows that $T v_i = 0$ for all i. Hence x belongs to $\ker(T) \otimes \mathscr{H}_2$, completing the proof of the lemma. □

Lemma 2.3.6. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces. If $B_1, \ldots, B_m$ are closed subspaces of $\mathscr{H}_1$, then*

$$\bigcap_{l=1}^{m} (B_l \otimes \mathscr{H}_2) = \Big(\bigcap_{l=1}^{m} B_l\Big) \otimes \mathscr{H}_2.$$

Proof. We only prove the non-trivial inclusion, namely, $\cap_{l=1}^{m}(B_l \otimes \mathscr{H}_2) \subset \big(\cap_{l=1}^{m} B_l\big) \otimes \mathscr{H}_2$.

Let $\{f_j\}_j$ be an orthonormal basis of $\mathscr{H}_2$ and x be an arbitrary element in $\mathscr{H}_1 \otimes \mathscr{H}_2$. Recall that x can be written uniquely as $\sum x_j \otimes f_j$, $x_j \in \mathscr{H}_1$.

Claim: If x belongs to $B_l \otimes \mathscr{H}_2$, then x_j belongs to B_l for all j.

To prove the claim, assume that $\{e_i\}_i$ is an orthonormal basis of B_l. Since $\{e_i \otimes f_j\}_{i,j}$ is an orthonormal basis of $B_l \otimes \mathscr{H}_2$ and x can be written as $\sum x_{ij} e_i \otimes f_j = \sum_j (\sum_i x_{ij} e_i) \otimes f_j$. Then, the uniqueness of the representation $x = \sum x_j \otimes f_j$, ensures that $x_j = \sum_i x_{ij} e_i$. In particular, x_j belongs to B_l for all j. Thus the claim is verified.

Now let y be any element in $\cap_{l=1}^{m}(B_l \otimes \mathscr{H}_2)$. Let $\sum y_j \otimes f_j$ be the unique representation of y in $\mathscr{H}_1 \otimes \mathscr{H}_2$. Then from the claim, it follows that $y_j \in \cap_{l=1}^{m} B_l$. Consequently, $y \in (\cap_{l=1}^{m} B_l) \otimes \mathscr{H}_2$. This completes the proof. □

The proof of the following lemma is straightforward and therefore it is omitted.

Lemma 2.3.7. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces. Let $A : \mathscr{H}_1 \to \mathscr{H}_1$ be a bounded linear operator and $B : \mathscr{H}_1 \to \mathscr{H}_2$ be a unitary operator. Then*

$$\ker BAB^* = B(\ker A).$$

The lemma given below is a generalization of [21, Lemma 1.22 (i)] to commuting tuples. Recall that for a commuting m-tuple $\boldsymbol{T} = (T_1, \ldots, T_m)$, the operator $\boldsymbol{T}^{\boldsymbol{i}}$ is defined by $T_1^{i_1} \cdots T_m^{i_m}$, where $\boldsymbol{i} = (i_1, \ldots, i_m) \in \mathbb{Z}_+^m$.

Lemma 2.3.8. *If $K : \Omega \times \Omega \to \mathbb{C}$ is a positive definite kernel such that the m-tuple of multiplication operators $\boldsymbol{M}_z = (M_{z_1}, \ldots, M_{z_m})$ on $(\mathscr{H}, K)$ is bounded, then for $w \in \Omega$ and $\boldsymbol{i} = (i_1, \ldots, i_m)$, $\boldsymbol{j} = (j_1, \ldots, j_m)$ in $\mathbb{Z}_+^m$,*

(i) $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{j}} K(\cdot, w) = 0$ *if* $|\boldsymbol{i}| > |\boldsymbol{j}|$,

(ii) $(\mathbf{M}_z^* - \bar{w})^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{j}} K(\cdot, w) = \boldsymbol{j}! \delta_{\boldsymbol{i}\boldsymbol{j}} K(\cdot, w)$ *if* $|\boldsymbol{i}| = |\boldsymbol{j}|$.

Proof. First, we claim that if $i_l > j_l$ for some $1 \le l \le m$, then $(M_{z_l}^* - \bar{w}_l)^{i_l} \bar{\partial}_l^{j_l} K(\cdot, w) = 0$. The claim is verified by induction on j_l. The case $j_l = 0$ holds trivially since $(M_{z_l}^* - \bar{w}_l) K(\cdot, w) = 0$.

Now assume that the claim is valid for $j_l = p$. We have to show that it is true for $j_l = p+1$ also. Suppose $i_l > p+1$. Then $i_l - 1 > p$. Hence, by the induction hypothesis, $(M_{z_l}^* - \bar{w}_l)^{i_l - 1}\bar{\partial}_l^p K(\cdot, w) = 0$. Differentiating this with respect to $\bar{w}_l$, we see that

$$(i_l - 1)(M_{z_l}^* - \bar{w}_l)^{i_l-2}(-1)\bar{\partial}_l^p K(\cdot, w) + (M_{z_l}^* - \bar{w}_l)^{i_l-1}\bar{\partial}_l^{p+1} K(\cdot, w) = 0.$$

Applying $(M_{z_l}^* - \bar{w}_l)$ to both sides of the equation above, we obtain

$$(i_l - 1)(M_{z_l}^* - \bar{w}_l)^{i_l-1}(-1)\bar{\partial}_l^p K(\cdot, w) + (M_{z_l}^* - \bar{w}_l)^{i_l}\bar{\partial}_l^{p+1} K(\cdot, w) = 0.$$

Using the induction hypothesis once again, we conclude that $(M_{z_l}^* - \bar{w}_l)^{i_l}\bar{\partial}_l^{p+1} K(\cdot, w) = 0$. Hence the claim is verified.

Now, to prove the first part of the lemma, assume that $|\boldsymbol{i}| > |\boldsymbol{j}|$. Then there exists a l such that $i_l > j_l$. Hence from the claim, we have $(M_{z_l}^* - \bar{w}_l)^{i_l}\bar{\partial}_l^{j_l} K(\cdot, w) = 0$. Differentiating with respect to all other variables except $\bar{w}_l$, we get $(M_{z_l}^* - \bar{w}_l)^{i_l}\bar{\partial}^{\boldsymbol{j}} K(\cdot, w) = 0$. Applying the operator $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i} - i_l e_l}$, where e_l is the lth standard unit vector of $\mathbb{C}^m$, we see that $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{j}} K(\cdot, w) = 0$, completing the proof of the first part.

For the second part, assume that $|\boldsymbol{i}| = |\boldsymbol{j}|$ and $\boldsymbol{i} \neq \boldsymbol{j}$. Then there is atleast one l such that $i_l > j_l$. Hence by the argument used in the last paragraph, we conclude that $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{j}} K(\cdot, w) = 0$. Finally, if $\boldsymbol{i} = \boldsymbol{j}$, we use induction on $\boldsymbol{i}$ to proof the lemma. There is nothing to prove if $\boldsymbol{i} = 0$. For the proof by induction, now, assume that $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{i}} K(\cdot, w) = \boldsymbol{i}! K(\cdot, w)$ for some $\boldsymbol{i} \in \mathbb{Z}_+^m$. To complete the induction step, we have to prove that $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}+e_l}\bar{\partial}^{\boldsymbol{i}+e_l} K(\cdot, w) = (\boldsymbol{i} + e_l)! K(\cdot, w)$. By the first part of the lemma, we have $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}+e_l}\bar{\partial}^{\boldsymbol{i}} K(\cdot, w) = 0$. Differentiating with respect to $\bar{w}_l$, we get that

$$(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}+e_l}\bar{\partial}^{\boldsymbol{i}+e_l} K(\cdot, w) - (i_l + 1)(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{i}} K(\cdot, w) = 0.$$

Hence, by the induction hypothesis, $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{i}+e_l}\bar{\partial}^{\boldsymbol{i}+e_l} K(\cdot, w) = (\boldsymbol{i} + e_l)! K(\cdot, w)$. This completes the proof. □

Corollary 2.3.9. *Let $K : \Omega \times \Omega \to \mathbb{C}$ be a positive definite kernel. Suppose that the m-tuple of multiplication operators $\boldsymbol{M}_z$ on $(\mathscr{H}, K)$ is bounded. Then, for all $w \in \Omega$, the set $\{\bar{\partial}^{\boldsymbol{i}} K(\cdot, w) : \boldsymbol{i} \in \mathbb{Z}_+^m\}$ is linearly independent. Consequently, the matrix $\big(\partial^{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{j}} K(w, w)\big)_{\boldsymbol{i},\boldsymbol{j}\in\Lambda}$ is positive definite for any finite subset Λ of $\mathbb{Z}_+^m$.*

Proof. Let w be an arbitrary point in Ω. It is enough to show that the set $\{\bar{\partial}^{\boldsymbol{i}} K(\cdot, w) : \boldsymbol{i} \in \mathbb{Z}_+^m, |\boldsymbol{i}| \leq k\}$ is linearly independent for each non-negative integer k. Since K is positive definite, there is nothing to prove if $k = 0$. To complete the proof by induction on k, assume that the set $\{\bar{\partial}^{\boldsymbol{i}} K(\cdot, w) : \boldsymbol{i} \in \mathbb{Z}_+^m, |\boldsymbol{i}| \leq k\}$ is linearly independent for some non-negative integer k. Suppose that $\sum_{|\boldsymbol{i}|\leq k+1} a_{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{i}} K(\cdot, w) = 0$ for some $a_{\boldsymbol{i}}$'s in $\mathbb{C}$. Then $(\boldsymbol{M}_z^* - \bar{w})^{\boldsymbol{q}}(\sum_{|\boldsymbol{i}|\leq k+1} a_{\boldsymbol{i}}\bar{\partial}^{\boldsymbol{i}} K(\cdot, w)) = 0$, for all $\boldsymbol{q} \in \mathbb{Z}_+^m$ with $|\boldsymbol{q}| \leq k+1$. If $|\boldsymbol{q}| = k+1$, by Lemma 2.3.8, we have that $a_{\boldsymbol{q}}\ q! K(\cdot, w) = 0$.

Consequently, $a_{\boldsymbol{q}} = 0$ for all $\boldsymbol{q} \in \mathbb{Z}_+^m$ with $|\boldsymbol{q}| = k+1$. Hence, by the induction hypothesis, we conclude that $a_{\boldsymbol{i}} = 0$ for all $\boldsymbol{i} \in \mathbb{Z}_+^m$, $|\boldsymbol{i}| \le k+1$ and the set $\left\{ \bar\partial^{\boldsymbol{i}} K(\cdot, w) : \boldsymbol{i} \in \mathbb{Z}_+^m, |\boldsymbol{i}| \le k+1 \right\}$ is linearly independent, completing the proof of the first part of the corollary.

If Λ is a finite subset of $\mathbb{Z}_+^m$, then it follows form the linear independence of the vectors $\{\bar\partial^{\boldsymbol{i}} K(\cdot, w) : \boldsymbol{i} \in \Lambda\}$ that the matrix $\left(\langle \bar\partial^{\boldsymbol{j}} K(\cdot, w), \bar\partial^{\boldsymbol{i}} K(\cdot, w)\rangle\right)_{\boldsymbol{i},\boldsymbol{j} \in \Lambda}$ is positive definite. Now the proof is complete since $\langle \bar\partial^{\boldsymbol{j}} K(\cdot, w), \bar\partial^{\boldsymbol{i}} K(\cdot, w)\rangle = \partial^{\boldsymbol{i}} \bar\partial^{\boldsymbol{j}} K(w,w)$ (see Proposition 2.1.3). □

The following proposition is also a generalization to the multi-variate setting of [21, Lemma 1.22 (ii)](see also [22]).

Proposition 2.3.10. *If $K : \Omega \times \Omega \to \mathbb{C}$ is a sharp kernel, then for every $w \in \Omega$*

$$\bigcap_{|\boldsymbol{j}|=k+1} \ker\, (\boldsymbol{M}_z^* - \bar w)^{\boldsymbol{j}} = \bigvee \left\{\bar\partial^{\boldsymbol{j}} K(\cdot, w) : |\boldsymbol{j}| \le k\right\}.$$

Proof. The inclusion $\bigvee\{\bar\partial^{\boldsymbol{j}} K(\cdot, w) : |\boldsymbol{j}| \le k\} \subseteq \bigcap_{|\boldsymbol{j}|=k+1} \ker\, (\boldsymbol{M}_z^* - \bar w)^{\boldsymbol{j}}$ follows from part (i) of Lemma 2.3.8. We use induction on k for the opposite inclusion. From the definition of sharp kernel, this inclusion is evident if $k = 0$. Assume that

$$\bigcap_{|\boldsymbol{j}|=k+1} \ker\, (\boldsymbol{M}_z^* - \bar w)^{\boldsymbol{j}} \subseteq \bigvee \left\{\bar\partial^{\boldsymbol{j}} K(\cdot, w) : |\boldsymbol{j}| \le k\right\}$$

for some non-negative integer k. To complete the proof by induction, we show that the inclusion remains valid for $k+1$ as well. Let f be an arbitrary element of $\bigcap_{|\boldsymbol{i}|=k+2} \ker(\boldsymbol{M}_z^* - \bar w)^{\boldsymbol{i}}$. Fix a $\boldsymbol{j} \in \mathbb{Z}_+^m$ with $|\boldsymbol{j}| = k+1$. Then it follows that $(\boldsymbol{M}_z^* - \bar w)^{\boldsymbol{j}} f$ belongs to $\cap_{l=1}^m \ker(M_{z_l}^* - \bar w_l)$. Since K is sharp, we see that $(\boldsymbol{M}_z^* - \bar w)^{\boldsymbol{j}} f = c_{\boldsymbol{j}} K(\cdot, w)$ for some constant $c_{\boldsymbol{j}}$ depending on w. Therefore

$$\begin{aligned}(\boldsymbol{M}_z^* - \bar w)^{\boldsymbol{j}} \Big(f - \sum_{|\boldsymbol{q}|=k+1} \frac{c_{\boldsymbol{q}}}{\boldsymbol{q}!} \bar\partial^{\boldsymbol{q}} K(\cdot, w)\Big) =& c_{\boldsymbol{j}} K(\cdot, w) - \sum_{|\boldsymbol{q}|=k+1} \frac{c_{\boldsymbol{q}}}{\boldsymbol{q}!} (\boldsymbol{M}_z^* - \bar w)^{\boldsymbol{j}} \bar\partial^{\boldsymbol{q}} K(\cdot, w)\\ =& c_{\boldsymbol{j}} K(\cdot, w) - \sum_{|\boldsymbol{q}|=k+1} c_{\boldsymbol{q}} \delta_{\boldsymbol{j}\boldsymbol{q}} \frac{\boldsymbol{j}!}{\boldsymbol{q}!} K(\cdot, w)\\ =& 0,\end{aligned}$$

where the last equality follows from Lemma 2.3.8. Hence the element $f - \sum_{|\boldsymbol{q}|=k+1} \frac{c_{\boldsymbol{q}}}{\boldsymbol{q}!} \bar\partial^{\boldsymbol{q}} K(\cdot, w)$ belongs to $\bigcap_{|\boldsymbol{j}|=k+1} \ker(\boldsymbol{M}_z^* - \bar w)^{\boldsymbol{j}}$. Thus by the induction hypothesis, $f - \sum_{|\boldsymbol{q}|=k+1} \frac{c_{\boldsymbol{q}}}{\boldsymbol{q}!} \bar\partial^{\boldsymbol{q}} K(\cdot, w) = \sum_{|\boldsymbol{j}| \le k} d_{\boldsymbol{j}} \bar\partial^{\boldsymbol{j}} K(\cdot, w)$. Hence f belongs to $\bigvee\{\bar\partial^{\boldsymbol{j}} K(\cdot, w) : |\boldsymbol{j}| \le k+1\}$. This completes the proof. □

For a m-tuple of bounded operators $\mathbf{T} = (T_1, \ldots, T_m)$ on a Hilbert space $\mathscr{H}$, we define an operator $D^{\mathbf{T}} : \mathscr{H} \oplus \cdots \oplus \mathscr{H} \to \mathscr{H}$ by

$$D^{\boldsymbol{T}}(x_1, \ldots, x_m) = \sum_{i=1}^m T_i x_i, \;\; x_1, \ldots, x_m \in \mathscr{H}.$$

A routine verification shows that $(D_{\boldsymbol{T}})^* = D^{\boldsymbol{T}^*}$. The following lemma is undoubtedly well known, however, we provide a proof for the sake of completeness.

Lemma 2.3.11. *Let $K : \Omega \times \Omega \to \mathbb{C}$ be a positive definite kernel such that the m-tuple of multiplication operators $\boldsymbol{M}_z$ on $(\mathscr{H}, K)$ is bounded. Let $w = (w_1, \ldots, w_m)$ be a fixed but arbitrary point in Ω and let $\mathcal{V}_w$ be the subspace given by $\{f \in (\mathscr{H}, K) : f(w) = 0\}$. Then K is a generalized Bergman kernel if and only if for every $w \in \Omega$,*

$$\mathcal{V}_w = \Big\{ \textstyle\sum_{i=1}^m (z_i - w_i) g_i : g_i \in (\mathscr{H}, K) \Big\}. \tag{2.26}$$

Proof. First, observe that the right-hand side of (2.26) is equal to ran $D^{\boldsymbol{M}_z - w}$. Hence it suffices to show that K is a generalized Bergman kernel if and only if $\mathcal{V}_w = \text{ran } D^{\boldsymbol{M}_z - w}$. In any case, we have the following inclusions

$$\begin{aligned} \text{ran } D^{\boldsymbol{M}_z - w} = \text{ran } (D_{(\boldsymbol{M}_z - w)^*})^* \subseteq \overline{\text{ran } (D_{(\boldsymbol{M}_z - w)^*})^*} &= \ker D_{(\boldsymbol{M}_z - w)^*}{}^{\perp} \\ &\subseteq \{cK(\cdot, w) : c \in \mathbb{C}\}^{\perp} \\ &= \mathcal{V}_w. \end{aligned} \tag{2.27}$$

Hence it follows that $\mathcal{V}_w = \text{ran } D^{\boldsymbol{M}_z - w}$ if and only if equality is forced everywhere in these inclusions, that is, ran $(D_{(\boldsymbol{M}_z - w)^*})^* = \overline{\text{ran } (D_{(\boldsymbol{M}_z - w)^*})^*}$ and $\ker D_{(\boldsymbol{M}_z - w)^*}{}^{\perp} = \{cK(\cdot, w) : c \in \mathbb{C}\}^{\perp}$. Now ran $(D_{(\boldsymbol{M}_z - w)*})^* = \overline{\text{ran } (D_{(\boldsymbol{M}_z - w)^*})^*}$ if and only if ran $(D_{(\boldsymbol{M}_z - w)^*})^*$ is closed. Recall that, if $\mathscr{H}_1, \mathscr{H}_2$ are two Hilbert spaces, and an operator $T : \mathscr{H}_1 \to \mathscr{H}_2$ has closed range, then T^* also has closed range. Therefore, ran $(D_{(\boldsymbol{M}_z - w)^*})^*$ is closed if and only if ran $D_{(\boldsymbol{M}_z - w)^*}$ is closed. Finally, note that $\ker D_{(\boldsymbol{M}_z - w)^*}{}^{\perp} = \{cK(\cdot, w) : c \in \mathbb{C}\}^{\perp}$ holds if and only if $\ker D_{(\boldsymbol{M}_z - w)^*} = \{cK(\cdot, w) : c \in \mathbb{C}\}$. This completes the proof. □

Notation 2.3.12. *Recall that for $1 \le i \le m$, $M_i^{(1)}, M_i^{(2)}, J_k M_i$ denote the operators of multiplication by the coordinate function z_i on the Hilbert spaces $(\mathscr{H}, K_1), (\mathscr{H}, K_2)$ and $(\mathscr{H}, J_k(K_1, K_2)_{|\text{res}\,\Delta})$, respectively. Set $\boldsymbol{M}^{(1)} = (M_1^{(1)}, \ldots, M_m^{(1)})$, $\boldsymbol{M}^{(2)} = (M_1^{(2)}, \ldots, M_m^{(2)})$ and $\boldsymbol{J}_k \boldsymbol{M} = (J_k M_1, \ldots, J_k M_m)$. Also, for the sake of brevity, let $\mathscr{H}_1$ and $\mathscr{H}_2$ be the Hilbert spaces $(\mathscr{H}, K_1)$ and $(\mathscr{H}, K_2)$, respectively for the rest of this section.*

The following lemma is the main tool to prove that the kernel $J_k(K_1, K_2)_{|\text{res}\,\Delta}$ is sharp whenever K_1 and K_2 are sharp.

Lemma 2.3.13. *If $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ are two sharp kernels, then for all $w = (w_1, \ldots, w_m) \in \Omega$,*

$$\begin{aligned} \bigcap_{p=1}^m \ker\Big(\big((M_p^{(1)} - w_p)^* \otimes I\big)_{|\mathscr{A}_k^{\perp}} \Big) &= \bigcap_{|\boldsymbol{i}|=1} \ker\big(\boldsymbol{M}^{(1)} - w\big)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker\big(\boldsymbol{M}^{(2)} - w\big)^{*\boldsymbol{i}} \\ &= \bigvee \big\{ K_1(\cdot, w) \otimes \bar{\partial}^{\boldsymbol{i}} K_2(\cdot, w) : |\boldsymbol{i}| \le k \big\}. \end{aligned}$$

Proof. Since K_1 and K_2 are sharp kernels, by Proposition 2.3.10, it follows that

$$\bigcap_{|\boldsymbol{i}|=1} \ker(\boldsymbol{M}^{(1)}-w)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker(\boldsymbol{M}^{(2)}-w)^{*\boldsymbol{i}} = \bigvee\{K_1(\cdot,w)\otimes\bar\partial^{\boldsymbol{j}}K_2(\cdot,w) : |\boldsymbol{j}|\le k\}. \tag{2.28}$$

Therefore, if we can show that

$$\bigcap_{p=1}^{m} \ker\Big(\big((M_p^{(1)}-w_p)^*\otimes I\big)_{|\mathscr{A}_k{}^{\perp}}\Big) = \bigcap_{|\boldsymbol{i}|=1} \ker(\boldsymbol{M}^{(1)}-w)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker(\boldsymbol{M}^{(2)}-w)^{*\boldsymbol{i}}, \tag{2.29}$$

then we will be done. To prove this, first note that

$$\begin{aligned}
\bigcap_{p=1}^{m} \ker\Big(\big((M_p^{(1)}-w_p)^*\otimes I\big)_{|\mathscr{A}_k^{\perp}}\Big) &= \bigcap_{p=1}^{m}\Big(\ker\big((M_p^{(1)}-w_p)^*\otimes I\big)\bigcap \mathscr{A}_k^{\perp}\Big)\\
&= \Big(\bigcap_{p=1}^{m}\ker\big((M_p^{(1)}-w_p)^*\otimes I\big)\Big)\bigcap \mathscr{A}_k^{\perp}\\
&= \Big(\bigcap_{p=1}^{m}\big(\ker(M_p^{(1)}-w_p)^*\otimes \mathscr{H}_2\big)\Big)\bigcap \mathscr{A}_k^{\perp}\\
&= \Big(\Big(\bigcap_{p=1}^{m}\ker(M_p^{(1)}-w_p)^*\Big)\otimes \mathscr{H}_2\Big)\bigcap \mathscr{A}_k^{\perp}\\
&= \Big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*}\otimes \mathscr{H}_2\Big)\bigcap \mathscr{A}_k^{\perp}.
\end{aligned}$$

Here the third equality follows from Lemma 2.3.5 and the forth equality follows from Lemma 2.3.6. In view of the above computation, to verify (2.29), it is enough to show that

$$\Big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*}\otimes \mathscr{H}_2\Big)\bigcap \mathscr{A}_k^{\perp} = \bigcap_{|\boldsymbol{i}|=1} \ker(\boldsymbol{M}^{(1)}-w)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker(\boldsymbol{M}^{(2)}-w)^{*\boldsymbol{i}}. \tag{2.30}$$

Since K_1 is a sharp kernel, $\ker D_{(\boldsymbol{M}^{(1)}-w)^*}$ is spanned by the vector $K_1(\cdot,w)$. Hence, by (2.24), the vector $K_1(\cdot,w)\otimes\bar\partial^{\boldsymbol{j}}K_2(\cdot,w)$ belongs to $\Big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*}\otimes \mathscr{H}_2\Big)\bigcap \mathscr{A}_k^{\perp}$ for all $\boldsymbol{j}$ in $\mathbb{Z}_+^m$ with $|\boldsymbol{j}|\le k$. Therefore, by (2.28), we have the inclusion

$$\bigcap_{|\boldsymbol{i}|=1} \ker(\boldsymbol{M}^{(1)}-w)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker(\boldsymbol{M}^{(2)}-w)^{*\boldsymbol{i}} \subseteq \Big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*}\otimes \mathscr{H}_2\Big)\bigcap \mathscr{A}_k^{\perp}. \tag{2.31}$$

Now to prove the opposite inclusion, note that an arbitrary vector of $\big(\ker D_{(\boldsymbol{M}^{(1)}-w)^*}\otimes\mathscr{H}_2\big)\bigcap\mathscr{A}_k^{\perp}$ can be taken to be of the form $K_1(\cdot,w)\otimes g$, where $g\in\mathscr{H}_2$ is such that $K_1(\cdot,w)\otimes g\in\mathscr{A}_k^{\perp}$. We claim that such a vector g must be in $\bigcap_{|\boldsymbol{i}|=k+1}\ker(\boldsymbol{M}^{(2)}-w)^{*\boldsymbol{i}}$.

As before, we realize the vectors of $\mathscr{H}_1\otimes\mathscr{H}_2$ as functions in $z=(z_1,\ldots,z_m), \zeta=(\zeta_1,\ldots,\zeta_m)$ in Ω. Fix any $\boldsymbol{i}\in\mathbb{Z}_+^m$ with $|\boldsymbol{i}|=k+1$. Then $(\zeta-z)^{\boldsymbol{i}}=(\zeta_{q_1}-z_{q_1})(\zeta_{q_2}-z_{q_2})\cdots(\zeta_{q_{k+1}}-z_{q_{k+1}})$ for some $1\le q_1,q_2,\ldots,q_{k+1}\le m$. Since $M_i^{(1)}$ and $M_i^{(2)}$ are bounded for $1\le i\le m$, for any $h\in\mathscr{H}_1\otimes\mathscr{H}_2$, we see that the function $(\zeta - z)^{\boldsymbol{i}} h$ belongs to $\mathscr{H}_1 \otimes \mathscr{H}_2$. Then

$$\begin{aligned}
&\langle K_1(\cdot, w) \otimes g, (\zeta_{q_1} - z_{q_1})(\zeta_{q_2} - z_{q_2}) \cdots (\zeta_{q_{k+1}} - z_{q_{k+1}}) h \rangle \\
&\quad = \left\langle M^*_{(\zeta_{q_1} - z_{q_1})} (K_1(\cdot, w) \otimes g), (\zeta_{q_2} - z_{q_2}) \cdots (\zeta_{q_{k+1}} - z_{q_{k+1}}) h \right\rangle \\
&\quad = \left\langle (I \otimes M_{q_1}^{(2)*} - M_{q_1}^{(1)*} \otimes I) K_1(\cdot, w) \otimes g, (\zeta_{q_2} - z_{q_2}) \cdots (\zeta_{q_{k+1}} - z_{q_{k+1}}) h \right\rangle \\
&\quad = \left\langle K_1(\cdot, w) \otimes M_{q_1}^{(2)*} g - \bar{w}_{q_1} K_1(\cdot, w) \otimes g, (\zeta_{q_2} - z_{q_2}) \cdots (\zeta_{q_{k+1}} - z_{q_{k+1}}) h \right\rangle \\
&\quad = \left\langle K_1(\cdot, w) \otimes (M_{q_1}^{(2)} - w_{q_1})^* g, (\zeta_{q_2} - z_{q_2}) \cdots (\zeta_{q_{k+1}} - z_{q_{k+1}}) h \right\rangle.
\end{aligned}$$

Repeating this process, we get

$$\langle K_1(\cdot, w) \otimes g, (\zeta - z)^{\boldsymbol{i}} h \rangle = \left\langle K_1(\cdot, w) \otimes (\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}} g, h \right\rangle.$$

Since $|\boldsymbol{i}| = k+1$, it follows that the element $(\zeta - z)^{\boldsymbol{i}} h$ belongs to $\mathscr{A}_k$. Furthermore, since $K_1(\cdot, w) \otimes g \in \mathscr{A}_k^{\perp}$, from the above equality, we have

$$\left\langle K_1(\cdot, w) \otimes (\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}} g, h \right\rangle = 0$$

for any $h \in \mathscr{H}_1 \otimes \mathscr{H}_2$. Taking $h = K_1(\cdot, w) \otimes K_2(\cdot, u)$, $u \in \Omega$, we get $K_1(w, w)\big((\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}} g\big)(u) = 0$ for all $u \in \Omega$. Since $K_1(w, w) > 0$, it follows that $(\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}} g = 0$. Since this is true for all $\boldsymbol{i} \in \mathbb{Z}_+^m$ with $|\boldsymbol{i}| = k+1$, it follows that $g \in \bigcap_{|\boldsymbol{i}|=k+1} \ker (\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}}$. Hence $K_1(\cdot, w) \otimes g$ belongs to

$$\bigcap_{|\boldsymbol{i}|=1} \ker (\boldsymbol{M}^{(1)} - w)^{*\boldsymbol{i}} \otimes \bigcap_{|\boldsymbol{i}|=k+1} \ker (\boldsymbol{M}^{(2)} - w)^{*\boldsymbol{i}},$$

proving the opposite inclusion of the one appearing in (2.31). This completes the proof of equality in (2.29). □

Theorem 2.3.14. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. If $K_1, K_2 : \Omega \times \Omega \to \mathbb{C}$ are two sharp kernels, then so is the kernel $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$, $k \geq 0$.*

Proof. Since the tuple $\boldsymbol{M}^{(1)}$ is bounded, by Corollary 2.3.3, it follows that the tuple $\boldsymbol{J}_k \boldsymbol{M}$ is also bounded. Now we will show that the kernel $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$ is positive definite on $\Omega \times \Omega$. Since K_2 is positive definite, by Corollary 2.3.9, we obtain that the matrix $\big(\partial^{\boldsymbol{i}} \bar{\partial}^{\boldsymbol{j}} K_2(w, w)\big)_{|\boldsymbol{i}|, |\boldsymbol{j}|=0}^{k}$ is positive definite for $w \in \Omega$. Moreover, since K_1 is also positive definite, we conclude that $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}(w, w)$ is positive definite for $w \in \Omega$. Hence, by Proposition 2.1.14, we conclude that the kernel $J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}$ is positive definite.

To complete the proof, we need to show that

$$\ker D_{(\boldsymbol{J}_k \boldsymbol{M} - w)^*} = \operatorname{ran} J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}(\cdot, w), \ w \in \Omega.$$

Note that, by the definition of R and J_k (see the discussion before Theorem 1.1.4), we have

$$RJ_k(K_1(\cdot,w)\otimes\bar{\partial}^{\boldsymbol{i}}K_2(\cdot,w)) = J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}(\cdot,w)e_{\boldsymbol{i}},\ \boldsymbol{i}\in\mathbb{Z}_+^m,\ |\boldsymbol{i}|\leq k. \tag{2.32}$$

In the computation below, the third equality follows from Lemma 2.3.7, the injectivity of the map $RJ_{k|\mathscr{A}_k^\perp}$ implies the fourth equality, the fifth equality follows from Lemma 2.3.13 and finally the last equality follows from (2.32):

$$\begin{aligned}
\ker D_{(\boldsymbol{J}_k\boldsymbol{M}-w)^*} &= \bigcap_{p=1}^m \ker(J_kM_p-w_p)^*\\
&= \bigcap_{p=1}^m \ker\Big((RJ_k)P_{\mathscr{A}_k^\perp}\big((M_p^{(1)}-w_p)^*\otimes I\big)_{|\mathscr{A}_k^\perp}(RJ_k)^*\Big)\\
&= \bigcap_{p=1}^m (RJ_k)\Big(\ker\big(P_{\mathscr{A}_k^\perp}\big((M_p^{(1)}-w_p)^*\otimes I\big)_{|\mathscr{A}_k^\perp}\big)\Big)\\
&=(RJ_k)\Big(\bigcap_{p=1}^m \ker\big(P_{\mathscr{A}_k^\perp}\big((M_p^{(1)}-w_p)^*\otimes I\big)_{|\mathscr{A}_k^\perp}\big)\Big)\\
&=(RJ_k)\Big(\bigvee\big\{K_1(\cdot,w)\otimes\bar{\partial}^{\boldsymbol{i}}K_2(\cdot,w):|\boldsymbol{j}|\leq k\big\}\Big)\\
&=\mathrm{ran}\, J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}(\cdot,w).
\end{aligned}$$

This completes the proof. □

The lemma given below is the main tool to prove Theorem 2.3.16.

Lemma 2.3.15. *Let $K_1,K_2:\Omega\times\Omega\to\mathbb{C}$ be two generalized Bergman kernels, and let $w=(w_1,\ldots,w_m)$ be an arbitrary point in Ω. Suppose that f is a function in $\mathscr{H}_1\otimes\mathscr{H}_2$ satisfying $\Big(\big(\frac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}}f(z,\zeta)\Big)_{|z=\zeta=w}=0$ for all $\boldsymbol{i}\in\mathbb{Z}_+^m$, $|\boldsymbol{i}|\leq k$. Then*

$$f(z,\zeta)=\sum_{j=1}^m(z_j-w_j)f_j(z,\zeta)+\sum_{|\boldsymbol{q}|=k+1}(z-\zeta)^{\boldsymbol{q}}f_{\boldsymbol{q}}^\sharp(z,\zeta)$$

for some functions $f_j, f_{\boldsymbol{q}}^\sharp$ in $\mathscr{H}_1\otimes\mathscr{H}_2$, $j=1,\ldots,m$, $\boldsymbol{q}\in\mathbb{Z}_+^m, |\boldsymbol{q}|=k+1$.

Proof. Since K_1 and K_2 are generalized Bergman kernels, by Theorem 1.1.8, we have that $K_1\otimes K_2$ is also a generalized Bergman kernel. Therefore, if f is a function in $\mathscr{H}_1\otimes\mathscr{H}_2$ vanishing at (w,w), then using Lemma 2.3.11, we find functions $f_1,\ldots,f_m$, and $g_1,\ldots,g_m$ in $\mathscr{H}_1\otimes\mathscr{H}_2$ such that

$$f(z,\zeta)=\sum_{j=1}^m(z_j-w_j)f_j+\sum_{j=1}^m(\zeta_j-w_j)g_j.$$

Equivalently, we have

$$f(z,\zeta)=\sum_{j=1}^{m}(z_j-w_j)(f_j+g_j)+\sum_{j=1}^{m}(z_j-\zeta_j)(-g_j).$$

Thus the statement of the lemma is verified for $k=0$. To complete the proof by induction on k, assume that the statement is valid for some non-negative integer k. Let f be a function in $\mathscr{H}_1\otimes\mathscr{H}_2$ such that $\big((\frac{\partial}{\partial\zeta})^{\boldsymbol{i}}f(z,\zeta)\big)_{|z=\zeta=w}=0$ for all $\boldsymbol{i}\in\mathbb{Z}_+^m$, $|\boldsymbol{i}|\le k+1$. By induction hypothesis, we can write

$$f(z,\zeta)=\sum_{j=1}^{m}(z_j-w_j)f_j(z,\zeta)+\sum_{|\boldsymbol{q}|=k+1}(z-\zeta)^{\boldsymbol{q}}f^{\sharp}_{\boldsymbol{q}}(z,\zeta) \tag{2.33}$$

for some $f_j, f^{\sharp}_{\boldsymbol{q}}\in\mathscr{H}_1\otimes\mathscr{H}_2$, $j=1,\dots,m$, $\boldsymbol{q}\in\mathbb{Z}_+^m$, $|\boldsymbol{q}|=k+1$. Fix a $\boldsymbol{i}\in\mathbb{Z}_+^m$ with $|\boldsymbol{i}|=k+1$. Applying $(\frac{\partial}{\partial\zeta})^{\boldsymbol{i}}$ to both sides of (2.33), we see that

$$\begin{aligned}\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}}f(z,\zeta)&=\sum_{j=1}^{m}(z_j-w_j)\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}}f_j(z,\zeta)+\sum_{|\boldsymbol{q}|=k+1}\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}}\big((z-\zeta)^{\boldsymbol{q}}f^{\sharp}_{\boldsymbol{q}}(z,\zeta)\big)\\&=\sum_{j=1}^{m}(z_j-w_j)\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}}f_j(z,\zeta)+\sum_{|\boldsymbol{q}|=k+1}\sum_{\boldsymbol{p}\le\boldsymbol{i}}\binom{\boldsymbol{i}}{\boldsymbol{p}}\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{p}}(z-\zeta)^{\boldsymbol{q}}\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}-\boldsymbol{p}}f^{\sharp}_{\boldsymbol{q}}(z,\zeta).\end{aligned}$$

Putting $z=\zeta=w$, we obtain

$$\big(\big(\tfrac{\partial}{\partial\zeta}\big)^{\boldsymbol{i}}f(z,\zeta)\big)_{|z=\zeta=w}=(-1)^{|\boldsymbol{i}|}\boldsymbol{i}!\,f^{\sharp}_{\boldsymbol{i}}(w,w),$$

where we have used the simple identity: $\Big(\big(\frac{\partial}{\partial\zeta}\big)^{\boldsymbol{p}}(z-\zeta)^{\boldsymbol{q}}\Big)_{|z=\zeta=w}=\delta_{\boldsymbol{p}\boldsymbol{q}}(-1)^{|\boldsymbol{p}|}p!$.

Since $\big((\frac{\partial}{\partial\zeta})^{\boldsymbol{i}}f(z,\zeta)\big)_{|z=\zeta=w}=0$, we conclude that $f^{\sharp}_{\boldsymbol{i}}(w,w)=0$. Since the statement of the lemma has been shown to be valid for $k=0$, it follows that

$$f^{\sharp}_{\boldsymbol{i}}(z,\zeta)=\sum_{j=1}^{m}(z_j-w_j)\big(f^{\sharp}_{\boldsymbol{i}}\big)_j(z,\zeta)+\sum_{j=1}^{m}(z_j-\zeta_j)\big(f^{\sharp}_{\boldsymbol{i}}\big)^{\sharp}_j(z,\zeta) \tag{2.34}$$

for some $\big(f^{\sharp}_{\boldsymbol{i}}\big)_j, \big(f^{\sharp}_{\boldsymbol{i}}\big)^{\sharp}_j\in\mathscr{H}_1\otimes\mathscr{H}_2$, $j=1,\dots,m$. Since (2.34) is valid for any $\boldsymbol{i}\in\mathbb{Z}_+^m$, $|\boldsymbol{i}|=k+1$, replacing the $f^{\sharp}_{\boldsymbol{q}}$'s in (2.33) by $\sum_{j=1}^m(z_j-w_j)\big(f^{\sharp}_{\boldsymbol{q}}\big)_j(z,\zeta)+\sum_{j=1}^m(z_j-\zeta_j)\big(f^{\sharp}_{\boldsymbol{q}}\big)^{\sharp}_j(z,\zeta)$, we obtain the desired conclusion after some straightforward algebraic manipulation. □

Theorem 2.3.16. *Let $\Omega\subset\mathbb{C}^m$ be a bounded domain. If $K_1,K_2:\Omega\times\Omega\to\mathbb{C}$ are generalized Bergman kernels, then so is the kernel $J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}$, $k\ge 0$.*

Proof. By Theorem 2.3.14, we will be done if we can show that ran $D_{(\boldsymbol{J}_k\boldsymbol{M}-w)^*}$ is closed for every $w\in\Omega$. Fix a point $w=(w_1,\dots,w_m)$ in Ω. Let $\boldsymbol{X}:=\big(P_{\mathscr{A}_k^\perp}(M_1^{(1)}\otimes I)_{|\mathscr{A}_k^\perp},\dots,P_{\mathscr{A}_k^\perp}(M_m^{(1)}\otimes I)_{|\mathscr{A}_k^\perp}\big)$. By Corollary 2.3.3, we see that ran $D_{(\boldsymbol{J}_k\boldsymbol{M}-w)^*}$ is closed if and only if ran $D_{(\boldsymbol{X}-w)^*}$ is closed.

Moreover, since $(D_{(\boldsymbol{X}-w)^*})^* = D^{(\boldsymbol{X}-w)}$, we conclude that ran $D_{(\boldsymbol{X}-w)^*}$ is closed if and only if ran $D^{(\boldsymbol{X}-w)}$ is closed. Note that X satisfies the following equality:

$$\ker D_{(\boldsymbol{X}-w)^*}{}^{\perp} = \overline{\operatorname{ran}\,(D_{(\boldsymbol{X}-w)^*})^*} = \overline{\operatorname{ran}\,D^{(\boldsymbol{X}-w)}}.$$

Therefore, to prove ran $D^{(\boldsymbol{X}-w)}$ is closed, it is enough to show that $\ker D_{(\boldsymbol{X}-w)^*}{}^{\perp} \subseteq \operatorname{ran} D^{\boldsymbol{X}-w}$. To prove this, note that

$$D^{(\boldsymbol{X}-w)}(g_1 \oplus \cdots \oplus g_m) = P_{\mathscr{A}_k^{\perp}}\Big(\sum_{i=1}^{m}(z_i - w_i)g_i\Big),\ g_i \in \mathscr{A}_k^{\perp}, i = 1,\ldots,m.$$

Thus

$$\operatorname{ran} D^{(\boldsymbol{X}-w)} = \Big\{P_{\mathscr{A}_k^{\perp}}\Big(\sum_{i=1}^{m}(z_i - w_i)g_i : g_1,\ldots,g_m \in \mathscr{A}_k^{\perp}\Big\}. \tag{2.35}$$

Now, let f be an arbitrary element of $\ker D_{(\boldsymbol{X}-w)^*}{}^{\perp}$. Then, by Lemma 2.3.13 and Proposition 2.1.3, we have $\big((\frac{\partial}{\partial \zeta})^{\boldsymbol{i}} f(z,\zeta)\big)_{|z=\zeta=w} = 0$ for all $\boldsymbol{i} \in \mathbb{Z}_+^m$, $|\boldsymbol{i}| \le k$. By Lemma 2.3.15,

$$f(z,\zeta) = \sum_{j=1}^{m}(z_j - w_j)f_j(z,\zeta) + \sum_{|\boldsymbol{q}|=k+1}(z-\zeta)^{\boldsymbol{q}} f_{\boldsymbol{q}}^{\sharp}(z,\zeta)$$

for some functions $f_j, f_{\boldsymbol{q}}^{\sharp}$ in $\mathscr{H}_1 \otimes \mathscr{H}_2$, $j = 1,\ldots,m$ and $\boldsymbol{q} \in \mathbb{Z}_+^m, |\boldsymbol{q}| = k+1$. Note that the element $\sum_{|\boldsymbol{q}|=k+1}(z-\zeta)^{\boldsymbol{q}} f_{\boldsymbol{q}}^{\sharp}$ belongs to $\mathscr{A}_k$. Hence $f = P_{\mathscr{A}_k^{\perp}}(f) = P_{\mathscr{A}_k^{\perp}}\big(\sum_{j=1}^{m}(z_j - w_j)f_j\big)$. Furthermore, since the subspace $\mathscr{A}_k$ is invariant under $(M_j^{(1)} - w_j)$, $j = 1,\ldots,m$, we see that

$$\begin{aligned} f = P_{\mathscr{A}_k^{\perp}}\big(\textstyle\sum_{j=1}^{m}(z_j - w_j)f_j\big) &= P_{\mathscr{A}_k^{\perp}}\Big(\textstyle\sum_{j=1}^{m}(z_j - w_j)\big(P_{\mathscr{A}_k^{\perp}} f_j + P_{\mathscr{A}_k} f_j\big)\Big) \\ &= P_{\mathscr{A}_k^{\perp}}\big(\textstyle\sum_{j=1}^{m}(z_j - w_j)(P_{\mathscr{A}_k^{\perp}} f_j)\big). \end{aligned}$$

Therefore, from (2.35), we conclude that $f \in \operatorname{ran} D^{(\boldsymbol{X}-w)}$. This completes the proof. □

2.3.2 The class $\mathscr{F}B_2(\Omega)$

In this subsection, first we will use Theorem 2.3.16 to prove that, if $\Omega \subset \mathbb{C}$, and K^{α}, K^{β}, defined on $\Omega \times \Omega$, are generalized Bergman kernels, then so is the kernel $\mathbb{K}^{(\alpha,\beta)}$. The following proposition, which is interesting on its own right, is an essential tool in proving this theorem. The notation below is chosen to be close to that of [37].

Proposition 2.3.17. *Let $\Omega \subset \mathbb{C}$ be a bounded domain. Let T be a bounded linear operator of the form $\begin{bmatrix} T_0 & S \\ 0 & T_1 \end{bmatrix}$ on $H_0 \bigoplus H_1$. Suppose that T belongs to $B_2(\Omega)$ and T_0 belongs to $B_1(\Omega)$. Then T_1 belongs to $B_1(\Omega)$.*

Proof. First, note that, for $w \in \Omega$,

$$(T - w)(x \oplus y) = ((T_0 - w)x + Sy) \oplus (T_1 - w)y. \tag{2.36}$$

Since $T \in B_2(\mathbb{D})$, $T - w$ is onto. Hence, from the above equality, it follows that $(T_1 - w)$ is onto.

Now we claim that $\dim\ker(T_1 - w) = 1$ for all $w \in \Omega$. From (2.36), we see that $(x \oplus y)$ belongs to $\ker(T - w)$ if and only if $(T_0 - w)x + Sy = 0$ and $y \in \ker(T_1 - w)$. Therefore, if $\dim\ker(T_1 - w)$ is 0, it must follow that $\ker(T - w) = \ker(T_0 - w)$, which is a contradiction. Hence $\dim\ker(T_1 - w)$ is atleast 1. Now assume that $\dim\ker(T_1 - w) > 1$. Let $v_1(w)$ and $v_2(w)$ be two linearly independent vectors in $\ker(T_1 - w)$. Since $(T_0 - w)$ is onto, there exist $u_1(w), u_2(w) \in H_0$ such that $(T_0 - w)u_i(w) + Sv_i(w) = 0$, $i = 1, 2$. Hence the vectors $(u_1(w) \oplus v_1(w)), (u_2(w) \oplus v_2(w))$ belong to $\ker(T - w)$. Also, since $\dim\ker(T_0 - w) = 1$, there exists $\gamma(w) \in H_0$, such that $(\gamma(w) \oplus 0)$ belongs to $\ker(T - w)$. It is easy to verify that the vectors $\{(u_1(w) \oplus v_1(w)), (u_2(w) \oplus v_2(w)), (\gamma(w) \oplus 0)\}$ are linearly independent. This is a contradiction since $\dim\ker(T - w) = 2$. Therefore $\dim\ker(T_1 - w) \leq 1$. In consequence, $\dim\ker(T_1 - w) = 1$.

Finally, to show that $\bigvee_{w \in \Omega} \ker(T_1 - w) = H_1$, let y be an arbitrary vector in H_1 which is orthogonal to $\bigvee_{w \in \Omega} \ker(T_1 - w)$. Then it follows that $(0 \oplus y)$ is orthogonal to $\ker(T - w)$, $w \in \Omega$. Consequently, $y = 0$. This completes the proof. □

Theorem 2.3.18. *Let $\Omega \subset \mathbb{C}$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are positive definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. Suppose that the operators ${M^{(\alpha)}}^*$ on $(\mathscr{H}, K^\alpha)$ and ${M^{(\beta)}}^*$ on $(\mathscr{H}, K^\beta)$ belong to $B_1(\Omega^*)$. Then the operator ${\mathbb{M}^{(\alpha,\beta)}}^*$ on $(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$ belongs to $B_1(\Omega^*)$. Equivalently, if K^α and K^β are generalized Bergman kernels, then so is the kernel $\mathbb{K}^{(\alpha,\beta)}$.*

Proof. Since the operators ${M^{(\alpha)}}^*$ and ${M^{(\beta)}}^*$ belong to $B_1(\Omega^*)$, by Theorem 2.3.16, the kernel $J_1(K^\alpha, K^\beta)_{|\text{res}\,\Delta}$ is a generalized Bergman kernel. Therefore, from corollary 2.3.4, we deduce that the operator $\begin{pmatrix} {M^{(\alpha+\beta)}}^* & \eta\, \mathsf{inc}^* \\ 0 & {\mathbb{M}^{(\alpha,\beta)}}^* \end{pmatrix}$ belongs to $B_2(\Omega^*)$, where $\eta = \frac{\beta}{\sqrt{\alpha\beta(\alpha+\beta)}}$ and inc is the inclusion operator from $(\mathscr{H}, K^{\alpha+\beta})$ into $(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$. Also, by Theorem 1.1.8, the operator ${M^{(\alpha+\beta)}}^*$ on $(\mathscr{H}, K^{\alpha+\beta})$ belongs to $B_1(\Omega^*)$. Proposition 2.3.17, therefore shows that the operator ${\mathbb{M}^{(\alpha,\beta)}}^*$ on $(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$ belongs to $B_1(\Omega^*)$. □

A smaller class of operators $\mathscr{F}B_n(\Omega)$ from $B_n(\Omega)$, $n \geq 2$, was introduced in [37]. A set of tractable complete unitary invariants and concrete models were given for operators in this class. We give below examples of a large class of operators in $\mathscr{F}B_2(\Omega)$. In case Ω is the unit disc $\mathbb{D}$, these examples include the homogeneous operators of rank 2 in $B_2(\mathbb{D})$ which are known to be in $\mathscr{F}B_2(\mathbb{D})$.

Definition 2.3.19. *An operator T on $H_0 \bigoplus H_1$ is said to be in $\mathscr{F}B_2(\Omega)$ if it is of the form $\begin{bmatrix} T_0 & S \\ 0 & T_1 \end{bmatrix}$, where $T_0, T_1 \in B_1(\Omega)$ and S is a non-zero operator satisfying $T_0 S = S T_1$.*

Theorem 2.3.20. *Let $\Omega \subset \mathbb{C}$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are positive definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. Suppose that the operators $M^{(\alpha)*}$ on $(\mathscr{H}, K^\alpha)$ and $M^{(\beta)*}$ on $(\mathscr{H}, K^\beta)$ belong to $B_1(\Omega^*)$. Then the operator $(J_1 M)^*$ on $(\mathscr{H}, J_1(K^\alpha, K^\beta)_{|\mathrm{res}\,\Delta})$ belongs to $\mathscr{F}B_2(\Omega^*)$.*

Proof. By Theorem 2.3.16, the operator $(J_1M)^*$ on $(\mathscr{H}, J_1(K^\alpha, K^\beta)_{|\mathrm{res}\,\Delta})$ belongs to $B_2(\Omega^*)$, and by Corollary 2.3.4, it is unitarily equivalent to $\begin{pmatrix} M^{(\alpha+\beta)*} & \eta\, \mathrm{inc}^* \\ 0 & \mathbb{M}^{(\alpha,\beta)*} \end{pmatrix}$ on $(\mathscr{H}, K^{\alpha+\beta}) \bigoplus (\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$. By Theorem 1.1.8, the operator $M^{(\alpha+\beta)*}$ on $(\mathscr{H}, K^{\alpha+\beta})$ belongs to $B_1(\Omega^*)$ and by Theorem 2.3.18, the operator $\mathbb{M}^{(\alpha,\beta)*}$ on $(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$ belongs to $B_1(\Omega^*)$. The adjoint of the inclusion operator inc clearly intertwines $M^{(\alpha+\beta)*}$ and $\mathbb{M}^{(\alpha,\beta)*}$. Therefore the operator $(J_1M)^*$ on $(\mathscr{H}, J_1(K^\alpha, K^\beta)_{|\mathrm{res}\,\Delta})$ belongs to $\mathscr{F}B_2(\Omega^*)$. □

Let $\Omega \subset \mathbb{C}$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions $K^{\alpha_1}, K^{\alpha_2}, K^{\beta_1}$ and K^{β_2} are positive definite on $\Omega \times \Omega$ for some $\alpha_i, \beta_i > 0$, $i = 1, 2$. Suppose that the operators $M^{(\alpha_i)*}$ on $(\mathscr{H}, K^{\alpha_i})$ and $M^{(\beta_i)*}$ on $(\mathscr{H}, K^{\beta_i})$, $i = 1, 2$, belong to $B_1(\Omega^*)$. Let $\mathscr{A}_1(\alpha_i, \beta_i)$ be the subspace $\mathscr{A}_1$ of the Hilbert space $(\mathscr{H}, K^{\alpha_i}) \otimes (\mathscr{H}, K^{\beta_i})$ for $i = 1, 2$. Then we have the following corollary.

Corollary 2.3.21. *The operators $\big(M^{(\alpha_1)} \otimes I\big)^*_{|\mathscr{A}_1(\alpha_1,\beta_1)^\perp}$ and $\big(M^{(\alpha_2)} \otimes I\big)^*_{|\mathscr{A}_1(\alpha_2,\beta_2)^\perp}$ are unitarily equivalent if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.*

Proof. If $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then there is nothing to prove. For the converse, assume that the operators $\big(M^{(\alpha_1)} \otimes I\big)^*_{|\mathscr{A}_1(\alpha_1,\beta_1)^\perp}$ and $\big(M^{(\alpha_2)} \otimes I\big)^*_{|\mathscr{A}_1(\alpha_2,\beta_2)^\perp}$ are unitarily equivalent. Then, by Corollary 2.2.10, we see that the operators $\begin{pmatrix} M^{(\alpha_1+\beta_1)*} & \eta_1\, (\mathrm{inc})_1^* \\ 0 & \mathbb{M}^{(\alpha_1,\beta_1)*} \end{pmatrix}$ on $(\mathscr{H}, K^{\alpha_1+\beta_1}) \bigoplus (\mathscr{H}, \mathbb{K}^{(\alpha_1,\beta_1)})$ and $\begin{pmatrix} M^{(\alpha_2+\beta_2)*} & \eta_2\, (\mathrm{inc})_2^* \\ 0 & \mathbb{M}^{(\alpha_2,\beta_2)*} \end{pmatrix}$ on $(\mathscr{H}, K^{\alpha_2+\beta_2}) \bigoplus (\mathscr{H}, \mathbb{K}^{(\alpha_2,\beta_2)})$ are unitarily equivalent, where $\eta_i = \frac{\beta_i}{\sqrt{\alpha_i\beta_i(\alpha_i+\beta_i)}}$ and $(\mathrm{inc})_i$ is the inclusion operator from $(\mathscr{H}, K^{\alpha_i+\beta_i})$ into $(\mathscr{H}, \mathbb{K}^{(\alpha_i,\beta_i)})$, $i = 1, 2$. Since $M^{(\alpha_i)*}$ on $(\mathscr{H}, K^{\alpha_i})$ and $M^{(\beta_i)*}$ on $(\mathscr{H}, K^{\beta_i})$, $i = 1, 2$, belong to $B_1(\Omega^*)$, by Theorem 2.3.20, we conclude that the operator $\begin{pmatrix} M^{(\alpha_i+\beta_i)*} & \eta_i\, (\mathrm{inc})_i^* \\ 0 & \mathbb{M}^{(\alpha_i,\beta_i)*} \end{pmatrix}$ belongs to $\mathscr{F}B_2(\Omega^*)$ for $i = 1, 2$. Therefore, by [37, Theorem 2.10], we obtain that

$$\mathscr{K}_{M^{(\alpha_1+\beta_1)*}} = \mathscr{K}_{M^{(\alpha_2+\beta_2)*}} \quad \text{and} \quad \frac{\eta_1 \,\|(\mathrm{inc})_1^*(t_1)\|^2}{\|t_1\|^2} = \frac{\eta_2 \,\|(\mathrm{inc})_2^*(t_2)\|^2}{\|t_2\|^2}, \tag{2.37}$$

where $\mathscr{K}_{M^{(\alpha_i+\beta_i)*}}$, $i = 1, 2$, is the curvature of the operator $M^{(\alpha_i+\beta_i)*}$, and t_1 and t_2 are two non-vanishing holomorphic sections of the vector bundles $E_{\mathbb{M}^{(\alpha_1,\beta_1)*}}$ and $E_{\mathbb{M}^{(\alpha_2,\beta_2)*}}$, respectively. Note that, for $i = 1, 2$, $t_i(w) = \mathbb{K}^{(\alpha_i,\beta_i)}(\cdot, w)$ is a holomorphic non-vanishing section of the vector bundle $E_{\mathbb{M}^{(\alpha_i,\beta_i)*}}$, and also $(\mathrm{inc})_i^*(\mathbb{K}^{(\alpha_i,\beta_i)}(\cdot, w)) = K^{\alpha_i+\beta_i}(\cdot, w)$, $w \in \Omega$. Therefore the second equality in (2.37) implies that

$$\frac{\eta_1 K^{\alpha_1+\beta_1}(w,w)}{K^{\alpha_1+\beta_1}(w,w)\partial\bar{\partial}\log K(w,w)} = \frac{\eta_2 K^{\alpha_2+\beta_2}(w,w)}{K^{\alpha_2+\beta_2}(w,w)\partial\bar{\partial}\log K(w,w)}, \quad w \in \Omega,$$

or equivalently $\eta_1 = \eta_2$. Furthermore, it is easy to see that $\mathscr{K}_{M^{(\alpha_1+\beta_1)*}} = \mathscr{K}_{M^{(\alpha_2+\beta_2)*}}$ if and only if $\alpha_1+\beta_1 = \alpha_2+\beta_2$. Hence, from (2.37), we see that

$$\alpha_1+\beta_1 = \alpha_2+\beta_2 \quad \text{and} \quad \eta_1 = \eta_2. \tag{2.38}$$

Then a simple calculation shows that (2.38) is equivalent to $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, completing the proof. □

2.4 Some Applications

2.4.1 A limit computation

In this subsection, we give an alternative computation for the curvature of an operator in the Cowen - Douglas class $B_1(\Omega)$.

Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega\times\Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are non-negative definite on $\Omega\times\Omega$ for some $\alpha,\beta > 0$. For a non-negative integer p, let $K^{\otimes}_{\mathscr{A}_p}$ be the reproducing kernel of $\mathscr{A}_p$, where $\mathscr{A}_p$ is defined in (2.22).

One way to prove both of the following two lemmas is to make the change of variables

$$u_1 = \tfrac{1}{2}(z_1-\zeta_1),\ldots,u_m = \tfrac{1}{2}(z_m-\zeta_m);\ v_1 = \tfrac{1}{2}(z_1+\zeta_1),\ldots,v_m = \tfrac{1}{2}(z_m+\zeta_m).$$

We give the details for the proof of the first lemma. The proof for the second one follows by similar arguments.

Lemma 2.4.1. *Suppose that $f : \Omega\times\Omega \to \mathbb{C}$ is a holomorphic function satisfying $f_{|\Delta} = 0$. Then for each $z_0 \in \Omega$, there exists a neighbourhood $\Omega_0 \subset \Omega$ (independent of f) of z_0 and holomorphic functions $f_1, f_2,\ldots,f_m$ on $\Omega_0\times\Omega_0$ such that*

$$f(z,\zeta) = \sum_{i=1}^{m}(z_i-\zeta_i)f_i(z,\zeta),\ z,\zeta \in \Omega_0.$$

Lemma 2.4.2. *Suppose that $f : \Omega\times\Omega \to \mathbb{C}$ is a holomorphic function satisfying $f_{|\Delta} = 0$ and $\big((\frac{\partial}{\partial\zeta_j})f(z,\zeta)\big)_{|\Delta} = 0,\ j = 1,\ldots,m$. Then for each $z_0 \in \Omega$, there exists a neighbourhood $\Omega_0 \subset \Omega$ (independent of f) of z_0 and holomorphic functions f_{ij}, $1 \le i \le j \le m$, on $\Omega_0\times\Omega_0$ such that*

$$f(z,\zeta) = \sum_{1\le i\le j\le m}(z_i-\zeta_i)(z_j-\zeta_j)f_{ij}(z,\zeta),\ z,\zeta \in \Omega_0.$$

Note that the image of the diagonal set $\Delta \subseteq \Omega\times\Omega$ under the map $(\boldsymbol{u},\boldsymbol{v}) : \Omega\times\Omega \to \mathbb{C}^{2m}$, where $(\boldsymbol{u},\boldsymbol{v}) = (u_1,\ldots,u_m,v_1,\ldots,v_m)$, is the set $\{(0,v) : v \in \Omega\}$. Therefore we may choose a neighbourhood of $(0,z_0)$ which is a polydisc contained in $\widehat{\Omega} := (\boldsymbol{u},\boldsymbol{v})(\Omega\times\Omega)$. Let f be a holomorphic function on $\Omega \times \Omega$ vanishing on the set Δ. Setting $g := f \circ (\boldsymbol{u}, \boldsymbol{v})^{-1}$ on $\hat{\Omega}$, we see that g is a holomorphic function on $\hat{\Omega}$ vanishing on the set $\{(0, v) : v \in \Omega\}$. Therefore g has a power series representation around $(0, z_0)$ of the form $\sum_{\boldsymbol{i},\boldsymbol{j}} a_{\boldsymbol{i}\boldsymbol{j}} u^{\boldsymbol{i}} (v - z_0)^{\boldsymbol{j}}$, where $\sum_{\boldsymbol{j}} a_{0\boldsymbol{j}} (v - z_0)^{\boldsymbol{j}} = 0$ and $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}_+^m$, on the chosen polydisc. Hence $a_{0\boldsymbol{j}} = 0$ for all $\boldsymbol{j} \in \mathbb{Z}_+^m$, and the power series is of the form $\sum_{\ell=1}^m u_\ell g_\ell(u, v)$, where

$$g_\ell(u, v) = \sum_{\boldsymbol{i}\boldsymbol{j}} a_{\boldsymbol{i}\boldsymbol{j}} u^{\boldsymbol{i}-e_\ell} (v - z_0)^{\boldsymbol{j}},\ 1 \le \ell \le m.$$

Here the sum is over all multi-indices $\boldsymbol{i}$ satisfying $i_1 = 0, \ldots, i_{\ell-1} = 0, i_\ell \ge 1$ while $\boldsymbol{j}$ remains arbitrary. Pulling this expression back to $\Omega \times \Omega$ under the bi-holomorphic map $(\boldsymbol{u}, \boldsymbol{v})$, we obtain the expansion of f in a neighbourhood of (z_0, z_0) as prescribed in the Lemma 2.4.1.

Theorem 2.4.3. *For z in Ω and $1 \le i, j \le m$, we have*

$$\lim_{\substack{\zeta_i \to z_i \\ \zeta_j \to z_j}} \left(\frac{K^{\otimes}_{\mathscr{A}_0}(z, \zeta; z, \zeta)}{(z_i - \zeta_i)(\bar{z}_j - \bar{\zeta}_j)} \Big|_{\zeta_l = z_l, l \ne i, j} \right) = \tfrac{\alpha\beta}{(\alpha+\beta)} K(z, z)^{\alpha+\beta} \partial_i \bar{\partial}_j \log K(z, z),$$

where $\frac{K^{\otimes}_{\mathscr{A}_0}(z,\zeta;z,\zeta)}{(z_i-\zeta_i)(\bar{z}_j-\bar{\zeta}_j)}\Big|_{\zeta_l = z_l, l \ne i, j}$ *is the restriction of the function* $\frac{K^{\otimes}_{\mathscr{A}_0}(z,\zeta;z,\zeta)}{(z_i-\zeta_i)(\bar{z}_j-\bar{\zeta}_j)}$ *to the set* $\{(z, \zeta) \in \Omega \times \Omega : z_l = \zeta_l, l = 1, \ldots, m, l \ne i, j\}$.

Proof. Let $K^{\otimes}_{\mathscr{A}_0 \ominus \mathscr{A}_1}(z, \zeta; w, v)$ be the reproducing kernels of $\mathscr{A}_0 \ominus \mathscr{A}_1$. Fix a point z_0 in Ω. Choose a neighbourhood Ω_0 of z_0 in Ω such that the conclusions of Lemma 2.4.1 and Lemma 2.4.2 are valid. Now we restrict the kernels K^α and K^β to $\Omega_0 \times \Omega_0$.

Let f be an arbitrary function in $\mathscr{A}_1$. Then, by definition, f satisfies the hypothesis of Lemma 2.4.2, and therefore, it follows that

$$\lim_{\zeta_i \to z_i} \left(\frac{f(z, \zeta)}{(z_i - \zeta_i)} \Big|_{z_l = \zeta_l, l \ne i} \right) = 0,\ \ i = 1, \ldots, m. \tag{2.39}$$

Let $\{h_n\}_{n \in \mathbb{Z}_+}$ be an orthonormal basis of $\mathscr{A}_1$. Since the series $\sum_{n=0}^\infty h_n(z, \zeta) \overline{h_n(z, \zeta)}$ converges uniformly to $K^{\otimes}_{\mathscr{A}_1}(z, \zeta; z, \zeta)$ on the compact subsets of $\Omega_0 \times \Omega_0$, using (2.39) we see that

$$\begin{aligned}\lim_{\substack{\zeta_i \to z_i \\ \zeta_j \to z_j}} \left(\frac{K^{\otimes}_{\mathscr{A}_1}(z, \zeta; z, \zeta)}{(z_i - \zeta_i)(\bar{z}_j - \bar{\zeta}_j)} \Big|_{\zeta_l = z_l, l \ne i, j} \right) &= \sum_{n=0}^\infty \lim_{\zeta_i \to z_i} \left(\frac{h_n(z, \zeta)}{(z_i - \zeta_i)} \Big|_{z_l = \zeta_l, l \ne i} \right) \lim_{\zeta_j \to z_j} \overline{\left(\frac{h_n(z, \zeta)}{(z_j - \zeta_j)} \Big|_{z_l = \zeta_l, l \ne j} \right)} \\ &= 0.\end{aligned}$$

Since $K^{\otimes}_{\mathscr{A}_0} = K^{\otimes}_{\mathscr{A}_0 \ominus \mathscr{A}_1} + K^{\otimes}_{\mathscr{A}_1}$, the above equality leads to

$$\lim_{\substack{\zeta_i \to z_i \\ \zeta_j \to z_j}} \left(\frac{K^{\otimes}_{\mathscr{A}_0}(z, \zeta; z, \zeta)}{(z_i - \zeta_i)(\bar{z}_j - \bar{\zeta}_j)} \Big|_{\zeta_l = z_l, l \ne i, j} \right) = \lim_{\substack{\zeta_i \to z_i \\ \zeta_j \to z_j}} \left(\frac{K^{\otimes}_{\mathscr{A}_0 \ominus \mathscr{A}_1}(z, \zeta; z, \zeta)}{(z_i - \zeta_i)(\bar{z}_j - \bar{\zeta}_j)} \Big|_{\zeta_l = z_l, l \ne i, j} \right).$$

Now let $\{e_n\}_{n\in\mathbb{Z}_+}$ be an orthonormal basis of $\mathscr{A}_0 \ominus \mathscr{A}_1$. Since each $e_n \in \mathscr{A}_0$, by Lemma 2.4.1, there exist holomorphic functions $e_{n,i}$, $1 \le i \le m$, on $\Omega_0 \times \Omega_0$ such that

$$e_n(z,\zeta) = \sum_{i=1}^{m} (z_i - \zeta_i) e_{n,i}(z,\zeta),\ z,\zeta \in \Omega_0.$$

Thus for $1 \le i \le m$, we have

$$\lim_{\zeta_i \to z_i} \left(\frac{e_n(z,\zeta)}{(z_i - \zeta_i)} \Big|_{z_l=\zeta_l, l\neq i} \right) = e_{n,i}(z,z),\ z \in \Omega_0. \tag{2.40}$$

Since the series $\sum_{n=0}^{\infty} e_n(z,\zeta)\overline{e_n(z,\zeta)}$ converges to $K^{\otimes}_{\mathscr{A}_0\ominus\mathscr{A}_1}$ uniformly on compact subsets of $\Omega_0 \times \Omega_0$, using (2.40), we see that

$$\lim_{\substack{\zeta_i \to z_i \\ \zeta_j \to z_j}} \left(\frac{K^{\otimes}_{\mathscr{A}_0\ominus\mathscr{A}_1}(z,\zeta;z,\zeta)}{(z_i - \zeta_i)(\bar{z}_j - \bar{\zeta}_j)} \Big|_{\zeta_l = z_l, l\neq i,j} \right) = \sum_{n=0}^{\infty} e_{n,i}(z,z)\overline{e_{n,j}(z,z)},\ z \in \Omega_0. \tag{2.41}$$

Recall that by Theorem 2.2.3, the map $\mathscr{R}_1 : \mathscr{A}_0 \ominus \mathscr{A}_1 \to (\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$ given by

$$\mathscr{R}_1 f = \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \begin{pmatrix} (\beta\partial_1 f - \alpha\partial_{m+1} f)_{|\Delta} \\ \vdots \\ (\beta\partial_m f - \alpha\partial_{2m} f)_{|\Delta} \end{pmatrix},\ f \in \mathscr{A}_0 \ominus \mathscr{A}_1$$

is unitary. Hence $\{\mathscr{R}_1(e_n)\}_n$ is an orthonormal basis for $(\mathscr{H}, \mathbb{K}^{\alpha,\beta})$ and consequently

$$\sum_{n=0}^{\infty} \mathscr{R}_1(e_n)(z)\mathscr{R}_1(e_n)(w)^* = \mathbb{K}^{(\alpha,\beta)}(z,w),\ z,w \in \Omega_0. \tag{2.42}$$

A direct computation shows that

$$\big((\beta\partial_i - \alpha\partial_{m+i})e_n(z,\zeta)\big)_{|\Delta} = (\alpha+\beta)e_{n,i}(z,\zeta)_{|\Delta},\ 1 \le i \le m,\ n \ge 0.$$

Therefore $\mathscr{R}_1(e_n)(z) = \sqrt{\frac{\alpha+\beta}{\alpha\beta}} \begin{pmatrix} e_{n,1}(z,z) \\ \vdots \\ e_{n,m}(z,z) \end{pmatrix}$. Thus using (2.42) we obtain

$$\sum_{n=0}^{\infty} \begin{pmatrix} e_{n,1}(z,z) \\ \vdots \\ e_{n,m}(z,z) \end{pmatrix} \begin{pmatrix} e_{n,1}(z,z) \\ \vdots \\ e_{n,m}(z,z) \end{pmatrix}^* = \tfrac{\alpha\beta}{(\alpha+\beta)} \mathbb{K}^{(\alpha,\beta)}(z,z),\ \ z \in \Omega_0.$$

Now the proof is complete using (2.41). □

The following corollary is immediate by choosing $\alpha = 1 = \beta$. It also gives an alternative for computing the Gaussian curvature defined in (2.2) whenever the metric is of the form $K(z,z)^{-1}$ for some positive definite kernel K defined on $\Omega\times\Omega$, where $\Omega\subset\mathbb{C}$ is a bounded domain. Indeed, the assumption that $\boldsymbol{T}$ is in $B_1(\Omega)$ is not necessary to arrive at the formula in the corollary below.

Corollary 2.4.4. *Let $\boldsymbol{T}$ be a commuting m-tuple in the Cowen-Douglas class $B_1(\Omega)$ realized as the adjoint of the m-tuple of multiplication operators by coordinate functions on a Hilbert space $\mathscr{H}\subseteq \mathrm{Hol}(\Omega_0)$, for some open subset Ω_0 of Ω possessing a reproducing kernel K. The curvature $\mathscr{K}_{\boldsymbol{T}}(z)$ is then given by the formula*

$$\mathscr{K}_{\boldsymbol{T}}(z)_{i,j} = \frac{2}{K(z,z)^2}\lim_{\substack{\zeta_i\to z_i\\ \zeta_j\to z_j}}\left(\frac{K^{\otimes}_{\mathscr{A}_0}(z,\zeta;z,\zeta)}{(z_i-\zeta_i)(\bar z_j-\bar\zeta_j)}\Big|_{\zeta_l=z_l,l\neq i,j}\right),\ z\in\Omega,\ 1\le i,j\le m.$$

2.4.2 Some additional results

Continuing our investigation of the behaviour of a non-negative definite kernel K and the non-negative definite kernel $\big(K^2(z,w)\partial_i\bar\partial_j\log K(z,w)\big)_{i,j=1}^m$ obtained from it, we prove the following monotonicity property.

Proposition 2.4.5. *Let $\Omega\subset\mathbb{C}^m$ be a bounded domain. If $K_1,K_2:\Omega\times\Omega\to\mathbb{C}$ are two non-negative definite kernels satisfying $K_1\ge K_2$, then*

$$\big(K_1^2\partial_i\bar\partial_j\log K_1(z,w)\big)_{i,j=1}^m \ge \big(K_2^2\partial_i\bar\partial_j\log K_2(z,w)\big)_{i,j=1}^m.$$

Proof. Set $K_3 = K_1 - K_2$. By hypothesis, K_3 is non-negative definite on $\Omega\times\Omega$. For $1\le i,j\le m$, a straightforward computation shows that

$$\begin{aligned}K_1^2\partial_i\bar\partial_j\log K_1 &= K_2^2\partial_i\bar\partial_j\log K_2 + K_3^2\partial_i\bar\partial_j\log K_3\\ &\qquad + K_2\partial_i\bar\partial_j K_3 + K_3\partial_i\bar\partial_j K_2 - \partial_i K_2\bar\partial_j K_3 - \partial_i K_3\bar\partial_j K_2.\end{aligned}\tag{2.43}$$

Now set $\gamma_i(w) = K_2(\cdot,w)\otimes\bar\partial_i K_3(\cdot,w) - \bar\partial_i K_2(\cdot,w)\otimes K_3(\cdot,w)$, $1\le i\le m,\ w\in\Omega$. For $1\le i,j\le m$ and $z,w\in\Omega$, then we have

$$\begin{aligned}&\big\langle\gamma_j(w),\gamma_i(z)\big\rangle\\ &\quad= \big(K_2\partial_i\bar\partial_j K_3\big)(z,w) + \big(K_3\partial_i\bar\partial_j K_2\big)(z,w) - \big(\partial_i K_2\bar\partial_j K_3\big)(z,w) - \big(\partial_i K_3\bar\partial_j K_2\big)(z,w).\end{aligned}\tag{2.44}$$

Combining (2.43) and (2.44), we obtain

$$\begin{aligned}&\big(\big(K_1^2\partial_i\bar\partial_j\log K_1\big)(z,w)\big)_{i,j=1}^m\\ &\quad= \big(\big(K_2^2\partial_i\bar\partial_j\log K_2\big)(z,w)\big)_{i,j=1}^m + \big(\big(K_3^2\partial_i\bar\partial_j\log K_3\big)(z,w)\big)_{i,j=1}^m + \big(\big\langle\gamma_j(w),\gamma_i(z)\big\rangle\big)_{i,j=1}^m.\end{aligned}$$

It follows from Lemma 2.1.1 that $\big(\langle\gamma_j(w),\gamma_i(z)\rangle\big)_{i,j=1}^m$ is non-negative definite on $\Omega\times\Omega$. The proof is now complete since sum of two non-negative definite kernels remains non-negative definite. □

The theorem given below shows that the differential operator is bounded from $(\mathscr{H},K)$ to $(\mathscr{H},\mathbb{K})$.

Theorem 2.4.6. *Let $\Omega\subset\mathbb{C}^m$ be a bounded domain. Let $K:\Omega\times\Omega\to\mathbb{C}$ be a non-negative definite kernel. Suppose that the Hilbert space $(\mathscr{H},K)$ contains the constant function 1. Then the linear operator $\boldsymbol{\partial}:(\mathscr{H},K)\to(\mathscr{H},\mathbb{K})$, where $\boldsymbol{\partial}f=(\partial_1 f,\ldots,\partial_m f)^{\rm tr}$, $f\in(\mathscr{H},K)$, is bounded with $\|\boldsymbol{\partial}\|\le\|1\|_{(\mathscr{H},K)}$.*

Proof. It is easily verified that the map $\boldsymbol{\partial}$ is unitary from $\ker\boldsymbol{\partial}^\perp$ to $(\mathscr{H},(\partial_i\bar\partial_j K)_{i,j=1}^m)$, and therefore is contractive from $(\mathscr{H},K)$ to $(\mathscr{H},(\partial_i\bar\partial_j K)_{i,j=1}^m)$. Hence, to complete the proof, it suffices to show that $(\mathscr{H},(\partial_i\bar\partial_j K)_{i,j=1}^m)$ is contained in $(\mathscr{H},\mathbb{K})$ and the inclusion map is bounded by $\|1\|_{(\mathscr{H},K)}$.

Set $c=\|1\|^2_{(\mathscr{H},K)}$. Choose an orthonormal basis $\{e_n(z)\}_{n\ge0}$ of $(\mathscr{H},K)$ with $e_0(z)=\frac{1}{\sqrt{c}}$. Then

$$K(z,w)-\frac{1}{c}=\sum_{i=1}^{\infty}e_i(z)\overline{e_i(w)},\ z,w\in\Omega.$$

Hence $K(z,w)-\frac1c$ is non-negative definite on $\Omega\times\Omega$, or equivalently $cK-1$ is non-negative definite on $\Omega\times\Omega$. Therefore, by Corollary 2.1.5, it follows that $\big(\,(cK-1)^2\partial_i\bar\partial_j\log(cK-1)\,\big)_{i,j=1}^m$ is non-negative definite on $\Omega\times\Omega$. Note that, for $z,w\in\Omega$, we have

$$\begin{aligned}&\big((cK-1)^2\partial_i\bar\partial_j\log(cK-1)\big)(z,w)\\&\quad=(cK-1)(z,w)\big(\partial_i\bar\partial_j(cK-1)\big)(z,w)-\big(\partial_i(cK-1)\big)(z,w)\big(\bar\partial_j(cK-1)\big)(z,w)\\&\quad=c^2K(z,w)\partial_i\bar\partial_jK(z,w)-c\partial_i\bar\partial_jK(z,w)-c^2\partial_iK(z,w)\bar\partial_jK(z,w)\\&\quad=c^2K^2\partial_i\bar\partial_j\log K(z,w)-c\partial_i\bar\partial_jK(z,w).\end{aligned}$$

Hence we conclude that

$$\big(\,\partial_i\bar\partial_jK(z,w)\,\big)_{i,j=1}^m\preceq c\big(\,K^2\partial_i\bar\partial_j\log K(z,w)\,\big)_{i,j=1}^m.$$

The proof is now complete by using [47, Theorem 6.25]. □

Corollary 2.4.7. *Let $\Omega\subset\mathbb{C}^m$ be a bounded domain. Let $K:\Omega\times\Omega\to\mathbb{C}$ be a non-negative definite kernel. Suppose that K is normalized at the point $w_0\in\Omega$, that is, $K(\cdot,w_0)$ is the constant function 1. Then the linear operator $\boldsymbol{\partial}:(\mathscr{H},K)\to(\mathscr{H},\mathbb{K})$ is contractive.*

Proof. By hypothesis, we have $\|1\|^2_{(\mathscr{H},K)}=\langle K(\cdot,w_0),K(\cdot,w_0)\rangle_{(\mathscr{H},K)}=K(w_0,w_0)=1$. The proof now follows from Theorem 2.4.6. □

2.5 Relationship between the Jet construction and the Gamma construction

The motivation for Theorem 2.5.1 in this section comes from two different constructions for homogeneous operators. The homogeneous operators in the Cowen-Douglas class $B_1(\mathbb{D})$ are easily described using the curvature invariant. However, to determine which operators in $B_n(\mathbb{D})$ are homogeneous, the curvature is of very little use. From the beginning, two distinct methods were available to answer this question. The first of these used the jet construction of [28] and the second one used an intertwining operator called the *Gamma* map in [39]. It turns out that both of these methods succeed in identifying all the homogeneous operators in $B_2(\mathbb{D})$. The answer for $n > 2$ is more complicated. In this section, we establish a correspondence between the homogeneous operators obtained using the jet construction to those obtained via the Γ - map. Indeed the theorem goes beyond the homogeneous operators and establishes a relationship between these two constructions in much greater generality.

Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ be a sesqui-analytic function such that the functions K^α and K^β are non-negative definite on $\Omega \times \Omega$ for some $\alpha, \beta > 0$. The map $\Gamma : \big(\mathscr{H}, K^{\alpha+\beta}\big) \bigoplus \big(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)}\big) \to \text{Hol}\,(\Omega, \mathbb{C}^{m+1})$ is defined by

$$\Gamma(f \oplus g) = \begin{pmatrix} f \\ \frac{1}{\alpha+\beta}\boldsymbol{\partial} f \end{pmatrix} + \mu \begin{pmatrix} 0 \\ g \end{pmatrix}, \;\; f \oplus g \in \big(\mathscr{H}, K^{\alpha+\beta}\big) \bigoplus \big(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)}\big), \tag{2.45}$$

where $\boldsymbol{\partial} f$ is the vector $(\partial_1 f, \ldots, \partial_m f)^{\text{tr}}$, and $\mu > 0$ is arbitrary. Note that the map Γ is one-to-one. Define an inner product on the linear space ran Γ by requiring the map Γ to be a unitary. Pick any orthonormal basis $\{e'_n : n \geq 0\}$ in $(\mathscr{H}, K^{\alpha+\beta})$ and $\{e''_n : n \geq 0\}$ in $(\mathscr{H}, \mathbb{K}^{(\alpha,\beta)})$. Setting $e_n := e'_n \oplus e''_n$, $n \geq 0$, we see that $\sum_{n=0}^{\infty} (\Gamma e_n)(z) \overline{(\Gamma e_n)(w)^{\text{tr}}}$, $z, w \in \Omega$, which is

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha+\beta} I_m \end{pmatrix} \Big(\partial^i \bar{\partial}^j K^{\alpha+\beta}(z,w)\Big)_{i,j=0}^{m} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha+\beta} I_m \end{pmatrix} + \mu^2 \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{K}^{(\alpha,\beta)} \end{pmatrix}, \tag{2.46}$$

is the reproducing kernel for the Hilbert space ran Γ.

Since ran $\Gamma \subseteq \text{Hol}(\Omega, \mathbb{C}^{m+1})$, every vector in ran Γ is of the form $\binom{f}{g}$, where $f \in \text{Hol}(\Omega, \mathbb{C})$ and $g \in \text{Hol}(\Omega, \mathbb{C}^m)$. Let U be the linear map from ran Γ to $\text{Hol}(\Omega, \mathbb{C}^{m+1})$ given by

$$U \binom{f}{g} = \binom{f}{\beta g}, \;\; \binom{f}{g} \in \text{ran}\,\Gamma.$$

Note that U is also one-to-one. Therefore, as before, we identify ran U with ran Γ as a Hilbert space using the map U. Then we see that ran U possesses a reproducing kernel which is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{\beta}{\alpha+\beta} I_m \end{pmatrix} \Big(\partial^i \bar{\partial}^j K^{\alpha+\beta}(z,w)\Big)_{i,j=0}^{m} \begin{pmatrix} 1 & 0 \\ 0 & \frac{\beta}{\alpha+\beta} I_m \end{pmatrix} + \beta^2 \mu^2 \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{K}^{(\alpha,\beta)} \end{pmatrix}.$$

We recall that $\mathscr{S}_0 = \mathscr{A}_0^\perp$ and $\mathscr{S}_1 = \mathscr{A}_0 \ominus \mathscr{A}_1$, and $\mathscr{R}_0$ is the map from $(\mathscr{H}K^\alpha) \otimes (\mathscr{H}, K^\beta)$ to $(\mathscr{H}, K^{\alpha+\beta})$ given by $f \to f_{|\Delta}$. The map J_1 is defined in (1.3) and R is the map given by $R(\boldsymbol{h}) = \boldsymbol{h}_{|\Delta}$, $\boldsymbol{h} \in \operatorname{ran} J_1$. Finally, set $\hat{\mathscr{R}}_1 = -\mathscr{R}_1$, where $\mathscr{R}_1$ is defined in (2.17).

Theorem 2.5.1. *If* $\mu = \sqrt{\frac{\alpha}{\beta(\alpha+\beta)}}$*, then the following diagram of Hilbert modules is commutative:*

$$\begin{array}{ccccc} \mathscr{S}_0 \bigoplus \mathscr{S}_1 & \xrightarrow{\quad J_1 \quad} & J_1(\mathscr{S}_0 \bigoplus \mathscr{S}_1) & \xrightarrow{R} & RJ_1(\mathscr{S}_0 \bigoplus \mathscr{S}_1) \\ \mathscr{R}_0 \bigoplus \hat{\mathscr{R}}_1 \Big\downarrow & & & & \Big\downarrow id \\ (\mathscr{H}, K^{\alpha+\beta}) \bigoplus (\mathscr{H}, \mathbb{K}^{(\alpha,\beta)}) & \xrightarrow{\quad \Gamma \quad} & \operatorname{ran} \Gamma & \xrightarrow{\quad U \quad} & \operatorname{ran} U \end{array} \tag{2.47}$$

Proof. Let $f_0 \oplus f_1$ be an arbitrary element in $\mathscr{S}_0 \oplus \mathscr{S}_1$. Note that

$$RJ_1(f_0 \oplus f_1) = \begin{pmatrix} (f_0+f_1)_{|\Delta} \\ \big(\partial_{m+1}(f_0+f_1)\big)_{|\Delta} \\ \vdots \\ \big(\partial_{2m}(f_0+f_1)\big)_{|\Delta} \end{pmatrix} = \begin{pmatrix} (f_0)_{|\Delta} \\ \big(\partial_{m+1}(f_0+f_1)\big)_{|\Delta} \\ \vdots \\ \big(\partial_{2m}(f_0+f_1)\big)_{|\Delta} \end{pmatrix}, \tag{2.48}$$

where the last equality follows since $f_1 \in \mathscr{S}_1$. Computing $\mathscr{R}_0 \bigoplus \hat{\mathscr{R}}_1$ on $f_0 \oplus f_1$, we see that

$$\big(\mathscr{R}_0 \bigoplus \hat{\mathscr{R}}_1\big)(f_0 \oplus f_1) = \big((f_0)_{|\Delta}\big) \oplus \frac{1}{\sqrt{\alpha\beta(\alpha+\beta)}} \begin{pmatrix} (\alpha\partial_{m+1} f_1 - \beta\partial_1 f_1)_{|\Delta} \\ \vdots \\ (\alpha\partial_{2m} f_1 - \beta\partial_m f_1)_{|\Delta} \end{pmatrix}.$$

Therefore applying the map $(U \circ \Gamma)$ and using $\mu = \sqrt{\frac{\alpha}{\beta(\alpha+\beta)}}$, we obtain that

$$U\Gamma\big(\mathscr{R}_0 \bigoplus \hat{\mathscr{R}}_1\big)(f_0 \oplus f_1) = \begin{pmatrix} (f_0)_{|\Delta} \\ \frac{\beta}{\alpha+\beta}\partial_1\big((f_0)_{|\Delta}\big) \\ \vdots \\ \frac{\beta}{\alpha+\beta}\partial_m\big((f_0)_{|\Delta}\big) \end{pmatrix} + \frac{1}{(\alpha+\beta)} \begin{pmatrix} 0 \\ (\alpha\partial_{m+1} f_1 - \beta\partial_1 f_1)_{|\Delta} \\ \vdots \\ (\alpha\partial_{2m} f_1 - \beta\partial_m f_1)_{|\Delta} \end{pmatrix}. \tag{2.49}$$

Thus, in view of (2.48) and (2.49), we will be done if we can show

$$\tfrac{\beta}{\alpha+\beta}\partial_i\big((f_0)_{|\Delta}\big) + \tfrac{1}{\alpha+\beta}(\alpha\partial_{m+i} f_1 - \beta\partial_i f_1)_{|\Delta} = \big(\partial_{m+i}(f_0+f_1)\big)_{|\Delta},\ \ i = 1,\ldots,m.$$

To verify this, it suffices to show that

$$\tfrac{\beta}{\alpha+\beta}\partial_i\big((f_0)_{|\Delta}\big) = \big(\partial_{m+i} f_0\big)_{|\Delta} \text{ and } \tfrac{1}{\alpha+\beta}(\alpha\partial_{m+i} f_1 - \beta\partial_i f_1)_{|\Delta} = \big(\partial_{m+i} f_1\big)_{|\Delta},\ \ i = 1,\ldots,m. \tag{2.50}$$

Since $f_0 \in \mathscr{S}_0$ and $\mathscr{S}_0 \subseteq \ker \mathscr{R}_1^{\perp}$, it follows that $\big(\alpha\partial_{m+i} f_0 - \beta\partial_i f_0\big)_{|\Delta} = 0$. Therefore,

$$\begin{aligned}
\tfrac{\beta}{\alpha+\beta}\partial_i\big((f_0)_{|\Delta}\big) &= \tfrac{\beta}{\alpha+\beta}\big((\partial_i f_0)_{|\Delta} + (\partial_{m+i} f_0)_{|\Delta}\big) \\
&= \tfrac{\alpha}{\alpha+\beta}(\partial_{m+i} f_0)_{|\Delta} + \tfrac{\beta}{\alpha+\beta}(\partial_{m+i} f_0)_{|\Delta} \\
&= (\partial_{m+i} f_0)_{|\Delta},
\end{aligned}$$

where, for the first equality, we have used $\partial_i\big((f_0)_{|\Delta}\big) = (\partial_i f_0)\big)_{|\Delta} + (\partial_{m+i} f_0)_{|\Delta}$ (see the proof of Lemma 2.2.1). Finally, since $f_1 \in \mathscr{S}_1$, we have $(\partial_i f_1)_{|\Delta} + (\partial_{m+i} f_1)_{|\Delta} = 0$ by Lemma 2.2.1. Therefore

$$\begin{aligned}
\tfrac{1}{\alpha+\beta}(\alpha\partial_{m+i} f_1 - \beta\partial_i f_1)_{|\Delta} &= \tfrac{\alpha}{\alpha+\beta}(\partial_{m+i} f_1)_{|\Delta} + \tfrac{\beta}{\alpha+\beta}(\partial_{m+i} f_1)_{|\Delta} \\
&= (\partial_{m+i} f_1)_{|\Delta}.
\end{aligned}$$

This completes the proof. □

Chapter 3

The generalized Bergman metrics and the generalized Wallach set

In this chapter, we study the Generalized Bergman metrics and the generalized Wallach set. It is shown that if $\Omega \subset \mathbb{C}^m$ is a bounded domain and $K : \Omega \times \Omega \to \mathbb{C}$ is a quasi-invariant kernel, then $K^t\big(\partial_i \bar{\partial}_j \log K\big)_{i,j=1}^m$ is also a quasi-invariant kernel whenever t is in the generalized Wallach set $G\mathscr{W}_\Omega(K)$. The generalized Wallach set for the Bergman kernel of the open Euclidean unit ball in $\mathbb{C}^m$ is determined.

3.1 Introduction and background

Let Ω be a bounded domain in $\mathbb{C}^m$. Recall that the Bergman space $A^2(\Omega)$ is the Hilbert space of all square integrable analytic functions defined on Ω. The inner product of $A^2(\Omega)$ is given by the formula

$$\langle f, g \rangle := \int_\Omega f(z)\overline{g(z)}\ \mathrm{dV(z)},\ \mathrm{f, g} \in \mathrm{A}^2(\Omega),$$

where dV(z) is the area measure on $\mathbb{C}^m$. The evaluation linear functional $f \mapsto f(w)$ is bounded on $A^2(\Omega)$ for all $w \in \Omega$. Consequently, the Bergman space is a reproducing kernel Hilbert space. The reproducing kernel of the Bergman space $A^2(\Omega)$ is called the Bergman kernel of Ω and is denoted by B_Ω.

It is known that $B_\Omega(w, w) > 0$ for all $w \in \Omega$. The Bergman metric of Ω is defined to be $\big(\partial_i \bar{\partial}_j \log B_\Omega(w, w)\big)_{i,j=1}^m$, $w \in \Omega$, which is evidently non-negative definite. Finally, let us define a generalized Bergman metric of Ω to be the bilinear form $B_\Omega(w, w)^t\big(\partial_i \bar{\partial}_j \log B_\Omega(w, w)\big)_{i,j=1}^m, w \in \Omega,\ t \in \mathbb{R}$. Clearly, such a generalized Bergman metric is also non-negative definite at each point w in Ω. It is important to note that the notion of generalized Bergman metric introduced here is different from the one introduced in [26].

If $\Omega \subset \mathbb{C}^m$ is a bounded symmetric domain, then the ordinary Wallach set $\mathscr{W}_\Omega$ is defined as $\{t>0 : B_\Omega^t$ is non-negative definite$\}$. Here B_Ω^t, $t>0$, makes sense since every bounded symmetric domain Ω is simply connected and the Bergman kernel on it is non-vanishing. If Ω is the Euclidean unit ball $\mathbb{B}_m$, then the Bergman kernel is given by

$$B_{\mathbb{B}_m}(z,w) = (1-\langle z,w\rangle)^{-(m+1)},\ \ z,w\in B_{\mathbb{B}_m}, \tag{3.1}$$

and the Wallach set $\mathscr{W}_{\mathbb{B}_m} = \{t\in\mathbb{R} : t>0\}$. But, in general, there are examples of bounded symmetric domains Ω, like the open unit ball in the space of all $m\times n$ matrices, $m,n>1$, with respect to the operator norm, where the Wallach set $\mathscr{W}_\Omega$ is a proper subset of $\{t\in\mathbb{R} : t>0\}$. For any bounded symmetric domain Ω, an explicit description of $\mathscr{W}_\Omega$ is given in [30].

Replacing the Bergman kernel in the definition of the Wallach set by an arbitrary scalar valued non-negative definite kernel K, we define the ordinary Wallach set $\mathscr{W}_\Omega(K)$ to be the set

$$\{t>0 : K^t \text{ is non-negative definite}\}.$$

Here we have assumed that there exists a continuous branch of logarithm of K on $\Omega\times\Omega$ and therefore K^t, $t>0$, makes sense. Clearly, every natural number belongs to the Wallach set $\mathscr{W}_\Omega(K)$. In [13], it is shown that K^t is non-negative definite for all $t>0$ if and only if the function $\big(\partial_i\bar\partial_j \log K(z,w)\big)_{i,j=1}^m$ is non-negative definite. Therefore it follows from the discussion in the previous paragraph that there are non-negative definite kernels K on $\Omega\times\Omega$ for which $\big(\partial_i\bar\partial_j \log K(z,w)\big)_{i,j=1}^m$ need not define a non-negative definite kernel on $\Omega\times\Omega$. However, it follows from Proposition 2.1.4 that $K^{t_1+t_2}\big(\partial_i\bar\partial_j \log K(z,w)\big)_{i,j=1}^m$ is a non-negative kernel on $\Omega\times\Omega$ as soon as t_1 and t_2 are in the Wallach set $\mathscr{W}_\Omega(K)$. Therefore it is natural to introduce the generalized Wallach set for any scalar valued kernel K defined on $\Omega\times\Omega$ as follows:

$$G\mathscr{W}_\Omega(K) := \big\{t\in\mathbb{R} :\ K^{t-2}\mathbb{K} \text{ is non-negative definite}\big\}, \tag{3.2}$$

where, as before, we have assumed that K^t is well defined for all $t\in\mathbb{R}$ and $\mathbb{K}$ is the function $K^2\big(\partial_i\bar\partial_j \log K\big)_{i,j=1}^m$ as in chapter 2. Clearly, we have the following inclusion

$$\big\{t_1+t_2 :\ t_1,t_2\in\mathscr{W}_\Omega(K)\big\} \subseteq G\mathscr{W}_\Omega(K).$$

3.2 Generalized Wallach set for the Bergman kernel of the Euclidean unit ball in $\mathbb{C}^m$

In this section, we compute the generalized Wallach set for the Bergman kernel of the Euclidean unit ball in $\mathbb{C}^m$. In the case of the unit disc $\mathbb{D}$, the Bergman kernel $B_\mathbb{D}(z,w) = (1-z\bar w)^{-2}$ and $\partial\bar\partial \log B_\mathbb{D}(z,w) = 2(1-z\bar w)^{-2}$, $z,w\in\mathbb{D}$. Therefore t is in $G\mathscr{W}_\mathbb{D}(B_\mathbb{D})$ if and only if $(1-z\bar w)^{-(2t+2)}$ is non-negative definite on $\mathbb{D}\times\mathbb{D}$. Consequently,

$$G\mathscr{W}_\mathbb{D}(B_\mathbb{D}) = \{t\in\mathbb{R} : t\geq -1\}.$$

For the case of the Bergman kernel $B_{\mathbb{B}_m}$ of the Euclidean unit ball $\mathbb{B}_m$, $m \geq 2$, we have shown that $G\mathscr{W}_{\mathbb{B}_m}(B_{\mathbb{B}_m}) = \{t \in \mathbb{R} : t \geq 0\}$. We need some lemmas to prove this statement.

As before, we write $K \succeq 0$ to denote that K is a non-negative definite kernel. For two non-negative definite kernels $K_1, K_2 : \Omega \times \Omega \to \mathscr{M}_k(\mathbb{C})$, we write $K_1 \preceq K_2$ if $K_2 - K_1$ is a non-negative definite kernel on $\Omega \times \Omega$. Analogously, we write $K_1 \succeq K_2$ if $K_1 - K_2$ is non-negative definite.

Lemma 3.2.1. *Let Ω be a bounded domain in $\mathbb{C}^m$, and $\lambda_0 > 0$ be an arbitrary constant. Let $\{K_\lambda\}_{\lambda \geq \lambda_0}$ be a family of non-negative definite kernels, defined on $\Omega \times \Omega$, taking values in $\mathscr{M}_k(\mathbb{C})$ such that the followings hold:*

(i) *if $\lambda \geq \lambda' \geq \lambda_0$, then $K_{\lambda'} \preceq K_\lambda$,*

(ii) *for $z, w \in \Omega$, $K_\lambda(z, w)$ converges to $K_{\lambda_0}(z, w)$ entrywise as $\lambda \to \lambda_0$.*

Suppose that $f : \Omega \to \mathbb{C}^k$ is a holomorphic function which is in $(\mathscr{H}, K_\lambda)$ for all $\lambda > \lambda_0$. Then $f \in (\mathscr{H}, K_{\lambda_0})$ if and only if $\sup_{\lambda > \lambda_0} \|f\|_{(\mathscr{H}, K_\lambda)} < \infty$.

Proof. Recall that if K and K' are two non-negative definite kernels satisfying $K \preceq K'$, then $(\mathscr{H}, K) \subseteq (\mathscr{H}, K')$ and $\|h\|_{(\mathscr{H}, K')} \leq \|h\|_{(\mathscr{H}, K)}$ for $h \in (\mathscr{H}, K)$ (see [47, Theorem 6.25]). Therefore, by the hypothesis, we have that

$$(\mathscr{H}, K_{\lambda'}) \subseteq (\mathscr{H}, K_\lambda) \quad \text{and} \quad \|h\|_{(\mathscr{H}, K_\lambda)} \leq \|h\|_{(\mathscr{H}, K_{\lambda'})}, \tag{3.3}$$

whenever $\lambda \geq \lambda' \geq \lambda_0$ and $h \in (\mathscr{H}, K_{\lambda'})$.

Now assume that $f \in (\mathscr{H}, K_{\lambda_0})$. Then, clearly $\|f\|_{(\mathscr{H}, K_\lambda)} \leq \|f\|_{(\mathscr{H}, K_{\lambda_0})}$ for all $\lambda > \lambda_0$. Consequently, $\sup_{\lambda > \lambda_0} \|f\|_{(\mathscr{H}, K_\lambda)} \leq \|f\|_{(\mathscr{H}, K_{\lambda_0})} < \infty$.

For the converse, assume that $\sup_{\lambda > \lambda_0} \|f\|_{(\mathscr{H}, K_\lambda)} < \infty$. Then, from (3.3), it follows that $\lim_{\lambda \to \lambda_0} \|f\|_{(\mathscr{H}, K_\lambda)}$ exists and is equal to $\sup_{\lambda > \lambda_0} \|f\|_{(\mathscr{H}, K_\lambda)}$. Since $f \in (\mathscr{H}, K_\lambda)$ for all $\lambda > \lambda_0$, by [47, Theorem 6.23], we have that

$$f(z)f(w)^* \preceq \|f\|^2_{(\mathscr{H}, K_\lambda)} K_\lambda(z, w).$$

Taking limit as $\lambda \to \lambda_0$ and using part (ii) of the hypothesis, we obtain

$$f(z)f(w)^* \preceq \sup_{\lambda > \lambda_0} \|f\|^2_{(\mathscr{H}, K_\lambda)} K_{\lambda_0}(z, w).$$

Hence, using [47, Theorem 6.23] once again, we conclude that $f \in (\mathscr{H}, K_{\lambda_0})$. □

If $m \geq 2$, then from (3.1), we have

$$\Big(\big(B^t_{\mathbb{B}_m}\partial_i\bar\partial_j \log B_{\mathbb{B}_m}\big)(z,w)\Big)_{i,j=1}^m = \frac{m+1}{(1-\langle z,w\rangle)^{t(m+1)+2}}\begin{pmatrix} 1-\sum_{j\neq 1} z_j\bar w_j & z_2\bar w_1 & \cdots & z_m\bar w_1 \\ z_1\bar w_2 & 1-\sum_{j\neq 2} z_j\bar w_j & \cdots & z_m\bar w_2 \\ \vdots & \vdots & \vdots & \vdots \\ z_1\bar w_m & z_2\bar w_m & \cdots & 1-\sum_{j\neq m} z_j\bar w_j \end{pmatrix}. \tag{3.4}$$

For $m \geq 2$, $\lambda \in \mathbb{R}$ and $z, w \in \mathbb{B}_m$, set

$$\mathbb{K}_\lambda(z,w) := \frac{1}{(1-\langle z,w\rangle)^{\lambda}}\begin{pmatrix} 1-\sum_{j\neq 1} z_j\bar w_j & z_2\bar w_1 & \cdots & z_m\bar w_1 \\ z_1\bar w_2 & 1-\sum_{j\neq 2} z_j\bar w_j & \cdots & z_m\bar w_2 \\ \vdots & \vdots & \vdots & \vdots \\ z_1\bar w_m & z_2\bar w_m & \cdots & 1-\sum_{j\neq m} z_j\bar w_j \end{pmatrix}. \tag{3.5}$$

In view (3.4) and (3.5), for $\lambda > 2$, we have

$$\mathbb{K}_\lambda = \frac{2}{t(m+1)}\Big(\,(B^{\frac{t}{2}}_{\mathbb{B}_m})^2\partial_i\bar\partial_j \log B^{\frac{t}{2}}_{\mathbb{B}_m}\Big)_{i,j=1}^m,$$

where $t = \frac{\lambda-2}{m+1} > 0$. Since $B^{t/2}_{\mathbb{B}_m}$ is positive definite on $\mathbb{B}_m \times \mathbb{B}_m$ for $t > 0$, it follows from Corollary 2.1.5 that $\mathbb{K}_\lambda$ is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$ for $\lambda > 2$. Since $\mathbb{K}_\lambda(z,w) \to \mathbb{K}_2(z,w)$, $z, w \in \mathbb{B}_m$, entrywise as $\lambda \to 2$, we conclude that $\mathbb{K}_2$ is also non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$.

Let $\{e_1,\ldots,e_m\}$ be the standard basis of $\mathbb{C}^m$. The lemma given below finds the norm of the vector $z_2 \otimes e_1$ in $(\mathscr{H},\mathbb{K}_\lambda)$ when $\lambda > 2$.

Lemma 3.2.2. *For each $\lambda > 2$, the vector $z_2 \otimes e_1$ belongs to $(\mathscr{H},\mathbb{K}_\lambda)$ and $\|z_2 \otimes e_1\|_{(\mathscr{H},\mathbb{K}_\lambda)} = \sqrt{\frac{\lambda-1}{\lambda(\lambda-2)}}$.*

Proof. By a straight forward computation, we obtain

$$\bar\partial_1\mathbb{K}_\lambda(\cdot,0)e_2 = z_2\otimes e_1 + (\lambda-1)z_1\otimes e_2$$

and

$$\bar\partial_2\mathbb{K}_\lambda(\cdot,0)e_1 = (\lambda-1)z_2\otimes e_1 + z_1\otimes e_2.$$

Thus we have

$$(\lambda-1)\bar\partial_2\mathbb{K}_\lambda(\cdot,0)e_1 - \bar\partial_1\mathbb{K}_\lambda(\cdot,0)e_2 = (\lambda^2-2\lambda)z_2\otimes e_1. \tag{3.6}$$

By Proposition 2.1.3, the vectors $\bar\partial_2\mathbb{K}_\lambda(\cdot,0)e_1$ and $\bar\partial_1\mathbb{K}_\lambda(\cdot,0)e_2$ belong to $(\mathscr{H},\mathbb{K}_\lambda)$. Since $\lambda>2$, from (3.6), it follows that the vector $z_2\otimes e_1$ belongs to $(\mathscr{H},\mathbb{K}_\lambda)$. Now, taking norm in both sides of (3.6) and using Proposition 2.1.3 a second time, we obtain

$$\begin{aligned}(\lambda^2-2\lambda)^2\|z_2\otimes e_1\|^2&\\ &= (\lambda-1)^2\langle\partial_2\bar\partial_2\mathbb{K}_\lambda(0,0)e_1,e_1\rangle-(\lambda-1)\langle\partial_1\bar\partial_2\mathbb{K}_\lambda(0,0)e_1,e_2\rangle\\ &\quad-(\lambda-1)\langle\bar\partial_1\partial_2\mathbb{K}_\lambda(0,0)e_2,e_1\rangle+\langle\partial_1\bar\partial_1\mathbb{K}_\lambda(0,0)e_2,e_2\rangle\end{aligned}\tag{3.7}$$

By a routine computation, we obtain

$$\partial_i\bar\partial_j\mathbb{K}_\lambda(0,0)=(\lambda-1)\delta_{ij}I_m+E_{ji},$$

where δ_{ij} is the Kronecker delta function, I_m is the identity matrix of order m, and E_{ji} is the matrix whose (j,i)th entry is 1 and all other entries are 0. Hence, from (3.7), we see that

$$\begin{aligned}(\lambda^2-2\lambda)^2||z_2\otimes e_1||^2&\\ &=(\lambda-1)^2(\lambda-1)-2(\lambda-1)+(\lambda-1)\\ &=(\lambda-1)(\lambda^2-2\lambda).\end{aligned}$$

Hence $||z_2\otimes e_1||=\sqrt{\frac{\lambda-1}{\lambda(\lambda-2)}}$, completing the proof of the lemma. □

Lemma 3.2.3. *The multiplication operator by the coordinate function z_2 on $(\mathscr{H},\mathbb{K}_2)$ is not bounded.*

Proof. Since $\mathbb{K}_2(\cdot,0)e_1=e_1$, we have that the constant function e_1 is in $(\mathscr{H},\mathbb{K}_2)$. Hence, to prove that M_{z_2} is not bounded on $(\mathscr{H},\mathbb{K}_2)$, it suffices to show that the vector $z_2\otimes e_1$ does not belong to $(\mathscr{H},\mathbb{K}_2)$.

Consider the family of non-negative definite kernels $\{\mathbb{K}_\lambda\}_{\lambda\geq 2}$. Observe that for $\lambda\geq\lambda'\geq 2$,

$$\mathbb{K}_\lambda(z,w)-\mathbb{K}_{\lambda'}(z,w)=\Big((1-\langle z,w\rangle)^{-(\lambda-\lambda')}-1\Big)\mathbb{K}_{\lambda'}(z,w).\tag{3.8}$$

It is easy to see that if $\lambda\geq\lambda'$, then $(1-\langle z,w\rangle)^{-(\lambda-\lambda')}-1\geq 0$. Thus the right hand side of (3.8), being a product of a scalar valued non-negative definite kernel with a matrix valued non-negative definite kernel, is non-negative definite. Consequently, $K_{\lambda'}\preceq K_\lambda$. Also since $\mathbb{K}_\lambda(z,w)\to\mathbb{K}_2(z,w)$ entry-wise as $\lambda\to 2$, by Lemma 3.2.1, it follows that $z_2\otimes e_1\in(\mathscr{H},\mathbb{K}_2)$ if and only if $\sup_{\lambda>2}\|z_2\otimes e_1\|_{(\mathscr{H},\mathbb{K}_\lambda)}<\infty$. By lemma 3.2.2, we have $\|z_2\otimes e_1\|_{(\mathscr{H},\mathbb{K}_\lambda)}=\sqrt{\frac{\lambda-1}{\lambda(\lambda-2)}}$. Thus $\sup_{\lambda>2}||z_2\otimes e_1||_{(\mathscr{H},\mathbb{K}_\lambda)}=\infty$. Hence the vector $z_2\otimes e_1$ does not belong to $(\mathscr{H},\mathbb{K}_2)$ and the operator M_{z_2} on $(\mathscr{H},\mathbb{K}_\lambda)$ is not bounded. □

The following theorem describes the generalized Wallach set for the Bergman kernel of the Euclidean unit ball in $\mathbb{C}^m$, $m\geq 2$.

Theorem 3.2.4. *If $m \geq 2$, then $G\mathscr{W}_{\mathbb{B}_m}(B_{\mathbb{B}_m}) = \{t \in \mathbb{R} : t \geq 0\}$.*

Proof. In view of (3.4) and (3.5), we see that $t \in G\mathscr{W}_{\mathbb{B}_m}(B_{\mathbb{B}_m})$ if and only if $\mathbb{K}_{t(m+1)+2}$ is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$. Hence we will be done if we can show that $\mathbb{K}_\lambda$ is non-negative if and only if $\lambda \geq 2$.

From the discussion preceding Lemma 3.2.2, we have that $\mathbb{K}_\lambda$ is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$ for $\lambda \geq 2$.

To prove the converse, assume that $\mathbb{K}_\lambda$ is non-negative definite for some $\lambda < 2$. Note that $\mathbb{K}_2$ can be written as the product

$$\mathbb{K}_2(z,w) = (1-\langle z,w\rangle)^{-(2-\lambda)}\mathbb{K}_\lambda(z,w),\ z,w \in \mathbb{B}_m. \tag{3.9}$$

Note that the multiplication operator M_{z_2} on $(\mathscr{H}, (1-\langle z,w\rangle)^{-(2-\lambda)})$ is bounded. Hence, by Lemma 2.1.10, there exists a constant $c > 0$ such that $(c^2 - z_2\bar{w}_2)(1-\langle z,w\rangle)^{-(2-\lambda)}$ is non-negative definite. Consequently, the product $(c^2 - z_2\bar{w}_2)(1-\langle z,w\rangle)^{-(2-\lambda)}\mathbb{K}_\lambda$, which is $(c^2 - z_2\bar{w}_2)\mathbb{K}_2$, is non-negative. Hence, again by Lemma 2.1.10, it follows that the operator M_{z_2} is bounded on $(\mathscr{H}, \mathbb{K}_2)$. This is a contradiction to the Lemma 3.2.3. Hence our assumption that $\mathbb{K}_\lambda$ is non-negative for some $\lambda < 2$, is not valid. This completes the proof.

□

The theorem given below finds all $\lambda \in \mathbb{R}$ such that $\mathbb{K}_\lambda^{\mathrm{tr}}$ is non-negative definite.

Theorem 3.2.5. *For $m \geq 2$, ${\mathbb{K}_\lambda}^{\mathrm{tr}}(z,w)$ is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$ if and only if $\lambda \geq 1$.*

Proof. Note that

$${\mathbb{K}_\lambda}^{\mathrm{tr}}(z,w) = (1-\langle z,w\rangle)^{-(\lambda-1)}I_m + (1-\langle z,w\rangle)^{-\lambda}\big(z_i\bar{w}_j\big)_{i,j=1}^m,\ \ z,w \in \mathbb{B}_m.$$

It is easily verified that $\big(z_i\bar{w}_j\big)_{i,j=1}^m$ is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$. Assume $\lambda \geq 1$. Then ${\mathbb{K}_\lambda}^{\mathrm{tr}}$ is the sum of two non-negative definite kernels and therefore is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$. Conversely, assume that ${\mathbb{K}_\lambda}^{\mathrm{tr}}$ is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$. Then, by [13, Lemma 3.2] , we see that $\langle {\mathbb{K}_\lambda}^{\mathrm{tr}}(z,w)e_1, e_1\rangle$, which is equal to

$$(1-\sum_{j\neq 1} z_j\bar{w}_j)(1-\langle z,w\rangle)^{-\lambda},$$

is non-negative definite on $\mathbb{B}_m \times \mathbb{B}_m$. Hence, by an argument similar to the proof of Lemma 2.1.11, it follows that

$$\sum_{j\neq 1} M_{z_j}M_{z_j}^* \leq I$$

on the Hilbert space $(\mathscr{H}, (1-\langle z,w\rangle)^{-\lambda})$. In particular, we have $M_{z_2}M_{z_2}^* \leq I$ on $(\mathscr{H}, (1-\langle z,w\rangle)^{-\lambda})$. It is easily verified that $\|M_{z_2}\|_{(\mathscr{H},(1-\langle z,w\rangle)^{-\lambda})} \leq 1$ if and only if $\lambda \geq 1$. Hence $\lambda \geq 1$, completing the proof of the theorem. □

3.3 Quasi-invariant kernels

Let $\Omega \subset \mathbb{C}^m$ be a bounded domain and let $\mathrm{Aut}(\Omega)$ denote the group of all biholomorphic automorphisms of Ω. Let $J : \mathrm{Aut}(\Omega) \times \Omega \to \mathrm{GL_k}(\mathbb{C})$ be a function such that $J(\varphi, \cdot)$ is holomorphic for each φ in $\mathrm{Aut}(\Omega)$. A non-negative definite kernel $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ is said to be quasi-invariant with respect to J if K satisfies the following transformation rule:

$$J(\varphi, z)K(\varphi(z), \varphi(w))J(\varphi, w)^* = K(z, w),\ z, w \in \Omega,\ \varphi \in \mathrm{Aut}(\Omega). \tag{3.10}$$

The following proposition is a basic tool in defining unitary representations of the automorphism group $\mathrm{Aut}(\Omega)$ and the straightforward proof for the case of unit disc $\mathbb{D}$ appears in [39]. The proof for the general domain Ω follows in exactly the same way.

For a fixed but arbitrary $\varphi \in \mathrm{Aut}(\Omega)$, let U_φ be the linear map on $\mathrm{Hol}(\Omega, \mathbb{C}^k)$ defined by

$$U_\varphi(f) = J\big(\varphi^{-1}, \cdot\big) f \circ \varphi^{-1},\ \ f \in \mathrm{Hol}(\Omega, \mathbb{C}^k).$$

Proposition 3.3.1. *The linear map U_φ is unitary on $(\mathcal{H}, K)$ for all φ in $\mathrm{Aut}(\Omega)$ if and only if the kernel K is quasi-invariant with respect to J.*

Remark 3.3.2. *If $K : \Omega \times \Omega \to \mathcal{M}_k(\mathbb{C})$ is a quasi-invariant kernel with respect to some J and the commuting tuple $\boldsymbol{M}_z = (M_{z_1}, \ldots, M_{z_m})$ on $(\mathcal{H}, K)$ is bounded, then the commuting tuple $\boldsymbol{M}_\varphi := (M_{\varphi_1}, \ldots, M_{\varphi_m})$ is unitarily equivalent to $\boldsymbol{M}_z$ via the unitary map U_φ, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ is in* $\mathrm{Aut}(\Omega)$.

The lemma given below, which will be used in the proof of the Proposition 3.3.4, follows from applying the chain rule [48, page 8] twice.

Lemma 3.3.3. *Let $\phi = (\phi_1, \ldots, \phi_m) : \Omega \to \mathbb{C}^m$ be a holomorphic map and $g : \mathrm{ran}\, \phi \to \mathbb{C}$ be a real analytic function. If $h = g \circ \phi$, then*

$$\Big(\big(\partial_i \bar\partial_j h\big)(z)\Big)_{i,j=1}^m = (D\phi(z))^{\mathrm{tr}} \Big(\big(\partial_i \bar\partial_j g\big)(\varphi(z))\Big)_{i,j=1}^m \overline{(D\phi(z))},$$

where $(D\phi)(z)^{\mathrm{tr}}$ is the transpose of the derivative of ϕ at z.

The following proposition shows that if K a is quasi-invariant kernel with respect to some J, then $K^{t-2}\mathbb{K}$ is also a quasi-invariant kernel with respect to some $\mathbb{J}$ whenever t is in the generalized Wallach set $G\mathcal{W}_\Omega(K)$.

Proposition 3.3.4. *Let $\Omega \subset \mathbb{C}^m$ be a bounded domain. Let $K : \Omega \times \Omega \to \mathbb{C}$ be a non-negative definite kernel and $J : \mathrm{Aut}(\Omega) \times \Omega \to \mathbb{C} \setminus \{0\}$ be a function such that $J(\varphi, \cdot)$ is holomorphic for each φ in $\mathrm{Aut}(\Omega)$. Suppose that K is quasi-invariant with respect to J. Then the kernel $K^{t-2}\mathbb{K}$ is also quasi-invariant with respect to $\mathbb{J}$ whenever $t \in G\mathcal{W}_\Omega(K)$, where $\mathbb{J}(\varphi, z) = J(\varphi, z)^t D\varphi(z)^{\mathrm{tr}}$, $\varphi \in \mathrm{Aut}(\Omega),\ \mathrm{z} \in \Omega$.*

Proof. Since K is quasi-invariant with respect to J, we have

$$\log K(z,z) = \log|J(\varphi,z)|^2 + \log K(\varphi(z),\varphi(z)),\ \varphi\in \mathrm{Aut}(\Omega),\ z\in\Omega.$$

Also, $J(\varphi,\cdot)$ is a non-vanishing holomorphic function on Ω, therefore $\partial_i\bar\partial_j \log|J(\varphi,z)|^2 = 0$. Hence

$$\partial_i\bar\partial_j \log K(z,z) = \partial_i\bar\partial_j \log K(\varphi(z),\varphi(z)),\ \varphi\in \mathrm{Aut}(\Omega),\ z\in\Omega. \tag{3.11}$$

Any biholomorphic automorphism φ of Ω is of the form $(\varphi_1,\ldots,\varphi_m)$, where $\varphi_i:\Omega\to\mathbb{C}$ is holomorphic, $i=1,\ldots,m$. By setting $g(z)=\log K(z,z)$, $z\in\Omega$, and using Lemma 3.3.3, we obtain

$$\big(\partial_i\bar\partial_j \log K(\varphi(z),\varphi(z))\big)_{i,j=1}^m = D\varphi(z)^{\mathrm{tr}}\big(\big(\partial_l\bar\partial_p \log K\big)(\varphi(z),\varphi(z))\big)_{l,p=1}^m\overline{D\varphi(z)}.$$

Combining this with (3.11), we obtain

$$\big(\partial_i\bar\partial_j \log K(z,z)\big)_{i,j=1}^m = D\varphi(z)^{\mathrm{tr}}\big(\big(\partial_l\bar\partial_p \log K\big)(\varphi(z),\varphi(z))\big)_{l,p=1}^m\overline{D\varphi(z)}.$$

Multiplying $K(z,z)^t$ both sides and using the quasi-invariance of K, a second time, we obtain

$$\begin{aligned}&\big(\,K(z,z)^t\partial_i\bar\partial_j \log K(z,z)\,\big)_{i,j=1}^m\\ &\quad= J(\varphi,z)^tD\varphi(z)^{\mathrm{tr}}K(\varphi(z),\varphi(z))^t\big(\big(\partial_l\bar\partial_p \log K\big)(\varphi(z),\varphi(z)\,\big)_{l,p=1}^m\overline{J(\varphi,z)^tD\varphi(z)}.\end{aligned}$$

Equivalently, we have

$$K^{t-2}(z,z)\mathbb{K}(z,z) = \mathbb{J}(\varphi,z)K^{t-2}(\varphi(z),\varphi(z))\mathbb{K}(\varphi(z),\varphi(z))\mathbb{J}(\varphi,z)^*, \tag{3.12}$$

where $\mathbb{J}(\varphi,z)=J(\varphi,z)^tD\varphi(z)^{\mathrm{tr}}$, $\varphi\in\mathrm{Aut}(\Omega)$, $z\in\Omega$. Therefore, polarizing both sides of the above equation, we have the desired conclusion. □

Remark 3.3.5. *The function J in the definition of quasi-invariant kernel is said to be a projective cocycle if it is a Borel map satisfying*

$$J(\varphi\psi,z) = m(\varphi,\psi)J(\psi,z)J(\varphi,\psi z),\ \varphi,\psi\in\mathrm{Aut}(\Omega), z\in\Omega, \tag{3.13}$$

where $m:\mathrm{Aut}(\Omega)\times\mathrm{Aut}(\Omega)\to\mathbb{T}$ is a multiplier, that is, m is Borel and satisfies the following properties:

(i) $m(e,\varphi)=m(\varphi,e)=1$, *where* $\varphi\in\mathrm{Aut}(\Omega)$ *and e is the identity in* $\mathrm{Aut}(\Omega)$

(ii) $m(\varphi_1,\varphi_2)m(\varphi_1\varphi_2,\varphi_3)=m(\varphi_1,\varphi_2\varphi_3)m(\varphi_2,\varphi_3)$, $\varphi_1,\varphi_2,\varphi_3\in\mathrm{Aut}(\Omega)$.

J is said to be a cocycle if it is a projective cocycle with $m(\varphi,\psi)=1$ for all φ,ψ in $\mathrm{Aut}(\Omega)$.

If we assume that $J:\mathrm{Aut}(\Omega)\times\Omega\to\mathbb{C}\setminus\{0\}$ in the Proposition 3.3.4 is a cocycle, then it is verified that the function $\mathbb{J}$ is a projective co-cycle. Moreover, if t is a positive integer, then $\mathbb{J}$ is also a cocycle. □.

For the preceding to be useful, one must exhibit non-negative definite kernels which are quasi-invariant. It is known that the Bergman kernel B_Ω of any bounded domain Ω is quasi-invariant with respect to J, where $J(\varphi,z)=\det D\varphi(z)$, $\varphi\in\mathrm{Aut}(\Omega), z\in\Omega$ [40, Proposition 1.4.12]. We provide the easy proof here for the sake of completeness.

Lemma 3.3.6. *Let $\Omega\subset\mathbb{C}^m$ be a bounded domain and $\varphi:\Omega\to\Omega$ be a biholomorphic map. Then*

$$B_\Omega(z,w)=\det D\varphi(z)B_\Omega(\varphi(z),\varphi(w))\overline{\det D\varphi(w)},\ z,w\in\Omega.$$

Proof. For $\varphi\in\mathrm{Aut}(\Omega)$, consider the operator V_φ on $A^2(\Omega)$ given by

$$V_\varphi(f)(z)=\det D\varphi(z)(f\circ\varphi)(z),\ f\in A^2(\Omega).$$

Using the change of variable formula, it follows that the operator V_φ is unitary on $A^2(\Omega)$. Therefore, if $\{f_n\}_{n=0}^\infty$ is an orthonormal basis of $A^2(\Omega)$, so is $\{V_\varphi(f_n)\}_{n=0}^\infty$. Hence

$$\begin{aligned}B_\Omega(z,w)&=\sum_{n=0}^\infty V_\varphi(f_n)(z)\overline{V_\varphi(f_n)(w)}\\&=\sum_{n=0}^\infty \det D\varphi(z)(f_n\circ\varphi)(z)\ \overline{\det D\varphi(w)(f_n\circ\varphi)(w)}\\&=\det D\varphi(z)\Big(\sum_{n=0}^\infty f_n(\varphi(z))\overline{f_n(\varphi(w))}\Big)\ \overline{\det D\varphi(w)}\\&=\det D\varphi(z)B_\Omega(\varphi(z),\varphi(w))\overline{\det D\varphi(w)},\end{aligned}$$

completing the proof of the lemma. □

The following proposition follows from combining Proposition 3.3.4 and Lemma 3.3.6, and therefore the proof is omitted.

Proposition 3.3.7. *Let Ω be a bounded domain $\mathbb{C}^m$. If t is in $GW_\Omega(B_\Omega)$, then the kernel*

$$\big(B_\Omega^t(z,w)\partial_i\bar\partial_j\log B_\Omega(z,w)\big)_{i,j=1}$$

is quasi-invariant with respect to $(\det D\varphi(z))^tD\varphi(z)^{\mathrm{tr}},\ \varphi\in\mathrm{Aut}(\Omega),\ z\in\Omega$.

In case of the Bergman kernel $B_{\mathbb{B}_m}$ of the Euclidean unit ball in $\mathbb{C}^m$, we have

$$\big(B_{\mathbb{B}_m}^t\partial_i\bar\partial_j\log B_{\mathbb{B}_m}\big)_{i,j=1}^m=\tfrac{2}{t}\big((B_{\mathbb{B}_m}^{\frac{t}{2}})^2\partial_i\bar\partial_j\log B_{\mathbb{B}_m}^{\frac{t}{2}}\big)_{i,j=1}^m,\ t>0.$$

Since the multiplication tuple $\boldsymbol{M}_z$ on $\big(\mathscr{H}, B_{\mathbb{B}_m}^{t/2}\big)$ is bounded, it follows from the above equation together with Theorem 2.1.16 that the multiplication tuple $\boldsymbol{M}_z$ on $\big(\mathscr{H}, (B_{\mathbb{B}_m}^{t}\partial_i\bar{\partial}_j \log B_{\mathbb{B}_m})_{i,j=1}^{m}\big)$ is also bounded. Also, by Proposition 3.3.7, the kernel $\big(B_{\mathbb{B}_m}^{t}\partial_i\bar{\partial}_j \log B_{\mathbb{B}_m}\big)_{i,j=1}^{m}$ is quasi-invariant. Hence, by Remark 3.3.2, it follows that the multiplication tuple $\boldsymbol{M}_z$ and the tuple $\boldsymbol{M}_\varphi$ are unitarily equivalent on $\big(\mathscr{H}, (B_{\mathbb{B}_m}^{t}\partial_i\bar{\partial}_j \log B_{\mathbb{B}_m})_{i,j=1}^{m}\big)$ for all $\varphi \in \mathrm{Aut}(\mathbb{B}_m)$. Therefore, in the language of [46], we conclude that the multiplication tuple $\boldsymbol{M}_z$ on $\big(\mathscr{H}, (B_{\mathbb{B}_m}^{t}\partial_i\bar{\partial}_j \log B_{\mathbb{B}_m})_{i,j=1}^{m}\big)$, $t > 0$, is homogeneous with respect to the group $\mathrm{Aut}(\mathbb{B}_m)$.

Chapter 4

Weakly homogeneous operators

In this chapter, we study weakly homogeneous operators. First, some elementary properties of weakly homogeneous operators are discussed. In section 4.2, we show that the weak homogeneity of the multiplication operator M_z on reproducing kernel Hilbert spaces is equivalent to the existence of certain bounded and invertible weighted composition operators. We use this to show that the weak homogeneity of M_z is preserved under the jet construction. Next, in section 4.3, weakly homogeneous operators in the class $\mathscr{F}B_2(\mathbb{D})$ are studied. In section 4.4, we discuss Möbius bounded operators. It is shown that the Shields' conjecture on this class of operators has an affirmative answer in the class of quasi-homogeneous operators. Finally, in section 4.5, we show that there exists a Möbius bounded weakly homogeneous operator which is not similar to any homogeneous operator. This answers a question of Bagchi and Misra in the negative.

4.1 Definition and elementary properties

Throughout this chapter, we let Möb denote the group of all biholomorphic automorphisms $\{\varphi_{\theta,a} : \theta \in [0,2\pi),\, a \in \mathbb{D}\}$ of the unit disc $\mathbb{D}$, where $\varphi_{\theta,a}(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z},\ z \in \mathbb{D}$. It is a topological group with the topology induced by $\mathbb{T} \times \mathbb{D}$.

For an operator $T \in B(\mathscr{H})$ with $\sigma(T) \subseteq \bar{\mathbb{D}}$, recall that the operator $\varphi(T)$, $\varphi \in$ Möb, is defined by using the Riesz functional calculus since φ is holomorphic in a neighbourhood of $\bar{\mathbb{D}}$. Such an operator T is said to be homogeneous if $\varphi(T)$ is unitarily equivalent to T for all $\varphi \in$ Möb. Weakly homogeneous operators are straightforward generalization of homogeneous operators, see [16], [10].

Definition 4.1.1. *An operator $T \in B(\mathscr{H})$ is said to be weakly homogeneous if $\sigma(T) \subseteq \bar{\mathbb{D}}$ and $\varphi(T)$ is similar to T for all φ in* Möb.

Suppose that T is an operator which is similar to a homogeneous operator, that is, $T =$

XSX^{-1} for some homogeneous operator S and an invertible operator X. Clearly, $\sigma(T) = \sigma(S)$. Thus, using homogeneity of S, we have that $\sigma(T)$ is either $\mathbb{T}$ or $\bar{\mathbb{D}}$, and consequently the operator $\varphi(T)$ is well-defined for any $\varphi \in$ Möb. Note that $\varphi(T) = X\varphi(S)X^{-1}$. Using homogeneity of S a second time, we have that $\varphi(S) = U_\varphi S U_\varphi^{-1}$ for some unitary operator U_φ. Hence

$$\varphi(T) = (XU_\varphi)S(XU_\varphi)^{-1} = (XU_\varphi X^{-1})T(XU_\varphi X^{-1})^{-1}. \tag{4.1}$$

Thus $\varphi(T)$ is similar to T for all $\varphi \in$ Möb and consequently, T is weakly homogeneous. Hence every operator which is similar to a homogeneous operator is weakly homogeneous. The converse of this is not true, that is, a weakly homogeneous operator need not be similar to any homogeneous operator (see Corollary 4.3.9 and section 4.5).

The following lemma, which shows that the spectrum of a weakly homogeneous operator is either $\mathbb{T}$ or $\bar{\mathbb{D}}$, is a straightforward generalization of [9, Lemma 2.2]. Therefore the proof is omitted.

Lemma 4.1.2. *Let T be an operator in $B(\mathscr{H})$. If the operators T and $\varphi(T)$ are similar for all φ in a neighbourhood of the identity in* Möb*, then $\sigma(T)$ is either $\mathbb{T}$ or $\bar{\mathbb{D}}$, and T is similar to $\varphi(T)$ (which makes sense for all φ in* Möb *since $\sigma(T)$ is either $\bar{\mathbb{D}}$ or $\mathbb{T}$) for all φ in* Möb*. In particular, T is weakly homogeneous.*

It is easy to see that an operator T is weakly homogeneous if and only if the operator T^* is weakly homogeneous.

Since two normal operators are similar if and only if they are unitarily equivalent, the proof of the following proposition is evident.

Proposition 4.1.3. *A normal operator N is homogeneous if and only if it is weakly homogeneous.*

4.2 Jet construction and weak homogeneity

In this section, we show that the weak homogeneity of the multiplication operators M_z on reproducing kernel Hilbert spaces is preserved under the jet construction.

Throughout this section, we assume that $\Omega \subset \mathbb{C}$ is a bounded domain. By Hol(Ω), we denote the space of all holomorphic functions from Ω to $\mathbb{C}$. Let Ω' denote one of the four domains: Ω, $\mathbb{C}^k$, $GL_k(\mathbb{C})$ and $\mathscr{M}_k(\mathbb{C})$. By Hol$(\Omega, \Omega')$, we denote the space of all holomorphic functions from Ω to Ω'. As before, Aut(Ω) denotes the group of all biholomorphic automorphisms of Ω.

Let $K : \Omega \times \Omega \to \mathscr{M}_k(\mathbb{C})$ be a non-negative definite kernel. Let ψ be a holomorphic function on Ω taking values in $\mathscr{M}_k(\mathbb{C})$. Let M_ψ be the linear map on Hol$(\Omega, \mathbb{C}^k)$ defined by point-wise multiplication:

$$(M_\psi f)(\cdot) = \psi(\cdot)f(\cdot), f \in \text{Hol}(\Omega, \mathbb{C}^k).$$

For a holomorphic self map φ of Ω, let C_φ be the linear map on $\mathrm{Hol}(\Omega,\mathbb{C}^k)$ defined by composition:

$$(C_\varphi f)(\cdot) = (f\circ\varphi)(\cdot),\, f\in \mathrm{Hol}(\Omega,\mathbb{C}^k).$$

In general, neither M_ψ nor C_φ maps $(\mathscr{H},K)$ into $(\mathscr{H},K)$. However, by the Closed graph theorem, if they do, then they are bounded. Whenever the map $M_\psi C_\varphi$ is bounded on $(\mathscr{H},K)$, it is called a weighted composition operator on $(\mathscr{H},K)$. Fix a $w\in\Omega$ and $\eta\in\mathbb{C}^k$. Then, for $h\in(\mathscr{H},K)$, we see that

$$\begin{aligned}\big\langle (M_\psi C_\varphi)^* K(\cdot,w)\eta, h\big\rangle &= \big\langle K(\cdot,w)\eta, \psi(\cdot)h(\varphi(\cdot))\big\rangle\\ &= \big\langle \eta, \psi(w)h(\varphi(w))\big\rangle\\ &= \big\langle \psi(w)^*\eta, h(\varphi(w))\big\rangle\\ &= \big\langle K\big(\cdot,\varphi(w)\big)\big(\psi(w)^*\eta\big), h\big\rangle.\end{aligned}$$

Therefore,

$$(M_\psi C_\varphi)^* K(\cdot,w)\eta = K(\cdot,\varphi(w))(\psi(w)^*\eta),\;\; w\in\Omega, \eta\in\mathbb{C}^k. \tag{4.2}$$

We now recall the jet construction from chapter 2. Suppose that $K_1,K_2:\Omega\times\Omega\to\mathbb{C}$ are two non-negative definite kernels. As before we realize the vectors of the Hilbert space $(\mathscr{H},K_1)\otimes(\mathscr{H},K_2)$ as holomorphic functions in z and ζ, $z,\zeta\in\Omega$. Recall that the subspaces $\mathscr{A}_k$, $k\geq 0$, of $(\mathscr{H},K_1)\otimes(\mathscr{H},K_2)$ are defined as following:

$$\mathscr{A}_k := \Big\{f\in(\mathscr{H},K_1)\otimes(\mathscr{H},K_2) : \Big(\big(\tfrac{\partial}{\partial\zeta}\big)^i f(z,\zeta)\Big)_{|\Delta} = 0,\, 0\leq i\leq k\Big\}, \tag{4.3}$$

where Δ is the diagonal set $\{(z,z): z\in\Omega\}$. Also recall that the map $J_k : (\mathscr{H},K_1)\otimes(\mathscr{H},K_2)\to \mathrm{Hol}(\Omega\times\Omega,\mathbb{C}^{k+1})$ is given by the following formula

$$(J_k f)(z,\zeta) = \sum_{i=0}^{k}\big(\tfrac{\partial}{\partial\zeta}\big)^i f(z,\zeta)\otimes e_i,\; f\in(\mathscr{H},K_1)\otimes(\mathscr{H},K_2),$$

where $\{e_i\}_{i=0}^k$ is the standard orthonormal basis of $\mathbb{C}^{k+1}$. The map $R:\mathrm{ran}\,J_k\to\mathrm{Hol}(\Omega,\mathbb{C}^{k+1})$ is the restriction map, that is, $R(\mathbf{h}) = \mathbf{h}_{|\Delta}$, $\mathbf{h}\in\mathrm{ran}\,J_k$. By Theorem 1.1.4, we have that ran RJ_k is a reproducing kernel Hilbert space determined by the non-negative definite kernel $J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}$, where

$$J_k(K_1,K_2)_{|\mathrm{res}\,\Delta} = \big(K_1(z,w)\partial^i\bar\partial^j K_2(z,w)\big)_{i,j=0}^k,\;\; z,w\in\Omega.$$

For any $\psi\in\mathrm{Hol}(\Omega)$, let $\psi^{(i)}(z)$, $i\in\mathbb{Z}_+$, denote the ith derivative of ψ at the point z. Let $(\mathscr{J}_k\psi)(z)$, $z\in\Omega$, be the $(k+1)\times(k+1)$ lower triangular matrix given by the following formula:

$$(\mathscr{J}_k\psi)(z) := \begin{pmatrix} \psi(z) & 0 & 0 & \dots & 0 \\ \binom{1}{0}\psi^{(1)}(z) & \psi(z) & 0 & \dots & 0 \\ \binom{2}{0}\psi^{(2)}(z) & \binom{2}{1}\psi^{(1)}(z) & \psi(z) & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \binom{k}{0}\psi^{(k)}(z) & \dots & \dots & \binom{k}{k-1}\psi^{(1)}(z) & \psi(z) \end{pmatrix}$$

Recall that for any $f \in \mathrm{Hol}(\Omega)$ and $\varphi \in \mathrm{Hol}(\Omega,\Omega)$, the Faà di Bruno's formula (cf. [17, page 139]) for the ith derivative of the composition function $f\circ\varphi$ is the following:

$$(f\circ\varphi)^{(i)}(z) = \sum_{j=1}^{i} f^{(j)}\big(\varphi(z)\big)B_{i,j}\big(\varphi^{(1)}(z),\dots,\varphi^{(i-j+1)}(z)\big),\ \ z\in\Omega, \tag{4.4}$$

where $B_{i,j}(z_1,\dots,z_{i-j+1})$, $i \geq j \geq 1$, are the Bell's polynomials. Furthermore, let $(\mathscr{B}_k\varphi)(z)$ denote the $(k+1)\times(k+1)$ lower triangular matrix of the form

$$(\mathscr{B}_k\varphi)(z) := \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \Big(\!\Big(B_{i,j}\big(\varphi^{(1)}(z),\dots,\varphi^{(i-j+1)}(z)\big)\Big)\!\Big)_{i,j=1}^{k} \end{pmatrix}, z\in\Omega,$$

where $B_{i,j}$, $1 \leq i < j \leq k$, is set to be 0.

The main result of this subsection is the Theorem below identifying the compression of the tensor product of two weighted composition operators with another weighted composition operator.

Theorem 4.2.1. *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain, $\psi_1,\psi_2 \in \mathrm{Hol}(\Omega)$ and $\varphi \in \mathrm{Hol}(\Omega,\Omega)$. Suppose that the weighted composition operators $M_{\psi_1}C_\varphi$ and $M_{\psi_2}C_\varphi$ are bounded on $(\mathscr{H},K_1)$ and $(\mathscr{H},K_2)$ respectively. Then the operator $P_{\mathscr{A}_k^\perp}(M_{\psi_1}C_\varphi \otimes M_{\psi_2}C_\varphi)_{|\mathscr{A}_k^\perp}$ is unitarily equivalent to the operator $M_{\psi_1(\mathscr{J}_k\psi_2)(\mathscr{B}_k\varphi)}C_\varphi$ on $\big(\mathscr{H}, J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}\big)$.*

In particular, the operator $M_{\psi_1(\mathscr{J}_k\psi_2)(\mathscr{B}_k\varphi)}C_\varphi$ is bounded on $\big(\mathscr{H}, J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}\big)$ and

$$\|M_{\psi_1(\mathscr{J}_k\psi_2)(\mathscr{B}_k\varphi)}C_\varphi\| \leq \|M_{\psi_1}C_\varphi\|\,\|M_{\psi_2}C_\varphi\|.$$

Before, we give the proof of Theorem 4.2.1, we state a second theorem refining some of the statements in it. In this refined form, it will be a useful tool in finding new weakly homogeneous operators.

Theorem 4.2.2. *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain, $\psi_1,\psi_2 \in \mathrm{Hol}(\Omega)$ and $\varphi \in \mathrm{Aut}(\Omega)$. Then*

(i) *if the operators $M_{\psi_1}C_\varphi$ and $M_{\psi_2}C_\varphi$ are bounded and invertible on $(\mathscr{H},K_1)$ and $(\mathscr{H},K_2)$ respectively, then so is the operator $M_{\psi_1(\mathscr{J}_k\psi_2)(\mathscr{B}_k\varphi)}C_\varphi$ on $\big(\mathscr{H}, J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}\big)$.*

(ii) *if the operators $M_{\psi_1}C_\varphi$ and $M_{\psi_2}C_\varphi$ are unitary on $(\mathscr{H}, K_1)$ and $(\mathscr{H}, K_2)$ respectively, then so is the operator $M_{\psi_1(\mathscr{J}_k\psi_2)(\mathscr{B}_k\varphi)}C_\varphi$ on $\big(\mathscr{H}, J_k(K_1, K_2)_{|\mathrm{res}\,\Delta}\big)$.*

The following lemma will be an essential ingredient in the proof of Theorem 4.2.2.

Lemma 4.2.3. *Let H be a Hilbert space and $X : H \to H$ be a bounded, invertible operator. Suppose that H_0 be a closed subspace of H which is invariant under both X and X^{-1}. Then the operators $X_{|H_0}$ and $P_{H_0^\perp}X_{|H_0^\perp}$ are invertible. Moreover, if X is unitary, then $H_0^\perp$ is also invariant under X, and the operators $X_{|H_0}$ and $X_{|H_0^\perp}$ are unitary.*

Proof. Since H_0 is invariant under both X and X^{-1}, we have

$$X = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{and} \quad X^{-1} = \begin{pmatrix} P & Q \\ 0 & R \end{pmatrix} \tag{4.5}$$

on $H_0 \oplus H_0^\perp$, for some A, B, C and P, Q, R. A routine calculation shows that $AP = PA = I$ and $CR = RC = I$. Hence A and C are invertible. If X is unitary, using $XX^* = I$, we see that

$$\begin{pmatrix} AA^* + BB^* & BC^* \\ CB^* & CC^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Thus $BC^* = 0$. By the first part of the lemma, we have that C is invertible. Therefore $B = 0$. Hence $AA^* = CC^* = I$. Since A and C are also invertible, it follows that A and C are unitary. □

Proof of Theorem 4.2.1. First, set

$$(\psi_1 \otimes \psi_2)(z, \zeta) := \psi_1(z)\psi_2(\zeta) \;\; \text{and} \;\; \boldsymbol{\varphi}(z, \zeta) := (\varphi(z), \varphi(\zeta)), z, \zeta \in \Omega.$$

Then $\psi_1 \otimes \psi_2 \in \mathrm{Hol}(\Omega \times \Omega)$ and $\boldsymbol{\varphi} \in \mathrm{Hol}(\Omega \times \Omega, \Omega \times \Omega)$. Consequently, the operator $M_{\psi_1 \otimes \psi_2}C_{\boldsymbol{\varphi}}$ is a weighted composition operator on $(\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)$.

Recall that $(\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)$ is the reproducing kernel Hilbert space with the reproducing kernel $K_1 \otimes K_2$ where $K_1 \otimes K_2 : (\Omega \times \Omega) \times (\Omega \times \Omega) \to \mathbb{C}$ is given by

$$(K_1 \otimes K_2)(z, \zeta; w, \rho) = K_1(z, w)K_2(\zeta, \rho), \;\; z, \zeta, w, \rho \in \Omega.$$

By (4.2), we see that for $w, \rho \in \Omega$,

$$\begin{aligned}(M_{\psi_1}C_\varphi \otimes M_{\psi_2}C_\varphi)^*(K_1(\cdot, w) \otimes K_2(\cdot, \rho)) &= \overline{\psi_1(w)}K_1(\cdot, \varphi(w)) \otimes \overline{\psi_2(\rho)}K_2(\cdot, \varphi(\rho)) \\ &= \overline{(\psi_1 \otimes \psi_2)(w, \rho)}(K_1 \otimes K_2)(\cdot, (\varphi(w), \varphi(\rho))) \\ &= (M_{\psi_1 \otimes \psi_2}C_{\boldsymbol{\varphi}})^*(K_1(\cdot, w) \otimes K_2(\cdot, \rho)).\end{aligned}$$

Since $\bigvee \{K_1(\cdot, w) \otimes K_2(\cdot, \rho) : w, \rho \in \Omega\}$ is dense in $(\mathscr{H}, K_1) \otimes (\mathscr{H}, K_2)$, it follows that

$$M_{\psi_1}C_\varphi \otimes M_{\psi_2}C_\varphi = M_{\psi_1 \otimes \psi_2}C_{\boldsymbol{\varphi}}. \tag{4.6}$$

Recall that by Theorem 1.1.4, the operator $(RJ_k)_{|\mathscr{A}_k^\perp} : \mathscr{A}_k^\perp \to \big(\mathscr{H}, J_k(K_1,K_2)_{|\operatorname{res}\Delta}\big)$ is unitary. Therefore we will be done if we can show

$$\big((RJ_k)_{|\mathscr{A}_k^\perp}\big)P_{\mathscr{A}_k^\perp}(M_{\psi_1}C_\varphi \otimes M_{\psi_2}C_\varphi)_{|\mathscr{A}_k^\perp}\big((RJ_k)_{|\mathscr{A}_k^\perp}\big)^* = M_{\psi_1(\mathscr{J}_k\psi_2)(\mathscr{B}_k\varphi)}C_\varphi. \tag{4.7}$$

To verify this, let $(RJ_k)(f)$ be an arbitrary element in $\big(\mathscr{H}, J_k(K_1,K_2)_{|\operatorname{res}\Delta}\big)$ where $f \in \mathscr{A}_k^\perp$. Since $\ker(RJ_k) = \mathscr{A}_k$ (see the discussion before Theorem 1.1.4), it follows that

$$\big((RJ_k)_{|\mathscr{A}_k^\perp}\big)P_{\mathscr{A}_k^\perp}(M_{\psi_1}C_\varphi \otimes M_{\psi_2}C_\varphi)_{|\mathscr{A}_k^\perp}\big((RJ_k)_{|\mathscr{A}_k^\perp}\big)^*(RJ_kf) = \big((RJ_k)(M_{\psi_1}C_\varphi \otimes M_{\psi_2}C_\varphi)(f). \tag{4.8}$$

Using (4.6), we see that

$$\begin{aligned}\big((RJ_k)(M_{\psi_1}C_\varphi \otimes M_{\psi_2}C_\varphi)(f) &= (RJ_k)\big(\psi_1(z)\psi_2(\zeta)f(\varphi(z),\varphi(\zeta)\big)\\ &= \sum_{i=0}^{k}\Big(\big(\tfrac{\partial}{\partial\zeta}\big)^i\big(\psi_1(z)\psi_2(\zeta)f(\varphi(z),\varphi(\zeta))\big)\Big)_{|\Delta}\otimes e_i.\end{aligned} \tag{4.9}$$

Also a straightforward computation, noting that $\mathscr{J}_k\psi_2$ and $\mathscr{B}_k\varphi$ are lower triangular matrices, shows that

$$\begin{aligned}&\big(M_{\psi_1(\mathscr{J}_k\psi_2)(\mathscr{B}_k\varphi)}C_\varphi\big)\big((RJ_k)f\big)(z)\\ &\quad= \big(M_{\psi_1(\mathscr{J}_k\psi_2)(\mathscr{B}_k\varphi)}C_\varphi\big)\big(\textstyle\sum_{i=0}^k\big(\big(\tfrac{\partial}{\partial\zeta}\big)^i f(z,\zeta)\big)_{|\Delta}\otimes e_i\big)\\ &\quad= \psi_1(z)\textstyle\sum_{i=0}^k\Big(\sum_{j=0}^i\big((\mathscr{J}_k\psi_2)(\mathscr{B}_k\varphi)\big)_{i,j}(z)\big(\big(\tfrac{\partial}{\partial\zeta}\big)^j f(z,\zeta)\big)_{|\Delta}(\varphi(z),\varphi(z))\Big)\otimes e_i.\end{aligned} \tag{4.10}$$

Hence, in view of (4.8), (4.9) and (4.10), to verify (4.7), it suffices to show that

$$\begin{aligned}&\Big(\big(\tfrac{\partial}{\partial\zeta}\big)^i\big(\psi_2(\zeta)f(\varphi(z),\varphi(\zeta))\big)\Big)(z,z)\\ &\qquad= \textstyle\sum_{j=0}^i\big((\mathscr{J}_k\psi_2)(\mathscr{B}_k\varphi)\big)_{i,j}(z)\big(\big(\tfrac{\partial}{\partial\zeta}\big)^j f(z,\zeta)\big)(\varphi(z),\varphi(z)),\ \ 0 \le i \le k.\end{aligned} \tag{4.11}$$

Since $\big((\mathscr{J}_k\psi_2)(\mathscr{B}_k\varphi)\big)_{0,0}(z) = 1$, equality in both sides of (4.11) is easily verified for the case $i = 0$. For $1 \le i \le k$, we see that

$$\begin{aligned}&\Big(\big(\tfrac{\partial}{\partial\zeta}\big)^i\big(\psi_2(\zeta)f(\varphi(z),\varphi(\zeta))\big)\Big)(z,z)\\ &\quad= \Big(\psi_2^{(i)}(\zeta)f(\varphi(z),\varphi(\zeta)) + \textstyle\sum_{p=1}^i\binom{i}{p}\psi_2^{(i-p)}(\zeta)\big(\tfrac{\partial}{\partial\zeta}\big)^p\big(f(\varphi(z),\varphi(\zeta))\big)\Big)(z,z)\\ &\quad= \Big(\psi_2^{(i)}(\zeta)f(\varphi(z),\varphi(\zeta)) + \textstyle\sum_{p=1}^i\binom{i}{p}\psi_2^{(i-p)}(\zeta)\sum_{j=1}^p(\mathscr{B}_k\varphi)_{p,j}(\zeta)\big(\big(\tfrac{\partial}{\partial\zeta}\big)^j f(z,\zeta)\big)(\varphi(z),\varphi(\zeta))\Big)(z,z)\\ &\quad= \psi_2^{(i)}(z)f(\varphi(z),\varphi(z)) + \textstyle\sum_{p=1}^i\sum_{j=1}^p\binom{i}{p}\psi_2^{(i-p)}(z)(\mathscr{B}_k\varphi)_{p,j}(z)\big(\big(\tfrac{\partial}{\partial\zeta}\big)^j f(z,\zeta)\big)(\varphi(z),\varphi(z)).\end{aligned} \tag{4.12}$$

Here the first equality follows from the Leibniz rule for derivative of product while the second one follows from (4.4). Finally, we compute

$$\begin{aligned}
&\textstyle\sum_{j=0}^{i}\big((\mathscr{J}_k\psi_2)(\mathscr{B}_k\varphi)\big)_{i,j}(z)\big(\big(\frac{\partial}{\partial\zeta}\big)^j f(z,\zeta)\big)(\varphi(z),\varphi(z))\\
&\quad=\textstyle\sum_{j=0}^{i}\sum_{p=j}^{i}\binom{i}{p}\psi_2^{(i-p)}(z)(\mathscr{B}_k\varphi)_{p,j}(z)\big(\big(\frac{\partial}{\partial\zeta}\big)^j f(z,\zeta)\big)(\varphi(z),\varphi(z))\\
&\quad=\psi_2^{(i)}(z)f(\varphi(z),\varphi(z))+\textstyle\sum_{j=1}^{i}\sum_{p=j}^{i}\binom{i}{p}\psi_2^{(i-p)}(z)(\mathscr{B}_k\varphi)_{p,j}(z)\big(\big(\frac{\partial}{\partial\zeta}\big)^j f(z,\zeta)\big)(\varphi(z),\varphi(z))\\
&\quad=\psi_2^{(i)}(z)f(\varphi(z),\varphi(z))+\textstyle\sum_{p=1}^{i}\sum_{j=1}^{p}\binom{i}{p}\psi_2^{(i-p)}(z)(\mathscr{B}_k\varphi)_{p,j}(z)\big(\big(\frac{\partial}{\partial\zeta}\big)^j f(z,\zeta)\big)(\varphi(z),\varphi(z)).
\end{aligned}\tag{4.13}$$

Here the second equality follows since $(\mathscr{B}_k\varphi)_{q,0}=\delta_{q\,0}$, $0\le q\le k$.

The equality in (4.11) is therefore verified, completing the proof of the theorem. □

Remark 4.2.4. *From* (4.12)*, we see that if the hypothesis of Theorem 4.2.1 is in force, then the subspace $\mathscr{A}_k$ is invariant under the operator $M_{\psi_1}C_\varphi\otimes M_{\psi_2}C_\varphi$.*

Proof of Theorem 4.2.2 (i). By hypothesis the operators $M_{\psi_1}C_\varphi$ and $M_{\psi_2}C_\varphi$ are bounded and invertible on $(\mathscr{H},K_1)$ and $(\mathscr{H},K_2)$, respectively. It follows easily that $(M_{\psi_i}C_\varphi)^{-1}=M_{\chi_i}C_{\varphi^{-1}}$, where $\chi_i=\frac{1}{\psi_i\circ\varphi^{-1}}$. Consequently, $(M_{\psi_1}C_\varphi\otimes M_{\psi_2}C_\varphi)^{-1}=M_{\chi_1}C_{\varphi^{-1}}\otimes M_{\chi_2}C_{\varphi^{-1}}$. Therefore, by Remark 4.2.4, $\mathscr{A}_k$ is invariant under both $M_{\psi_1}C_\varphi\otimes M_{\psi_2}C_\varphi$ and $(M_{\psi_1}C_\varphi\otimes M_{\psi_2}C_\varphi)^{-1}$. Hence, by Lemma 4.2.3, the operator $P_{\mathscr{A}_k^\perp}(M_{\psi_1}C_\varphi\otimes M_{\psi_2}C_\varphi)_{|\mathscr{A}_k^\perp}$ is invertible. An application of Theorem 4.2.1 now completes the proof. □

Proof of Theorem 4.2.2 (ii). If $M_{\psi_1}C_\varphi$ and $M_{\psi_2}C_\varphi$ are unitary, then so is the operator $M_{\psi_1}C_\varphi\otimes M_{\psi_2}C_\varphi$. Hence, by the argument used in part (i) of this theorem together with Lemma 4.2.3, we see that $\mathscr{A}_k$ is reducing under $M_{\psi_1}C_\varphi\otimes M_{\psi_2}C_\varphi$. Consequently, the operator $(M_{\psi_1}C_\varphi\otimes M_{\psi_2}C_\varphi)_{|\mathscr{A}_k^\perp}$ is unitary. Hence, by Theorem 4.2.1, we conclude that the operator $M_{\psi_1(\mathscr{J}_k\psi_2)(\mathscr{B}_k\varphi)}C_\varphi$ on $\big(\mathscr{H},J_k(K_1,K_2)_{|\mathrm{res}\,\Delta}\big)$ is unitary. □

Recall that the compression of the operators $M_z\otimes I$ and $I\otimes M_z$ acting on $(\mathscr{H},K_1)\otimes(\mathscr{H},K_2)$ to the subspace $\mathscr{A}_0^\perp$ are unitarily equivalent to the operator M_z on the Hilbert space $(\mathscr{H},K_1K_2)$. The following corollary isolates the case of $\mathscr{A}_0$ from the Theorem 4.2.1 and Theorem 4.2.2 providing a similar model for the compression of the tensor product of the weighted composition operators on $(\mathscr{H},K_1)$ and $(\mathscr{H},K_2)$ to $\mathscr{A}_0^\perp$. As a consequence, we obtain the boundedness and invertibility of such weighted composition operators.

Corollary 4.2.5. *Let $\Omega\subseteq\mathbb{C}$ be a bounded domain, $\psi_1,\psi_2\in\mathrm{Hol}(\Omega)$ and $\varphi\in\mathrm{Hol}(\Omega,\Omega)$. Let K_1 and K_2 be two scalar valued non-negative definite kernels on $\Omega\times\Omega$. Suppose that the weighted composition operators $M_{\psi_1}C_\varphi$ and $M_{\psi_2}C_\varphi$ are bounded on $(\mathscr{H},K_1)$ and $(\mathscr{H},K_2)$, respectively. Then the operator $M_{\psi_1\psi_2}C_\varphi$ is bounded on $(\mathscr{H},K_1K_2)$ with*

$$\|M_{\psi_1\psi_2}C_\varphi\|\le\|M_{\psi_1}C_\varphi\|\,\|M_{\psi_2}C_\varphi\|.$$

Moreover, if the operators $M_{\psi_1}C_\varphi$ and $M_{\psi_2}C_\varphi$ are invertible (resp. unitary) on $(\mathscr{H},K_1)$ and $(\mathscr{H},K_2)$, respectively, $\varphi\in Aut(\Omega)$, then the operator $M_{\psi_1\psi_2}C_\varphi$ is also invertible (resp. unitary) on $(\mathscr{H},K_1K_2)$.

4.2.1 Weighted composition operators and weakly homogeneous operators

In this subsection, we show that the multiplication by the coordinate function z acting on a Hilbert space $\mathscr{H}$ possessing a sharp reproducing kernel K is weakly homogeneous if and only if there exist bounded invertible weighted composition operators, one for each $\varphi\in$ Möb, on the same Hilbert space $\mathscr{H}$. We begin with a useful Lemma. A version of this lemma, without involving the composition by φ, is in [26].

Lemma 4.2.6. *Let $K(z,w):\mathbb{D}\times\mathbb{D}\to\mathscr{M}_k(\mathbb{C})$ be a positive definite kernel. Suppose that the multiplication operator M_z is bounded on $(\mathscr{H},K)$. Let φ be a fixed but arbitrary function in* Möb *which is analytic in a neighbourhood of $\sigma(M_z)$. If X is a bounded invertible operator on $(\mathscr{H},K)$ of the from $M_{g_\varphi}C_{\varphi^{-1}}$, where $g_\varphi\in\mathrm{Hol}(\mathbb{D},GL_k(\mathbb{C}))$, then X intertwines M_z and $\varphi(M_z)$, that is, $M_zX=X\varphi(M_z)$.*
Moreover, if K is sharp and X is a bounded invertible operator on $(\mathscr{H},K)$ intertwining M_z and $\varphi(M_z)$, then $X=M_{g_\varphi}C_{\varphi^{-1}}$ for some $g_\varphi\in\mathrm{Hol}(\mathbb{D},GL_k(\mathbb{C}))$.

Proof. Suppose that X is a bounded invertible operator of the form $M_{g_\varphi}C_{\varphi^{-1}}$. Then for $f\in(\mathscr{H},K)$, we have

$$\begin{aligned} X\varphi(M_z)(f)=(M_{g_\varphi}C_{\varphi^{-1}})M_\varphi(f)&=M_{g_\varphi}(z(f\circ\varphi^{-1}))\\ &=(M_zM_{g_\varphi}C_{\varphi^{-1}})(f)\\ &=(M_zX)(f).\end{aligned}\tag{4.14}$$

Therefore X intertwines M_z and $\varphi(M_z)$, i.e. $X\varphi(M_z)=M_zX$.

Conversely, assume that X is a bounded invertible operator on $(\mathscr{H},K)$ such that $M_zX=X\varphi(M_z)$. Taking adjoint and acting on the vector $K(\cdot,w)\eta$, $w\in\mathbb{D},\eta\in\mathbb{C}^k$, we obtain

$$\varphi(M_z)^*X^*K(\cdot,w)\eta=X^*M_z^*K(\cdot,w)\eta=\overline{w}X^*K(\cdot,w)\eta.\tag{4.15}$$

Thus $X^*K(.,w)\eta\in\ker\big(\varphi(M_z)^*-\overline{w}\big)$.

claim: $\ker\big(\varphi(M_z)^*-\overline{w}\big)=\ker\big(M_z^*-\overline{\varphi^{-1}(w)}\big)$.

To verify the claim, set $\hat{\varphi}(z):=\overline{\varphi(\overline{z})}$, $z\in\mathbb{D}$. Clearly, $\hat{\varphi}\in$ Möb. It is also easy to see that $\varphi(M_z)^*=\hat{\varphi}(M_z^*)$. Therefore $\ker\big(\varphi(M_z)^*-\overline{w}\big)=\ker\big(\hat{\varphi}(M_z^*)-\overline{w}\big)$. Let f be an arbitrary vector in $\ker\big(\hat{\varphi}(M_z^*)-\overline{w}\big)$. Then we have $\hat{\varphi}(M_z^*)f=\overline{w}f$. Hence

$$M_z^*f=\big(\hat{\varphi}^{-1}(\hat{\varphi}(M_z^*))\big)f=\hat{\varphi}^{-1}\left(\overline{w}\right)f=\overline{\varphi^{-1}(w)}f.$$

Therefore $f \in \ker\big(M_z^* - \overline{\varphi^{-1}(w)}\big)$. Consequently, $\ker\big(\hat{\varphi}(M_z^*) - \overline{w}\big) \subseteq \ker\big(M_z^* - \overline{\varphi^{-1}(w)}\big)$. By the same argument, it also follows that $\ker\big(M_z^* - \overline{\varphi^{-1}(w)}\big) \subseteq \ker\big(\hat{\varphi}(M_z^*) - \overline{w}\big)$. Hence the claim is verified.

Since K is sharp and the vector $X^*K(\cdot, w)\eta \in \ker\big(\varphi(M_z)^* - \overline{w}\big)$, it follows from the claim that there exists a unique vector $h_\varphi(w)\eta \in \mathbb{C}^k$ such that

$$X^*K(\cdot, w)\eta = K(\cdot, \varphi^{-1}(w))h_\varphi(w)\eta. \tag{4.16}$$

The invertibility of the matrix $K(\varphi^{-1}(w), \varphi^{-1}(w))$ ensures the uniqueness of the vector $h_\varphi(w)\eta$. It is easily verified that for each $w \in \mathbb{D}$, the map $\eta \mapsto h_\varphi(w)\eta$ defines a linear map on $\mathbb{C}^k$. Since X is invertible, it follows from (4.16) that $h_\varphi(w)$ is invertible. Now for any $w \in \mathbb{D}$, $\eta \in \mathbb{C}^k$ and $f \in (\mathscr{H}, K)$, we see that

$$\begin{aligned}
\big\langle (Xf)(w), \eta \big\rangle &= \big\langle Xf, K(\cdot, w)\eta \big\rangle \\
&= \big\langle f, X^*K(\cdot, w)\eta \big\rangle \\
&= \big\langle f, K(\cdot, \varphi^{-1}(w))h_\varphi(w)\eta \big\rangle \\
&= \big\langle (f \circ \varphi^{-1})(w), h_\varphi(w)\eta \big\rangle \\
&= \big\langle h_\varphi(w)^*(f \circ \varphi^{-1})(w), \eta \big\rangle.
\end{aligned}$$

Hence $X = M_{g_\varphi}C_{\varphi^{-1}}$ where $g_\varphi(w) = h_\varphi(w)^*$, $w \in \mathbb{D}$. Since we have already shown that $g_\varphi(w)$, $w \in \mathbb{D}$, is invertible, to complete the proof, we only need to show that the map $w \mapsto g_\varphi(w)$ is holomorphic.

Let w_0 be a fixed but arbitrary point in $\mathbb{D}$. Since $K(\varphi^{-1}(w_0), \varphi^{-1}(w_0))$ is invertible, there exists a neighbourhood Ω_0 of w_0 such that $K(\varphi^{-1}(w_0), \varphi^{-1}(w))$ is invertible for all w in Ω_0. From (4.16), we have

$$(X^*K(\cdot, w)\eta)\varphi^{-1}(w_0) = K(\varphi^{-1}(w_0), \varphi^{-1}(w))h_\varphi(w)\eta, \ w \in \Omega_0.$$

Therefore

$$h_\varphi(w)\eta = K(\varphi^{-1}(w_0), \varphi^{-1}(w))^{-1}(X^*K(\cdot, w)\eta)\varphi^{-1}(w_0), \ w \in \Omega_0.$$

Since the right hand side of the above equality is anti-holomorphic on Ω_0, it follows that the function $h_\varphi(w)$ is anti-holomorphic on Ω_0 and therefore g_φ is holomorphic on Ω_0. Since w_0 is arbitrary, we conclude that g_φ is holomorphic on Ω. This completes the proof. □

Proposition 4.2.7. *Let $K(z, w) : \mathbb{D} \times \mathbb{D} \to \mathscr{M}_k(\mathbb{C})$ be a positive definite kernel. If for each $\varphi \in$ Möb, there exists a function $g_\varphi \in \mathrm{Hol}(\mathbb{D}, GL_k(\mathbb{C}))$ such that the operator $M_{g_\varphi}C_{\varphi^{-1}}$ is bounded and invertible on $(\mathscr{H}, K)$, then M_z on $(\mathscr{H}, K)$ is weakly homogeneous. Moreover, if K is sharp and the operator M_z on $(\mathscr{H}, K)$ is weakly homogeneous, then for each φ in Möb, there exists $g_\varphi \in \mathrm{Hol}(\mathbb{D}, GL_k(\mathbb{C}))$ such that the weighted composition operator $M_{g_\varphi}C_{\varphi^{-1}}$ is bounded and invertible on $(\mathscr{H}, K)$.*

Proof. Let U be a neighbourhood of the identity in Möb such that $\varphi(M_z)$ is well-defined for all $\varphi \in U$. By hypothesis, there exists $g_\varphi \in \text{Hol}(\mathbb{D}, GL_k(\mathbb{C}))$ such that the operator $M_{g_\varphi} C_{\varphi^{-1}}$ on $(\mathscr{H}, K)$ is bounded and invertible. Then by Lemma 4.2.6, it follows that the operator M_z satisfies $M_z X = \varphi(M_z) X$, $\varphi \in U$. Hence M_z is similar to $\varphi(M_z)$ for all $\varphi \in U$. Now an application of Lemma 4.1.2 completes the proof in the forward direction.

For the proof in the other direction, suppose that M_z on $(\mathscr{H}, K)$ is weakly homogeneous. Then, by definition, $\sigma(M_z) \subseteq \bar{\mathbb{D}}$ and $\varphi(M_z)$ is similar to M_z for all φ in Möb. Therefore, for each $\varphi \in$ Möb, there exists a bounded invertible operator X_φ satisfying $M_z X_\varphi = X_\varphi \varphi(M_z)$. By Lemma 4.2.6, X_φ is of the form $M_{g_\varphi} C_{\varphi^{-1}}$, $g_\varphi \in \text{Hol}(\mathbb{D}, GL_k(\mathbb{C}))$. This completes the proof. □

The theorem appearing below shows that the weak homogeneity of the multiplication operator is preserved under the jet construction.

Theorem 4.2.8. *Suppose that K_1 and K_2 are two scalar valued sharp positive definite kernels on $\mathbb{D} \times \mathbb{D}$. If the multiplication operators M_z on $(\mathscr{H}, K_1)$ and $(\mathscr{H}, K_2)$ are weakly homogeneous, then M_z on $\big(\mathscr{H}, J_k(K_1, K_2)_{|\text{res}\,\Delta}\big)$ is also weakly homogeneous.*

Proof. By hypothesis, the operator M_z on $(\mathscr{H}, K_1)$ as well as on $(\mathscr{H}, K_2)$ is weakly homogeneous. By Proposition 4.2.7, for each $\varphi \in$ Möb, there exist $g_\varphi, h_\varphi \in \text{Hol}(\mathbb{D}, \mathbb{C} \setminus \{0\})$ such that the weighted composition operators $M_{g_\varphi} C_{\varphi^{-1}}$ and $M_{h_\varphi} C_{\varphi^{-1}}$ are bounded and invertible on $(\mathscr{H}, K_1)$ and $(\mathscr{H}, K_2)$ respectively. Then by Theorem (4.2.2), it follows that the operator $M_{g_\varphi (\mathscr{J}_k h_\varphi)(\mathscr{B}_k \varphi^{-1})} C_{\varphi^{-1}}$ is bounded and invertible on $\big(\mathscr{H}, J_k(K_1, K_2)_{|\text{res}\,\Delta}\big)$. Again an application of Proposition 4.2.7 completes the proof. □

4.3 Weakly homogeneous operators in the class $\mathscr{F}B_2(\mathbb{D})$

In this section, we study weakly homogeneous operators in the class $\mathscr{F}B_2(\mathbb{D})$. The reader is referred to see section 2.3.2 for the definition of operators in $\mathscr{F}B_2(\mathbb{D})$. The following proposition will be an essential tool in this study.

Proposition 4.3.1. ([37, Proposition 3.3]) *Let T and $\tilde{T}$ be any two operators in $\mathscr{F}B_2(\Omega)$. If X is a bounded invertible operator which intertwines T and $\tilde{T}$, then X and X^{-1} are upper triangular.*

Corollary 4.3.2. *Let $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ on $\mathscr{H}_0 \oplus \mathscr{H}_1$ and $\tilde{T} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}$ on $\tilde{\mathscr{H}}_0 \oplus \tilde{\mathscr{H}}_1$ be two operators in $\mathscr{F}B_2(\Omega)$. Then T is similar to $\tilde{T}$ if and only if there exist bounded invertible operators $X : \mathscr{H}_0 \to \tilde{\mathscr{H}}_0$, $Y : \mathscr{H}_1 \to \tilde{\mathscr{H}}_1$ and a bounded operator $Z : \mathscr{H}_1 \to \tilde{\mathscr{H}}_0$ such that*

(i) $XT_0 = \tilde{T}_0 X$, $YT_1 = \tilde{T}_1 Y$,

(ii) $XS + ZT_1 = \tilde{T}_0 Z + \tilde{S} Y$.

Proof. Suppose that T is similar to $\tilde{T}$. Let $A = \begin{pmatrix} X & Z \\ W & Y \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_1$ be an invertible operator such that $AT = \tilde{T}A$. By Proposition 4.3.1, we have that $W = 0$. Further, we see that the intertwining relation is equivalent to

$$\begin{pmatrix} XT_0 & XS + ZT_1 \\ 0 & YT_1 \end{pmatrix} = \begin{pmatrix} \tilde{T}_0 X & \tilde{T}_0 Z + \tilde{S}Y \\ 0 & \tilde{T}_1 Y \end{pmatrix}.$$

Again applying Proposition 4.3.1, we see that A^{-1} is also upper triangular. Hence using Lemma 4.2.3, we conclude that X and Y are invertible.

Conversely, assume that there exist bounded invertible operators $X : \mathcal{H}_0 \to \tilde{\mathcal{H}}_0$, $Y : \mathcal{H}_1 \to \tilde{\mathcal{H}}_1$ and a linear operator $Z : \mathcal{H}_1 \to \tilde{\mathcal{H}}_0$ satisfying (i) and (ii) of this Corollary. Let A be the operator $\begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix}$. Since X and Y are invertible, it follows that A is invertible. The intertwining requirement $AT = \tilde{T}A$ is also easily verified. □

Lemma 4.3.3. *Let $T \in B(\mathcal{H})$ be an operator in $B_1(\mathbb{D})$ with $\sigma(T) = \bar{\mathbb{D}}$. Then the operator $\varphi(T)$ belongs to $B_1(\mathbb{D})$ for all φ in* Möb.

Proof. Let $\varphi_{\theta,a}$, $\theta \in [0, 2\pi)$, $a \in \mathbb{D}$, be an arbitrary biholomorphic automorphism in Möb. A routine calculation shows that

$$\begin{aligned} \varphi_{\theta,a}(T) - \varphi_{\theta,a}(w) &= e^{i\theta}\big((T-a)(I-\bar{a}T)^{-1} - (w-a)(1-\bar{a}w)^{-1}\big) \\ &= e^{i\theta}(1-|a|^2)(1-\bar{a}w)^{-1}(T-w)(I-\bar{a}T)^{-1}, \;\; w \in \mathbb{D}. \end{aligned} \tag{4.17}$$

Since $T \in B_1(\mathbb{D})$, $(T-w)$ is Fredholm for all $w \in \mathbb{D}$. Therefore, from (4.17), we see that the operator $\varphi_{\theta,a}(T) - \varphi_{\theta,a}(w)$ is the product of a Fredholm operator with an invertible operator. Hence it is a Fredholm operator (see [19]). Consequently, ran $(\varphi(T) - \varphi(w))$ is closed. Furthermore, since the operator $(I - \bar{a}T)^{-1}$ is invertible and commutes with $(T-w)$, using (4.17) once again, it follows that $\ker\big(\varphi_{\theta,a}(T) - \varphi_{\theta,a}(w)\big) = \ker(T-w)$. Consequently,

$$\dim\ker\big(\varphi_{\theta,a}(T) - \varphi_{\theta,a}(w)\big) = \dim\ker(T-w) = 1, \;\; w \in \mathbb{D}$$

and

$$\overline{\bigvee}_{w\in\Omega} \ker\big(\varphi_{\theta,a}(T) - \varphi_{\theta,a}(w)\big) = \overline{\bigvee}_{w\in\Omega} \ker(T-w) = \mathcal{H}.$$

This completes the proof. □

Lemma 4.3.4. *Let $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ be an operator in $\mathcal{F}B_2(\mathbb{D})$ with $\sigma(T) = \sigma(T_0) = \sigma(T_1) = \bar{\mathbb{D}}$. Then the operator $\varphi(T)$ belongs to $\mathcal{F}B_2(\mathbb{D})$ for all φ in* Möb.

Proof. A routine verification, using $T_0S = ST_1$, shows that

$$\varphi(T) = \begin{pmatrix} \varphi(T_0) & \varphi'(T_0)S \\ 0 & \varphi(T_1) \end{pmatrix}, \text{ and } \varphi(T_0)\varphi'(T_0)S = \varphi'(T_0)S\varphi(T_1).$$

Also by Lemma 4.3.3, we have that $\varphi(T_0)$ and $\varphi(T_1)$ belong to $B_1(\mathbb{D})$. Hence the operator $\varphi(T)$ belongs to $\mathscr{F}B_2(\mathbb{D})$. □

The Corollary given below, which gives a necessary and sufficient condition for an operator T in $\mathscr{F}B_2(\mathbb{D})$ to be weakly homogeneous, is a consequence of Corollary 4.3.2 combined with the Lemma we have just proved. Hence the proof is omitted.

Corollary 4.3.5. *Let* $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ *be an operator in* $\mathscr{F}B_2(\mathbb{D})$ *with* $\sigma(T) = \sigma(T_0) = \sigma(T_1) = \bar{\mathbb{D}}$. *Then* T *is weakly homogeneous if and only if for each* φ *in* Möb*, there exist bounded invertible operators* $X_\varphi : \mathscr{H}_0 \to \mathscr{H}_0$, $Y_\varphi : \mathscr{H}_1 \to \mathscr{H}_1$ *and a bounded operator* $Z_\varphi : \mathscr{H}_1 \to \mathscr{H}_0$ *such that the following holds:*

(i) $X_\varphi T_0 = \varphi(T_0)X_\varphi,\ Y_\varphi T_1 = \varphi(T_1)Y_\varphi$

(ii) $X_\varphi S + Z_\varphi T_1 = \varphi(T_0)Z_\varphi + \varphi'(T_0)SY_\varphi$.

4.3.1 A useful Lemma

Let $K_1, K_2 : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ be two positive definite kernels. As in the previous chapter, let $M^{(1)}$ and $M^{(2)}$ denote the operators of multiplication by the coordinate function z on $(\mathscr{H}, K_1)$ and $(\mathscr{H}, K_2)$, respectively. The following lemma which is a generalization of Lemma 4.2.6 will be used to construct operators in $\mathscr{F}B_2(\mathbb{D})$ that are not weakly homogeneous.

Lemma 4.3.6. *Let* φ *be a fixed but arbitrary function in* Möb *which is analytic in a neighbourhood of* $\sigma(M^{(1)})$. *Let* ψ *be a function in* Hol$(\mathbb{D})$ *such that the weighted composition operator* $M_\psi C_{\varphi^{-1}}$ *is bounded from* $(\mathscr{H}, K_1)$ *to* $(\mathscr{H}, K_2)$. *If* X *is a bounded linear operator from* $(\mathscr{H}, K_1)$ *to* $(\mathscr{H}, K_2)$ *such that* $X(f) = \psi(\varphi^{-1})'(f' \circ \varphi^{-1}) + \chi(f \circ \varphi^{-1})$, $f \in (\mathscr{H}, K_1)$ *for some* $\chi \in Hol(\mathbb{D})$, *then* X *satisfies*

$$X\varphi(M^{(1)}) - M^{(2)}X = M_\psi C_{\varphi^{-1}}. \tag{4.18}$$

Moreover, if K_1 *is sharp and* $X : (\mathscr{H}, K_1) \to (\mathscr{H}, K_2)$ *is a bounded linear operator satisfying* (4.18), *then there exists a function* $\chi \in Hol(\mathbb{D})$ *such that* $X(f) = \psi(\varphi^{-1})'(f' \circ \varphi^{-1}) + \chi(f \circ \varphi^{-1})$, $f \in (\mathscr{H}, K_1)$.

(Here $\psi(\varphi^{-1})'$ denotes the pointwise product of the two functions ψ and $(\varphi^{-1})'$. Similarly, $\psi(\varphi^{-1})'(f' \circ \varphi^{-1})$ denotes the pointwise product of $\psi(\varphi^{-1})'$ and $(f' \circ \varphi^{-1})$. Finally, $\chi(f \circ \varphi^{-1})$ is the pointwise product of χ and $f \circ \varphi^{-1}$. This convention is adopted throughout this chapter.)

Proof. Suppose that X is bounded linear operator taking f to $\psi(\varphi^{-1})'(f'\circ\varphi^{-1})+\chi(f\circ\varphi^{-1})$, $f\in(\mathscr{H},K_1)$. Then we see that

$$\begin{aligned}(X\varphi(M^{(1)})-M^{(2)}X)f =& X(\varphi f)-zXf\\ =&\psi(\varphi^{-1})'((\varphi f)'\circ\varphi^{-1})+\chi((\varphi f)\circ\varphi^{-1})-z\psi(\varphi^{-1})'(f'\circ\varphi^{-1})-z\chi(f\circ\varphi^{-1})\\ =&\psi(\varphi^{-1})'\big((\varphi'\circ\varphi^{-1})(f\circ\varphi^{-1})+(\varphi\circ\varphi^{-1})(f'\circ\varphi^{-1})\big)+z\chi(f\circ\varphi^{-1})\\ &-z\psi(\varphi^{-1})'(f'\circ\varphi^{-1})-z\chi(f\circ\varphi^{-1})\\ =&\psi(\varphi^{-1})'(\varphi'\circ\varphi^{-1})(f\circ\varphi^{-1})+\psi(\varphi^{-1})'z(f'\circ\varphi^{-1})-z\psi(\varphi^{-1})'(f'\circ\varphi^{-1})\\ =&\psi(f\circ\varphi^{-1}).\end{aligned}$$

Here for the last equality we have used the identity $(\varphi^{-1})'(\varphi'\circ\varphi^{-1})=1$.

For the converse, assume that K_1 is sharp and $X:(\mathscr{H},K_1)\to(\mathscr{H},K_2)$ is a bounded linear operator satisfying (4.18). Then taking adjoint and acting on $K_2(\cdot,z)$, $z\in\mathbb{D}$, we obtain

$$\begin{aligned}\varphi(M^{(1)})^*X^*K_2(\cdot,z)-\bar{z}X^*K_2(\cdot,z)&=(M_\psi C_{\varphi^{-1}})^*K_2(\cdot,z)\\ &=\overline{\psi(z)}K_1(\cdot,\varphi^{-1}z).\end{aligned}\tag{4.19}$$

Here the last equality follows from exactly the same argument as in (4.2). Further, since $\big(\varphi(M^{(1)})^*-\overline{\varphi(w)}\big)K_1(\cdot,w)=0$, $w\in\mathbb{D}$, differentiating with respect to $\overline{w}$, we see that

$$\big(\varphi(M^{(1)})^*-\overline{\varphi(w)}\big)\bar\partial K_1(\cdot,w)=\overline{\varphi'(w)}K_1(\cdot,w),\ \ w\in\mathbb{D}.\tag{4.20}$$

Replacing w by $\varphi^{-1}z$ in the above equation and combining it with (4.19), we see that

$$\big(\varphi(M^{(1)})^*-\bar{z}\big)X^*K_2(\cdot,z)=\big(\varphi(M^{(1)})^*-\bar{z}\big)\left(\frac{\overline{\psi(z)}}{\overline{\varphi'(\varphi^{-1}z)}}\bar\partial K_1(\cdot,\varphi^{-1}z)\right).\tag{4.21}$$

Consequently, the vector $X^*K_2(\cdot,z)-\frac{\overline{\psi(z)}}{\overline{\varphi'(\varphi^{-1}z)}}\bar\partial K_1(\cdot,\varphi^{-1}z)\in\ker\big(\varphi(M^{(1)})^*-\bar{z}\big)$. Since K_1 is sharp, we have that $\ker\big(\varphi(M^{(1)})^*-\bar{z}\big)=\bigvee\{K_1(\cdot,\varphi^{-1}z)\}$ (see the proof of Lemma 4.2.6). Therefore

$$X^*K_2(\cdot,z)-\frac{\overline{\psi(z)}}{\overline{\varphi'(\varphi^{-1}z)}}\bar\partial K_1(\cdot,\varphi^{-1}z)=\overline{\chi(z)}K_1(\cdot,\varphi^{-1}z),$$

for some $\chi\in\mathrm{Hol}(\mathbb{D})$ (the holomorphicity of χ can be proved by a similar argument used at the end of Lemma 4.2.6).

Finally, for $f\in(\mathscr{H},K_1)$ and $z\in\mathbb{D}$, we see that

$$\begin{aligned}\big(Xf\big)(z)&=\big\langle Xf,K_2(\cdot,z)\big\rangle\\ &=\big\langle f,X^*K_2(\cdot,z)\big\rangle\\ &=\left\langle f,\frac{\overline{\psi(z)}}{\overline{\varphi'(\varphi^{-1}z)}}\bar\partial K_1(\cdot,\varphi^{-1}z)+\overline{\chi(z)}K_1(\cdot,\varphi^{-1}z)\right\rangle\\ &=\psi(z)(\varphi^{-1})'(z)(f'\circ\varphi^{-1})(z)+\chi(z)(f\circ\varphi^{-1})(z).\end{aligned}$$

Here the identity $(\varphi' \circ \varphi^{-1})(z)(\varphi^{-1})'(z) = 1$, $z \in \mathbb{D}$, is used a second time for the last equality. This completes the proof. □

Notation 4.3.7. *For the rest of this section, let $\mathscr{H}^{(\lambda)}$, $\lambda > 0$, denote the Hilbert space determined by the positive definite kernel $K^{(\lambda)}$ where $K^{(\lambda)}(z,w) := \frac{1}{(1-z\bar{w})^\lambda}$, $z, w \in \mathbb{D}$. Note that*

$$K^{(\lambda)}(z,w) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}, \ \ z, w \in \mathbb{D}, \tag{4.22}$$

where $(\lambda)_n$ is the Pochhammer symbol given by $\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$.

For any $\gamma \in \mathbb{R}$, let $K_{(\gamma)}$ be the positive definite kernel given by

$$K_{(\gamma)}(z,w) := \sum_{n=0}^{\infty} (n+1)^\gamma (z\bar{w})^n, z, w \in \mathbb{D}. \tag{4.23}$$

Note that $K_{(0)} = K^{(1)}$ (the Szegö kernel of the unit disc $\mathbb{D}$) and $K_{(1)} = K^{(2)}$ (the Bergman kernel of the unit disc $\mathbb{D}$). The kernel $K_{(-1)}$ is known as the Dirichlet kernel of the unit disc $\mathbb{D}$.

For two sequences $\{a_n\}$ and $\{b_n\}$ of positive real numbers, we write $a_n \sim b_n$ if there exist constants $c_1, c_2 > 0$ such that $c_1 b_n \leq a_n \leq c_2 b_n$, $n \in \mathbb{Z}_+$. From (4.22) *and* (4.23)*, it is clear that $\|z^n\|^2_{\mathscr{H}^{(\lambda)}} = \frac{n!}{(\lambda)_n} = \frac{\Gamma(n+1)\Gamma(\lambda)}{\Gamma(n+\lambda)}$ and $\|z^n\|^2_{(\mathscr{H}, K_{(\gamma)})} = (n+1)^{-\gamma}$, $n \in \mathbb{Z}_+$. Using the identity* $\lim_{n\to\infty} \frac{\Gamma(n+a)}{\Gamma(n)n^a} = 1$, $a \in \mathbb{C}$, *we see that*

$$\|z^n\|^2_{\mathscr{H}^{(\lambda)}} \sim \|z^n\|^2_{(\mathscr{H}, K_{(\lambda-1)})}, \ \ \lambda > 0. \tag{4.24}$$

Therefore, for $\lambda > 0$, there exist constants $c_1, c_2 > 0$ such that $c_1 K_{(\lambda-1)} \preceq K^{(\lambda)} \preceq c_2 K_{(\lambda-1)}$ and consequently, $\mathscr{H}^{(\lambda)} = (\mathscr{H}, K_{(\lambda-1)})$.

Recall that a Hilbert space $\mathscr{H}$ consisting of holomorphic functions on the unit disc $\mathbb{D}$ is said to be Möbius invariant if for each $\varphi \in$ Möb, $f \circ \varphi \in \mathscr{H}$ whenever $f \in \mathscr{H}$. By an application of the closed graph Theorem, it follows that $\mathscr{H}$ is Möbius invariant if and only if the composition operator C_φ is bounded on $\mathscr{H}$ for each $\varphi \in$ Möb. If the multiplication operator M_z is bounded on some Möbius invariant Hilbert space $\mathscr{H}$, then by Proposition 4.2.7, it follows that M_z is weakly homogeneous on $\mathscr{H}$. It is known that the Hilbert spaces $\mathscr{H}^{(\lambda)}$, $\lambda > 0$, and $(\mathscr{H}, K_{(\gamma)})$, $\gamma \in \mathbb{R}$, are Möbius invariant (see [54], [20]). We record this fact as a Lemma for our later use.

Lemma 4.3.8. *The Hilbert spaces $\mathscr{H}^{(\lambda)}$, $\lambda > 0$ and $(\mathscr{H}, K_{(\gamma)})$, $\gamma \in \mathbb{R}$, are Möbius invariant. Consequently, the composition operator C_φ, $\varphi \in$ Möb, is bounded and invertible on $\mathscr{H}^{(\lambda)}$, $\lambda > 0$, as well as on $(\mathscr{H}, K_{(\gamma)})$, $\gamma \in \mathbb{R}$.*

Corollary 4.3.9. *For any $\gamma \in \mathbb{R}$, the operator M_z^* on $(\mathscr{H}, K_{(\gamma)})$ is a weakly homogeneous operator in $B_1(\mathbb{D})$. Moreover, it is similar to a homogeneous operator if and only if $\gamma > -1$. In particular, M_z^* on the Dirichlet space is a weakly homogeneous operator which is not similar to any homogeneous operator.*

Proof. Note that M_z on $(\mathscr{H}, K_{(\gamma)})$ is unitarily equivalent to the weighted shift with the weight sequence $\{w_n\}_{n\in\mathbb{Z}_+}$ where $w_n = \left(\frac{n+1}{n+2}\right)^{\frac{\gamma}{2}}$, $n \in \mathbb{Z}_+$. Since $\sup_{n\in\mathbb{Z}_+} w_n < \infty$, it follows that M_z is bounded on $(\mathscr{H}, K_{(\gamma)})$. By Lemma 4.3.8, C_φ is bounded and invertible on $(\mathscr{H}, K_{(\gamma)})$. Hence by Proposition 4.2.7, it follows that M_z on $(\mathscr{H}, K_{(\gamma)})$ is weakly homogeneous .

Recall that for an operator T, $r_1(T)$ is defined as $\lim_{n\to\infty}\big(m(T^n)\big)^{\frac{1}{n}}$ (which always exists, see [51]), where $m(T) = \inf\big\{\|Tf\| : \|f\| = 1\big\}$. For the multiplication operator M_z on $(\mathscr{H}, K_{(\gamma)})$, it is easily verified that $r_1(M_z) = r(M_z) = 1$, where $r(M_z)$ is the spectral radius of M_z. Hence by a theorem of Seddighi (cf. [50]), we conclude that M_z^* on $(\mathscr{H}, K_{(\gamma)})$ belongs to $B_1(\mathbb{D})$.

Finally assume that M_z^* on $(\mathscr{H}, K_{(\gamma)})$ is similar to a homogeneous operator, say S. Since $B_1(\mathbb{D})$ is closed under similarity, the operator S belongs to $B_1(\mathbb{D})$. Furthermore, since upto unitary equivalence, every homogeneous operator in $B_1(\mathbb{D})$ is of the form M_z^* on $(\mathscr{H}, K^{(\lambda)})$, $\lambda > 0$, it follows that M_z^* on $(\mathscr{H}, K_{(\gamma)})$ is similar to M_z^* on $\mathscr{H}^{(\lambda)}$ for some $\lambda > 0$. Hence by [51, Theorem 2′], γ satisfies $\|z^n\|^2_{(\mathscr{H},K_{(\gamma)})} \sim \|z^n\|^2_{\mathscr{H}^{(\lambda)}}$. Then by (4.24), we see that $\|z^n\|^2_{(\mathscr{H},K_{(\gamma)})} \sim \|z^n\|^2_{(\mathscr{H},K_{(\lambda-1)})}$. Hence $\gamma = \lambda - 1$. Since $\lambda > 0$, it follows that $\gamma > -1$.

For the converse, let $\gamma > -1$. Again using [51, Theorem 2′] and (4.24), it follows that M_z^* on $(\mathscr{H}, K_{(\gamma)})$ is similar to the homogeneous operator M_z^* on $\mathscr{H}^{(\gamma+1)}$. □

The lemma given below shows that the linear map $f \mapsto f'$ is bounded from $\mathscr{H}^{(\lambda)}$ to $\mathscr{H}^{(\lambda+2)}$.

Lemma 4.3.10. *Let $\lambda > 0$ and f be an arbitrary holomorphic function on the unit disc $\mathbb{D}$. Then $f \in \mathscr{H}^{(\lambda)}$ if and only if $f' \in \mathscr{H}^{(\lambda+2)}$. Moreover, if $f \in \mathscr{H}^{(\lambda)}$, then $\|f'\|_{\mathscr{H}^{(\lambda+2)}} \le \sqrt{\lambda(\lambda+1)}\|f\|_{\mathscr{H}^{(\lambda)}}$. Consequently, the differential operator D, that maps f to f', is bounded from $\mathscr{H}^{(\lambda)}$ to $\mathscr{H}^{(\lambda+2)}$ with $\|D\| \le \sqrt{\lambda(\lambda+1)}$.*

Proof. Let $\sum_{n=0}^\infty \alpha_n z^n$, $z \in \mathbb{D}$, be the power series representation of f. Then we have $f'(z) = \sum_{n=0}^\infty (n+1)\alpha_{n+1} z^n$, $z \in \mathbb{D}$. Recall that $\|z^n\|^2_{\mathscr{H}^{(\lambda)}} = \frac{n!}{\lambda(\lambda+1)\cdots(\lambda+n-1)}$. By a straightforward computation we see that

$$\lambda\|z^{n+1}\|^2_{\mathscr{H}^{(\lambda)}} \le (n+1)^2\|z^n\|^2_{\mathscr{H}^{(\lambda+2)}} \le \lambda(\lambda+1)\|z^{n+1}\|^2_{\mathscr{H}^{(\lambda)}},\ \ n \ge 0.$$

Consequently,

$$\lambda\sum_{n=0}^\infty |\alpha_{n+1}|^2\|z^{n+1}\|^2_{\mathscr{H}^{(\lambda)}} \le \sum_{n=0}^\infty |\alpha_{n+1}|^2(n+1)^2\|z^n\|^2_{\mathscr{H}^{(\lambda+2)}} \le \lambda(\lambda+1)\sum_{n=0}^\infty |\alpha_{n+1}|^2\|z^{n+1}\|^2_{\mathscr{H}^{(\lambda)}}. \tag{4.25}$$

From the above inequality, it follows that $\sum_{n=0}^\infty |\alpha_n|^2\|z^n\|^2_{\mathscr{H}^{(\lambda)}} < \infty$ if and only if $\sum_{n=0}^\infty |\alpha_{n+1}|^2(n+1)^2\|z^n\|^2_{\mathscr{H}^{(\lambda+2)}} < \infty$. Therefore, $f \in \mathscr{H}^{(\lambda)}$ if and only if $f' \in \mathscr{H}^{(\lambda+2)}$. From (4.25), it is also easy to see that if $f \in \mathscr{H}^{(\lambda)}$, then $\|f'\|^2_{\mathscr{H}^{(\lambda+2)}} \le \lambda(\lambda+1)\|f\|^2_{\mathscr{H}^{(\lambda)}}$. □

The proof of the corollary given below follows from the Lemma 4.3.10 together with the fact that the inclusion operator $f \mapsto f$ is bounded from $\mathscr{H}^{(\lambda+2)}$ to $\mathscr{H}^{(\mu)}$, $\mu - \lambda \geq 2$.

Corollary 4.3.11. *Let λ, μ be two positive real numbers such that $\mu - \lambda \geq 2$. Then the linear map $f \mapsto f'$ is bounded from $\mathscr{H}^{(\lambda)}$ to $\mathscr{H}^{(\mu)}$.*

Recall that for two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ consisting of holomorphic functions on the unit disc $\mathbb{D}$, the multiplier algebra Mult $(\mathscr{H}_1, \mathscr{H}_2)$ is defined as

$$\text{Mult}\,(\mathscr{H}_1, \mathscr{H}_2) := \left\{\psi \in \text{Hol}(\mathbb{D}) : \psi f \in \mathscr{H}_2 \text{ whenever } f \in \mathscr{H}_1\right\}.$$

When $\mathscr{H}_1 = \mathscr{H}_2$, we write Mult $(\mathscr{H}_1)$ instead of Mult $(\mathscr{H}_1, \mathscr{H}_1)$. By the closed graph theorem, it is easy to see that $\psi \in$ Mult $(\mathscr{H}_1, \mathscr{H}_2)$ if and only if the multiplication operator M_ψ is bounded from $\mathscr{H}_1$ to $\mathscr{H}_2$.

For $\mu \geq \lambda > 0$, since $\mathscr{H}^{(\lambda)} \subseteq \mathscr{H}^{(\mu)}$, it follows that $\psi f \in \mathscr{H}^{(\mu)}$ whenever $f \in \mathscr{H}^{(\lambda)}$ and $\psi \in$ Mult $(\mathscr{H}^{(\lambda)})$. Hence

$$\text{Mult}\,(\mathscr{H}^{(\lambda)}) \subseteq \text{Mult}\,(\mathscr{H}^{(\lambda)}, \mathscr{H}^{(\mu)}),\;\; 0 < \lambda \leq \mu. \tag{4.26}$$

It is known that for $\lambda \geq 1$, Mult $(\mathscr{H}^{(\lambda)}) = H^\infty(\mathbb{D})$, where $H^\infty(\mathbb{D})$ is the algebra of all bounded holomorphic functions on the unit disc $\mathbb{D}$. Thus, from (4.26), we conclude that

$$H^\infty(\mathbb{D}) \subseteq \text{Mult}(\mathscr{H}^{(\lambda)}, \mathscr{H}^{(\mu)}),\;\; 1 \leq \lambda \leq \mu. \tag{4.27}$$

On the other hand, if $\lambda > \mu$, then Mult$(\mathscr{H}^{(\lambda)}, \mathscr{H}^{(\mu)}) = \{0\}$, and hence we make the assumption $\lambda \leq \mu$ without loss of generality.

The proposition given below describes a class a weakly homogeneous operators in $\mathscr{F}B_2(\mathbb{D})$.

Proposition 4.3.12. *Let $0 < \lambda \leq \mu$ and $\psi \in$ Mult$(\mathscr{H}^{(\lambda)}, \mathscr{H}^{(\mu)})$. Let $T = \begin{pmatrix} M_z^* & M_\psi^* \\ 0 & M_z^* \end{pmatrix}$ on $\mathscr{H}^{(\lambda)} \oplus \mathscr{H}^{(\mu)}$. If M_ψ is bounded and invertible on $\mathscr{H}^{(\lambda)}$ as well as on $\mathscr{H}^{(\mu)}$, then T is weakly homogeneous.*

Proof. It suffices to show that T^* is weakly homogeneous. By a routine computation, we obtain $\varphi(T^*) = \begin{pmatrix} M_\varphi & 0 \\ M_{\psi\varphi'} & M_\varphi \end{pmatrix}$ on $\mathscr{H}^{(\lambda)} \oplus \mathscr{H}^{(\mu)}$. By Lemma 4.3.8, the operator $C_{\varphi^{-1}}$, $\varphi \in$ Möb, is bounded and invertible on $\mathscr{H}^{(\lambda)}$ as well as on $\mathscr{H}^{(\mu)}$. Also, by hypothesis, M_ψ is bounded and invertible on $\mathscr{H}^{(\lambda)}$ as well as on $\mathscr{H}^{(\mu)}$. Thus $M_\psi C_{\varphi^{-1}}$ is bounded and invertible on $\mathscr{H}^{(\mu)}$. For $\varphi \in$ Möb, set

$$L_\varphi := \begin{pmatrix} M_{(\psi\circ\varphi^{-1})(\varphi'\circ\varphi^{-1})} C_{\varphi^{-1}} & 0 \\ 0 & M_\psi C_{\varphi^{-1}} \end{pmatrix} \text{ on } \mathscr{H}^{(\lambda)} \oplus \mathscr{H}^{(\mu)}.$$

Using the equality $M_{\psi\circ\varphi^{-1}} = C_{\varphi^{-1}} M_\psi C_\varphi$, we see that the operator $M_{\psi\circ\varphi^{-1}}$ is bounded and invertible on $\mathscr{H}^{(\lambda)}$. Consequently, $M_{\psi\circ\varphi^{-1}} C_{\varphi^{-1}}$ is bounded and invertible on $\mathscr{H}^{(\lambda)}$. Therefore, to prove that L_φ is bounded and invertible, it suffices to show that the operator $M_{(\varphi'\circ\varphi^{-1})}$ is bounded and invertible on $\mathcal{H}^{(\lambda)}$. Take φ to be $\varphi_{\theta,a}^{-1}$ and note that

$$\big((\varphi_{\theta,a}^{-1})'\circ\varphi_{\theta,a}\big)(z)=\frac{1}{(\varphi_{\theta,a})'(z)}=e^{-i\theta}\frac{(1-\bar{a}z)^2}{(1-|a|^2)},\ \ z\in\mathbb{D}, \tag{4.28}$$

which is a polynomial. Hence $M_{((\varphi_{\theta,a}^{-1})'\circ\varphi_{\theta,a})}$ is bounded on $\mathcal{H}^{(\lambda)}$.

From (4.28), we see that $M_{((\varphi_{\theta,a}^{-1})'\circ\varphi_{\theta,a})}$ is invertible on $\mathcal{H}^{(\lambda)}$ if and only if the operator $M_{\frac{1}{(1-\bar{a}z)^2}}$ is bounded on $\mathcal{H}^{(\lambda)}$. By the closed graph theorem, this is equivalent to $\frac{1}{(1-\bar{a}z)^2}$ is in $\mathrm{Mult}(\mathcal{H}^{(\lambda)})$. To verify this, let $f\in\mathcal{H}^{(\lambda)}$. Note that

$$\Big(\frac{1}{(1-\bar{a}z)^2}f\Big)'(z)=\frac{1}{(1-\bar{a}z)^2}f'(z)+\frac{2\bar{a}}{(1-\bar{a}z)^3}f(z),\ \ z\in\mathbb{D}, \tag{4.29}$$

Since $f\in\mathcal{H}^{(\lambda)}$ and $\mathcal{H}^{(\lambda)}\subseteq\mathcal{H}^{(\lambda+2)}$, we have $f\in\mathcal{H}^{(\lambda+2)}$. Also by Lemma 4.3.10, $f'\in\mathcal{H}^{(\lambda+2)}$. Since the functions $\frac{1}{(1-\bar{a}z)^2}$ and $\frac{2\bar{a}}{(1-\bar{a}z)^3}$ belong to $H^\infty(\mathbb{D})$ and $\mathrm{Mult}(\mathcal{H}^{(\lambda+2)})=H^\infty(\mathbb{D})$, it follows that both of the functions $\frac{1}{(1-\bar{a}z)^2}f'$ and $\frac{2\bar{a}}{(1-\bar{a}z)^3}f$ belong to $\mathcal{H}^{(\lambda+2)}$. Thus, by (4.29), $\big(\frac{1}{(1-\bar{a}z)^2}f\big)'$ belongs to $\mathcal{H}^{(\lambda+2)}$. Hence, again applying Lemma 4.3.10, we conclude that $\frac{1}{(1-\bar{a}z)^2}f$ belongs to $\mathcal{H}^{(\lambda)}$. Hence the operator $M_{\frac{1}{(1-\bar{a}z)^2}}$ is bounded on $\mathcal{H}^{(\lambda)}$.

Finally, a straightforward calculation shows that

$$T^*L_\varphi=L_\varphi\varphi(T^*)=\begin{pmatrix}M_{z(\psi\circ\varphi^{-1})(\varphi'\circ\varphi^{-1})}C_{\varphi^{-1}} & 0\\ M_{\psi(\psi\circ\varphi^{-1})(\varphi'\circ\varphi^{-1})}C_{\varphi^{-1}} & M_{z\psi}C_{\varphi^{-1}}\end{pmatrix}, \tag{4.30}$$

completing the proof. □

Lemma 4.3.13. *Let $0<\lambda\le\mu<\lambda+2$ and ψ, χ be two holomorphic functions on the unit disc $\mathbb{D}$. Let X be the linear map given by $X(f)=\psi f'+\chi f$, $f\in\mathrm{Hol}(\mathbb{D})$. Suppose that X is bounded from $\mathcal{H}^{(\lambda)}$ to $\mathcal{H}^{(\mu)}$. Then ψ is identically zero.*

Proof. Let $\psi(z)=\sum_{j=0}^\infty\alpha_jz^j$ and $\chi(z)=\sum_{j=0}^\infty\beta_jz^j$ be the power series representations of ψ and χ, respectively. Then for $n\ge1$, we see that

$$\begin{aligned}\|X(z^n)\|^2_{\mathcal{H}^{(\mu)}}&=\|nz^{n-1}\psi(z)+z^n\chi(z)\|^2_{\mathcal{H}^{(\mu)}}\\&=\|nz^{n-1}\alpha_0+\sum_{j=1}^\infty(n\alpha_j+\beta_{j-1})z^{j+n-1}\|^2_{\mathcal{H}^{(\mu)}}\\&=|\alpha_0|^2n^2\|z^{n-1}\|^2_{\mathcal{H}^{(\mu)}}+\sum_{j=1}^\infty|n\alpha_j+\beta_{j-1}|^2\|z^{j+n-1}\|^2_{\mathcal{H}^{(\mu)}}.\end{aligned}$$

Since X is bounded from $\mathcal{H}^{(\lambda)}$ to $\mathcal{H}^{(\mu)}$, we have that $\|X(z^n)\|^2_{\mathcal{H}^{(\mu)}}\le\|X\|^2\|z^n\|^2_{\mathcal{H}^{(\lambda)}}$. Consequently, for $n\ge1$,

$$|\alpha_0|^2n^2\|z^{n-1}\|^2_{\mathcal{H}^{(\mu)}}+\sum_{j=1}^\infty|n\alpha_j+\beta_{j-1}|^2\|z^{j+n-1}\|^2_{\mathcal{H}^{(\mu)}}\le\|X\|^2\|z^n\|^2_{\mathcal{H}^{(\lambda)}}. \tag{4.31}$$

From (4.24), we have $\|z^n\|^2_{\mathscr{H}^{(\lambda)}} \sim n^{-(\lambda-1)}$ and $\|z^n\|^2_{\mathscr{H}^{(\mu)}} \sim n^{-(\mu-1)}$. Thus by (4.31), there exists a constant $c > 0$ such that $|\alpha_0|^2 n^{-(\mu-3)} \leq cn^{-(\lambda-1)}$. Equivalently, $|\alpha_0|^2 \leq cn^{\mu-\lambda-2}$. Since $\mu-\lambda-2 < 0$, taking limit as $n \to \infty$, we obtain $\alpha_0 = 0$.

For $j \geq 1$, using $\|z^{j+n-1}\|^2_{\mathscr{H}^{(\mu)}} \sim (j+n-1)^{-(\mu-1)} \sim n^{-(\mu-1)}$ in (4.31), we see that

$$\left|\alpha_j + \frac{\beta_{j-1}}{n}\right|^2 \leq dn^{(\mu-1)-(\lambda-1)-2} = dn^{(\mu-\lambda-2)}$$

for some constant $d > 0$. As before, since $\mu-\lambda-2 < 0$, taking $n \to \infty$, we obtain $\alpha_j = 0$ for $j \geq 1$. Hence ψ is identically zero, completing the proof of the lemma. □

Combining Corollary 4.3.11 and Lemma 4.3.13, we obtain the following corollary.

Corollary 4.3.14. *The linear map $f \mapsto f'$, $f \in \mathrm{Hol}(\mathbb{D})$, is bounded from $\mathscr{H}^{(\lambda)}$ to $\mathscr{H}^{(\mu)}$ if and only if $\mu - \lambda \geq 2$.*

As a consequence of the lemma 4.3.13, we also obtain the following proposition which is a strengthening of [38, Theorem 4.5 (2)] in the particular case of quasi-homogeneous operators of rank 2. Recall that an operator T is said to be strongly irreducible if XTX^{-1} is irreducible for all invertible operator X.

Proposition 4.3.15. *Let $0 < \lambda \leq \mu < \lambda + 2$ and $\psi \in \mathrm{Mult}(\mathscr{H}^{(\lambda)}, \mathscr{H}^{(\mu)})$. Let $T = \begin{pmatrix} M_z^* & M_\psi^* \\ 0 & M_z^* \end{pmatrix}$ on $\mathscr{H}^{(\lambda)} \oplus \mathscr{H}^{(\mu)}$. If ψ is non-zero, then T is strongly irreducible.*

Proof. Since $T \in \mathscr{F}B_2(\mathbb{D})$, by [37, Proposition 2.22], it follows that T is strongly reducible if and only if there exists a bounded operator $X : \mathscr{H}^{(\mu)} \to \mathscr{H}^{(\lambda)}$ satisfying $T_0X - XT_1 = M_\psi^*$ where $T_0 = M_z^*$ on $\mathscr{H}^{(\lambda)}$ and $T_1 = M_z^*$ on $\mathscr{H}^{(\mu)}$.

Suppose that ψ is non-zero and T is strongly reducible. Then there exists a bounded operator $X : \mathscr{H}^{(\mu)} \to \mathscr{H}^{(\lambda)}$ such that $X^*T_0^* - T_1^*X^* = M_\psi$. Since the kernel $K^{(\lambda)}$ is sharp, by Lemma 4.3.6 (with φ to be the identity map), there exists a function $\chi \in \mathrm{Hol}\,(\mathbb{D})$ such that $X^*(f) = \psi f' + \chi f$, $f \in \mathscr{H}^{(\lambda)}$. Since X is bounded and $0 < \lambda \leq \mu < \lambda + 2$, by Lemma 4.3.13, ψ is identically zero on $\mathbb{D}$. This is a contradiction to the assumption that ψ is non-zero. Hence T must be strongly irreducible, completing the proof. □

Let $C(\bar{\mathbb{D}})$ denote the space of all continuous functions on $\bar{D}$. If ψ is an arbitrary function in $C(\bar{\mathbb{D}}) \cap \mathrm{Hol}(\mathbb{D})$, then it is easy to see that $\psi \in H^\infty(\mathbb{D})$. Furthermore, if $1 \leq \lambda \leq \mu$, then by (4.27), we see that $\psi \in \mathrm{Mult}\,(\mathscr{H}^{(\lambda)}, \mathscr{H}^{(\mu)})$.

The theorem given below gives several examples and nonexamples of weakly homogeneous operators in the class $\mathscr{F}B_2(\mathbb{D})$.

Theorem 4.3.16. *Let $1 \le \lambda \le \mu < \lambda + 2$ and ψ be a non-zero function in $C(\bar{\mathbb{D}}) \cap \mathrm{Hol}(\mathbb{D})$. The operator $T = \begin{pmatrix} M_z^* & M_\psi^* \\ 0 & M_z^* \end{pmatrix}$ on $\mathscr{H}^{(\lambda)} \oplus \mathscr{H}^{(\mu)}$ is weakly homogeneous if and only if ψ is non-vanishing on $\bar{\mathbb{D}}$.*

Proof. Suppose that ψ is non-vanishing on $\bar{\mathbb{D}}$. Since ψ is continuous on $\bar{\mathbb{D}}$, ψ must be bounded below. Therefore $\frac{1}{\psi}$ is a bounded analytic function on $\mathbb{D}$. Further, since $\lambda, \mu \ge 1$, we have that Mult $(\mathscr{H}^{(\lambda)})$ = Mult $(\mathscr{H}^{(\mu)}) = H^\infty(\mathbb{D})$. Hence the operator $M_{\frac{1}{\psi}}$ is bounded on $\mathscr{H}^{(\lambda)}$ as well as on $\mathscr{H}^{(\mu)}$. Consequently, the operator M_ψ is bounded and invertible on $\mathscr{H}^{(\lambda)}$ as well as on $\mathscr{H}^{(\mu)}$. Hence, by Proposition 4.3.12, T is weakly homogeneous.

Conversely, assume that T is weakly homogeneous. It is easily verified that $T \in \mathscr{F}B_2(\mathbb{D})$ and T satisfies the hypothesis of Corollary 4.3.5. Therefore, for each φ in Möb, there exists bounded operators $X_\varphi : \mathscr{H}^{(\lambda)} \to \mathscr{H}^{(\lambda)}$, $Y_\varphi : \mathscr{H}^{(\mu)} \to \mathscr{H}^{(\mu)}$ and $Z_\varphi : \mathscr{H}^{(\mu)} \to \mathscr{H}^{(\lambda)}$, with X_φ, Y_φ invertible, such that the following holds:

$$\begin{aligned} X_\varphi T_0 = \varphi(T_0) X_\varphi,\ Y_\varphi T_1 = \varphi(T_1) Y_\varphi \\ X_\varphi M_\psi^* + Z_\varphi T_1 = \varphi(T_0) Z_\varphi + M_\psi^* \varphi'(T_1) Y_\varphi, \end{aligned} \tag{4.32}$$

where T_0 is M_z^* on $\mathscr{H}^{(\lambda)}$ and T_1 is M_z^* on $\mathscr{H}^{(\mu)}$. Note that $\varphi(T_0)^* = \hat{\varphi}(T_0^*)$ where $\hat{\varphi}(z) := \overline{\varphi(\bar{z})}$. Taking adjoint in the first equation of (4.32), we see that X_φ satisfies $T_0^* X_\varphi^* = X_\varphi^* \hat{\varphi}(T_0^*)$. Since $K^{(\lambda)}$ is sharp, by Lemma 4.2.6 (or Lemma 4.3.6), we obtain $X_\varphi^* = M_{g_\varphi} C_{\hat{\varphi}^{-1}}$ for some non-vanishing function g_φ in $\mathrm{Hol}(\mathbb{D})$. Furthermore, since $C_{\hat{\varphi}}$ is bounded and invertible on $\mathscr{H}^{(\lambda)}$ (see Lemma 4.3.8), it follows from the boundedness and invertibility of X_φ that the operator M_{g_φ} is bounded and invertible on $\mathscr{H}^{(\lambda)}$. Also, since Mult$(\mathscr{H}^{(\lambda)}) = \mathrm{H}^\infty(\mathbb{D})$, $\lambda \ge 1$, it follows that g_φ must be bounded above as well as bounded below on $\mathbb{D}$. By the same argument, we have $Y_\varphi^* = M_{h_\varphi} C_{\hat{\varphi}^{-1}}$ for some non-vanishing function h_φ in $\mathrm{Hol}(\mathbb{D})$ which is bounded above as well as bounded below on $\mathbb{D}$. Taking adjoint in the last equation of (4.32), we see that

$$M_\psi X_\varphi^* + T_1^* Z_\varphi^* = Z_\varphi^* \hat{\varphi}(T_0^*) + Y_\varphi^* \widehat{\varphi}'(T_1^*) M_\psi.$$

Equivalently,

$$\begin{aligned} Z_\varphi^* \hat{\varphi}(T_0^*) - T_1^* Z_\varphi^* &= M_\psi X_\varphi^* - Y_\varphi^* \widehat{\varphi}'(T_1^*) M_\psi \\ &= M_\psi M_{g_\varphi} C_{\hat{\varphi}^{-1}} - M_{h_\varphi} C_{\hat{\varphi}^{-1}} M_{\widehat{\varphi'}} M_\psi \\ &= M_{\ell_\varphi} C_{\hat{\varphi}^{-1}}, \end{aligned}$$

where $\ell_\varphi = \psi g_\varphi - h_\varphi (\widehat{\varphi'} \circ \hat{\varphi}^{-1})(\psi \circ \hat{\varphi}^{-1})$. Since the kernel $K^{(\lambda)}$ is sharp, by Lemma 4.3.6, it follows that

$$Z_\varphi^* f = \ell_\varphi (\hat{\varphi}^{-1})' (f' \circ \hat{\varphi}^{-1}) + \chi_\varphi (f \circ \hat{\varphi}^{-1}),\ \ f \in \mathscr{H}^{(\lambda)}, \tag{4.33}$$

for some $\chi_\varphi \in$ Hol($\mathbb{D}$). Furthermore, since the composition operator $C_{\hat{\varphi}}$ is bounded on $\mathscr{H}^{(\mu)}$ by Lemma 4.3.8, the operator $C_{\hat{\varphi}} Z_\varphi^*$ is bounded from $\mathscr{H}^{(\lambda)}$ to $\mathscr{H}^{(\mu)}$. Note that

$$C_{\hat{\varphi}} Z_\varphi^*(f) = (\ell_\varphi \circ \hat{\varphi})((\hat{\varphi}^{-1})' \circ \hat{\varphi}) f' + (\chi_\varphi \circ \hat{\varphi}) f, \ \ f \in \mathscr{H}^{(\lambda)}.$$

Since $\lambda \le \mu < \lambda + 2$, by Lemma 4.3.13, it follows that that $(\ell_\varphi \circ \hat{\varphi})((\hat{\varphi}^{-1})' \circ \hat{\varphi})$ is identically the zero function for each $\varphi \in$ Möb and therefore ℓ_φ is identically the zero function for each $\varphi \in$ Möb. Equivalently,

$$\psi(z) g_\varphi(z) = h_\varphi(z)(\widehat{\varphi'} \circ \hat{\varphi}^{-1})(z)(\psi \circ \hat{\varphi}^{-1})(z), \ \ z \in \mathbb{D}, \varphi \in \text{Möb}. \tag{4.34}$$

Now we show that ψ is non-vanishing on $\mathbb{D}$. If possible let $\psi(w_0) = 0$ for some $w_0 \in \mathbb{D}$ and w be a fixed but arbitrary point in $\mathbb{D}$. We will show that $\psi(w) = 0$. By transitivity of Möb, there exists a function φ_w in Möb such that $\widehat{\varphi_w}^{-1}(w_0) = w$. Putting $z = w_0$ and $\varphi = \varphi_w$ in (4.34), we see that

$$\psi(w_0) g_{\varphi_w}(w_0) = h_{\varphi_w}(w_0)(\widehat{\varphi'_w} \circ \widehat{\varphi_w}^{-1})(w_0)\psi(w). \tag{4.35}$$

Since the functions h_{φ_w} and $(\widehat{\varphi'_w} \circ \widehat{\varphi_w}^{-1})$ are non-vanishing on $\mathbb{D}$, it follows from (4.35) that $\psi(w) = 0$. Since this holds for an arbitrary $w \in \mathbb{D}$, we conclude that ψ vanishes on $\mathbb{D}$. Consequently, ψ vanishes on $\bar{\mathbb{D}}$, which contradicts that ψ is non-zero on $\bar{\mathbb{D}}$. Hence ψ is non-vanishing on $\mathbb{D}$.

Now we show that ψ is non-vanishing on the unit circle $\mathbb{T}$. Replacing φ by $\varphi_{\theta,0}$ (which is the rotation map $e^{i\theta} z$) in (4.35), we obtain

$$\psi(z) g_{\varphi_{\theta,0}}(z) = e^{-i\theta} h_{\varphi_{\theta,0}}(z)\psi(e^{i\theta} z), \ \ z \in \mathbb{D}. \tag{4.36}$$

Let $\{w_n\}$ be a sequence in $\mathbb{D}$ such that $w_n \to 1$ as $n \to \infty$. If possible let ψ vanishes at some point $e^{i\theta_0}$ on $\mathbb{T}$. Putting $z = e^{i\theta_0} w_n$ in (4.36), we obtain

$$\psi(e^{i\theta_0} w_n) g_{\varphi_{\theta,0}}(e^{i\theta_0} w_n) = e^{-i\theta} h_{\varphi_{\theta,0}}(e^{i\theta_0} w_n)\psi(e^{i(\theta_0+\theta)} w_n). \tag{4.37}$$

Since $\psi \in C(\bar{\mathbb{D}})$ and $g_{\varphi_{\theta,0}}, h_{\varphi_{\theta,0}}$ are bounded above as well as bounded below on $\mathbb{D}$, taking limit as $n \to \infty$, it follows that $\psi(e^{i(\theta_0+\theta)}) = 0$. Since this is true for any $\theta \in \mathbb{R}$, we conclude that ψ vanishes at all points on $\mathbb{T}$. Consequently, ψ is identically zero on $\bar{\mathbb{D}}$. This contradicts our hypothesis that ψ is non-zero on $\bar{\mathbb{D}}$. □

As an immediate consequence of the above theorem, we obtain a class of operators in $\mathscr{F}B_2(\mathbb{D})$ which are not weakly homogeneous.

Corollary 4.3.17. *Let $1 \le \lambda \le \mu < \lambda + 2$. If ψ is a non-zero function in $C(\bar{\mathbb{D}}) \cap \text{Hol}(\mathbb{D})$ with atleast one zero in $\bar{\mathbb{D}}$, then the operator $T = \begin{pmatrix} M_z^* & M_\psi^* \\ 0 & M_z^* \end{pmatrix}$ on $\mathscr{H}^{(\lambda)} \oplus \mathscr{H}^{(\mu)}$ is not weakly homogeneous.*

4.4 Möbius bounded operators

Recall that an operator T on a Hilbert space $\mathscr{H}$ is said to be power bounded if $\sup_{n\geq 0}\|T^n\| \leq c$, for some constant $c > 0$. The related notion of Möbius bounded operators was introduced by Shields in [52].

Definition 4.4.1. *An operator T on a Banach space $\mathscr{B}$ is said to be Möbius bounded if $\sigma(T) \subset \bar{\mathbb{D}}$ and*

$$\sup_{\varphi\in \text{Möb}} \|\varphi(T)\| < \infty.$$

We will only discuss Möbius bounded operators on Hilbert spaces. By the von Neumann's inequality, every contraction on a Hilbert space is Möbius bounded. If T is an operator which is similar to a homogeneous operator, then from (4.1), it follows that T is Möbius bounded. It is also easily verified that an operator T is Möbius bounded if and only if T^* is Möbius bounded.

In this section, we find some necessary conditions for Möbius boundedness of the multiplication operator M_z on the reproducing kernel Hilbert space $(\mathscr{H}, K)$, where $K(z,w)$ is form $\sum_{n=0}^{\infty} b_n(z\bar{w})^n$, $b_n > 0$, on $\mathbb{D}\times\mathbb{D}$. As a consequence, we show that the multiplication operator M_z on the Dirichlet space is not Möbius bounded. We begin with a preparatory lemma.

First we recall that, for $\theta \in [0,2\pi)$ and $a \in \mathbb{D}$, the biholomorphic automorphism $\varphi_{\theta,a}$ of the unit disc $\mathbb{D}$, is defined by $\varphi_{\theta,a}(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$, $z \in \mathbb{D}$. Note that the power series representation of $\varphi_{\theta,a}$ is $\sum_{n=0}^{\infty}\alpha_n z^n$, $z \in \mathbb{D}$, where

$$\alpha_0 = -e^{i\theta}a \text{ and } \alpha_n = e^{i\theta}(1-|a|^2)(\bar{a})^{n-1}, \quad n \geq 1. \tag{4.38}$$

Lemma 4.4.2. *Let $K(z,w) = \sum_{n=0}^{\infty} b_n(z\bar{w})^n$, $b_n > 0$, be a positive definite kernel on $\mathbb{D}\times\mathbb{D}$. Suppose that the multiplication operator M_z is bounded on $(\mathscr{H}, K)$ and $\sigma(M_z) = \bar{\mathbb{D}}$. If the sequence $\{nb_n\}_{n\in\mathbb{Z}_+}$ is bounded, then there exists a constant $c > 0$ such that*

$$\|\varphi_{\theta,a}(M_z)\| \geq \frac{K(a,a)}{c} - |a|, \;\; a \in \mathbb{D},\, \theta \in [0,2\pi).$$

Proof. Since $\{nb_n\}$ is bounded, there exists a constant $c > 0$ such that $nb_n < c$ for all $n \geq 0$. For $a \in \mathbb{D}, \theta \in [0,2\pi)$, setting $\tilde{\varphi}_{\theta,a}(z) = \varphi_{\theta,a}(z) - \varphi_{\theta,a}(0)$, $z \in \mathbb{D}$, we see that

$$\tilde{\varphi}_{\theta,a}(z)K(z,a) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\alpha_k b_{n-k}(\bar{a})^{n-k}\right)z^n, \; z \in \mathbb{D}. \tag{4.39}$$

Since M_z on $(\mathscr{H}, K)$ is bounded and $\sigma(M_z) = \bar{\mathbb{D}}$, the operator $\varphi_{\theta,a}(M_z)$ is bounded and hence the function $\tilde{\varphi}_{\theta,a}(\cdot)K(\cdot,a)$ belongs to $(\mathscr{H},K)$ for all $a\in\mathbb{D}$. Note that

$$\begin{aligned}\|\tilde{\varphi}_{\theta,a}(\cdot)K(\cdot,a)\|^2 &= \sum_{n=1}^{\infty}\Big|\sum_{k=1}^{n}\alpha_k b_{n-k}(\bar{a})^{n-k}\Big|^2\|z^n\|^2\\ &= (1-|a|^2)^2\sum_{n=1}^{\infty}|a|^{2(n-1)}\big(\sum_{j=0}^{n-1}b_j\big)^2\frac{1}{b_n} \qquad (4.40)\\ &= (1-|a|^2)^2\sum_{n=0}^{\infty}|a|^{2n}\big(\sum_{j=0}^{n}b_j\big)^2\frac{1}{b_{n+1}}\end{aligned}$$

Claim: For any $a\in\mathbb{D}$,

$$\sum_{n=0}^{\infty}|a|^{2n}\big(\sum_{j=0}^{n}b_j\big)^2\frac{1}{b_{n+1}}\geq\frac{1}{c}(1-|a|^2)^{-2}K(a,a)^2. \qquad (4.41)$$

Since $(1-|a|^2)^{-2}=\sum_{n=0}^{\infty}(n+1)|a|^{2n}$, $a\in\mathbb{D}$, setting $\beta_n=\sum_{j=0}^{n}(j+1)b_{n-j}$, $n\geq 0$, we see that

$$(1-|a|^2)^{-2}K(a,a)=\sum_{n=0}^{\infty}\beta_n|a|^{2n},\ \ a\in\mathbb{D}.$$

Furthermore, setting $\gamma_n=\sum_{j=0}^{n}\beta_j b_{n-j}$, $n\geq 0$, we see that

$$(1-|a|^2)^{-2}K(a,a)^2=\sum_{n=0}^{\infty}\gamma_n|a|^{2n},\ \ a\in\mathbb{D}. \qquad (4.42)$$

Note that

$$\beta_n=\sum_{j=0}^{n}(j+1)b_{n-j}\leq(n+1)\big(\sum_{j=0}^{n}b_j\big),\ \ n\geq 0.$$

Therefore

$$\gamma_n=\sum_{j=0}^{n}\beta_j b_{n-j}\leq\sum_{j=0}^{n}(j+1)\big(\sum_{p=0}^{j}b_p\big)b_{n-j}\leq(n+1)\big(\sum_{j=0}^{n}b_j\big)^2.$$

Consequently,

$$\begin{aligned}\sum_{n=0}^{\infty}\gamma_n|a|^{2n} &\leq\sum_{n=0}^{\infty}(n+1)\big(\sum_{j=0}^{n}b_j\big)^2|a|^{2n}\\ &\leq c\sum_{n=0}^{\infty}|a|^{2n}\big(\sum_{j=0}^{n}b_j\big)^2\frac{1}{b_{n+1}},\end{aligned}$$

where for the last inequality, we have used that $nb_n<c$, $n\geq 0$. Hence, by (4.42), the claim is verified.

Combining the claim with (4.40), it follows that

$$\|\tilde{\varphi}_{\theta,a}(\cdot)K(\cdot,a)\|^2\geq\frac{1}{c}K(a,a)^2.$$

Since $\|K(\cdot,a)\|^2 = K(a,a)$, it follows that

$$\|\tilde{\varphi}_{\theta,a}(M_z)\|^2 \geq \frac{\|\tilde{\phi}_{\theta,a}(\cdot)K(\cdot,a)\|^2}{\|K(\cdot,a)\|^2} \geq \frac{1}{c}K(a,a).$$

Finally, note that for $a \in \mathbb{D}$,

$$\|\varphi_{\theta,a}(M_z)\| = \|\tilde{\varphi}_{\theta,a}(M_z) - \tilde{\varphi}_{\theta,a}(0)I\| \geq \|\tilde{\varphi}_{\theta,a}(M_z)\| - |a| \geq \frac{1}{c}K(a,a) - |a|.$$

This completes the proof. □

We reproduce below the easy half of the statement of [15, Lemma 2] along with its proof, which is all we need.

Lemma 4.4.3. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $a_n \geq 0$. *If* $f(x) \leq c(1-x)^{-\alpha}$, $0 \leq x < 1$, *for some constants* $\alpha, c > 0$, *then there exists* $c' > 0$ *such that*

$$a_0 + a_1 + \ldots + a_n \leq c'(n+1)^{\alpha}, \;\; n \geq 0.$$

Proof. For $0 \leq x < 1$ and $n \geq 0$, we have

$$x^n(a_0 + a_1 + \ldots + a_n) \leq \sum_{j=0}^{n} a_j x^j \leq f(x) \leq c(1-x)^{-\alpha}.$$

Taking $x = e^{-\frac{1}{n}}$ in the above inequality, we obtain

$$(a_0 + a_1 + \ldots + a_n) \leq ce(1 - e^{-\frac{1}{n}})^{-\alpha}. \tag{4.43}$$

The proof is now complete since $\lim_{n\to\infty} n(1-e^{-\frac{1}{n}}) = 1$. □

The following lemma will be used in the proof of Theorem 4.4.5.

Lemma 4.4.4 (cf. [42])**.** *If* $\{b_n\}_{n\in\mathbb{Z}_+}$ *is a sequence of positive real numbers such that* $\sum_{n=0}^{\infty} b_n < \infty$, *then*

$$\sum_{n=0}^{\infty} \frac{n+1}{\frac{1}{b_0} + \frac{1}{b_1} + \ldots + \frac{1}{b_n}} \leq 2\sum_{n=0}^{\infty} b_n.$$

Theorem 4.4.5. *Let* $K(z,w) = \sum_{n=0}^{\infty} b_n (z\bar{w})^n$, $b_n > 0$, *be a positive definite kernel on* $\mathbb{D}\times\mathbb{D}$. *If the multiplication operator* M_z *on* $(\mathscr{H}, K)$ *is Möbius bounded, then*

$$\sum_{n=0}^{\infty} b_n = \infty.$$

Proof. Note that for any $\theta \in [0,2\pi)$, $a \in \mathbb{D}$ and $j \in \mathbb{Z}_+$,

$$\begin{aligned}\left\|\varphi_{\theta,a}(M_z)\left(\frac{z^j}{\|z^j\|}\right)\right\|^2 &= \frac{1}{\|z^j\|^2}\|\varphi_{\theta,a}(z)z^j\|^2 \\ &= \frac{1}{\|z^j\|^2}\Big\|\sum_{n=0}^{\infty}\alpha_n z^{n+j}\Big\|^2 \\ &= \frac{1}{\|z^j\|^2}\Big(|a|^2\|z^j\|^2+(1-|a|^2)^2\sum_{n=1}^{\infty}|a|^{2(n-1)}\|z^{n+j}\|^2\Big)\end{aligned} \tag{4.44}$$

If M_z on $(\mathscr{H},K)$ is Möbius bounded, then there exists a constant $c>0$ such that

$$\sup_{\theta\in[0,2\pi),a\in\mathbb{D},j\in\mathbb{Z}_+}\left\|\varphi_{\theta,a}(M_z)\left(\frac{z^j}{\|z^j\|}\right)\right\|^2 \le c.$$

Therefore, from (4.44), we see that

$$(1-|a|^2)^2\sum_{n=1}^{\infty}|a|^{2(n-1)}\frac{\|z^{n+j}\|^2}{\|z^j\|^2}\le c,\ \ a\in\mathbb{D}, j\in\mathbb{Z}_+.$$

Replacing $|a|^2$ by x, we obtain

$$\sum_{n=0}^{\infty}c_{n,j}x^n \le \frac{c}{(1-x)^2},\ \ x\in[0,1), \tag{4.45}$$

where $c_{n,j}=\frac{\|z^{n+j+1}\|^2}{\|z^j\|^2}$, $n,j\in\mathbb{Z}_+$. Hence, applying Lemma 4.4.3, we see that there exists a constant $c'>0$ such that for all $n,j\in\mathbb{Z}_+$,

$$(c_{0,j}+c_{1,j}+\cdots+c_{n,j})\le c'(n+1)^2.$$

Since $b_n=\frac{1}{\|z^n\|^2}$, $n\in\mathbb{Z}_+$, putting $j=0$ in the above inequality, we obtain

$$\left(\frac{1}{b_1}+\frac{1}{b_2}+\cdots+\frac{1}{b_{n+1}}\right)\le\frac{c'}{b_0}(n+1)^2,\ n\in\mathbb{Z}_+.$$

Therefore

$$\sum_{n=0}^{\infty}\frac{n+1}{\frac{1}{b_1}+\ldots+\frac{1}{b_{n+1}}}\ge\frac{b_0}{c'}\sum_{n=0}^{\infty}\frac{1}{n+1}.$$

Consequently, $\sum_{n=0}^{\infty}\frac{n+1}{\frac{1}{b_1}+\frac{1}{b_1}+\cdots+\frac{1}{b_{n+1}}}=\infty$. Hence, by Lemma 4.4.4, we conclude that $\sum_{n=0}^{\infty}b_n=\infty$. This completes the proof. □

Theorem 4.4.6. *Let $K(z,w)=\sum_{n=0}^{\infty}b_n(z\bar{w})^n$, $b_n>0$, be a positive definite kernel on $\mathbb{D}\times\mathbb{D}$. If the multiplication operator M_z on $(\mathscr{H},K)$ is Möbius bounded, then the sequence $\{nb_n\}_{n\in\mathbb{Z}_+}$ is unbounded.*

Proof. Assume that M_z on $(\mathscr{H}, K)$ is Möbius bounded. If possible, let the sequence $\{nb_n\}_{n\in\mathbb{Z}_+}$ is bounded. Then by Lemma (4.4.2), there exists a constant $c > 0$ such that

$$\sup_{a\in\mathbb{D}}\Big(\frac{K(a,a)}{c} - |a|\Big) < \infty.$$

Therefore $\sup_{a\in\mathbb{D}} K(a,a) < \infty$. Since Abel summation method is totally regular, it follows that $\sum_n^\infty b_n < \infty$ (see [35, page 10]). By Theorem 4.4.5, this contradicts the assumption that M_z is Möbius bounded. Hence the sequence $\{nb_n\}_{n\in\mathbb{Z}_+}$ is unbounded, completing the proof. □

Corollary 4.4.7. *Let $K(z,w) = \sum_{n=0}^\infty b_n(z\bar{w})^n$, $b_n > 0$, be a positive definite kernel on $\mathbb{D}\times\mathbb{D}$. Suppose that $b_n \sim (n+1)^\gamma$ for some $\gamma \in \mathbb{R}$. Then the multiplication operator M_z on $(\mathscr{H}, K)$ is Möbius bounded if and only if $\gamma > -1$. In particular, the multiplication operator M_z on the Dirichlet space is not Möbius bounded.*

Proof. By [51, Theorem 2′], it follows that the operator M_z on $(\mathscr{H}, K)$ is similar to the operator M_z on $(\mathscr{H}, K_{(\gamma)})$. Since similarity preserves Möbius boundedness, it suffices to show that M_z on $(\mathscr{H}, K_{(\gamma)})$ is Möbius bounded if and only if $\gamma > -1$. If $\gamma > -1$, then by Corollary 4.3.9, M_z on $(\mathscr{H}, K_{(\gamma)})$ is similar to a homogeneous operator and therefore is Möbius bounded. If $\gamma \le -1$, then note that the sequence $\{n.(n+1)^\gamma\}_{n\in\mathbb{Z}_+}$ is bounded. Hence by Theorem 4.4.6, M_z on $(\mathscr{H}, K_{(\gamma)})$ is not Möbius bounded. □

4.4.1 Shields' Conjecture

We have already mentioned that a Möbius bounded operator need not be power bounded. Shields proved that if T is a Möbius bounded operator on a Banach space, then $\|T^n\| \le c(n+1)$, $n \in \mathbb{Z}_+$, for some constant $c > 0$. But in case of Hilbert spaces, he made the following conjecture.

Conjecture 4.4.8 (Shields, [52])**.** *If T is a Möbius bounded operator on a Hilbert space, then there exists a constant $c > 0$ such that*

$$\|T^n\| \le c(n+1)^{\frac{1}{2}},\ \ n \in \mathbb{Z}_+.$$

The following theorem shows that Shields conjecture has an affirmative answer in a small class of weighted shifts containing the non-contractive homogeneous operators in $B_1(\mathbb{D})$.

Theorem 4.4.9. *Let $K(z,w) = \sum_{n=0}^\infty b_n(z\bar{w})^n$ be a positive definite kernel on $\mathbb{D}\times\mathbb{D}$. Assume that the sequence $\{b_n\}_{n\in\mathbb{Z}_+}$ is decreasing. If the multiplication operator M_z on $(\mathscr{H}, K)$ is Möbius bounded, then there exists a constant $c > 0$ such that $\|M_z^n\| \le c(n+1)^{\frac{1}{2}}$, $n \in \mathbb{Z}_+$.*

Proof. It suffices to show that $\|M_z^{n+1}\| \le c(n+1)^{\frac{1}{2}}$, $n \in \mathbb{Z}_+$. By, a straightforward computation, we see that

$$\|M_z^{n+1}\|^2 = \sup_{j\in\mathbb{Z}_+}\frac{\|z^{n+j+1}\|^2}{\|z^j\|^2},\ \ n\in\mathbb{Z}_+. \tag{4.46}$$

From (4.45), we already have that

$$\sum_{n=0}^{\infty} c_{n,j} x^n \leq \frac{c}{(1-x)^2}, \;\; x \in [0,1),$$

where $c_{n,j} = \frac{\|z^{n+j+1}\|^2}{\|z^j\|^2}$. Multiplying both sides by $1-x$, we see that

$$c_{0,j} + \sum_{n=1}^{\infty} (c_{n,j} - c_{n-1,j}) x^n \leq \frac{c}{1-x}, \;\; x \in [0,1), j \in \mathbb{Z}_+.$$

Since $\|z^n\|^2 = \frac{1}{b_n}$ and $\{b_n\}_{n\in\mathbb{Z}_+}$ is decreasing, the sequence $\{c_{n,j}\}_{n\in\mathbb{Z}_+}$ is increasing. Consequently, $(c_{n,j} - c_{n-1,j}) \geq 0$ for $n \geq 1, j \geq 0$. Therefore, using Lemma 4.4.3, we conclude that there exists a constant $c' > 0$ (independent of n and j) such that for all $n, j \in \mathbb{Z}_+$,

$$c_{0,j} + (c_{1,j} - c_{0,j}) + \cdots + (c_{n,j} - c_{n-1,j}) \leq c'(n+1),$$

that is,

$$c_{n,j} \leq c'(n+1).$$

Hence, in view of (4.46), we conclude that $\|M_z^{n+1}\|^2 \leq c'(n+1)$, $n \in \mathbb{Z}_+$, completing the proof. □

4.4.2 Möbius bounded quasi-homogeneous operators

In this subsection we identify all quasi-homogeneous operators which are Möbius bounded. We start with the following theorem which gives a necessary condition for a class of operators in $\mathcal{F}B_2(\mathbb{D})$ to be Möbius bounded.

Theorem 4.4.10. *Let* $0 < \lambda \leq \mu$ *and* ψ *be a non-zero function in* $\mathrm{Mult}(\mathcal{H}^{(\lambda)}, \mathcal{H}^{(\mu)})$. *Let* T *be the operator* $\begin{pmatrix} M_z^* & M_\psi^* \\ 0 & M_z^* \end{pmatrix}$ *on* $\mathcal{H}^{(\lambda)} \oplus \mathcal{H}^{(\mu)}$. *If* T *is Möbius bounded, then* $\mu - \lambda \geq 2$.

Proof. Note that T is Möbius bounded if and only if T^* is Möbius bounded. Therefore, it suffices to show that if T^* is Möbius bounded, then $\mu - \lambda \geq 2$. Since $\sigma(M_z) = \bar{\mathbb{D}}$ on both $\mathcal{H}^{(\lambda)}$ and $\mathcal{H}^{(\mu)}$, it is easily verified that $\sigma(T) = \bar{\mathbb{D}}$. As before, for $\varphi \in$ Möb, we have

$$\varphi(T^*) = \begin{pmatrix} M_\varphi & 0 \\ M_{\psi\varphi'} & M_\varphi \end{pmatrix} \text{ on } \mathcal{H}^{(\lambda)} \oplus \mathcal{H}^{(\mu)}.$$

Observe that for an operator of the form $\left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)$, $\|B\| \leq \left\|\left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)\right\| \leq (\|A\| + \|B\| + \|C\|)$. Therefore, we have

$$\|M_{\psi\varphi'}\|_{\mathcal{H}^{(\lambda)} \to \mathcal{H}^{(\mu)}} \leq \|\varphi(T^*)\| \leq \|M_\varphi\|_{\mathcal{H}^{(\lambda)}} + \|M_\varphi\|_{\mathcal{H}^{(\mu)}} + \|M_{\psi\varphi'}\|_{\mathcal{H}^{(\lambda)} \to \mathcal{H}^{(\mu)}} \quad (4.47)$$

Since the multiplication operator M_z on $\mathscr{H}^{(\lambda)}$ as well as on $\mathscr{H}^{(\mu)}$ is Möbius bounded, in view of (4.47), it follows that T^* is Möbius bounded if and only if

$$\sup_{\varphi\in\text{Möb}} \|M_{\psi\varphi'}\|_{\mathscr{H}^{(\lambda)}\to\mathscr{H}^{(\mu)}} < \infty.$$

Now for all w in $\mathbb{D}$, we have

$$\begin{aligned}\|M_{\psi\varphi'}\|^2_{\mathscr{H}^{(\lambda)}\to\mathscr{H}^{(\mu)}} = \|(M_{\psi\varphi'})^*\|^2_{\mathscr{H}^{(\mu)}\to\mathscr{H}^{(\lambda)}} &\geq \frac{\|(M_{\psi\varphi'})^*(K^{(\mu)}(\cdot,w))\|^2}{\|K^{(\mu)}(\cdot,w)\|^2}\\ &= |\psi(w)\varphi'(w)|^2\frac{\|K^{(\lambda)}(\cdot,w)\|^2}{\|K^{(\mu)}(\cdot,w)\|^2}\\ &= |\psi(w)\varphi'(w)|^2(1-|w|^2)^{\mu-\lambda}.\end{aligned}$$

Note that $\varphi'_{\theta,a}(w) = e^{i\theta}\frac{1-|a|^2}{(1-\bar{a}w)^2}$, $w\in\mathbb{D}$. Thus, if T^* is Möbius bounded, then there exists a constant $c>0$ such that

$$\sup_{a,w\in\mathbb{D}} \frac{|\psi(w)|^2(1-|a|^2)^2}{|1-\bar{a}w|^4}(1-|w|^2)^{\mu-\lambda} \leq c.$$

Taking $a = w$, we obtain

$$|\psi(w)|^2 \leq c(1-|w|^2)^{-(\mu-\lambda-2)}. \tag{4.48}$$

If possible, assume that $\mu-\lambda-2<0$. Then by an application of maximum modulus principle, it follows from (4.48) that ψ is identically zero, which is a contradiction to our assumption that ψ is non-zero. Hence $\mu-\lambda\geq 2$. □

Quasi-homogeneous operators

Suppose that $0<\lambda_0\leq\lambda_1\leq\ldots\leq\lambda_{n-1}$, $n\geq 1$, are n positive numbers such that the difference $\lambda_{i+1}-\lambda_i$, $0\leq i\leq n-2$, is a fixed number Λ. As before, let $\mathscr{H}^{(\lambda_i)}$, $i=0,1,\cdots,n-2$, be the Hilbert space determined by the kernel $K^{(\lambda_i)} = \frac{1}{(1-z\bar{w})^{\lambda_i}}$, $z,w\in\mathbb{D}$. Let T_i, $0\leq i\leq n-1$, denote the adjoint M_z^* of the multiplication operator by the coordinate function z on $\mathscr{H}^{(\lambda_i)}$. Furthermore, let $S_{i,j}, 0\leq i<j\leq n-1$, be the linear map given by the formula

$$S_{i,j}(K^{(\lambda_j)}(\cdot,w)) = m_{i,j}\bar{\partial}^{(j-i-1)}K^{(\lambda_i)}(\cdot,w),\ m_{i,j}\in\mathbb{C},\ 0\leq i<j\leq n-1.$$

Note that if $S_{i,j}$ defines a bounded linear operator from $\mathscr{H}^{(\lambda_j)}$ to $\mathscr{H}^{(\lambda_i)}$, then $(S_{i,j})^*(f) = \overline{m}_{i,j}f^{(j-i-1)}$, $f\in\mathscr{H}^{(\lambda_i)}$.

A quasi-homogeneous operator T of rank n is a bounded operator of the form

$$\begin{pmatrix} T_0 & S_{0,1} & S_{0,2} & \cdots & S_{0,n-1}\\ 0 & T_1 & S_{1,2} & \cdots & S_{1,n-1}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & T_{n-2} & S_{n-2,n-1}\\ 0 & 0 & \cdots & 0 & T_{n-1}\end{pmatrix} \tag{4.49}$$

on $\mathscr{H}^{(\lambda_0)} \oplus \mathscr{H}^{(\lambda_1)} \oplus \ldots \oplus \mathscr{H}^{(\lambda_{n-1})}$. For a quasi-homogeneous operator T, let $\Lambda(T)$ denote the fixed difference Λ. When $\Lambda(T) \geq 2$, a repeated application of Lemma 4.3.10 shows that each $S_{i,j}$, $0 \leq i < j \leq n-1$, is bounded from $\mathscr{H}^{(\lambda_i)}$ to $\mathscr{H}^{(\lambda_j)}$ and consequently, an operator of the form (4.49) is also bounded. In case of $\Lambda(T) < 2$, the boundedness criterion for T was obtained in terms of $\Lambda(T)$, n and $m_{i,j}$'s in [38, Proposition 3.2].

It is easily verified that, a quasi-homogeneous operator T satisfies $T_i S_{i,i+1} = S_{i,i+1} T_{i+1}$, $0 \leq i \leq n-2$. Therefore T belongs to the class $\mathscr{F}B_{n+1}(\mathbb{D}) \subseteq B_n(\mathbb{D})$ (see [37]).

The theorem given below describes all quasi-homogeneous operators which are Möbius bounded.

Theorem 4.4.11. *A quasi-homogeneous operator T is Möbius bounded if and only if* $\Lambda(T) \geq 2$.

Proof. If $\Lambda(T) \geq 2$, then by [38, Theorem 4.2 (1)], T is similar to the direct sum $T_0 \oplus T_1 \oplus \cdots \oplus T_{n-1}$. Hence T is Möbius bounded if and only if $T_0 \oplus T_1 \oplus \cdots \oplus T_{n-1}$ is Möbius bounded. Note that each T_i, $0 \leq i \leq n-1$, is homogeneous and therefore is Möbius bounded. Consequently, the operator $T_0 \oplus T_1 \oplus \cdots \oplus T_{n-1}$ is also Möbius bounded.

To prove the converse, assume that T is Möbius bounded. By a straightforward computation using the intertwining relation $T_i S_{i,i+1} = S_{i,i+1} T_{i+1}$, $0 \leq i \leq n-2$, we obtain

$$\varphi(T) = \begin{pmatrix} \varphi(T_0) & \varphi'(S_{0,1}) & * & \cdots & * \\ 0 & \varphi(T_1) & \varphi'(S_{1,2}) & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \varphi(T_{n-2}) & \varphi'(S_{n-3,n-2}) \\ 0 & 0 & \cdots & 0 & \varphi(T_{n-1}) \end{pmatrix}$$

on $\mathscr{H}^{(\lambda_0)} \oplus \mathscr{H}^{(\lambda_1)} \oplus \ldots \oplus \mathscr{H}^{(\lambda_{n-1})}$. Since

$$\|\varphi(T)\| \geq \left\| \begin{pmatrix} \varphi(T_0) & \varphi'(S_{0,1}) \\ 0 & \varphi(T_1) \end{pmatrix} \right\| = \left\| \varphi \begin{pmatrix} T_0 & S_{0,1} \\ 0 & T_1 \end{pmatrix} \right\|,$$

it follows that the operator $\begin{pmatrix} T_0 & S_{0,1} \\ 0 & T_1 \end{pmatrix}$ is Möbius bounded. Note that this operator is of the form $\begin{pmatrix} M_z^* & M_\psi^* \\ 0 & M_z^* \end{pmatrix}$ on $\mathscr{H}^{(\lambda_0)} \oplus \mathscr{H}^{(\lambda_1)}$, where ψ is the constant function $\overline{m}_{0,1}$. Hence, by Theorem 4.4.10, we conclude that $\lambda_1 - \lambda_0 \geq 2$. Consequently, $\Lambda(T) \geq 2$, completing the proof. □

Corollary 4.4.12. *The Shields' conjecture has an affirmative answer for the class of quasi-homogeneous operators.*

Proof. First note that, by Theorem 4.4.11, a quasi-homogeneous operator T is Möbius bounded if and only if $\Lambda(T) \geq 2$. Second, if $\Lambda(T) \geq 2$, then T is similar to $T_0 \oplus T_1 \oplus \cdots \oplus T_{n-1}$, (see [38, Theorem 4.2 (1)]). Shields' conjecture is easily verified for these operators using the explicit weights (see [10, section 7.2]). Therefore, its validity for T follows via the similarity. □

Corollary 4.4.13. *A quasi-homogeneous operator T is Möbius bounded if and only if it is similar to a homogeneous operator.*

Proof. The proof in the forward direction is exactly the same as the proof given in the previous corollary. In the other direction, an operator similar to a homogeneous operator is clearly Möbius bounded. □

The corollary given below follows immediately from Proposition 4.3.12. Therefore the proof is omitted.

Corollary 4.4.14. *Every quasi-homogeneous operator T of rank 2 is weakly homogeneous.*

4.5 A Möbius bounded weakly homogeneous operator not similar to any homogeneous operator

We recall that every operator which is similar to a homogeneous operator is weakly homogeneous. Corollary 4.3.9 gives examples of a continuum of weakly homogeneous operators that are not similar to any homogeneous operator. In [10], two more classes of examples, distinct from the ones given in Corollary 4.3.9 have appeared. Among these two classes of examples, we recall the one due to M. Ordower.

For an arbitrary homogeneous operator T on a Hilbert space $\mathscr{H}$, let $\tilde{T}$ be the operator $\begin{pmatrix} T & I \\ 0 & T \end{pmatrix}$. Let U_φ be a unitary operator on $\mathscr{H}$ such that $\varphi(T) = U_\varphi T U_\varphi^*$. A routine verification taking L_φ to be the invertible operator $\varphi'(T)^{\frac{1}{2}} U_\varphi \bigoplus \varphi'(T)^{-\frac{1}{2}} U_\varphi$ shows that $L_\varphi \tilde{T} L_\varphi^{-1} = \varphi(\tilde{T})$. Thus $\tilde{T}$ is weakly homogeneous. Since $\varphi(\tilde{T}) = \begin{pmatrix} \varphi(T) & \varphi'(T) \\ 0 & \varphi(T) \end{pmatrix}$, it follows that $\|\varphi(\tilde{T})\| \geq \|\varphi'(T)\|$. Moreover,

$$\|\varphi'_{\theta,a}(T)\| \geq r\big(\varphi'_{\theta,a}(T)\big) = \sup_{z\in\sigma(T)} |\varphi'_{\theta,a}(z)| = \sup_{z\in\bar{\mathbb{D}}} |\varphi'_{\theta,a}(z)| \geq \frac{1}{(1-|a|^2)^2}, \tag{4.50}$$

where $r\big(\varphi'_{\theta,a}(T)\big)$ is the spectral radius of the operator $\varphi'_{\theta,a}(T)$. Note that $\sigma(T)$ is either $\bar{\mathbb{D}}$ or $\mathbb{T}$, and hence by the maximum modulus principle, we see that $\sup_{z\in\sigma(T)} |\varphi'_{\theta,a}(z)|$ equals $\sup_{z\in\bar{\mathbb{D}}} |\varphi'_{\theta,a}(z)|$. From (4.50), it is clear that $\sup_{\varphi\in \text{Möb}} \|\varphi'(T)\| = \infty$. Consequently, $\tilde{T}$ is not Möbius bounded. Since an operator, which is similar to a homogeneous operator, is necessarily

Möbius bounded, we conclude that $\tilde{T}$ is not similar to any homogeneous operator. Therefore, it is natural to ask the following question.

Question 4.5.1 (Bagchi-Misra, [10, Question 10])**.** *Is it true that every Möbius bounded weakly homogeneous operator is similar to a homogeneous operator?*

The following lemma, which will be used for the proof of the main theorem of this section, provides a sufficient condition on K to determine if the multiplication operator M_z on $(\mathscr{H}, K)$ is Möbius bounded.

Lemma 4.5.2. *Let $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ be a positive definite kernel. Suppose that there exists a constant $\lambda > 0$ such that $(1 - z\bar{w})^\lambda K(z, w)$ is non-negative definite on $\mathbb{D} \times \mathbb{D}$. Then the multiplication operator M_z on $(\mathscr{H}, K)$ is bounded and $\sigma(M_z) = \bar{\mathbb{D}}$. Moreover, M_z on $(\mathscr{H}, K)$ is Möbius bounded.*

Proof. Let $M^{(\lambda)}$ denote the multiplication operator M_z on the Hilbert space $(\mathscr{H}, K^{(\lambda)})$ and set $\hat{K}(z, w) = (1 - z\bar{w})^\lambda K(z, w),\ z, w \in \mathbb{D}$. Then, by hypothesis, $\hat{K}$ is non-negative definite on $\mathbb{D} \times \mathbb{D}$ and K can be written as the product $K^{(\lambda)}\hat{K}$. Since the operator $M^{(\lambda)}$ on $(\mathscr{H}, K^{(\lambda)})$ is Möbius bounded, by Lemma 2.1.10, there exists a constant $c > 0$ such that $(c^2 - \varphi(z)\overline{\varphi(w)})K^{(\lambda)}$ is non-negative definite on $\mathbb{D} \times \mathbb{D}$ for all φ in Möb. Hence $(c^2 - \varphi(z)\overline{\varphi(w)})K$, being a product of two non-negative definite kernels $(c^2 - \varphi(z)\overline{\varphi(w)})K^{(\lambda)}$ and $\hat{K}$, is non-negative definite. Therefore, again by Lemma 2.1.10, it follows that M_φ, $\varphi \in$ Möb, is uniformly bounded on $(\mathscr{H}, K)$.

To show that the spectrum of M_z on $(\mathscr{H}, K)$ is $\bar{\mathbb{D}}$, let a be an arbitrary point in $\mathbb{C} \setminus \bar{\mathbb{D}}$. Since $\sigma(M^{(\lambda)}) = \bar{\mathbb{D}}$, the operator M_{z-a} is invertible on $\mathscr{H}^{(\lambda)}$. Consequently, the operator $M_{(z-a)^{-1}}$ is bounded on $\mathscr{H}^{(\lambda)}$. Then, by the same argument used in the last paragraph, it follows that $M_{(z-a)^{-1}}$ is bounded on $(\mathscr{H}, K)$ and therefore $a \notin \sigma(M_z)$. Since each $K(\cdot, w)$, $w \in \mathbb{D}$, is an eigenvector of M_z^* on $(\mathscr{H}, K)$, it follows that $\bar{\mathbb{D}} \subseteq \sigma(M_z)$. Therefore we conclude that $\sigma(M_z) = \bar{\mathbb{D}}$.

Since $\sigma(M_z) = \bar{D}$ and $\|M_\varphi\| \le c$, $\varphi \in$ Möb, for some constant $c > 0$, it follows that M_z on $(\mathscr{H}, K)$ is Möbius bounded. □

The theorem given below answers Question 4.5.1 in the negative.

Theorem 4.5.3. *Let $K(z, w) = \sum_{n=0}^\infty b_n (z\bar{w})^n$, $b_n > 0$, be a positive definite kernel on $\mathbb{D} \times \mathbb{D}$ such that for each $\gamma \in \mathbb{R}$, $\lim_{|z| \to 1}(1 - |z|^2)^\gamma K(z, z)$ is either 0 or ∞. Assume that the adjoint M_z^* of the multiplication operator by the coordinate function z on $(\mathscr{H}, K)$ is in $B_1(\mathbb{D})$ and is weakly homogeneous. Then the multiplication operator M_z on $(\mathscr{H}, KK^{(\lambda)})$, $\lambda > 0$, is a Möbius bounded weakly homogeneous operator which is not similar to any homogeneous operator.*

Proof. Since the operator M_z^* on $(\mathscr{H}, K)$ is weakly homogeneous, so is the operator M_z on $(\mathscr{H}, K)$. Furthermore, since the operator M_z on $(\mathscr{H}, K^{(\lambda)})$ is homogeneous, by Theorem 4.2.8, it follows that M_z on $(\mathscr{H}, KK^{(\lambda)})$ is weakly homogeneous. Also, by Lemma 4.5.2, we see that

M_z on $(\mathscr{H}, KK^{(\lambda)})$ has spectrum $\bar{\mathbb{D}}$ and is Möbius bounded. Therefore, to complete the proof, it remains to show that M_z on $(\mathscr{H}, KK^{(\lambda)})$ is not similar to any homogeneous operator.

Suppose that M_z on $(\mathscr{H}, KK^{(\lambda)})$ is similar to a homogeneous operator, say T. Since the operators M_z^* on $(\mathscr{H}, K^{(\lambda)})$ and M_z^* on $(\mathscr{H}, K)$ belong to $B_1(\mathbb{D})$, by Theorem 1.1.8, the operator M_z^* on $(\mathscr{H}, KK^{(\lambda)})$ belongs to $B_1(\mathbb{D})$. Furthermore, since the class $B_1(\mathbb{D})$ is closed under similarity, the operator T^* belongs to $B_1(\mathbb{D})$. Also since T is homogeneous, the operator T^* is homogeneous. Recall that, upto unitary equivalence, every homogeneous operator in $B_1(\mathbb{D})$ is of the form M_z^* on $(\mathscr{H}, K^{(\mu)})$ for some $\mu > 0$ (cf. [44]). Therefore, the operator T^* is unitarily equivalent to M_z^* on $(\mathscr{H}, K^{(\mu)})$ for some $\mu > 0$. Consequently, M_z on $(\mathscr{H}, KK^{(\lambda)})$ is similar to M_z on $(\mathscr{H}, K^{(\mu)})$. Hence, by [51, Theorem 2′], there exist constants $c_1, c_2 > 0$ such that

$$c_1 \leq \frac{K(z,z)K^{(\lambda)}(z,z)}{K^{(\mu)}(z,z)} \leq c_2, \ \ z \in \mathbb{D}.$$

Equivalently,

$$c_1 \leq (1-|z|^2)^{\mu-\lambda} K(z,z) \leq c_2, z \in \mathbb{D}.$$

This is a contradiction to our hypothesis that for each $\gamma \in \mathbb{R}$, $\lim_{|z|\to 1}(1-|z|^2)^{\gamma} K(z,z)$ is either 0 or ∞. Hence the operator M_z on $(\mathscr{H}, KK^{(\lambda)})$ is not similar to any homogeneous operator, completing the proof of the theorem. □

Below we give one example which satisfy the hypothesis of the Theorem 4.5.3. Recall that the Dirichlet kernel $K_{(-1)}$ is defined by

$$K_{(-1)}(z,w) = \sum_{n=0}^{\infty} \frac{1}{n+1}(z\bar{w})^n = \frac{1}{z\bar{w}} \log \frac{1}{1-z\bar{w}}, \ \ z, w \in \mathbb{D}.$$

By corollary 4.3.9, the operator M_z^* on $(\mathscr{H}, K_{(-1)})$ is weakly homogeneous and belongs to $B_1(\mathbb{D})$. Let γ be a fixed but arbitrary real number. We will be done if we can show that $\lim_{|z|\to 1} \frac{(1-|z|^2)^{\gamma}}{|z|^2} \log \frac{1}{1-|z|^2}$ is either 0 or ∞. To see that, we observe that

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^{\gamma}}{|z|^2} \log \frac{1}{1-|z|^2} = \lim_{x\to 0} \frac{-x^{\gamma}\log x}{1-x}.$$

Since

$$\lim_{x\to 0} -x^{\gamma}\log x = \begin{cases} \infty & (\text{if } \gamma \leq 0) \\ 0 & (\text{if } \gamma > 0), \end{cases}$$

we conclude that the kernel $K_{(-1)}$ satisfies the hypothesis of Theorem 4.5.3. Consequently, we have the following corollary.

Corollary 4.5.4. *The multiplication operator M_z on $(\mathscr{H}, K_{(-1)}K^{(\lambda)})$, $\lambda > 0$, is a Möbius bounded weakly homogeneous operator which is not similar to any homogeneous operator.*

Chapter 5

On sum of two subnormal kernels

In this chapter, we study the subnormality of the multiplication operator M_z on the Hilbert space determined by the sum of two positive definite kernels. In section 5.1, several different counter-examples settling a recent conjecture of Gregory T. Adams, Nathan S. Feldman and Paul J. McGuire, in the negative, are given. Some examples, where the conjecture has an affirmative answer, are also discussed. In section 5.2, we investigate these questions for a class of weighted multi-shifts. Almost all of the material presented in this chapter is from [32].

5.1 Sum of two subnormal reproducing kernels need not be subnormal

The reader is referred to chapter 1 for the basic definitions and preliminaries related to subnormal operators and completely monotone sequences. The following conjecture regarding the subnormality of the multiplication operator M_z on the Hilbert space determined by the the sum of two positive definite kernels appeared in [1, page 22]. Although we have stated the conjecture in chapter 1, it would be useful for the reader to recall it here once again.

Conjecture 5.1.1 (Adams-Feldman-McGuire, [1, page 22])**.** *Let* $K_1(z,w) = \sum_{k\in\mathbb{Z}_+} a_k(z\bar{w})^k$ *and* $K_2(z,w) = \sum_{k\in\mathbb{Z}_+} b_k(z\bar{w})^k$ *be any two reproducing kernels satisfying:*

(a) $\lim \frac{a_k}{a_{k+1}} = \lim \frac{b_k}{b_{k+1}} = 1$

(b) $\lim a_k = \lim b_k = \infty$

(c) $\frac{1}{a_k} = \int_{[0,1]} t^k d\nu_1(t)$ *and* $\frac{1}{b_k} = \int_{[0,1]} t^k d\nu_2(t)$ *for all* $k \in \mathbb{Z}_+$*, where* ν_1 *and* ν_2 *are two positive measures supported in* $[0,1]$.

Then the multiplication operator M_z *on* $(\mathscr{H}, K_1 + K_2)$ *is a subnormal operator.*

For the construction of counter-examples to the conjecture, we make use of the following result, borrowed from [3, Proposition 4.3].

Proposition 5.1.2. *For distinct positive real numbers $a_0,\ldots,a_n$ and non-zero real numbers $b_0,\ldots,b_n$, consider the polynomial $p(x)=\Pi_{k=0}^{n}(x+a_k+ib_k)(x+a_k-ib_k)$. Then the sequence $\{\frac{1}{p(l)}\}_{l\in\mathbb{Z}_+}$ is never a Hausdorff moment sequence.*

For $r>0$, let K_r be a positive definite kernel given by

$$K_r(z,w):=\sum_{k\in\mathbb{Z}_+}\frac{k+r}{r}(z\bar{w})^k \qquad (z,w\in\mathbb{D}).$$

The case $r=1$ corresponds to the Bergman kernel. It is easy to see that the multiplication operator M_z on $(\mathscr{H},K_r)$ is contractive subnormal and the representing measure is $rx^{r-1}dx$.

For $s,t>0$, consider the multiplication operator M_z on $(\mathscr{H},K_{s,t})$, where

$$K_{s,t}(z,w):=\sum_{k\in\mathbb{Z}_+}\frac{(k+s)(k+t)}{st}(z\bar{w})^k \qquad (z,w\in\mathbb{D}).$$

The case $s=1$ and $t=2$, corresponds to the kernel $(1-z\bar{w})^{-3}$. Note that M_z on $(\mathscr{H},K_{s,t})$ is contractive subnormal and the representing measure ν is given by

$$d\nu(x) \;=\; \begin{cases} -s^2x^{s-1}\log x\,dx & \text{if } s=t\\ st\frac{x^{t-1}-x^{s-1}}{s-t}dx & \text{if } s\neq t.\end{cases}$$

One easily verifies that K_r and $K_{s,t}$ both satisfy all the conditions $(a),(b)$ and (c) of the Conjecture 5.1.1. But the multiplication operator on their sum need not be subnormal for all possible choices of $r,s,t>0$. This follows from the following theorem.

Theorem 5.1.3. *The multiplication operator M_z on $(\mathscr{H},K_r+K_{s,t})$ is subnormal if and only if*

$$(rs+st+tr)^2\geq 8r^2st. \tag{5.1}$$

Proof. Notice that

$$(K_r+K_{s,t})(z,w)=\sum_{k\in\mathbb{Z}_+}\frac{k^2+(s+t+\frac{st}{r})k+2st}{st}(z\bar{w})^k \qquad (z,w\in\mathbb{D}).$$

The roots of the polynomial $x^2+(s+t+\frac{st}{r})x+2st$ are

$$x_1 \;:=\; \frac{-(s+t+\frac{st}{r})+\sqrt{(s+t+\frac{st}{r})^2-8st}}{2} \text{ and}$$

$$x_2 \;:=\; \frac{-(s+t+\frac{st}{r})-\sqrt{(s+t+\frac{st}{r})^2-8st}}{2}.$$

Suppose that $(rs+st+tr)^2 \geq 8r^2st$. Then the kernel $K_r+K_{s,t}$ is of the form $2K_{s',t'}$ where $s'=-x_1$ and $t'=-x_2$. Hence M_z on $(\mathscr{H}, K_r+K_{s,t})$ is a subnormal operator.

Conversely, assume that $(rs+st+tr)^2 < 8r^2st$. Then it follows from Proposition 5.1.2 that M_z on $(\mathscr{H}, K_r+K_{s,t})$ cannot be subnormal. □

Remark 5.1.4. *If we choose $s=1$, $t=2$ and $r>2$, then the inequality* (5.1) *is not valid.*

Recall that an operator T in $B(\mathscr{H})$ is said to be hyponormal if $T^*T-TT^* \geq 0$. It is not hard to verify that a weighted shift T with weight sequence $\{w_n\}_{n\in\mathbb{Z}_+}$ is hyponormal if and only if $\{w_n\}_{n\in\mathbb{Z}_+}$ is increasing (cf. [51]). We point out that if K_1 and K_2 are two reproducing kernels such that the multiplication operators M_z on $(\mathscr{H}, K_1)$ and $(\mathscr{H}, K_2)$ are hyponormal, then the multiplication operator M_z on $(\mathscr{H}, K_1+K_2)$ need not be hyponormal. An example illustrating this is given below.

Example 5.1.5. *For any $s,t>0$, consider the reproducing kernel $K^{s,t}$ given by*

$$K^{s,t}(z,w) := 1+sz\bar{w}+s^2(z\bar{w})^2+t\frac{(z\bar{w})^3}{1-z\bar{w}}.$$

Note that $K^{s,t}$ defines a reproducing kernel on the unit disc $\mathbb{D}\times\mathbb{D}$ and the multiplication operator M_z on $(\mathscr{H}, K^{s,t})$ may be realized as a weighted shift operator with weight sequence $\left(\sqrt{\frac{1}{s}}, \sqrt{\frac{1}{s}}, \sqrt{\frac{s^2}{t}}, 1, 1, \cdots\right)$. Hence M_z on $(\mathscr{H}, K^{s,t})$ is hyponormal if and only if $s^2 \leq t \leq s^3$.

Observe that M_z on $(\mathscr{H}, K^{s,t}+K^{s',t'})$ may be realized as a weighted shift operator with weight sequence $(\sqrt{\frac{2}{s+s'}}, \sqrt{\frac{s+s'}{s^2+s'^2}}, \sqrt{\frac{s^2+s'^2}{t+t'}}, 1, 1, \cdots)$. For the hyponormality of this weighted shift operator, it is necessary that $\frac{2}{s+s'} \leq \frac{s+s'}{s^2+s'^2}$, which is true only when $s=s'$.

We remark that this is different from the case of the product of two kernels, where, the hyponormality of the multiplication operator on the Hilbert space $(\mathscr{H}, K_1K_2)$ follows as soon as we assume they are hyponormal on the two Hilbert spaces $(\mathscr{H}, K_1)$ and $(\mathscr{H}, K_2)$, see [8].

If $T \in B(\mathscr{H})$ is left invertible, then the operator T', given by $T' = T(T^*T)^{-1}$, is said to be the operator *Cauchy dual* to T. The following result has been already recorded in [6, Proposition 6], which may be paraphrased as follows:

Theorem 5.1.6. *Let $K(z,w)=\sum_{k\in\mathbb{Z}_+} a_k(z\bar{w})^k$ be a positive definite kernel on $\mathbb{D}\times\mathbb{D}$ and M_z be the multiplication operator on $(\mathscr{H}, K)$. Assume that M_z is left invertible. Then the followings are equivalent:*

(i) $\{a_k\}_{k\in\mathbb{Z}_+}$ *is a completely alternating sequence.*

(ii) *The Cauchy dual M_z' of M_z is completely hyperexpansive.*

(iii) *For all $t > 0$, $\{\frac{1}{t(a_k-1)+1}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence.*

(iv) *For all $t > 0$, the multiplication operator M_z on $(\mathscr{H}, tK + (1-t)\mathbb{S}_{\mathbb{D}})$ is contractive subnormal, where $\mathbb{S}_{\mathbb{D}}(z,w) = \frac{1}{1-z\bar{w}}$ is the Szegö kernel of the unit disc $\mathbb{D}$.*

Remark 5.1.7. *If $\{a_k\}_{k\in\mathbb{Z}_+}$ is a completely alternating sequence, then putting $t = 1$ in part (iii) of Theorem 5.1.6, it follows that $\{\frac{1}{a_k}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence.*

Corollary 5.1.8. *Let $K_1(z,w) = \sum_{k\in\mathbb{Z}_+} a_k(z\bar{w})^k$ and $K_2(z,w) = \sum_{k\in\mathbb{Z}_+} b_k(z\bar{w})^k$ be any two reproducing kernels such that $\{a_k\}_{k\in\mathbb{Z}_+}$ and $\{b_k\}_{k\in\mathbb{Z}_+}$ are completely alternating sequences, then the multiplication operator M_z on $(\mathscr{H}, K_1 + K_2)$ is subnormal.*

Proof. It is easy to verify that the sum of two completely alternating sequences is completely alternating. The desired conclusion follows immediately from Remark 5.1.7. □

Remark 5.1.9. *Note that $\{\frac{k+r}{r}\}_{k\in\mathbb{Z}_+}$ is a completely alternating sequence, but the sequence $\{\frac{(k+s)(k+t)}{st}\}_{k\in\mathbb{Z}_+}$ is not completely alternating. Thus the reproducing kernels K_r and $K_{s,t}$, discussed in Theorem 5.1.3, does not satisfy the hypothesis of Corollary 5.1.8.*

Proposition 5.1.10. *Let $K(z,w) = \sum_{k\in\mathbb{Z}_+} a_k(z\bar{w})^k$ be a positive definite kernel such that the multiplication operator M_z on $(\mathscr{H}, \mathbb{S}_{\mathbb{D}} + K)$ is subnormal. Then the multiplication operator M_z on $(\mathscr{H}, K)$ is subnormal.*

Proof. From the subnormality of M_z on $(\mathscr{H}, \mathbb{S}_{\mathbb{D}} + K)$, it follows that $\{\frac{1}{1+a_k}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence. Thus the sequence $\{1 - \frac{1}{1+a_k}\}_{k\in\mathbb{Z}_+}$ is completely alternating. Note that

$$\begin{aligned}(a_k)^{-1} &= (1+a_k)^{-1}(1 - \tfrac{1}{1+a_k})^{-1} \\ &= \sum_{j=1}^{\infty} \frac{1}{(1+a_k)^j}.\end{aligned}$$

Observe that $\{\frac{1}{(1+a_k)^j}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence for all $j \geq 1$. Hence so is the sequence of partial sums $\{\sum_{j=1}^n \frac{1}{(1+a_k)^j}\}_{k\in\mathbb{Z}_+}$ for each $n \geq 1$. Now, being the limit of completely monotone sequences, the sequence $\{a_k^{-1}\}_{k\in\mathbb{Z}_+}$ is completely monotone. □

Remark 5.1.11. *We have the following remarks:*

(i) The converse of the Proposition 5.1.10 is not true (see the example discussed in part (ii) of the Remark 5.1.14).

(ii) If we replace the Szegö kernel $\mathbb{S}_{\mathbb{D}}$ by the Bergman kernel $B_{\mathbb{D}}$, then the conclusion of the Proposition 5.1.10 need not be true. For example, by using Proposition 5.1.2, one may choose $\alpha > 0$ such that the sequence $\{\frac{1}{k^2+\alpha k+1}\}_{k\in\mathbb{Z}_+}$ is not completely monotone, but the sequence $\{\frac{1}{k^2+(\alpha+1)k+2}\}_{k\in\mathbb{Z}_+}$ is completely monotone.

For $\lambda, \mu > 0$, consider the positive definite kernel

$$K_{\lambda,\mu}(z,w) = \sum_{k\in\mathbb{Z}_+} \frac{(\lambda)_k}{(\mu)_k}(z\bar{w})^k \qquad (z,w\in\mathbb{D}),$$

where $(x)_k$ is the Pochhammer symbol given by $\frac{\Gamma(x+k)}{\Gamma(x)}$. It is easy to see that the case $\mu = 1$ corresponds to the kernel $(1-z\bar{w})^{-\lambda}$. Note that the multiplication operator M_z on $(\mathscr{H}, K_{\lambda,\mu})$ may be realized as a weighted shift operator with weight sequence $\{\sqrt{\frac{k+\mu}{k+\lambda}}\}_{k\in\mathbb{Z}_+}$.

The first part of the following theorem is proved in [25] and the representing measure is given in [23, Lemma 2.2]. Here, we provide a proof for the second part only.

Theorem 5.1.12. *The multiplication operator M_z on $(\mathscr{H}, K_{\lambda,\mu})$ is*

(i) subnormal if and only if $\lambda \geq \mu$. In the case of subnormality, the representing measure ν of M_z is given by

$$d\nu(x) \;=\; \begin{cases} \frac{\Gamma(\lambda)}{\Gamma(\mu)\Gamma(\lambda-\mu)} x^{\mu-1}(1-x)^{\lambda-\mu-1}dx & \text{if } \lambda > \mu \\ \delta_1(x)dx & \text{if } \lambda = \mu, \end{cases}$$

where δ_1 is the Dirac delta function.

(ii) completely hyperexpansive if and only if $\lambda \leq \mu \leq \lambda + 1$.

Proof. The multiplication operator M_z on $(\mathscr{H}, K_{\lambda,\mu})$ is completely hyperexpansive if and only if the sequence $\{\frac{(\mu)_k}{(\lambda)_k}\}_{k\in\mathbb{Z}_+}$ is completely alternating. Here

$$\begin{aligned}\Delta\left(\frac{(\mu)_k}{(\lambda)_k}\right) = \frac{(\mu)_{k+1}}{(\lambda)_{k+1}} - \frac{(\mu)_k}{(\lambda)_k} &= \frac{(\mu)_{k+1} - (\mu)_k(\lambda+k-1)}{(\lambda)_{k+1}} \\ &= \frac{\mu-\lambda}{\lambda}\frac{(\mu)_k}{(\lambda+1)_k}.\end{aligned}$$

By the first part of this theorem, it follows that $\{\frac{\mu-\lambda}{\lambda}\frac{(\mu)_k}{(\lambda+1)_k}\}_{k\in\mathbb{Z}_+}$ is completely monotone if and only if $\lambda \leq \mu \leq \lambda+1$. □

The following proposition gives a sufficient condition for the subnormality of the multiplication operator on Hilbert space determined by the sum of two kernels belonging to the class $K_{\lambda,\mu}$.

Proposition 5.1.13. *Let $0 < \mu \leq \lambda' \leq \lambda \leq \lambda' + 1$. Then the multiplication operator M_z on $(\mathscr{H}, K_{\lambda,\mu} + K_{\lambda',\mu})$ is contractive subnormal.*

Proof. Observe that

$$(K_{\lambda,\mu}+K_{\lambda',\mu})(z,w)=\sum_{k\in\mathbb{Z}_+}\frac{(\lambda)_k+(\lambda')_k}{(\mu)_k}(z\bar{w})^k \qquad (z,w\in\mathbb{D})$$

and

$$\frac{(\mu)_k}{(\lambda)_k+(\lambda')_k}=\frac{(\mu)_k}{(\lambda')_k}\frac{1}{1+\frac{(\lambda)_k}{(\lambda')_k}}.$$

Since $\mu\le\lambda'$, it follows from part (i) of Theorem 5.1.12 that $\{\frac{(\mu)_k}{(\lambda')_k}\}_{k\in\mathbb{Z}_+}$ is completely monotone.

If $\lambda'\le\lambda\le\lambda'+1$, then by part (ii) of Theorem 5.1.12, the sequence $\{\frac{(\lambda)_k}{(\lambda')_k}\}_{k\in\mathbb{Z}_+}$ is completely alternating. Thus so is the sequence $\{1+\frac{(\lambda)_k}{(\lambda')_k}\}_{k\in\mathbb{Z}_+}$. Hence, by Remark 5.1.7, we see that $\{(1+\frac{(\lambda)_k}{(\lambda')_k})^{-1}\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence. Therefore, being a product of two completely monotone sequences, it follows that $\{\frac{(\mu)_k}{(\lambda)_k+(\lambda')_k}\}_{k\in\mathbb{Z}_+}$ is completely monotone. This completes the proof. □

Remark 5.1.14. *Here are some remarks:*

(i) *Let $\mu<\lambda'$ and $\lambda=\lambda'+1$. The representing measure for the sequence $\{(1+\frac{(\lambda)_k}{(\lambda')_k})^{-1}\}_{k\in\mathbb{Z}_+}$ is $\lambda' x^{2\lambda'-1}dx$. Also the representing measure for the sequence $\{\frac{(\mu)_k}{(\lambda')_k}\}_{k\in\mathbb{Z}_+}$ is given in part (i) of Theorem 5.1.12. Thus, using Remark* 2.4 *of [3], one may obtain the representing measure for M_z on $(\mathscr{H},K_{\lambda,\mu}+K_{\lambda',\mu})$ to be given by*

$$d\nu(x)=\frac{\lambda'\Gamma(\lambda')}{\Gamma(\mu)\Gamma(\lambda'-\mu)}x^{2\lambda'-1}\left(\int_0^{1-x}t^{\lambda'-\mu-1}(1-t)^{\mu-2\lambda'-1}dt\right)dx.$$

But in general, when $\lambda<\lambda'+1$, we do not know the representing measure for the sequence $\left\{(1+\frac{(\lambda)_k}{(\lambda')_k})^{-1}\right\}_{k\in\mathbb{Z}_+}$ as well as for the sequence $\left\{\frac{(\mu)_k}{(\lambda)_k+(\lambda')_k}\right\}_{k\in\mathbb{Z}_+}$.

(ii) *Note that $(K_{1,1}+K_{3,1})(z,w)=\sum_{k\in\mathbb{Z}_+}\frac{k^2+3k+4}{2}(z\bar{w})^k$ for all $z,w\in\mathbb{D}$. It follows from Proposition* 5.1.2 *that the sequence $\{\frac{2}{k^2+3k+4}\}_{k\in\mathbb{Z}_+}$ is not completely monotone. Consequently, the multiplication operator M_z on $(\mathscr{H},K_{1,1}+K_{3,1})$ is not subnormal. For $\lambda>1$, consider the kernel $K_{\lambda,1}+K_{3,1}$. We claim that there exists a $\lambda_0>1$ such that $\{\frac{(1)_k}{(\lambda_0)_k+(3)_k}\}_{k\in\mathbb{Z}_+}$ is not completely monotone. If not, assume that $\{\frac{(1)_k}{(\lambda)_k+(3)_k}\}_{k\in\mathbb{Z}_+}$ is completely monotone for all $\lambda>1$. As λ goes to* 1*, one may get that $\{\frac{2}{k^2+3k+4}\}_{k\in\mathbb{Z}_+}$ is completely monotone, which is a contradiction. Therefore, we conclude that there exists a $\lambda_0>1$ such that the multiplication operator M_z on $(\mathscr{H},K_{\lambda_0,1}+K_{3,1})$ is not subnormal. By using the properties of the gamma function, one may verify that $K_{\lambda_0,1}$ and $K_{3,1}$ both satisfy $(a),(b)$ and (c) of the Conjecture 5.1.1. Hence this also provides a class of counterexamples for the Conjecture 5.1.1.*

Proposition 5.1.15. *Let $0 < p \le q \le p+1$. Suppose $K_1(z,w) = \sum_{k\in\mathbb{Z}_+} a_k^p (z\bar{w})^k$ and $K_2(z,w) = \sum_{k\in\mathbb{Z}_+} a_k^q (z\bar{w})^k$ are any two reproducing kernels such that $\{a_k\}_{k\in\mathbb{Z}_+}$ is a completely alternating sequence. Then the multiplication operator M_z on $(\mathscr{H}, K_1+K_2)$ is subnormal.*

Proof. Note that
$$\frac{1}{a_k^p + a_k^q} = \frac{1}{a_k^p(1+a_k^{q-p})}, \quad k \in \mathbb{Z}_+.$$
Since $0 \le q-p \le 1$ and $\{a_k\}_{k\in\mathbb{Z}_+}$ is completely alternating, it follows from [7, Corollary 1] that $\{a_k^{q-p}\}_{k\in\mathbb{Z}_+}$ is also completely alternating. Thus so is $\{1+a_k^{q-p}\}_{k\in\mathbb{Z}_+}$. Hence, by Remark 5.1.7, $\{(1+a_k^{q-p})^{-1}\}_{k\in\mathbb{Z}_+}$ is completely monotone. Also, by [11, Corollary 4.1], $\{a_k^{-p}\}_{k\in\mathbb{Z}_+}$ is completely monotone. Now the proof follows as the product of two completely monotone sequences is also completely monotone. □

Example 5.1.16. *Recall that any $p > 0$, the positive definite kernel $K_{(p)}$ is defined by*
$$K_{(p)}(z,w) := \sum_{k\in\mathbb{Z}_+} (k+1)^p (z\bar{w})^k \qquad (z,w \in \mathbb{D}).$$
It is known that the multiplication operator M_z on $(\mathscr{H}, K_{(p)})$ is subnormal with the representing measure $d\nu(x) = \frac{(-\log x)^{p-1}}{\Gamma(p)} dx$ (see [24, Theorem 4.3]). By Proposition 5.1.15, it follows that M_z on $(\mathscr{H}, K_{(p)} + K_{(q)})$ is subnormal if $p \le q \le p+1$.

The next result also provides a class of counter-examples to the Conjecture 5.1.1.

Theorem 5.1.17. *The multiplication operator M_z on $(\mathscr{H}, K_{(p)} + K_{(p+2)})$ is subnormal if and only if $p \ge 1$.*

Proof. For $x \in (0,1]$, let $g(x) := \frac{1}{\Gamma(p)} \int_0^{-\log x} (-\log x - y)^{p-1} \sin y \, dy$ and $d\nu(x) = g(x)dx$. Then
$$\begin{aligned}
\int_0^1 x^k d\nu(x) &= \frac{1}{\Gamma(p)} \int_{y=0}^{\infty} \int_{x=0}^{e^{-y}} x^k (-\log x - y)^{p-1} dx \sin y \, dy \\
&= \frac{1}{\Gamma(p)} \int_{y=0}^{\infty} e^{-(k+1)y} \int_{u=0}^{\infty} e^{-(k+1)u} u^{p-1} du \sin y \, dy \\
&= \frac{1}{\Gamma(p)} \int_{y=0}^{\infty} e^{-(k+1)y} \frac{\Gamma(p)}{(k+1)^p} \sin y \, dy \\
&= \frac{1}{(k+1)^p} \frac{1}{(k+1)^2+1}.
\end{aligned}$$
Thus the sequence $\left\{\frac{1}{(k+1)^p}\frac{1}{(k+1)^2+1}\right\}_{k\in\mathbb{Z}_+}$ is completely monotone if and only if the function $g(x)$ is non-negative a.e. Note that the function $g(x)$ is non-negative on $(0,1]$ a.e. if and only if the function $h(x) := g(e^{-x})$ is non-negative a.e. on $(0,\infty)$. Now
$$h(x) = \frac{1}{\Gamma(p)} \int_0^x (x-y)^{p-1} \sin y \, dy = \frac{x^p}{\Gamma(p)} \int_0^1 (1-y)^{p-1} sin(xy) \, dy.$$

By [33, Chapter 3, pp 439], we have $h(x) = \frac{\sqrt{x}}{\Gamma(p)} s_{p-\frac{1}{2},\frac{1}{2}}(x)$, where $s_{p-\frac{1}{2},\frac{1}{2}}(x)$ is the Lommel function of first kind. Thus the sequence $\{\frac{1}{(k+1)^p}\frac{1}{(k+1)^2+1}\}_{k\in\mathbb{Z}_+}$ being completely monotone is equivalent to the non-negativity of the function $s_{p-\frac{1}{2},\frac{1}{2}}(x)$ on $(0,\infty)$. If $p \geq 1$, then by [53, Theorem A], we get that $s_{p-\frac{1}{2},\frac{1}{2}}(x) \geq 0$ for all $x > 0$. The converse also follows from [53, Theorem 2], which completes the proof. □

5.2 Multi-variable case

Given a commuting m-tuple $\boldsymbol{T} = (T_1, \dots, T_m)$ of bounded linear operators on $\mathscr{H}$, set

$$Q_{\boldsymbol{T}}(X) := \sum_{i=1}^{m} T_i^* X T_i \qquad (X \in B(\mathscr{H})).$$

For $X \in B(\mathscr{H})$ and $k \geq 1$, one may define $Q_{\boldsymbol{T}}^k(X) := Q_{\boldsymbol{T}}(Q_{\boldsymbol{T}}^{k-1}(X))$, where $Q_{\boldsymbol{T}}^0(X) = X$.

Recall that $\boldsymbol{T}$ is said to be

(i) *spherical contraction* if $Q_{\boldsymbol{T}}(I) \leq I$.

(ii) *jointly left invertible* if there exists a positive number c such that $Q_{\boldsymbol{T}}(I) \geq cI$.

For a jointly left invertible m-tuple $\boldsymbol{T}$, the spherical Cauchy dual $\boldsymbol{T}^{\mathfrak{s}}$ of $\boldsymbol{T}$ is the m-tuple $(T_1^{\mathfrak{s}}, T_2^{\mathfrak{s}} \cdots, T_m^{\mathfrak{s}})$, where $T_i^{\mathfrak{s}} := T_i(Q_{\boldsymbol{T}}(I))^{-1}$ $(i = 1, 2, \cdots, m)$. We say that $\boldsymbol{T}$ is a *joint complete hyperexpansion* if

$$B_n(\boldsymbol{T}) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} Q_{\boldsymbol{T}}^k(I) \leq 0 \qquad (n \geq 1).$$

Recall that for $m \geq 1$, $\mathbb{B}_m$ denotes the Euclidean unit ball $\{z \in \mathbb{C}^m : |z_1|^2 + \cdots + |z_m|^2 < 1\}$. For the rest of the section, we write $\mathbb{B}$ instead of $\mathbb{B}_m$. Also let $\partial\mathbb{B}$ denote the unit sphere $\{z \in \mathbb{C}^m : |z_1|^2 + \cdots + |z_m|^2 = 1\}$ in $\mathbb{C}^m$.

Let $\{\beta_\alpha\}_{\alpha\in\mathbb{Z}_+^m}$ be a multi-sequence of positive numbers. Consider the Hilbert space $H^2(\beta)$ of formal power series $f(z) = \sum_{\alpha\in\mathbb{Z}_+^m} \hat{f}(\alpha) z^\alpha$ such that

$$\|f\|_{H^2(\beta)}^2 = \sum_{\alpha\in\mathbb{Z}_+^m} |\hat{f}(\alpha)|^2 \beta_\alpha^2 < \infty.$$

The Hilbert space $H^2(\beta)$ is said to be *spherically balanced* if the norm on $H^2(\beta)$ admits the slice representation $[\nu, H^2(\gamma)]$, that is, there exist a *Reinhardt measure* ν and a Hilbert space $H^2(\gamma)$ of formal power series in one variable such that

$$\|f\|_{H^2(\beta)}^2 = \int_{\partial\mathbb{B}} \|f_z\|_{H^2(\gamma)}^2 \, d\nu(z) \qquad (f \in H^2(\beta)),$$

where $\gamma = \{\gamma_k\}_{k\in\mathbb{Z}_+}$ is given by the relation $\beta_\alpha = \gamma_{|\alpha|}\|z^\alpha\|_{L^2(\partial\mathbb{B},\nu)}$ for all $\alpha \in \mathbb{Z}_+^m$. Here, by the Reinhardt measure, we mean a $\mathbb{T}^m$-invariant finite positive Borel measure supported in $\partial\mathbb{B}$, where $\mathbb{T}^m$ denotes the unit m-torus $\{z \in \mathbb{C}^m : |z_1| = 1, \cdots, |z_m| = 1\}$. For more details on spherically balanced Hilbert spaces, we refer to [14].

The following lemma has been already recorded in [14, Lemma 4.3]. We include a statement for ready reference.

Lemma 5.2.1. *Let $H^2(\beta)$ be a spherically balanced Hilbert space and let $[\nu, H^2(\gamma)]$ be the slice representation for the norm on $H^2(\beta)$. Consider the m-tuple $\boldsymbol{M}_z = (M_{z_1}, \cdots, M_{z_m})$ of multiplication by the coordinate functions $z_1, \cdots, z_m$ on $H^2(\beta)$. Then for every $n \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^m$,*

$$\langle B_n(\boldsymbol{M}_z)z^\alpha, z^\alpha\rangle = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\langle Q^k_{\boldsymbol{M}_z}(I)z^\alpha, z^\alpha\rangle = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\gamma^2_{k+|\alpha|}\|z^\alpha\|^2_{L^2(\partial\mathbb{B},\nu)}.$$

If the interior of the point spectrum $\sigma_p(\boldsymbol{M}_z^*)$ of $\boldsymbol{M}_z^*$ is non-empty, then $H^2(\beta)$ may be realized as a reproducing kernel Hilbert space $(\mathscr{H}, K)$ [36, Propositions 19 and 20], where the reproducing kernel K is given by

$$K(z,w) = \sum_{\alpha\in\mathbb{Z}_+^m}\frac{z^\alpha\,\bar{w}^\alpha}{\beta_\alpha^2} \qquad (z, w \in \sigma_p(\boldsymbol{M}_z^*)).$$

This leads to the following definition.

Definition 5.2.2. *Let $(\mathscr{H}, K)$ be a reproducing kernel Hilbert space defined on the open unit ball $\mathbb{B}$ with reproducing kernel $K(z,w) = \sum_{\alpha\in\mathbb{Z}_+^m} a_\alpha z^\alpha\,\bar{w}^\alpha$ for all $z, w \in \mathbb{B}$. We say that K is a balanced kernel if $(\mathscr{H}, K)$ is a spherically balanced Hilbert space. Further, the multiplication m-tuple $\boldsymbol{M}_z$ on $(\mathscr{H}, K)$ may be called a balanced multiplication tuple.*

Remark 5.2.3. *The spherical Cauchy dual $\boldsymbol{M}_z^{\mathfrak{s}}$ of a jointly left invertible balanced multiplication tuple $\boldsymbol{M}_z$ can be seen as a multiplication m-tuple $\boldsymbol{M}_z^{\mathfrak{s}} = (M_{z_1}^{\mathfrak{s}}, \cdots, M_{z_m}^{\mathfrak{s}})$ of multiplication by the coordinate functions $z_1, \cdots, z_m$ on $H^2(\beta^{\mathfrak{s}})$, where*

$$\beta_\alpha^{\mathfrak{s}} = \frac{1}{\gamma_{|\alpha|}}\|z^\alpha\|_{L^2(\partial\mathbb{B},\nu)} \qquad (\alpha \in \mathbb{Z}_+^m).$$

In other words, the norm on $H^2(\beta^{\mathfrak{s}})$ admits the slice representation $[\nu, H^2(\gamma')]$, where $\gamma'_k = 1/\gamma_k$ for all $k \in \mathbb{Z}_+$.

Proposition 5.2.4. *If $K_1(z,w) = \sum_{\alpha\in\mathbb{Z}_+^m} a_\alpha z^\alpha\,\bar{w}^\alpha$ and $K_2(z,w) = \sum_{\alpha\in\mathbb{Z}_+^m} b_\alpha z^\alpha\,\bar{w}^\alpha$ are any two balanced kernels with the slice representations $[\nu, H^2(\gamma_1)]$ and $[\nu, H^2(\gamma_2)]$ respectively, then $K_1 + K_2$ is a balanced kernel with the slice representation $[\nu/2, H^2(\gamma)]$, where $\gamma = \{\gamma_k\}$ is given by the relation*

$$\gamma_k = \frac{\sqrt{2}\gamma_{k,1}\gamma_{k,2}}{(\gamma_{k,1}^2 + \gamma_{k,2}^2)^{1/2}} \qquad (k \in \mathbb{Z}_+).$$

Proof. For every $\alpha \in \mathbb{Z}_+^m$, we have

$$a_\alpha + b_\alpha = \frac{1}{\gamma_{|\alpha|,1}^2 \|z^\alpha\|^2_{L^2(\partial\mathbb{B},\nu)}} + \frac{1}{\gamma_{|\alpha|,2}^2 \|z^\alpha\|_{L^2(\partial\mathbb{B},\nu)}} = \frac{(\gamma_{|\alpha|,1}^2 + \gamma_{|\alpha|,2}^2)}{\gamma_{|\alpha|,1}^2 \gamma_{|\alpha|,2}^2 \|z^\alpha\|^2_{L^2(\partial\mathbb{B},\nu)}}.$$

Therefore

$$\|z^\alpha\|^2_{(\mathscr{H},K_1+K_2)} = \frac{2\gamma_{|\alpha|,1}^2\gamma_{|\alpha|,2}^2}{(\gamma_{|\alpha|,1}^2 + \gamma_{|\alpha|,2}^2)} \|z^\alpha\|^2_{L^2(\partial\mathbb{B},\nu/2)} = \gamma_{|\alpha|}^2 \|z^\alpha\|^2_{L^2(\partial\mathbb{B},\nu/2)}$$

for all $\alpha \in \mathbb{Z}_+^m$. Since $\{z^\alpha\}_{\alpha\in\mathbb{Z}_+^m}$ forms an orthogonal subset of $L^2(\partial\mathbb{B},\nu/2)$, the conclusion follows immediately. □

Remark 5.2.5. *The conclusion of the Proposition 5.2.4 still holds even if we choose two different Reinhardt measures ν_1 and ν_2 in the slice representations of K_1 and K_2 such that for some sequence of positive real numbers $\{h_k\}_{k\in\mathbb{Z}_+}$, $\|z^\alpha\|_{L^2(\partial\mathbb{B},\nu_1)} = h_{|\alpha|}\|z^\alpha\|_{L^2(\partial\mathbb{B},\nu_2)}$ for all $\alpha \in \mathbb{Z}_+^m$. For every $j = 1,2$, it is easy to verify that*

$$\sum_{i=1}^{m} \frac{\|z^{\alpha+e_i}\|^2_{L^2(\partial\mathbb{B},\nu_j)}}{\|z^\alpha\|^2_{L^2(\partial\mathbb{B},\nu_j)}} = 1,$$

where e_i is the ith standard basis vector of $\mathbb{C}^m$. This implies that $\{h_k\}_{k\in\mathbb{Z}_+}$ is a constant sequence, say c. Now, by a routine argument, using the Stone-Weierstrass theorem, we conclude that $\nu_1 = c^2\nu_2$.

For a fixed Reinhardt measure ν, let $BK(\nu)$ denote the class of all balanced kernels with the following properties:

(i) For all $K \in BK(\nu)$, the norm on $(\mathscr{H},K)$ admits the slice representations with fixed Reinhardt measure ν.

(ii) For every member K of $BK(\nu)$, the multiplication operator $\boldsymbol{M}_z$ on $(\mathscr{H},K)$ is jointly left invertible.

(iii) The Cauchy dual tuple $\boldsymbol{M}_z^{\mathfrak{s}}$ of $\boldsymbol{M}_z$ is a joint complete hyperexpansion.

Lemma 5.2.6. *For every member K of $BK(\nu)$, the multiplication operator $\boldsymbol{M}_z$ on $(\mathscr{H},K)$ is a subnormal spherical contraction.*

Proof. Let $K \in BK(\nu)$ and $[\nu, H^2(\gamma)]$ be the slice representation for the norm on $(\mathscr{H},K)$. Note that the Cauchy dual $\boldsymbol{M}_z^{\mathfrak{s}}$ of $\boldsymbol{M}_z$ is a balanced multiplication tuple with slice representation $[\nu, H^2(1/\gamma)]$ (see Remark 5.2.3). Since $\boldsymbol{M}_z^{\mathfrak{s}}$ is a joint complete hyperexpansion, it follows from Lemma 5.2.1 that $\{1/\gamma_k^2\}_{k\in\mathbb{Z}_+}$ is a completely alternating sequence. Therefore, by Remark 5.1.7, $\{\gamma_k^2\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence. Now again by applying Lemma 5.2.1, we conclude that the multiplication operator $\boldsymbol{M}_z$ is a subnormal spherical contraction. □

Theorem 5.2.7. *If K_1 and K_2 are any two members of $BK(\nu)$, then the multiplication operator $\boldsymbol{M}_z$ on $(\mathscr{H}, K_1+K_2)$ is a subnormal spherical contraction.*

Proof. Note that the norm on $(\mathscr{H}, K_1+K_2)$ admits the slice representation $[\nu/2, H^2(\gamma)]$, where $\gamma_k^2 = 2\frac{\gamma_{k,1}^2\gamma_{k,2}^2}{\gamma_{k,1}^2+\gamma_{k,2}^2}$ for all $k \in \mathbb{Z}_+$ (see Proposition 5.2.4). It follows from the proof of Lemma 5.2.6 that $\{1/\gamma_{k,1}^2\}_{k\in\mathbb{Z}_+}$ and $\{1/\gamma_{k,2}^2\}_{k\in\mathbb{Z}_+}$ are completely alternating. So their sum, and hence $\{1/\gamma_k^2\}_{k\in\mathbb{Z}_+}$ is a completely alternating sequence. Now the conclusion follows by imitating the argument given in Lemma 5.2.6. □

For $\lambda > 0$, consider the positive definite kernel K_λ given by

$$K_\lambda(z,w) = \frac{1}{(1-\langle z, w\rangle)^\lambda} \qquad (z, w \in \mathbb{B}).$$

The norm on (H, K_λ) admits the slice representation $[\sigma, H^2(\gamma)]$, where σ denotes the normalized surface area measure on $\partial\mathbb{B}$ and $\gamma_k^2 = \frac{(m)_k}{(\lambda)_k}$ for all $k \in \mathbb{Z}_+$. It is well known that the multiplication operator $\boldsymbol{M}_{z,\lambda}$ on (H, K_λ) is a subnormal contraction if and only if $\lambda \geq m$. The same can also be verified by using Lemma 5.2.1 and part (i) of Theorem 5.1.12. Similarly, by using Lemma 5.2.1 and part (ii) of Theorem 5.1.12, one may conclude that the Cauchy dual tuple $\boldsymbol{M}_{z,\lambda}^{\mathfrak{s}}$ is a joint complete hyperexpansion if and only if $m \leq \lambda \leq m+1$. Thus, if we choose λ and λ' are such that $m \leq \lambda, \lambda' \leq m+1$. Then K_λ and $K_{\lambda'}$ are in $BK(\sigma)$. It now follows from Theorem 5.2.7 that the multiplication operator $\boldsymbol{M}_z$ on $(\mathscr{H}, K_\lambda + K_{\lambda'})$ is subnormal. This is also included in the following example.

Example 5.2.8. *Let $0 < m \leq \lambda' \leq \lambda \leq \lambda'+1$. Note that the norm on $(\mathscr{H}, K_\lambda + K_{\lambda'})$ admits the slice representation $[\sigma/2, H^2(\gamma)]$, where $\gamma_k^2 = \frac{2(m)_k}{(\lambda)_k+(\lambda')_k}$ for all $k \in \mathbb{Z}_+$. From the proof of Proposition 5.1.13, it is clear that $\{\gamma_k^2\}_{k\in\mathbb{Z}_+}$ is completely monotone. Hence the multiplication operator $\boldsymbol{M}_z$ on $(\mathscr{H}, K_\lambda + K_{\lambda'})$ is subnormal.*

A m-tuple $\boldsymbol{S} = (S_1, \ldots, S_m)$ of commuting bounded linear operators $S_1, \ldots, S_m$ in $B(\mathscr{H})$ is said to be a *spherical isometry* if $S_1^*S_1 + \cdots + S_m^*S_m = I$. In other words, $Q_{\boldsymbol{S}}(I) = I$. The most interesting example of a spherical isometry is the Szegö m-shift, that is, the m-tuple $\boldsymbol{M}_z$ of multiplication operators $M_{z_1}, \cdots, M_{z_m}$ on the Hardy space $H^2(\partial\mathbb{B})$ of the unit ball.

Let ν be a Reinhardt measure. Consider the multiplication tuple $\boldsymbol{M}_z$ on the reproducing kernel Hilbert space $(\mathscr{H}, K^\nu)$ where

$$K^\nu(z,w) = \sum_{\alpha\in\mathbb{Z}_+^m} \|z^\alpha\|^{-2}_{L^2(\partial\mathbb{B},\nu)} z^\alpha \bar{w}^\alpha \qquad (z, w \in \mathbb{B}). \tag{5.2}$$

Note that $\boldsymbol{M}_z$ is a spherical isometry. In this case, the norm on $(\mathscr{H}, K^\nu)$ admits the slice representation $[\nu, H^2(\mathbb{D})]$, where $H^2(\mathbb{D})$ is the Hardy space of the unit disc.

Theorem 5.2.9. *Let K^ν be the reproducing kernel given as in equation* (5.2) *and $\tilde{K}$ be any balanced kernel with the slice representation* $[\nu, H^2(\tilde{\gamma})]$. *Assume that the multiplication tuple $\boldsymbol{M}_z$ on $(\mathscr{H}, K^\nu + \tilde{K})$ is subnormal. Then the multiplication tuple $\boldsymbol{M}_z$ on $(\mathscr{H}, \tilde{K})$ is subnormal.*

Proof. Observe that the norm on $(\mathscr{H}, K^\nu + \tilde{K})$ admits the slice representation $[\nu/2, H^2(\gamma)]$, where $\gamma_k^2 = 2(1+1/\tilde{\gamma}_k^2)^{-1}$ for all $k \in \mathbb{Z}_+$. Since M_z on $(\mathscr{H}, K^\nu + \tilde{K})$ is subnormal, it follows from Lemma 5.2.1 that $\{\gamma_k^2\}_{k\in\mathbb{Z}_+}$ is a completely monotone sequence. Thus the sequence $\{(1+1/\tilde{\gamma}_k^2)^{-1}\}_{k\in\mathbb{Z}_+}$ is completely monotone. If we replace a_k by $1/\tilde{\gamma}_k^2$ in the proof of the Proposition 5.1.10, we get that $\{\tilde{\gamma}_k^2\}_{k\in\mathbb{Z}_+}$ is completely monotone. Now, by applying Lemma 5.2.1, we conclude that the multiplication operator on $(\mathscr{H}, \tilde{K})$ is subnormal. □

We conclude this chapter with the following questions:

Question 5.2.10. *In view of Proposition 5.1.13 and Theorem 5.1.17, it is natural to ask that*

(i) what is the necessary and sufficient condition for the multiplication operator M_z on $(\mathscr{H}, K_{\lambda,\mu} + K_{\lambda',\mu})$ to be subnormal?

(ii) what is the necessary and sufficient condition for the multiplication operator M_z on $(\mathscr{H}, K_{(p)} + K_{(q)})$ to be subnormal?

Question 5.2.11. *Let K^ν be the reproducing kernel given as in equation* (5.2) *and $\tilde{K}$ be any positive definite kernel given by*

$$\tilde{K}(z,w) := \sum_{\alpha\in\mathbb{Z}_+^m} a_\alpha z^\alpha \bar{w}^\alpha \qquad (z,w \in \mathbb{B}).$$

If the m-tuple $\boldsymbol{M}_z = (M_{z_1}, \ldots, M_{z_m})$ on $(\mathscr{H}, K^\nu + \tilde{K})$ is subnormal, then is it necessary that the m-tuple $\boldsymbol{M}_z$ on $(\mathscr{H}, \tilde{K})$ is also subnormal?

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