Trace Estimate For The Determinant Operator And $\mathbb{K}$- Homogeneous Operators

A Dissertation
submitted in partial fulfilment
of the requirements for the award of the
degree of
Doctor of Philosophy

by

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October 2020

Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Prof. Gadadhar Misra at the Department of Mathematics, Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

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Dedicated to my Parents

Acknowledgement

Foremost, I would like to express my sincere gratitude to my advisor Prof. Gadadhar Misra for his continuous guidance, active participation and stimulating discussions on various aspects of operator theory. A lot of his time and effort has gone into improving the presentation of this thesis. His support and guidance not only helped me grow as a researcher but also shaped me into a better human being.

I first learnt the proof, due to Voiculescu, of the Berger-Shaw theorem from Professor Kalyan Sinha. He explained the subtleties involved in this proof very clearly and proposed the norm estimates of the class $BS_{m,\vartheta}(\Omega)$ that is essential in finding the bound for the trace of the determinant operator $\mathrm{dEt}\big(\llbracket \boldsymbol{T}^*, \boldsymbol{T} \rrbracket\big)$. He has been always there to bail me out whenever I was stuck with a proof during the course of my work. Indeed most of the results in Chapters 2 and 3 have been obtained working jointly with him. I sincerely thank him for his generous help throughout my stay at IISc.

I thank Dr. Cherian Varughese who has helped me in working out the details of several proofs. In particular, without his help, it would have been impossibly difficult for me to complete the proof that the generalized commutator $\mathrm{GC}\big(\llbracket \boldsymbol{T}^*, \boldsymbol{T} \rrbracket\big)$ equals $\mathrm{dEt}\big(\llbracket \boldsymbol{T}^*, \boldsymbol{T} \rrbracket\big)$. Having the opportunity to discuss mathematics with him has been a non-trivial part of my learning process. I am very grateful to him for his unconditional support over the past several years.

I had the good fortune of learning some of the most intricate tools and methods of Jordan algebra and their application to the theory of bounded symmetric domains from Professor Harald Upmeier. I have also learnt a lot from the series of lectures he has delivered at our department. He has taken a keen interest in some of the problems I have been investigating resulting in his solution of a conjecture included in this thesis. I thank him for the many hours of pleasant discussions I had with him spread over the last couple of years.

I would like to thank one of the reviewers for many detailed comments and suggestions. Most of these have been included in this thesis.

A note of thanks goes to Professor Kallol Paul for encouraging me in learning Mathematics.

I also thank Dr. Surjit Kumar and Dr. Soumitra Ghara for many hours of delightful mathematical discussions.

My special thanks go to Amardeep Sarkar, Samarpita Ray, Prodip Howlader, Priyanka De, Sourav Hait, Samrat Sen, Subhojit Ghosh, Monojit Bhattacharya, Somnath Hazra, Annapurna Banik and Debmalya Sain for all the great moments that we had here together.

My heart-felt thanks go to Priya, Pousali, Deepika, Deblina and Sahil for being there for me as always. I am really privileged to have them in my life.

I must acknowledge Indian Institute of Science (IISc) and the National Board for Higher Mathematics (NBHM) for providing me the funding and fellowship to pursue my research at IISc.

Last but not the least, I would like to thank my family members for their love, patience and continuous support.

Abstract

Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a d- tuple of commuting operators on a Hilbert space $\mathscr{H}$. Assume that $\boldsymbol{T}$ is hyponormal, that is, $[\![\boldsymbol{T}^*, \boldsymbol{T}]\!] := (\!(\,[T_j^*, T_i]\,)\!)$ acting on the d - fold direct sum of the Hilbert space $\mathscr{H}$ is non-negative definite. The commutator $[T_j^*, T_i]$, $1 \le i, j \le d$, of a finitely ctyclic and hyponormal d - tuple is not necessarily compact and therefore the question of finding trace inequalities for such a d- tuple does not arise.

A generalization of the Berger-Shaw theorem for a commuting tuple $\boldsymbol{T}$ of hyponormal operators was obtained by Douglas and Yan decades ago. We discuss several examples of this generalization in an attempt to understand if the crucial hypothesis in their theorem requiring the Krull dimension of the Hilbert module over the polynomial ring defined by the map $p \to p(\boldsymbol{T})$, $p \in \mathbb{C}[\boldsymbol{z}]$, is optimal or not. Indeed, we find examples $\boldsymbol{T}$ to show that there is a large class of operators for which trace $[T_j^*, T_i]$, $1 \le j, i \le d$, is finite but the d - tuple is not finitely polynomially cyclic, which is one of the hypotheses of the Douglas-Yan theorem. We also introduce the weaker notion of "projectively hyponormal operators" and show that the Douglas-Yan thorem remains valid even under this weaker hypothesis.

We introduce the determinant operator $\mathrm{dEt}\,([\![\boldsymbol{T}^*, \boldsymbol{T}]\!])$, which coincides with the generalized commutator introduced by Helton and Howe earlier. We identify a class $BS_{m,\vartheta}(\Omega)$ consisting of commuting d- tuples of hyponormal operators $\boldsymbol{T}$, $\sigma(\boldsymbol{T}) = \overline{\Omega}$, satisfying a growth condition for which the dEt is a non-negative definite operator. We then obtain the trace estimate given in the Theorem below.

Theorem. *Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a commuting tuple of operators on a Hilbert space $\mathscr{H}$ such that $\boldsymbol{T}$ is in the class $BS_{m,\vartheta}(\Omega)$. Then the determinant operator dEt$\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$ is in trace-class and*

$$\mathit{trace}\Big(\mathit{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)\Big) \le m\,\vartheta\, d! \prod_{i=1}^{d} \|T_i\|^2.$$

In the case of a commuting d - tuple $\boldsymbol{T}$ of operators, where $\sigma(\boldsymbol{T})$ is of the form $\overline{\Omega}_1 \times \cdots \times \overline{\Omega}_d$, we obtain a slightly different but a related estimate for the trace of $\mathrm{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$. Explicit computation of $\mathrm{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$ in several examples and based on some numerical evidence, we make the following conjecture refining the estimate from the Theorem:

Conjecture. Let $\boldsymbol{T} = (T_1, \dots, T_d)$ be a commuting tuple of operators on a Hilbert space $\mathscr{H}$ such that $\boldsymbol{T}$ is in the class $BS_{m,\vartheta}(\Omega)$. Then the determinant operator $\mathrm{dEt}\big(\,[\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$ is in trace-class, and

$$\mathrm{trace}\big(\mathrm{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)\big) \le \frac{md!}{\pi^d} \nu(\overline{\Omega}),$$

where ν is the Lebesgue measure.

Let Ω be an irreducible classical bounded symmetric domain of rank r in $\mathbb{C}^d$. Let $\mathbb{K}$ be the maximal compact subgroup of the identity component G of the biholomorphic automorphism group of the domain Ω. The group $\mathbb{K}$ consisting of linear transformations acts naturally on any d-tuple $\boldsymbol{T}$ of commuting bounded linear operators by the rule:

$$k \cdot \boldsymbol{T} := \big(k_1(T_1, \dots, T_d), \dots, k_d(T_1, \dots, T_d)\big),\ k \in \mathbb{K},$$

where $k_1(\boldsymbol{z}), \dots, k_d(\boldsymbol{z})$ are linear polynomials. If the orbit of this action modulo unitary equivalence is a singleton, then we say that $\boldsymbol{T}$ is $\mathbb{K}$-homogeneous. We realize a certain class of $\mathbb{K}$-homogeneous d-tuples $\boldsymbol{T}$ as a d -tuple of multiplication by the coordinate functions $z_1, \dots, z_d$ on a reproducing kernel Hilbert space $\mathscr{H}_K$. (The Hilbert space $\mathscr{H}_K$ consisting of holomorphic functions defined on Ω and K is the reproducing kernel.) Using this model we obtain a criterion for (i) boundedness, (ii) membership in the Cowen-Douglas class, (iii) unitary equivalence and similarity of these d-tuples. In particular, we show that the adjoint of the d-tuple of multiplication by the coordinate functions on the weighted Bergman spaces are in the Cowen-Douglas class $B_1(\Omega)$. For an irreducible bounded symmetric domain Ω of rank 2, an explicit description of the operator $\sum_{i=1}^d T_i^* T_i$ is given. Based on this formula, a conjecture giving the form of this operator in any rank $r \ge 1$ was made. This conjecture was recently verified by H. Upmeier.

Conventions and Notations

The following conventions and notations will be in force throughout.

1. All Hilbert spaces will be assumed to be complex and separable.
2. An operator T, unless otherwise specified, will be assumed to be linear and bounded.
3. The algebra of bounded linear operators is denoted by $\mathscr{B}(\mathscr{H})$.
4. The symbol $\boldsymbol{T}$ denotes a commuting d - tupe of operators $(T_1,\dots,T_d)$.
5. For brevity, we write, hyponormal d - tuple for *jointly* hyponormal d - tuple of commuting operators.
6. $\mathbb{N}_0$ denotes the set of non-negative integers.
7. $\boldsymbol{\alpha}\in\mathbb{N}_0^d$ is called a multi-index.
8. $\boldsymbol{\alpha}+\epsilon_i := (\alpha_1,\dots,\alpha_i+1,\dots,\alpha_d)$
9. $\boldsymbol{\alpha}-\epsilon_i := (\alpha_1,\dots,\alpha_i-1,\dots,\alpha_d)$ whenever $\alpha_i>0$.
10. $|\boldsymbol{\alpha}| := \alpha_1+\cdots+\alpha_d$ and $\boldsymbol{\alpha}! = \alpha_1!\cdots\alpha_d!$.
11. We will use the term “finitely polynomially cyclic”, "m - cyclic" or even “m - polynomially cyclic" interchangeably, while “cyclic” always means 1 - cyclic.
12. An r- tuple $\underline{s}=(s_1,\dots,s_r)$, $s_1\geq\cdots\geq s_r\geq 0$ stands for a signature.
13. $\overrightarrow{\mathbb{N}}_0^r$ denotes the set of all signatures.
14. $\boldsymbol{M}=(M_1,\dots,M_d)$ is the commuting d- tuple of the operator of multiplications by the co-ordinate functions.
15. $\sigma(\boldsymbol{T})$ denotes the Taylor joint spectrum. However, the term spectrum, or joint spectrum would always mean the Taylor joint spectrum.

16. $\mathbb{C}[\boldsymbol{z}]$ is the ring of complex polynomials in d variables.

17. $\|\cdot\|_1$ is the trace norm.

18. $(\lambda)_n := \lambda(\lambda+1)\cdots(\lambda+n-1)$ is the Pochhammer symbol.

19. $(\lambda)_{\underline{s}} := \prod_{j=1}^{r}\left(\lambda - \frac{a}{2}(j-1)\right)_{s_j} = \prod_{j=1}^{r}\prod_{l=1}^{s_j}\left(\lambda - \frac{a}{2}(j-1) + l - 1\right)$ is the generalized Pochhammer symbol.

Contents

Chapter 1

Introduction

The spectral theorem provides both a complete set of unitary invariants and a canonical model for normal operators on a complex separable Hilbert space. Therefore, it is natural to ask if there are operators which one may study using techniques developed for studying normal operators. It is natural to expect that operators close to the class of normal operators may be somewhat more tractable by these methods than an arbitrary operator. Several notions of "close" have been around in the literature. For instance, (i) an operator T such that the commutator $[N^* - T^*, N - T] = K$ for some compact operator K, the quantitative version of this would be to require that K is in a Schatten p - class (ii) the hyponormal operators, namely, an operator T such that the commutator $[T^*, T]$ is a non-negative operator, (iii) the subnormal operators, the operators which are obtained by restricting a normal operator to an invariant subspace, etc. The study of each of these classes of operators has been vigorous over the past few decades. An impressive body of results have been accumulated. However, the study of commuting tuples of operators in each of these classes remains widely open. Among many other results from operator theory in one variable, the Berger-Shaw theorem has provided the impetus for a number of exciting developments in multi-variate operator theory [7]. Indeed, it shows that an operator T that is hyponormal and m-cyclic is close to a normal operator, that is,

$$\text{trace}[T^*, T] \leq \tfrac{m}{\pi}\text{Area}(\sigma(T)),$$

where $\sigma(T)$ is the spectrum of the operator T. Thus a very large class of operators is close to the class of normal operators. In this thesis, first we obtain trace inequalities similar to the one of Berger and Shaw for a certain small class of commuting tuples of operators. However, there are several instances, where all of the commutators $[T_j^*, T_i]$, $1 \leq i, j \leq d$, of a commuting tuple $\boldsymbol{T} := (T_1, \ldots, T_d)$ are not necessarily compact even after assuming that they are jointly hyponormal and cyclic. So, the question of finding trace inequalities for such d- tuples does not arise. However, we define an operator valued determinant of a $d \times d$- block operator

$\boldsymbol{B} := \big(\!\big(B_{ij}\big)\!\big)$ by the formula

$$\mathrm{dEt}\big(\boldsymbol{B}\big) := \sum_{\sigma,\tau\in\mathfrak{S}_d} \mathrm{sgn}(\sigma) B_{\tau(1),\sigma(\tau(1))} B_{\tau(2),\sigma(\tau(2))}, \ldots, B_{\tau(n),\sigma(\tau(d))}.$$

In this thesis we investigate the properties of the operator $\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$, where $[\![\boldsymbol{T}^*,\boldsymbol{T}]\!] := \big(\!\big([T_j^*, T_i]\big)\!\big)$, i.e., in this case, $B_{ij} = [T_j^*, T_i]$. We show that the operator $\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$ equals the generalized commutator of $\boldsymbol{T}$ introduced earlier by Helton and Howe [24]. Among other things, we find a trace inequality for the operator $\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$, after imposing certain growth and cyclicity conditions on the operator $\boldsymbol{T}$. We give explicit examples illustrating the abstract inequality and show that it is sharp in a number of examples.

In this introductory chapter, we first recall some definitions, namely that of operators in the trace class (Schatten p -class) and jointly hyponormal operators and its multiplicity followed by that of a Hilbert modules over a ring and finally the list of classical bounded symmetric domains. Since we use some of the basic tools from commutative algebra, we recall them here. This includes the notion of a Noetherian ring, the Krull dimension and the Noether normalization lemma. Our search for explicit examples, where the conditions of our main theorem are met and we obtain a trace inequality for the determinant operator $\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$, takes us to the class of the d- tuple of multiplication operators M_{z_i} on the weighted Bergman space of a bounded symmetric domain. We therefore recall some of the basic tools like the Peter-Weyl decomposition, Fischer-Fock inner product, the Wallach set etc. that will be essential to some of the proofs. Finally, the chapter ends with a brief description of the results in each of the following chapters.

1.1 Trace, Multiplicity and Hyponormal Operators

Let us recall the definition of trace of a bounded linear operator defined on a complex separable Hilbert space.

Definition 1.1 (Trace-class)**.** An operator $T : \mathscr{H} \to \mathscr{H}$ is said to be in trace-class if there is an orthonormal basis $\{e_n\}$ such that $\sum_n \langle |T| e_n, e_n\rangle < \infty$, where $|T|$ is the unique square root of T^*T. The trace norm of an operator T in the trace class is set to be

$$\|T\|_1 = \sum_n \langle |T| e_n, e_n\rangle,$$

where the sum $\sum_n \langle |T| e_n, e_n\rangle$ is independent of the choice of basis. The p - norm of an operator T is defined, similarly, by setting

$$\|T\|_p^p = \sum_n \langle |T|^p e_n, e_n\rangle.$$

An operator T with $\|T\|_p < \infty$ is said to be in the Schatten p - class.

Definition 1.2 (m- cyclic)**.** Let $\boldsymbol{\xi}(m)$ denote a set of linearly independent vectors $\xi_1,\ldots,\xi_m$ in $\mathscr{H}$. For a commuting tuple of operators $\boldsymbol{T} = (T_1,\ldots,T_d)$, we say that $\boldsymbol{\xi}(k)$ is cyclic for $\boldsymbol{T}$ if the linear span of the vectors

$$\left\{T_1^{i_1}T_2^{i_2}\ldots T_d^{i_d}v|\ v\in\boldsymbol{\xi}(k)\text{ and } i_1,i_2,\ldots,i_d\geq 0\right\}$$

is dense in $\mathscr{H}$. The commuting tuple $\boldsymbol{T}$ is said to be m- *polynomially cyclic* if

$$m = \min\{k : \boldsymbol{\xi}(k)\text{ is cyclic for }\boldsymbol{T}\}.$$

The set $\boldsymbol{\xi}_{\boldsymbol{T}}(m)$ is then said to be m-cyclic for $\boldsymbol{T}$. For a m-cyclic d- tuple $\boldsymbol{T}$, let

$$\mathscr{H}_N := \bigvee\left\{T_1^{i_1}T_2^{i_2}\ldots T_d^{i_d}v|\ v\in\boldsymbol{\xi}[m]\text{ and } 0\leq i_1+i_2+\ldots i_d\leq N\right\}$$

and P_N be the orthogonal projection onto $\mathscr{H}_N$.

In what follows, we will use the term “finitely polynomially cyclic”, "m - cyclic" or even “m - polynomially cyclic" interchangeably, while “cyclic” always means 1 - cyclic.

For any two operators T_1 and T_2 on a Hilbert space $\mathscr{H}$, the commutator $[T_1,T_2]$ of T_1 and T_2 is $T_1T_2 - T_2T_1$. An operator T on a Hilbert space $\mathscr{H}$ is said to be *hyponormal* if the commutator $[T^*,T]$ is non-negative definite. An operator T is *pure* if it has no reducing subspace $\mathscr{H}_0$ such that T restricted to $\mathscr{H}_0$ is normal. A bounded operator T on $\mathscr{H}$ is said to be subnormal if T has a normal extension. There are many possible notions of hyponormality for a d- tuple $\boldsymbol{T} = (T_1,\ldots,T_d)$ of commuting operators acting on a Hilbert space $\mathscr{H}$.

For instance, a commuting d- tuple $\boldsymbol{T}$ of operators acting on a Hilbert space $\mathscr{H}$ is said to be *jointly hyponormal* if

$$[\![\boldsymbol{T}^*,\boldsymbol{T}]\!] := \big(\!\big([T_j^*,T_i]\big)\!\big)_{i,j=1}^d : \bigoplus_d\mathscr{H}\longrightarrow\bigoplus_d\mathscr{H}$$

is non-negative definite, that is, for each $x\in\bigoplus_d\mathscr{H}$,

$$\left\langle[\![\boldsymbol{T}^*,\boldsymbol{T}]\!]x,x\right\rangle\geq 0.$$

Or, equivalently, if

$$\big(\!\big(T_j^*T_i\big)\!\big)_{i,j=0}^d : \bigoplus_{d+1}\mathscr{H}\longrightarrow\bigoplus_{d+1}\mathscr{H},$$

where $T_0 := I$, is non-negative definite (see [31]).

On the other hand, a commuting d- tuple $\boldsymbol{T}$ of operators acting on a Hilbert space $\mathscr{H}$ is said to be *projectively hyponormal* if, for each vector $(\alpha_1,\ldots,\alpha_d)\in\mathbb{C}^d\setminus\{0\}$, the sum $\sum_{i=1}^d\alpha_iT_i$ is a hyponormal operator on $\mathscr{H}$. In the literature, these have been called weakly hyponormal operators. However, the adjective "projective" describes better the way this class is related to the class of jointly hyponormal operators.

Remark 1.3. For $1 \le i, j \le d$, let $B_{ij} : \mathscr{H} \to \mathscr{H}$ be a bounded linear operator. Define

$$(\!(B_{ij})\!) : \mathscr{H} \otimes \mathbb{C}^d \longrightarrow \mathscr{H} \otimes \mathbb{C}^d$$

to be the operator

$$(\!(B_{ij})\!) x \otimes \alpha = (\!(B_{ij} x)\!) \cdot \alpha := \sum_{i=1}^{d} \big(\sum_{j=1}^{d} \alpha_j B_{ij} x \otimes e_i \big).$$

If $\langle (\!(B_{ij})\!) x \otimes \alpha, x \otimes \alpha \rangle \ge 0$ for every vector $x \otimes \alpha$ in $\mathscr{H} \otimes \mathbb{C}^d$, then we say that the operator $(\!(B_{ij})\!)$ is projectively positive.

Now, it easy to verify that a tuple of operators $\boldsymbol{T} = (T_1, \ldots, T_d)$ is projectively hyponormal if and only if $[\![\boldsymbol{T}^*, \boldsymbol{T}]\!]$ is projectively positive on $\mathscr{H} \otimes \mathbb{C}^d$.

It is evident that a hyponormal tuple is automatically projectively hyponormal. But the converse is not true in general.

Example 1.4. (R.Curto) For $x \ge 0$ let T_x be the weighted shift whose weight sequence is given by $\alpha_0 = x, \alpha_n = \sqrt{\frac{n+1}{n+2}}\,(n \ge 0)$. If $\frac{3}{4} \le x \le \sqrt{\frac{2}{3}}$ then (T_x, T_x^2) is projectively hyponormal but not hyponormal.

For a single hyponormal operator T, one of the most celebrated results is the trace inequality due to Berger and Shaw reproduced below.

Theorem 1.5 (Berger-Shaw, [7])**.** *If T is an m- cyclic hyponormal operator, then $[T^*, T]$ is in trace-class and*

$$\mathit{trace}[T^*, T] \le \frac{m}{\pi} \nu\big(\sigma(T)\big),$$

where $\sigma(T)$ is the spectrum of T and ν is the Lebesgue measure.

In what follows, unless explicitly indicated otherwise, we drop the adjective "jointly" and say "hyponormal" instead of jointly hyponormal.

1.2 Preliminaries on Hilbert modules

A Hilbert module $\mathscr{H}$ over a ring $\mathscr{R}$ is an ordinary module over the ring $\mathscr{R}$ except that $\mathscr{H}$ is a Hilbert space and the module multiplication is assumed to be continuous in the second variable. This notion was introduced by R. G. Douglas, see [17]. However, the original definition required the ring to be a complete normed algebra and the module multiplication to be continuous in both variables. Over the years, it has become apparent that these additional requirements are of no significant value. So, they are not included in the definition of a Hilbert module. Now, we give the formal definition of a Hilbert module over a normed ring.

Definition 1.6 (Hilbert Module)**.** A complex separable Hilbert space $\mathscr{H}$ is said to be a Hilbert module over a (complex) unital ring $\mathscr{R}$ if there exists a map $(p,h) \mapsto p \cdot h$ from $\mathscr{R} \times \mathscr{H}$ to $\mathscr{H}$ satisfying the following conditions:

1. $1 \cdot h = h$,
2. $(pq) \cdot h = p \cdot (q \cdot h)$,
3. $(p+q) \cdot h = p \cdot h + q \cdot h$, and
4. $p \cdot (\alpha h_1 + \beta h_2) = \alpha(p \cdot h_1) + \beta(p \cdot h_2)$.

where $p, q \in \mathscr{R}, h, h_i \in \mathscr{H}$, $i = 1,2$, and $\alpha, \beta \in \mathbb{C}$ and the linear operator $m_p : \mathscr{H} \mapsto \mathscr{H}$, $m_p(h) = p \cdot h$, $p \in \mathscr{R}$, is bounded.

Thus the notion of a Hilbert module $\mathscr{H}$ over a ring $\mathscr{R}$ is determined by the unital homomorphism $p \to m_p$, $p \in \mathscr{R}$, from the ring $\mathscr{R}$ to the algebra of bounded linear operators $\mathscr{B}(\mathscr{H})$ and conversely.

Apart from the boundedness of the operator m_p for each $p \in \mathscr{R}$, if we also require that the ring $\mathscr{R}$ complete with respect to some Banach algebra norm and impose the condition that the map

$$(p,h) \to p \cdot h, \; p \in \mathscr{R}, \; h \in \mathscr{H},$$

is continuous in both the variables, then from the uniform boundedness Principle, it follows that

$$\|m_p\| \le K\|p\|, \; p \in \mathscr{R},$$

for some constant K independent of p. Thus continuity in both the variables of the module map is equivalent to the boundedness of the homomorphism $p \to m_p$.

Now, we remark that if the ring $\mathscr{R}$ is the polynomial ring $\mathbb{C}[\boldsymbol{z}]$ in d variables, then any homomorphism $m_p : \mathbb{C}[\boldsymbol{z}] \to \mathscr{B}(\mathscr{H})$ is evidently determined by a commuting tuple $\boldsymbol{T} := (T_1, \dots T_d)$ of operators in $\mathscr{B}(\mathscr{H})$ by setting $m_p(h) := p(\boldsymbol{T})h$, $p \in \mathbb{C}[\boldsymbol{z}]$ and $h \in \mathscr{H}$. The other way round, given a homomomorphism $m_p : \mathbb{C}[\boldsymbol{z}] \to \mathscr{B}(\mathscr{H})$, it determines a d- tuple of commuting operators, namely, $T_i := m_{z_i}$, $i = 1, \dots, d$. We will therefore use the three notions, i.e., the Hilbert module $\mathscr{H}$, the homomorphism m_p and the d- tuple of commuting operators without making any distinction depending on the context.

If $\mathscr{H}$ is a Hilbert module over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$, then $\boldsymbol{\xi}(m)$ is said to be a *generating set* for $\mathscr{H}$. According to [17, Definition 1.25], the minimal cardinality of this generating set is defined to be the *rank* of the module $\mathscr{H}$, which coincides with the notion of m- polynomial cyclicity as in Definition 1.2.

Now, we recall some basic definitions from commutative algebra.

Definition 1.7 (Algebra)**.** Let $f : \mathscr{A} \to \mathscr{B}$ be a ring homomorphism. The scalar multiplication,

$$ab = f(a)b,\ a \in \mathscr{A} \text{ and } b \in \mathscr{B},$$

makes the ring $\mathscr{B}$ into an $\mathscr{A}$- module. Thus $\mathscr{B}$ has an (compatible) $\mathscr{A}$- module structure along with a ring structure. The ring equipped with this $\mathscr{A}$- module structure, is said to be an $\mathscr{A}$-algebra.

Definition 1.8 (Noetherian ring)**.** A ring $\mathscr{R}$ is said to be Noetherian if it satisfies the following three equivalent conditions:

(i) Every non-empty set of ideals in $\mathscr{R}$ has a maximal element.

(ii) Every assending chain of ideals in $\mathscr{R}$ is stationary.

(iii) Every ideal in $\mathscr{R}$ is finitely generated.

Definition 1.9 (Krull dimension)**.** The Krull dimension of a ring $\mathscr{R}$ is defined to be the maximum of those positive integers n for which there is an ascending chain of prime ideals of the form

$$\{0\} = P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n \subsetneq \mathscr{R}.$$

For example, $\mathbb{C}[z_1, \ldots, z_n]$ has Krull dimension n. A chain of prime ideals in $\mathbb{C}[z_1, \ldots, z_n]$ of length n is easy to exhibit:

$$(0) \subsetneq (z_1) \subsetneq (z_1, z_2) \subsetneq \cdots \subsetneq (z_1, \ldots, z_n) \subsetneq \mathbb{C}[z_1, \ldots, z_n].$$

One can show that it is maximal [20].

The *height* (or *co-dimension*) of a prime ideal P in $\mathscr{R}$ is the maximum length of the chain of prime ideals such that

$$P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n \subseteq P.$$

Height of an ideal I is the infimum of the heights of all prime ideals containing I.

Lemma 1.10 (Ideals in quotient rings)**.** *Let I be an ideal in a ring $\mathscr{R}$. Then contraction and extension by the quotient map $\phi : \mathscr{R} \to \mathscr{R}/I$ give a one to one correspondence between ideals in $\mathscr{R}/I$ and ideals containing I in $\mathscr{R}$:*

$$\{\text{ideals in } \mathscr{R}/I\} \longleftrightarrow \{\text{ideals containing } I \text{ in } \mathscr{R}\}$$
$$J \longrightarrow J^c = (\phi)^{-1}(J)$$
$$\phi(J) = J^e \longleftarrow J.$$

If I is an ideal in an integral domain $\mathscr{R}$, then $\dim(\mathscr{R}/I) + \text{height}(I) = \dim \mathscr{R}$.

Theorem 1.11 (Noether's normalization theorem)**.** *Let $\mathscr{R}$ be a Noetherian ring over $\mathbb{C}$ having dim$(\mathscr{R}) = n$ with generators $\{x_1,\ldots,x_m\}$.*

1. *Then there exists a complex $n \times m$ matrix $((a_{ij}))$ such that $\mathscr{R}$ is integral over $\mathbb{C}[\sum_{j=1}^m a_{1j}x_j,\ldots,\sum_{j=1}^m a_{nj}x_j]$.*

2. *If E is the linear span in $\mathbb{C}^m$ of the row vectors of the $n \times m$ matrix $((a_{ij}))$ such that $\mathscr{R}$ is integral over $\mathbb{C}[\sum_{j=1}^m a_{1j}x_j,\ldots,\sum_{j=1}^m a_{nj}x_j]$, then dim $(E) = m$.*

Further, if $\mathscr{R}$ is the co-ordinate ring $\mathbb{C}[z_1,\ldots,z_d]/I$, where $\sqrt{I} = I(U)$ and U is an algebraic curve, then there exists $((a_{ij}))_{i=1,j=1}^{k,m}$ such that if $g_i = \sum_{j=1}^m a_{ij}z_j$, $i = 1,\ldots,k$, then

1. *$\mathbb{C}[z_1,\ldots,z_d]$ is integral over $\mathbb{C}[I,g_i]$ and*

2. *each z_j is a linear combination of the g_i's.*

It is easy to adapt the definition of Shatten p - class and hyponornormality to a Hilbert module defined over a commutative Banach algbra.

Definition 1.12. Let $\mathscr{A}$ be a commutative Banach algebra containing a dense subalgebra $\mathscr{B}$. A Hilbert module $\mathscr{H}$ over $\mathscr{A}$ is said to be p-reductive for $\mathscr{B}$ if $[m_{b_1}^*, m_{b_2}]$ is in the Schatten p-class S_p for every $b_1, b_2 \in \mathscr{B}$.

Definition 1.13. Let $\mathscr{H}$ be a Hilbert module over a Banach algebra $\mathscr{A}$. Assume there is a dense Noetherian sub-algebra $\mathscr{B}$ of $\mathscr{A}$, that $\{b_1,b_2,\ldots,b_n\}$ generates $\mathscr{B}$ and that the commuting tuple of operators $(m_{b_1}, m_{b_2},\ldots,m_{b_n})$ on $\mathscr{H}$ is hyponormal (resp. subnormal). Then the Hilbert module $\mathscr{H}$ is said to be a hyponormal (resp. subnormal) module over $\mathscr{A}$.

We are now ready to state the Theorem of Douglas and Yan which was an attempt to find a generalization of the Berger-Shaw theorem for a commuting d- tuple of operators.

Theorem 1.14 (Douglas-Yan)**. (Module):** *Let $\mathscr{H}$ be a finitely generated hyponormal Hilbert module over $\mathscr{A}$ with the dense subalgebra $\mathscr{B}$. If dim$\mathscr{B} = 1$, then $\mathscr{H}$ is 1-reductive for $\mathscr{B}$.*

(d- tuple): *Let $\boldsymbol{T} = (T_1,\ldots,T_d)$ be a hyponormal d- tuple of operators on the Hilbert space $\mathscr{H}$ such that $\boldsymbol{T}$ is finitely polynomially cyclic. Let I be the ideal of $\mathbb{C}[z_1,\ldots,z_d]$ defined by*

$$I = \{p \in \mathbb{C}[z_1,\ldots,z_d] : p(\boldsymbol{T}) = 0\}.$$

If dim$\big(\mathbb{C}[z_1,\ldots,z_d]/I\big) = 1$, then $[T_j^*, T_i]$ is in trace-class for all i, j.

Another important topic that provides impetus for the results in Chapter 3 is the Arveson-Douglas Conjecture which we now describe briefly. First, we recall the definition of the weighted Bergman spaces.

Let $\mathscr{H}^{(\lambda)}(\mathbb{B}_d)$, $\lambda > 0$, be the weighted Bergman spaces of the unit Euclidean ball $\mathbb{B}_d$. In particular, $\lambda = d$ is the Hardy space $\mathscr{H}^2(\mathbb{B}_d)$. These spaces are determined by the orthonormal set of vectors:

$$\{c_{\boldsymbol{\alpha}}^{(\lambda)} z_1^{\alpha_1} \cdots z_d^{\alpha_d} : \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d\},$$

where $c_{\boldsymbol{\alpha}}^{(\lambda)} = \frac{(\lambda)_{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!}$. Here

$$(\lambda)_n := \lambda(\lambda+1)\cdots(\lambda+n-1)$$

is the Pochhammer symbol.

Conjecture 1.15 (Arveson-Douglas). Assume I is a homogeneous ideal of the polynomial ring $\mathbb{C}[z_1, \dots, z_d]$ and $[I]$ is the closure of I in $\mathscr{H}^{(\lambda)}(\mathbb{B}_d)$. Then for all $r > \dim \mathcal{Z}(I)$, where $\mathcal{Z}(I)$ is the common zero set of I, the quotient module $[I]^\perp$ is r-essentially normal.

In many cases, $[I] = \{f \in \mathscr{H}^{(\lambda)}(\mathbb{B}_d) : f_{|\mathcal{Z}(I)} = 0\}$. Assuming this is the case, Q is the closed linear span of the vectors $\{K^{(\lambda)}(z, w) : w \in \mathcal{Z}(I)\}$, where $K^{(\lambda)}(z, w)$, for every fixed $w \in \mathbb{B}_d$, is the unique vector in $\mathscr{H}^{(\lambda)}(\mathbb{B}_d)$ determined by evaluation at w. It is not hard to verify that the commutators $[M_j^*, M_i]$, $1 \le i \le d$, on the weighted Bergman spaces $\mathscr{H}^{(\lambda)}(\mathbb{B}_d)$, $\lambda \ge 1$, are in Schatten p - class for $p > d + \epsilon, \epsilon > 0$. It follows that the Arveson-Douglas conjecture is true for the trivial ideal $\{0\}$.

1.3 Bounded Symmetric Domains

Bounded symmetric domains are the natural generalization of the open unit disc in one complex variable and the open Euclidean unit ball in several complex variables. A bounded domain $\Omega \subset \mathbb{C}^d$ is said to be *symmetric* if for every $z \in \Omega$, there exists a biholomorphic automorphism of Ω of period two, having z as isolated fixed point. The domain Ω is said to be *irreducible* if it is not biholomorphically equivalent to a product of two non-trivial domains. We refer to [30], [1] for the definition and basic properties of bounded symmetric domains.

Let Ω be an irreducible bounded symmetric domain in $\mathbb{C}^d$ and let $\mathrm{Aut}(\Omega)$ denote the group of biholomorphic automorphisms of Ω, equipped with the topology of uniform convergence on compact subsets of Ω. Let G denote the connected component of the identity in $\mathrm{Aut}(\Omega)$. It is known that G acts transitively on Ω. Let $\mathbb{K}$ be the subgroup of linear automorphisms in G. By Cartan's theorem [37, Proposition 2, pp. 67], $\mathbb{K} = \{\phi \in G : \phi(0) = 0\}$ is a maximal compact subgroup of G and Ω is isomorphic to $G/\mathbb{K}$. Note that the unitary group $\mathscr{U}(d)$ is the subgroup of linear biholomorphic automorphisms of $\mathrm{Aut}(\mathbb{B}^{\mathrm{d}})$. Therefore, it is natural to replace $\mathscr{U}(d)$ with the subgroup $\mathbb{K}$ of linear biholomorphic automorphisms of an irreducible bounded symmetric domain Ω and study all commuting d- tuples $\boldsymbol{T}$ such that $k \cdot \boldsymbol{T}$ is unitarily equivalent to $\boldsymbol{T}$ for all $k \in \mathbb{K}$. The action of the group $\mathbb{K}$ on the d- tuples is defined below. The group $\mathbb{K}$ acts on Ω by the rule

$$k \cdot \boldsymbol{z} := \big(k_1(\boldsymbol{z}), \ldots, k_d(\boldsymbol{z})\big), \qquad k \in \mathbb{K} \text{ and } \boldsymbol{z} \in \Omega.$$

Note that $k_1(\boldsymbol{z}), \ldots, k_d(\boldsymbol{z})$ are linear polynomials. Thus $k \in \mathbb{K}$ acts on any commuting d- tuple of bounded linear operators $\boldsymbol{T} = (T_1, \ldots, T_d)$, defined on a complex separable Hilbert space $\mathscr{H}$, naturally, via the map

$$k \cdot \boldsymbol{T} := \big(k_1(T_1, \ldots, T_d), \ldots, k_d(T_1, \ldots, T_d)\big).$$

Definition 1.16. A d- tuple $\boldsymbol{T} = (T_1, \ldots, T_d)$ of commuting bounded linear operators on $\mathscr{H}$ is said to be $\mathbb{K}$-homogeneous if for all k in $\mathbb{K}$ the operators $\boldsymbol{T}$ and $k \cdot \boldsymbol{T}$ are unitarily equivalent, that is, for all k in $\mathbb{K}$ there exists a unitary operator $\Gamma(k)$ on $\mathscr{H}$ such that

$$T_j \Gamma(k) = \Gamma(k) k_j(T_1, \ldots, T_d), \qquad j = 1, 2, \ldots, d. \tag{1.1}$$

For brevity, we will write

$$\boldsymbol{T}\Gamma(k) = \Gamma(k)(k \cdot \boldsymbol{T}).$$

While a d- tuple of $\mathbb{K}$-homogeneous operators is clearly modeled after a spherical tuple, it is a much more intricate notion, in general. For instance, spherical tuples in the class $B_1(\mathbb{B}^d)$, introduced by Cowen and Douglas in the very influential paper [12], are necessarily joint weighted shifts. On the other hand, the structure of $\mathbb{K}$-homogeneous operator tuples in $B_1(\Omega)$, where Ω is a bounded symmetric domain of rank > 1, is much more complex. In particular, they are not joint weighted shifts. Also, recall that the commuting operator tuples $\boldsymbol{T} = (T_1, \ldots, T_d)$ such that $\boldsymbol{T}$ and $g(\boldsymbol{T})$ are unitarily equivalent for all g in G, called *homogeneous* tuples, have been studied extensively over the past few years, see [33], [35], [28]. In the case of open unit disc $\mathbb{D}$, all homogeneous operators in $B_1(\mathbb{D})$ were classified by Misra in [32]. As a corollary of his abstract classification theorem, Wilkins provided an explicit model for all homogeneous operators in $B_2(\mathbb{D})$, see [45]. Later in 2011, using techniques from complex geometry and representation theory, a complete classification of homogeneous operators in the Cowen-Douglas class $B_n(\mathbb{D})$ was obtained by Misra and Korányi in [27]. Homogeneous operators on an irreducible bounded symmetric domain of type I, discussed below, were studied by Misra and Bagchi in [6]. Later in [2], their results were generalized for an arbitrary irreducible bounded symmetric domain by Arazy and Zhang. A comparison of the class of d- tuples of homogeneous operators with $\mathbb{K}$-homogeneous operator tuples might reveal interesting connections with the inducing construction, which we intend to study in future.

Every irreducible bounded symmetric domain Ω of rank r can be realized as an open unit ball of a *Cartan factor* $Z = \mathbb{C}^d$. For a fixed frame $e_1, \cdots, e_r$ of pairwise orthogonal minimal tripotents, let

$$Z = \sum_{0\le i\le j\le r} Z_{ij}$$

be the *joint Peirce decomposition* of Z (see [44, pp. 57]). Note that $Z_{00} = \{0\}$ and $Z_{ii} = \mathbb{C}e_i$ for all $i = 1, \ldots, r$. Moreover,

$$a := \dim Z_{ij}, \qquad 1 \le i < j \le r$$

is independent of i, j and

$$b := \dim Z_{0j}, \qquad 1 \le j \le r$$

is independent of j. The parameters a, b are called the *characterstic multiplicities* of Z and the numerical invariants (r, a, b) determine the domain Ω uniquely upto biholomorphic equivalence (see [1]). The dimension d is related to the numerical invariants (r, a, b) as follows:

$$d = r + \frac{a}{2} r(r-1) + rb.$$

According to the classification due to E. Cartan [9], there are six types of irreducible bounded symmetric domains upto biholomorphic equivalence (see also [30]). The first four types of these domains are called the classical Cartan domains, while other two types are known as the exceptional domains. In what follows, we consider only the classical domains, that is, an irreducible bounded symmetric domain of one of the following four types:

(i) **Type I** : $n \times m$ $(m \ge n)$ complex matrices $\boldsymbol{z}$ with $\|\boldsymbol{z}\| < 1$. These domains are determined by the numerical invariants $(n, 2, m-n)$.

(ii) **Type II** : Symmetric complex matrices $\boldsymbol{z}$ of order n with $\|\boldsymbol{z}\| < 1$. In this case, the numerical invariants $(n, 1, 0)$ are complete biholomorphic invariant.

(iii) **Type III** : $n \times n$ anti-symmetric complex matrices $\boldsymbol{z}$ of order n with $\|\boldsymbol{z}\| < 1$. Here $r = \left[\frac{n}{2}\right]$, $a = 4$ and $b = 0$ if n is even and $b = 2$ if n is odd.

(iv) **Type IV** (*The Lie ball*) : All $\boldsymbol{z} \in \mathbb{C}^d$ $(d \ge 5)$ such that $1 + |\frac{1}{2}\boldsymbol{z}^t\boldsymbol{z}|^2 > \overline{\boldsymbol{z}}^t\boldsymbol{z}$ and $\overline{\boldsymbol{z}}^t\boldsymbol{z} < 2$, where $\overline{\boldsymbol{z}}^t$ is the complex conjugate of the transpose $\boldsymbol{z}^t$. The numerical invariants $(2, d-2, 0)$ are complete biholomorphic invariant for these domains.

Throughout the thesis, let $\mathbb{N}_0$ denote the set of all non-negative integers. Let $\mathscr{P}$ be the space of all analytic polynomials on Z, and let $\mathscr{P}_n$, $n \in \mathbb{N}_0$, denote the subspace of $\mathscr{P}$ consisting of all homogeneous polynomials of degree n. Clearly, as a vector space, $\mathscr{P}$ can be written as the direct sum $\sum_{n=0}^{\infty} \mathscr{P}_n$. The group $\mathbb{K}$ acts on the space $\mathscr{P}$ by composition, that is, $(k \cdot p)(\boldsymbol{z})$

$= p(k^{-1}\boldsymbol{z}),\ k \in \mathbb{K},\ p \in \mathscr{P}$. Below we describe the irreducible components of this action. An r- tuple $\underline{s} = (s_1, \ldots, s_r)$ is called a *signature* if $s_1 \geq \cdots \geq s_r \geq 0$. Let $\overrightarrow{\mathbb{N}}_0^r$ denote the set of all signatures. For all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$, we associate the *conical polynomial* $\Delta_{\underline{s}}$, see [44, pp. 128] for the definition, where

$$\Delta_{\underline{s}}(\boldsymbol{z}) = \Delta_1^{s_1 - s_2}(\boldsymbol{z}) \ldots \Delta_{r-1}^{s_{r-1} - s_r}(\boldsymbol{z}) \Delta_r^{s_r}(\boldsymbol{z})$$

and the polynomial space $\mathscr{P}_{\underline{s}}$ is the linear span of $\{\Delta_{\underline{s}} \circ k : k \in \mathbb{K}\}$. It is known that the polynomial spaces $\{\mathscr{P}_{\underline{s}}\}_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r}$ are precisely the $\mathbb{K}$-invariant, irreducible subspaces of $\mathscr{P}$ which are mutually $\mathbb{K}$-inequivalent, and

$$\mathscr{P} = \sum_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r} \mathscr{P}_{\underline{s}}.$$

The Fischer-Fock inner product on $\mathscr{P}$, defined by $\langle p, q \rangle_{\mathscr{F}} := \frac{1}{\pi^d} \int_{\mathbb{C}^d} p(\boldsymbol{z}) \overline{q(\boldsymbol{z})} e^{-|\boldsymbol{z}|^2} dm(\boldsymbol{z})$, is $\mathbb{K}$-invariant. The reproducing kernel of the space $\mathscr{P}_{\underline{s}}$ with respect to the Fischer-Fock inner product is denoted by $K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w})$. Note that $K_{\underline{s}}$ is $\mathbb{K}$-invariant and

$$\sum_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w}) = e^{\boldsymbol{z} \cdot \overline{\boldsymbol{w}}}.$$

Further any $\mathbb{K}$-invariant Hilbert space $\mathscr{H}$ of analytic functions on Ω has the decomposition

$$\mathscr{H} = \oplus_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r} \mathscr{P}_{\underline{s}}.$$

This decomposition is called *Peter-Weyl decomposition* [43].

Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a commuting d- tuple of bounded linear operators acting on a complex separable Hilbert space $\mathscr{H}$. Also, let $D_{\boldsymbol{T}} : \mathscr{H} \to \mathscr{H} \oplus \cdots \oplus \mathscr{H}$ be the operator

$$D_{\boldsymbol{T}} h := (T_1 h, \ldots, T_d h), \qquad h \in \mathscr{H}.$$

We note that $\ker D_T = \cap_{i=1}^d \ker T_i$ is the *joint kernel* and $\sigma_p(\boldsymbol{T}) = \{\boldsymbol{w} \in \mathbb{C}^d : \ker D_{\boldsymbol{T} - \boldsymbol{w} I} \neq \boldsymbol{0}\}$ is the *joint point spectrum* of the d- tuple $\boldsymbol{T} = (T_1, \ldots, T_d)$. The *Wallach set* $\mathscr{W}(\Omega)$ of a classical bounded symmetric domain Ω is of the form $\mathscr{W}_d(\Omega) \cup \mathscr{W}_c(\Omega)$, where

$$\mathscr{W}_d(\Omega) := \left\{0, \frac{a}{2}, \ldots, \frac{a}{2}(r-1)\right\},\ \mathscr{W}_c(\Omega) := \left(\frac{a}{2}(r-1), \infty\right),$$

see [22]. For $\lambda > 0$ consider the function $K^{(\lambda)} : \Omega \times \Omega \to \mathbb{C}$ given by the formula

$$K^{(\lambda)}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\underline{s}} (\lambda)_{\underline{s}} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w}), \qquad \boldsymbol{z}, \boldsymbol{w} \in \Omega,$$

where $(\lambda)_{\underline{s}}$ is the generalized Pochhammer symbol

$$(\lambda)_{\underline{s}} := \prod_{j=1}^r \left(\lambda - \frac{a}{2}(j-1)\right)_{s_j} = \prod_{j=1}^r \prod_{l=1}^{s_j} \left(\lambda - \frac{a}{2}(j-1) + l - 1\right).$$

The function $K^{(\lambda)}$ is non-negative definite if and only if λ is in the Wallach set $\mathscr{W}(\Omega)$. Let $\mathscr{H}^{(\lambda)}$ denote the Hilbert space determined by the non-negative definite kernel $K^{(\lambda)}$, $\lambda \in \mathscr{W}(\Omega)$. If $\lambda = \frac{d}{r}$ and $\lambda = \frac{a}{2}(r-1) + \frac{d}{r}$, then the Hilbert spaces $\mathscr{H}^{(\lambda)}$ coincide with the Hardy space $H^2(S)$ over the *Shilov boundary* S of Ω and the classical Bergman space $\mathbb{A}^2(\Omega)$ respectively. For this reason with a slight abuse of language, the Hilbert spaces $\mathscr{H}^{(\lambda)}$, $\lambda \in \mathscr{W}(\Omega)$, are called weighted Bergman spaces.

1.4 Main Results

This thesis is in two parts. The first part is an attempt to obtain an inequality for the trace of a suitable function of a d- tuple of operators $\boldsymbol{T}$. The second part is a detailed study of commuting d- tuples $\boldsymbol{T}$ that remain unitarily invariant under the action of a natural compact group acting on the joint spectrum $\sigma(\boldsymbol{T})$. We briefly describe the results of the thesis below.

Chapter 2 begins with elementary observations about tensor products of jointly (respectively, projectively) hyponormal operators. It is then shown that the commutator $[[\boldsymbol{T}^*, \boldsymbol{T}]]$, where $\boldsymbol{T}$ is of the form $(A_1 \otimes T, \ldots, A_d \otimes T)$, belongs to the trace class as soon as certain natural conditions are imposed on the commuting tuple $\boldsymbol{A}$ and the operator T. Next, a mild generalization of the Douglas-Yan theorem by replacing the strongly hyponormal d- tuples with projectively hyponormal d- tuples is obtained. This is followed by several explicit examples of commuting d- tuples of operators satisfying the hypothesis of the Douglas-Yan theorem. The chapter concludes by showing that the joint spectrum of a d - tuple of operators of the form $(A_1 \otimes T, \ldots, A_d \otimes T)$ is the countable union of thin sets whenever $(A_1, \ldots, A_d)$ is a commuting tuple consisting of normal and compact operators. The trace of the commutator $[(A_j \otimes T)^*, (A_i \otimes T)]$, $1 \le i, j \le d$, is finite while these commuting tuples need not be finitely polynomially cyclic. Therefore finding a different set of conditions on a commuting tuple of operators ensuring the membership of the commutators in the trace class remains an intriguing problem.

A different approach is to look for a function of $[\![\boldsymbol{T}^*, \boldsymbol{T}]\!]$ which may be in trace class. For this, we define an operator valued determinant of a $d \times d$- block operator $\boldsymbol{B} := (\!(B_{ij})\!)_{i,j=1}^d$ by the formula

$$\mathrm{dEt}(\boldsymbol{B}) := \sum_{\sigma, \tau \in \mathfrak{S}_d} \mathrm{sgn}(\sigma) B_{\tau(1),\sigma(\tau(1))} B_{\tau(2),\sigma(\tau(2))} \ldots B_{\tau(d),\sigma(\tau(d))}.$$

Setting $B_{ij} = [T_j^*, T_i]$, it is natural to investigate the properties of the operator $\mathrm{dEt}([\![\boldsymbol{T}^*, \boldsymbol{T}]\!])$. Indeed, we show that the operator $\mathrm{dEt}([\![\boldsymbol{T}^*, \boldsymbol{T}]\!])$ equals the generalized commutator $\mathrm{GC}(\boldsymbol{T}^*, \boldsymbol{T})$ introduced earlier by Helton and Howe [24, pp. 272], who had investigated the trace properties of the operator $\mathrm{GC}(\boldsymbol{T}^*, \boldsymbol{T})$. Among other things, we find a trace inequality for the operator $\mathrm{dEt}([\![\boldsymbol{T}^*, \boldsymbol{T}]\!])$, after imposing certain growth and cyclicity condition on the operator $\boldsymbol{T}$. We give explicit examples illustrating the abstract inequality and show that it is sharp in a number of examples. In Chapter 3, we explicitly compute $\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$ and $\mathrm{trace}\big(\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)\big)$ for some class of operators. Moreover, the class $BS_{m,\vartheta}(\Omega)$ of commuting d- tuple of operators $\boldsymbol{T}$ is introduced. For $d=1$, this class of operators is clearly included in the class of finitely polynomially cyclic hyponormal operators and therefore the trace of the commutator $[T^*,T]$ is finite by the Berger-Shaw theorem. In this case, $[T^*,T]=\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$ and we show that the trace of the operator $\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$ is finite in general for an arbitrary $d\in\mathbb{N}$. We give several examples of d- tuples in $BS_{m,\vartheta}(\Omega)$ when Ω is a ball of the form $\mathbb{B}_{p,q}:=\{(z_1,z_2):|z_1|^p+|z_2|^q<1\}$ with $p=2,q=2$ and $p=2,q=1$.

Definition 1.17. Fix a bounded domain $\Omega\subset\mathbb{C}^d$ such that $\overline{\Omega}$ is polynomially convex. A m-cyclic commuting d- tuple of operators with $\sigma(\boldsymbol{T})=\overline{\Omega}$ is said to be in the class $BS_{m,\vartheta}(\Omega)$, if

(i) $P_N T_j P_N^{\perp}=0$, $j=1,\ldots,d$.

(ii) $\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$ is non-negative definite.

(iii) For a fixed but arbitrary τ in the permutation group $\mathfrak{S}_d$ of d symbols, there exists $\vartheta\in\mathbb{N}$, independent of N, such that

$$\|P_N\big(\sum_{\eta\in\mathfrak{S}_d}\mathrm{Sgn}(\eta)\,T^*_{\eta(1)}T_{\tau(1)}T^*_{\eta(2)}\ldots T^*_{\eta(d)}\big)P_N^{\perp}T_{\tau(d)}P_N\|\le\vartheta\binom{N+d-1}{d-1}^{-1}\prod_{i=1}^{d}\|T_i\|^2.$$

The main theorem of Chapter 3 provides an estimate for $\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$ in terms of the operator norms $\|T_i\|$, $1\le i\le d$:

Theorem 1.18. *Let $\boldsymbol{T}=(T_1,\ldots,T_d)$ be a commuting tuple of operators on a Hilbert space $\mathscr{H}$ such that $\boldsymbol{T}$ is in the class $BS_{m,\vartheta}(\Omega)$. Then the determinant operator $dEt\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$ is in trace-class and*

$$\mathit{trace}\big(\mathit{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)\big)\le m\,\vartheta\,d!\prod_{i=1}^{d}\|T_i\|^2.$$

This is followed by explicit computation of $\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$ for the commuting tuple of multiplication by the coordinate function on the weighted Bergman space of the Euclidean ball, the generalized ellipsoid and the symmetrized bidisc. Based on these examples and some numerical evidence, we make the following conjecture. The conjecture is a refinement of the previous estimate of the trace. It connects the trace of $\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$ to the volume of the joint spectrum of the d- tuple $\boldsymbol{T}$:

Conjecture 1.19. Suppose that $\boldsymbol{T}=(T_1,\ldots,T_d)$ is a commuting tuple of operators, acting on a Hilbert space $\mathscr{H}$, is in $BS_{m,\vartheta}(\Omega)$. Then

$$\mathrm{trace}\Big(\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)\Big)\le\frac{md!}{\pi^d}\nu(\overline{\Omega}),$$

where ν is the Lebesgue measure.

We show that the inequality in the conjecture is sharp in the class $BS_{1,1}(\mathbb{B}_{2,2})$. Also, setting $\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)} := (T_1^{(1)} \otimes I, \ldots, T_{d_1}^{(1)} \otimes I, I \otimes T_1^{(2)}, \ldots I \otimes T_{d_2}^{(2)})$, we have

Theorem 1.20. *Assume that $\boldsymbol{T}^{(i)}$ is in the class $BS_{m_i,1}(\Omega_i)$, $i = 1,2$. Then the determinant operator*

$$dEt\big(\big[\!\big[(\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})^*, (\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})\big]\!\big]\big)$$

is non-negative definite and

$$trace\Big(dEt\big(\big[\!\big[(\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})^*, (\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})\big]\!\big]\big)\Big) \leq 2 d_1! d_2! m_1 m_2 \prod_{i=1}^{d_1} \| T_i^{(1)} \|^2 \prod_{i=1}^{d_2} \| T_i^{(2)} \|^2.$$

Finally, for the polydisc $\mathbb{D}^d$, the inequality proved in Theorem 1.20 is sharp as is easily verified by taking the example of the d- tuple of multiplication by the coordinate functions on the Hardy space of $\mathbb{D}^d$.

The results of Chapter 4 and 5 are in [23]. In Chapter 4, we introduce the class $\mathscr{A}\mathbb{K}(\Omega)$ consisting of commuting d- tuples of $\mathbb{K}$ - homogeneous operators $\boldsymbol{T}$ possessing a number of additional properties:

Definition 1.21. A commuting d- tuple of $\mathbb{K}$-homogeneous operators $\boldsymbol{T}$ possessing the following properties

(i) $\dim \ker D_{\boldsymbol{T}^*} = 1$,

(ii) any non-zero vector e in $\ker D_{\boldsymbol{T}^*}$ is cyclic for $\boldsymbol{T}$,

(iii) $\Omega \subseteq \sigma_p(\boldsymbol{T}^*)$

is said to be in the class $\mathscr{A}\mathbb{K}(\Omega)$.

Now, for $\boldsymbol{T} \in \mathscr{A}\mathbb{K}(\Omega)$, which are necessarily $\mathbb{K}$-homogeneous, we provide a model as multiplication by the coordinate functions $z_1, \ldots, z_d$ on a reproducing kernel Hilbert space $\mathscr{H}_K$ of holomorphic functions defined on Ω. We describe the kernel K in terms of the $\mathbb{K}$-invariant kernels $K_{\underline{s}}$ of the spaces $\mathscr{P}_{\underline{s}}$.

Theorem 1.22. *If $\boldsymbol{T}$ is a d- tuple of operators in $\mathscr{A}\mathbb{K}(\Omega)$, then $\boldsymbol{T}$ is unitarily equivalent to a d- tuple $\boldsymbol{M} = (M_1, \ldots, M_d)$ of multiplication by the coordinate functions $z_1, \ldots, z_d$ on a reproducing kernel Hilbert space H_K of holomorphic functions defined on Ω with $K(\boldsymbol{z}, \boldsymbol{w}) = \sum \alpha_{\underline{s}}^{-1} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w})$, $\boldsymbol{z}, \boldsymbol{w} \in \Omega$, for some choice of positive real numbers $\alpha_{\underline{s}}$ with $\alpha_{\underline{0}} = 1$.*

Having described the model, we obtain a criterion for boundedness of these operators. Using this criterion, we determine which d- tuple of multiplication operators on the *weighted Bergman spaces* $\mathscr{H}^{(\lambda)}$ are bounded. The boundedness criterion for the multiplication operators on the weighted Bergman spaces has appeared before in [6] and [2]. For any $\boldsymbol{T}$ in the class $\mathscr{A}\mathbb{K}(\Omega)$, we point out that the operators $\sum_{i=1}^{d} T_i^* T_i$ and $\sum_{i=1}^{d} T_i T_i^*$ restricted to the subspace $\mathscr{P}_{\underline{s}}$ are scalar times the identity. In particular, for the weighted Bergman spaces $\mathscr{H}^{(\lambda)}$, [2, Proposition 4.4] provides an explicit form for the operator $\sum_{i=1}^{d} T_i T_i^*$. We extend this formula for any $\boldsymbol{T}$ in the class $\mathscr{A}\mathbb{K}(\Omega)$. We also obtain criterion for the adjoint of the d- tuple of operators in $\mathscr{A}\mathbb{K}(\Omega)$ to be in the Cowen-Douglas class $B_1(\Omega_0)$ for some neighbourhood $\Omega_0 \subset \Omega$ of $0 \in \Omega$. In case of weighted Bergman spaces $\mathscr{H}^{(\lambda)}$, we prove that the adjoint of the d- tuple of multiplication operators by the coordinate functions are in the Cowen-Douglas class $B_1(\Omega)$.

Throughout the PhD thesis, let $K^{(a)} : \Omega \times \Omega \to \mathbb{C}$ denote the kernel function given by the formula $K^{(a)}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\underline{s} \in \vec{\mathbb{N}}_0^r} a_{\underline{s}} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w})$, $\boldsymbol{z}, \boldsymbol{w} \in \Omega$, for some choice of positive real numbers $a_{\underline{s}}$. The positivity of the sequence $a_{\underline{s}}$ ensures that $K^{(a)}$ is a positive definite kernel. Thus it determines a unique Hilbert space $\mathscr{H}^{(a)} \subseteq \mathrm{Hol}(\Omega)$ with the reproducing property: $\langle f, K^{(a)}(\cdot, \boldsymbol{w}) \rangle = f(\boldsymbol{w})$, $f \in \mathscr{H}^{(a)}$, $\boldsymbol{w} \in \Omega$. From [1], it follows that the polynomial ring $\mathscr{P}$ is dense in $\mathscr{H}^{(a)}$ and $\mathscr{P}_{\underline{s}}$ is orthogonal to $\mathscr{P}_{\underline{s}'}$ whenever $\underline{s} \neq \underline{s}'$, that is, $\mathscr{H}^{(a)} = \oplus_{\underline{s} \in \vec{\mathbb{N}}_0^r} \mathscr{P}_{\underline{s}}$. It is then easy to see that the d- tuple of multiplication operators $\boldsymbol{M}^{(a)}$ on the reproducing kernel Hilbert space $\mathscr{H}^{(a)}$ are in the class $\mathscr{A}\mathbb{K}(\Omega)$. Finally, we study the question of unitary equivalence and similarity of d-tuples of operators in the class $\mathscr{A}\mathbb{K}(\Omega)$ using the model of d- tuple of multiplication operators $\boldsymbol{M}^{(a)}$ on the reproducing kernel Hilbert space $\mathscr{H}^{(a)}$.

Theorem 1.23. *Let $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ be two operator tuples in $\mathscr{A}\mathbb{K}(\Omega)$. Suppose that $\boldsymbol{T}_1 \sim_u \boldsymbol{M}^{(a)}$ and $\boldsymbol{T}_2 \sim_u \boldsymbol{M}^{(b)}$. Then the following statements are equivalent.*

(i) *$\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are unitarily equivalent.*

(ii) *$a_{\underline{s}} = b_{\underline{s}}$ for all $\underline{s} \in \vec{\mathbb{N}}_0^r$.*

(iii) *$K^{(a)} = K^{(b)}$.*

Theorem 1.24. *Let $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ be two operator tuples in $\mathscr{A}\mathbb{K}(\Omega)$. Suppose that $\boldsymbol{T}_1 \sim_u \boldsymbol{M}^{(a)}$ and $\boldsymbol{T}_2 \sim_u \boldsymbol{M}^{(b)}$. Then the following statements are equivalent.*

(i) *$\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are similar.*

(ii) *There exist constants $\alpha, \beta > 0$ such that $\alpha \|p\|_{\mathscr{H}^{(a)}} \leq \|p\|_{\mathscr{H}^{(b)}} \leq \beta \|p\|_{\mathscr{H}^{(a)}}$, $p \in \mathscr{P}$.*

(iii) *$\mathscr{H}^{(a)} = \mathscr{H}^{(b)}$.*

(iv) *There exist constants* $\alpha, \beta > 0$ *such that* $\alpha K^{(a)} \preceq K^{(b)} \preceq \beta K^{(a)}$.

(v) there exist constants $\alpha, \beta > 0$ *such that* $\alpha a_{\underline{s}} \leq b_{\underline{s}} \leq \beta a_{\underline{s}}$, $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$.

Corollary 1.25. *If* $\lambda_1, \lambda_2 > \frac{a}{2}(r-1)$, *then the* d *- tuple of multiplication operators* $\boldsymbol{M}^{(\lambda_1)}$ *on* $\mathscr{H}^{(\lambda_1)}$ *and* $\boldsymbol{M}^{(\lambda_2)}$ *on* $\mathscr{H}^{(\lambda_2)}$ *are similar if and only if* $\lambda_1 = \lambda_2$.

In Chapter 5, for the Hardy space $H^2(S_\Omega)$ of the Shilov boundary S_Ω of a classical bounded symmetric domain Ω, we show that $\sum_{i=1}^d M_i^* M_i$ is the rank times the identity. This is independently proved in [5]. Also, for any $\boldsymbol{T}$ in $\mathscr{A}\mathbb{K}(\Omega)$, we show that the commutators $[M_i^*, M_i]$, $i = 1, \ldots, d$ on the weighted Bergman spaces are compact if and only if $r = 1$. This follows from the explicit computation of the operator $\sum_{i=1}^d T_i^* T_i$ on certain subspaces of $\mathscr{H}^{(a)}$. For any classical bounded symmetric domain Ω of rank 2, an explicit description of the operator $\sum_{i=1}^d T_i^* T_i$ is found on all of $\mathscr{H}^{(a)}$. The computation involved in this description naturally lead to a conjecture, given below, independent of the rank of the domain. This conjecture was recently proved by Upmeier, see [42].

Conjecture 1.26. Let Ω be an irreducible bounded symmetric domain of rank r. Then, for any polynomial p in $\mathscr{P}_{\underline{s}}$, $\boldsymbol{M}^{(a)*}\boldsymbol{M}^{(a)} p = \delta(\underline{s}) p$ on the Hilbert space $\mathscr{H}^{(a)}$, $\delta(\underline{s})$ given by the formula

$$\delta(\underline{s}) = \sum_{j \in I^+(\underline{s})} \frac{a_{\underline{s}}}{a_{\underline{s}+\epsilon_j}} \frac{(\frac{d}{r})_{\underline{s}+\epsilon_j}}{(\frac{d}{r})_{\underline{s}}} c_{\underline{s}}(j) \tag{1.2}$$

where $I^+(\underline{s}) := \{j : 1 \leq j \leq r,\ \underline{s} + \epsilon_j \in \overrightarrow{\mathbb{N}}_0^r\}$ and $c_{\underline{s}}(j) = \prod_{k \neq j} \frac{s_j - s_k + \frac{a}{2}(k-j+1)}{s_j - s_k + \frac{a}{2}(k-j)}$, $j = 1, \ldots, r$.

Chapter 2

Multivariate Hyponormal tuples and Trace Theorems

Any attempt to generalize the Berger-Shaw inequality must take into account the behaviour of the pair of multiplication operators $\boldsymbol{M} := (M_{z_1}, M_{z_2})$ on the Hardy space $H^2(\mathbb{D}^2)$ of the bidisc $\mathbb{D}^2$. The pair $\boldsymbol{M}$ is hyponormal and 1- cyclic. However, an easy computation shows that $[M_{z_i}^*, M_{z_i}]$ is of the form $P \otimes I$ or $I \otimes P$ depending on whether $i = 1$ or $i = 2$, where P is a finite rank projection and I is the identity operator on the Hardy space $H^2(\mathbb{D})$ and $H^2(\mathbb{D}^2)$ is identified with $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$, see (3.3). It follows that neither of the commutators $[M_{z_i}^*, M_{z_i}]$, $i = 1, 2$ can be compact leave alone trace class. The challenge is to find suitable additional hypotheses to ensure that the self commutators are in trace class. As we have discussed in the Introduction, Athavale had introduced the notion of hyponormality for a commuting d-tuple of operators and proved that if apart from hyponormality, one makes a strong cyclicity requirement, namely, require that each operator in the d- tuple is individually cyclic, then the self commutators (and hence all the commutators) of the d- tuple are in trace class. Douglas and Yan arrive at a similar conclusion by assuming joint cyclicity but making the strong assumption of thinness of the Taylor joint spectrum.

In this chapter, we isolate a small class of operators for which neither of the strong assumptions of joint cyclicity or thinness of the spectrum is required to conclude that the commutators $[T_j^*, T_i]$ $1 \leq i, j \leq d$ are in trace class. This is Theorem 2.5. We also obtain a class of explicit examples, where the Douglas-Yan theorem applies. We can even do this with the weaker hypothesis of projectively hyponormal d- tuples, at the moment, only for a pair of operators, this is Theorem 2.16.

2.1 A class of hyponormal d- tuples

The following lemma is certainly well known, however for the shake of completeness, we provide a proof.

Lemma 2.1. *Let A, C be positive operators on some Hilbert space $\mathscr{H}_1$ and B be a nonzero operator on (possibly) some other Hilbert space $\mathscr{H}_2$. Assume that*

$$A \otimes B^* B \geq C \otimes BB^*.$$

Then $A \geq C$.

Proof. By our hypothesis, for unit vectors $x \in \mathscr{H}_1$ and $y \in \mathscr{H}_2$, we have

$$\langle Ax, x\rangle \|By\|^2 = \langle A \otimes B^* Bx \otimes y, x \otimes y\rangle \tag{2.1}$$
$$\geq \langle C \otimes BB^* x \otimes y, x \otimes y\rangle = \langle Cx, x\rangle \|B^* y\|^2 \tag{2.2}$$

Since $\|B\|^2 \geq \|By\|^2$, $\sup\left\{\|B^* y\|^2 : \|y\| = 1\right\} = \|B^*\|^2$ and $\|B^*\| = \|B\|$, it follows that $A \geq C$. □

Proposition 2.2. *Let $\boldsymbol{A} = (A_1, \ldots, A_d)$ be d- tuple of operators on a Hilbert space $\mathscr{H}_1$ and T be a nonzero operator on (possibly) some other Hilbert space $\mathscr{H}_2$. Assume that the tuple of operators $\boldsymbol{A} \otimes T := (A_1 \otimes T, \ldots, A_d \otimes T)$ on $\mathscr{H}_1 \otimes \mathscr{H}_2$ is commuting and projectively hyponormal. Then we have the following.*

(i) Each of the operators $A_1, \ldots, A_d$ is hyponormal on $\mathscr{H}_1$ and the operator T is hyponormal on $\mathscr{H}_2$.

(ii) The tuple of operators $\boldsymbol{A}$ is commuting and projectively hyponormal on $\mathscr{H}_1$.

Proof. Since $\boldsymbol{A} \otimes T$ is projectively hyponormal, the commutators $[A_i^* \otimes T^*, A_i \otimes T]$ in the (i, i), $i = 1, \ldots, d$, positions of the matricial commutator $[\![(\boldsymbol{A} \otimes T)^*, \boldsymbol{A} \otimes T]\!]$ are hyponormal. In other words, for $i = 1, \ldots, d$, we have

$$[A_i^* \otimes T^*, A_i \otimes T] = (A_i^* A_i \otimes T^* T - A_i A_i^* \otimes TT^*) \geq 0.$$

Now, for a fixed i, $1 \leq i \leq d$, setting $A := A_i^* A_i$, $C := A_i A_i^*$ and $B := T$, we see that the hypotheses of Lemma 2.1 are satisfied. Therefore, we conclude that A_i, $1 \leq i \leq n$, is hyponormal.

A similar proof shows that T is hyponormal on $\mathscr{H}_2$. Next, we prove that A_i, A_j for $i \neq j$ commute. By hypothesis,

$$\begin{aligned} 0 &= [(A_i \otimes T), (A_j \otimes T)] \\ &= [A_i, A_j] \otimes T^2 \end{aligned}$$

whenever $i \neq j$. This implies: either $[A_i, A_j]$=0 or $T^2 = 0$. If $T^2 = 0$, then the spectral radius of T, $r(T) = 0$. Since T is a hyponormal operator we have $r(T) = \|T\|$ (see [41, Theorem 1]). So $\|T\| = 0$ which is a contradiction. Hence we have $[A_i, A_j]$=0.

To complete the proof, for an arbitrary choice of complex numbers $\alpha_1, \dots, \alpha_n$, let $\boldsymbol{A}_\alpha := \sum_{i=1}^n \alpha_i A_i$ and set

$$A := \boldsymbol{A}_\alpha^* \boldsymbol{A}_\alpha,\ C := \boldsymbol{A}_\alpha \boldsymbol{A}_\alpha^* \text{ and } B := T.$$

As before, the hypotheses of Lemma 2.1 are satisfied and it follows that $\boldsymbol{A}$ is projectively hyponormal. □

The following proposition is analogous to the previous one with hyponormality replacing the projectively hyponormality in it.

Proposition 2.3. *Let $\boldsymbol{A} = (A_1, \dots, A_d)$ be d - tuple of operators on a Hilbert space $\mathscr{H}_1$ and T be a nonzero operator on (possibly) some other Hilbert space $\mathscr{H}_2$. Assume that the tuple of operators $\boldsymbol{A} \otimes T := (A_1 \otimes T, \dots, A_d \otimes T)$ on $\mathscr{H}_1 \otimes \mathscr{H}_2$ is commuting and hyponormal. Then we have the following.*

(i) *Each of the operators $A_1, \dots, A_d$ is hyponormal on $\mathscr{H}_1$ and the operator T is hyponormal on $\mathscr{H}_2$.*

(ii) *The tuple of operators $\boldsymbol{A}$ is commuting and hyponormal on $\mathscr{H}_1$.*

Proof. Evidently, hyponormality implies projective hyponormality and thus the statement *(i)* and the first half of the statement in *(ii)* follow from the previous Proposition. To prove the second half of the statement in *(ii)*, we have to show that $[[\boldsymbol{A}^*, \boldsymbol{A}]] \geq 0$ on $\mathscr{H}_1 \otimes \mathbb{C}^d$. This is equivalent to showing

$$\begin{pmatrix} A_1^* A_1 & \dots & A_d^* A_1 \\ \vdots & \ddots & \vdots \\ A_1^* A_d & \dots & A_d^* A_d \end{pmatrix} \geq \begin{pmatrix} A_1 A_1^* & \dots & A_1 A_d^* \\ \vdots & \ddots & \vdots \\ A_d A_1^* & \dots & A_d A_d^* \end{pmatrix}$$

on $\mathscr{H}_1 \otimes \mathbb{C}^d$. By our hypothesis,

$$[[(\boldsymbol{A} \otimes T)^*, (\boldsymbol{A} \otimes T)]] \geq 0$$

on $\mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \mathbb{C}^d$. In other words,

$$\begin{pmatrix} A_1^* A_1 & \dots & A_1^* A_d \\ \vdots & \ddots & \vdots \\ A_d^* A_1 & \dots & A_d^* A_d \end{pmatrix} \otimes T^* T \geq \begin{pmatrix} A_1 A_1^* & \dots & A_d A_1^* \\ \vdots & \ddots & \vdots \\ A_1 A_d^* & \dots & A_d A_d^* \end{pmatrix} \otimes T T^*.$$

Now setting $A := \begin{pmatrix} A_1^*A_1 & \dots & A_d^*A_1 \\ \vdots & \ddots & \vdots \\ A_1^*A_d & \dots & A_d^*A_d \end{pmatrix}$, $C := \begin{pmatrix} A_1A_1^* & \dots & A_1A_d^* \\ \vdots & \ddots & \vdots \\ A_dA_1^* & \dots & A_dA_d^* \end{pmatrix}$, both acting on the Hilbert space $\mathscr{H}_1 \otimes \mathbb{C}^d$, and $B := T$ on the Hilbert space $\mathscr{H}_2$, it is easy to see that the hypotheses of Lemma 2.1 are met. Thus,

$$\begin{pmatrix} A_1^*A_1 & \dots & A_d^*A_1 \\ \vdots & \ddots & \vdots \\ A_1^*A_d & \dots & A_d^*A_d \end{pmatrix} \geq \begin{pmatrix} A_1A_1^* & \dots & A_1A_d^* \\ \vdots & \ddots & \vdots \\ A_dA_1^* & \dots & A_dA_d^* \end{pmatrix}$$

which completes the proof. □

The following Proposition provides a converse to the Propositions 2.2 and 2.3. For brevity, we have combined the hypotheses of the projective hyponormality and the (joint) hyponormality in a single statement. The proof is also given with the hypothesis of projective hyponormality with the understanding that the proof follows similarly in the case of hyponormal tuple.

Proposition 2.4. *Let $\boldsymbol{A} = (A_1, \dots, A_d)$ be a commuting projectively (resp. jointly) hyponormal tuple of operators on a Hilbert space $\mathscr{H}_1$ and T be a hyponormal operator on (possibly) some other Hilbert space $\mathscr{H}_2$. Then $\boldsymbol{A} \otimes T = (A_1 \otimes T, \dots, A_d \otimes T)$ is a commuting projectively (resp. jointly) hyponormal tuple of operators on $\mathscr{H}_1 \otimes \mathscr{H}_2$.*

Proof. For an arbitrary choice of $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{C}^d$, let $\boldsymbol{A}_\alpha := \sum_{i=1}^d \alpha_i A_i$ and $(\boldsymbol{A} \otimes T)_\alpha := \sum_{i=1}^d \alpha_i (A_i \otimes T)$. It is easy to see that $(\boldsymbol{A} \otimes T)_\alpha := \boldsymbol{A}_\alpha \otimes T$. Since $\boldsymbol{A}$ is commuting projectively hyponormal, we have for each $\alpha \in \mathbb{C}^d$, $[(\boldsymbol{A}_\alpha)^*, (\boldsymbol{A}_\alpha)] \geq 0$. Now

$$\begin{aligned} [(\boldsymbol{A}_\alpha \otimes T)^*, (\boldsymbol{A}_\alpha \otimes T)] &= (\boldsymbol{A}_\alpha)^*(\boldsymbol{A}_\alpha) \otimes T^*T - (\boldsymbol{A}_\alpha)(\boldsymbol{A}_\alpha)^* \otimes TT^* \\ &\geq (\boldsymbol{A}_\alpha)^*(\boldsymbol{A}_\alpha) \otimes TT^* - (\boldsymbol{A}_\alpha)(\boldsymbol{A}_\alpha)^* \otimes TT^* \\ &= [(\boldsymbol{A}_\alpha)^*, (\boldsymbol{A}_\alpha)] \otimes TT^* \\ &\geq 0. \end{aligned}$$

□

Theorem 2.5. *Let $\boldsymbol{A} := (A_1, \dots, A_d)$ be a d- tuple of commuting normal Hilbert-Schmidt operators on a Hilbert space $\mathscr{H}_1$ and T be a hyponormal operator on a second (possibly different) Hilbert space $\mathscr{H}_2$. Assume that T is m- polynomially cyclic. Then $\boldsymbol{A} \otimes T := (A_1 \otimes T, \dots, A_d \otimes T)$ is a commuting hyponormal d- tuple of operators and for all $1 \leq i, j \leq d$, $[A_i^* \otimes T^*, A_j \otimes T]$ is in trace class.*

Proof. Since T is a hyponormal and m- polynomially cyclic operator on $\mathscr{H}_2$, by the Berger-Shaw theorem, $[T^*, T]$ is in trace-class. Using Fuglede theorem for commuting normal operators we get, for $1 \leq i, j \leq d$,

$$[(A_i \otimes T)^*, (A_j \otimes T)] = A_i^* A_j \otimes [T^*, T] = A_j A_i^* \otimes [T^*, T] \tag{2.3}$$

Using Equation (2.3), we obtain the following string of equalities.

$$
\begin{aligned}
\big[\big[(\boldsymbol{A}\otimes T)^*,\boldsymbol{A}\otimes T\big]\big] &= \begin{pmatrix} A_1A_1^*\otimes[T^*,T] & \dots & A_dA_1^*\otimes[T^*,T]\\ \vdots & \ddots & \vdots\\ A_1A_d^*\otimes[T^*,T] & \dots & A_dA_d^*\otimes[T^*,T]\end{pmatrix}\\
&= \begin{pmatrix} A_1A_1^* & \dots & A_dA_1^*\\ \vdots & \ddots & \vdots\\ A_1A_d^* & \dots & A_dA_d^*\end{pmatrix}\otimes[T^*,T]\\
&= \begin{pmatrix} A_1\\ \vdots\\ A_d\end{pmatrix}\begin{pmatrix} A_1^* & \dots & A_d^*\end{pmatrix}\otimes[T^*,T]. \qquad (2.4)
\end{aligned}
$$

Since we have assumed that T is hyponormal, therefore from Equation (2.4), it follows that the commuting tuple $\boldsymbol{A}\otimes T$ is hyponormal.

Also, since A_i, is a Hilbert-Schmidt operator, it follows that $A_iA_j^*$, $1\le i,j\le d$, is in trace-class. The operator $[T^*,T]$ is in trace-class, consequently (using Equation (2.3)), we conclude that $[A_i^*\otimes T^*, A_j\otimes T]$, $1\le i,j\le d$, is in trace-class. □

2.2 Douglas-Yan Theorem

In this Section, we first recall the Douglas-Yan theorem from [18] in two different forms, which nevertheless are equivalent. We then give a proof for one of these theorems with a slightly weaker hypothesis. This proof is very similar to the original proof but we include it here for completeness. We also give some natural examples of commuting d- tuples of operators satisfying the hypothesis of the Douglas-Yan theorem. Next, for many of the d- tuple of operators discussed in the previous Section, we show that the conclusion of the Douglas-Yan theorem remains valid while they don't meet all the hypotheses of the theorem.

Theorem 2.6 (Douglas-Yan)**. (Module):** *Let $\mathscr{H}$ be a finitely generated hyponormal Hilbert module over $\mathscr{A}$ with the dense subalgebra $\mathscr{B}$. If dim$\mathscr{B}=1$, then $\mathscr{H}$ is 1-reductive for $\mathscr{B}$.*

(d- tuple): *Let $\boldsymbol{T}=(T_1,\dots,T_d)$ be a hyponormal d- tuple of operators on the Hilbert space $\mathscr{H}$ such that $\boldsymbol{T}$ is finitely polynomially cyclic. Let I be the ideal of $\mathbb{C}[z_1,\dots,z_d]$ defined by*

$$I=\big\{p\in\mathbb{C}[z_1,\dots,z_d]: p(\boldsymbol{T})=0\big\}.$$

If dim$\big(\mathbb{C}[z_1,\dots,z_d]/I\big)=1$, then $[T_i^*,T_j]$ is in trace-class for all i,j.

Remark 2.7. (i) Since $\mathscr{H}$ is a finitely generated hyponormal module, there exist generators $\phi_1,\dots,\phi_d$ for $\mathscr{B}$ (as an algebra) so that $(m_{\phi_1},\dots,m_{\phi_d})$ is a commuting finite polynomially cyclic hyponormal tuple of operators on $\mathscr{H}$. Since B is a subalgebra of finite type it follows that

$$\mathscr{B} \cong \mathbb{C}[z_1,\dots,z_d]/I.$$

Thus, by hypothesis, $\dim \mathscr{B} = \dim\big(\mathbb{C}[z_1,\dots,z_d]/I\big) = 1$.

(ii) In the special case, when the operator tuple $\boldsymbol{T}$ is subnormal, $\dim\big(\mathbb{C}[z_1,\dots,z_d]/I\big) = 1$ is equivalent to the fact that the Taylor joint spectrum of the d- tuple is thin.

2.2.1 Examples

For the sake of completeness, we first recall the definition of the joint spectrum $\sigma(\boldsymbol{T})$ of a commuting d - tuple $\boldsymbol{T}$ due to J. Taylor. The following paragraph describing the Taylor joint spectrum is taken verbatim from [16].

Let $\Lambda = \Lambda[e] = \Lambda_n[e]$ be the exterior algebra on n generators $e_1,\dots,e_n$, with identity $e_0 = 1$. Λ is the algebra of forms in $e_1,\dots,e_n$, with complex coefficients, subject to the collapsing property $e_i e_j + e_j e_i = 0\,(1 \le i, j \le n)$. Let $E_i : \Lambda \to \Lambda$ be given by $E_i x = e_i x\,(i = 1,\dots,n)$. $E_1,\dots,E_n$ are the "creation operators." Clearly $E_i E_j + E_j E_i = 0\,(1 \le i, j \le n)$. Λ can be regarded as a Hilbert space. If we declare $\{e_{i_1}\cdot\ldots\cdot e_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$ as an orthonormal basis. Then $E_i^* x = x'$, where $x = e_i x' + x''$ is the unique decomposition of a form x as the sum of an element in the range of E_i and an element in the kernel of E_i^*. Actually, each E_i is a partial isometry, and $E_i^* E_j + E_j E_i^* = \delta_{ij}\,(1 \le i, j \le n)$. For H a vector space and $A \subseteq \mathscr{L}(H)$, we define $D_A : \Lambda(H) \to \Lambda(H)$ $(\Lambda(H) = H \otimes_{\mathbb{C}} \Lambda)$ by

$$D_A := \sum_{i=1}^{n} A_i \otimes E_i.$$

Then

$$D_A^2(y \otimes x) = \sum_{i,j=1}^{n} A_j A_i y \otimes E_j E_i x = \sum_{i<j} A_i A_j y \otimes (E_i E_j + E_j E_i)x = 0,$$

so that $R(D_A) \subset N(D_A)$ (R and N denote range and kernel). We say that A is nonsingular on H if $R(D_A) = N(D_A)$. When $n = 1$, for instance, A is nonsingular if and only if A is one to one and onto. The Taylor spectrum of A on H is

$$\sigma(A,H) := \{\lambda \in \mathbb{C}^n : R(D_{A-\lambda}) \neq N(D_{A-\lambda})\}.$$

The Taylor joint spectrum $\sigma(\boldsymbol{T})$ of a commuting d - tuple $\boldsymbol{T}$ has several useful properties listed in [16, pp. 21]. These include

1. $\sigma(\boldsymbol{T})$ is non-empty and compact subset of $\mathbb{C}^d$.

2. The spectral mapping property: $\sigma(p(\boldsymbol{T})) = \{p(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \sigma(\boldsymbol{T})\}$ for any polynomial p in d variables. In particular, the joint spectrum has the projection property.

Note that the projection property shows that the joint spectrum $\sigma(\boldsymbol{T})$ of a commuting d - tuple $\boldsymbol{T}$ is a subset of the product $\sigma(T_1) \times \cdots \times \sigma(T_d)$. Now, we recall without proof, a useful proposition from [8] describing the Taylor joint spectrum of the direct sum of a countable set of commuting tuple of operators.

Proposition 2.8. *Let* $\{\boldsymbol{S}_i := (S_{i1}, \ldots, S_{id})\}_{i=1}^n$, *where* $n \in \mathbb{N} \cup \{\infty\}$, *be a set of commuting* d*- tuple of bounded linear operators on the Hilbert spaces* $\mathscr{H}_i$. *Then the Taylor joint spectrum* $\sigma(\oplus_{i=1}^n \boldsymbol{S}_i)$ *of the direct sum* $\oplus_{i=1}^n \boldsymbol{S}_i$ *equals* $\overline{\cup_{i=1}^n \sigma(\boldsymbol{S}_i)}$.

It is proved in Theorem 2.5 that the commutators $[(A_j \otimes T)^*, A_i \otimes T]$, $1 \leq i, j \leq d$, belongs to the trace class as soon as certain natural conditions are imposed on the commuting tuple $\boldsymbol{A}$ and the operator T. For this, the commuting tuple $\boldsymbol{T}$ need not be polynomially cyclic, which is one of the assumptions in the Douglas-Yan theorem. However, using the following theorem, we see that the joint spectrum of the d- tuples appearing in Theorem 2.5 is union of thin sets. A set $V \subseteq \mathbb{C}^d$ is said to be an algebraic curve if it is the common zero set of a set of polynomials in d variables. Let $I(V)$ denotes the set of polynomials that vanish on V. If $\dim\big(\mathbb{C}[z_1, \ldots, z_d]/I(V)\big) = 1$, then V is called an algebraic curve. In the paper [18], the joint spectrum $\sigma(\boldsymbol{T})$ of a commuting d - tuple $\boldsymbol{T}$ is said to be thin if it lies in an algebraic curve.

Theorem 2.9. *Let* $\boldsymbol{A} := (A_1, \ldots, A_d)$ *be a* $d (\geq 2)$*- tuple of commuting normal Hilbert-Schmidt operators on a Hilbert space* $\mathscr{H}_1$ *and* T *be an operator on a second (possibly different) Hilbert space* $\mathscr{H}_2$. *Then the joint spectrum of the commuting* d*- tuple of operators* $\boldsymbol{A} \otimes T := (A_1 \otimes T, \ldots, A_d \otimes T)$ *is a countable union of thin sets.*

Proof. Let $\boldsymbol{\lambda}^{(k)} = (\lambda_i^{(k)})_{i \in I}$, $1 \leq k \leq d$, be the countable set of eigenvalues of A_k, that is, I is at most a countable set. We can assume, without loss of generality, that $A_k = \sum \lambda_i^{(k)} P_i$, where P_i is the orthogonal projection to the eigenspace corresponding to $\lambda_i^{(k)}$. Consequently, the tuple $\boldsymbol{A} \otimes T$ is the direct sum

$$\bigoplus_{i \in I} \big(\lambda_i^{(1)} T, \ldots, \lambda_i^{(d)} T\big).$$

For a fixed $i \in I$, the spectrum of the operator $\big(\lambda_i^{(1)} T, \ldots, \lambda_i^{(d)} T\big)$ is the set $\{(\lambda_i^{(1)} z, \ldots, \lambda_i^{(d)} z) : z \in \sigma(T)\} \subseteq \mathbb{C}^d$. Now, applying Proposition 2.8, we conclude that the spectrum of the d- tuple of operators $\boldsymbol{A} \otimes T$ is the set

$$\sigma(\boldsymbol{A} \otimes T) = \bigcup_{i \in I} \big\{(\lambda_i^{(1)} z, \ldots, \lambda_i^{(d)} z) : z \in \sigma(T)\big\}.$$

Thus $\sigma(\boldsymbol{A} \otimes T)$ is a union of at most countably many thin sets. □

Remark 2.10. Abstract conditions on a commuting d - tuple of operators $\boldsymbol{T}$ are given in Theorem 2.6 to ensure that the commutators $[T_j^*, T_i]$, $1 \le i, j \le d$, are in trace class. In Theorem 2.5, it is shown that the commutators $[A_i^* \otimes T^*, A_j \otimes T]$, $1 \le i, j \le d$, are in trace class for any commuting d - tuple $(A_1, \ldots, A_d)$ of normal Hilbert-Schmidt operators and a polynomially cyclic hyponormal operator T. As shown in Theorem 2.9, the spectrum of these d - tuple of commuting operators, in general, is only a countable union of thin sets. For producing a different class of examples, let T be a m- polynomially cyclic pure hyponormal operator on a Hilbert space $\mathscr{H}$. Assume that the pair (T, T^2) is hyponormal. We claim that the Douglas-Yan theorem applies to the pair (T, T^2).

To verify this claim, we note that (T, T^2) is n- polynomially cyclic with $n \le m$ whenever T is m- polynomially cyclic. Thus it remains to show that the Krull dimension of $\mathbb{C}[z_1, z_2]/I$ is 1, where $I = \{p \in \mathbb{C}[z_1, z_2] | p(T, T^2) = 0\}$. By definition the ideal generated by the polynomial $p_T : p_T(z_1, z_2) = z_1^2 - z_2$ is included in I. If there is another polynomial p, which does not have p_T as a factor, with $p(T, T^2) = 0$, then $q(T) = 0$ for some non zero polynomial $q \in \mathbb{C}[z]$. But $q(T) = 0$ if and only if the spectrum of T is finite. Now, if T is a pure hyponormal operator, then the spectrum of T can not be discrete hence cannot be finite (see [41, Cor. 2]). Thus the polynomial p such that $p(T, T^2) = 0$ will have p_T as a factor. Therefore, the vanishing ideal of the pair (T, T^2) is the ideal generated by the polynomial ${z_1}^2 - z_2$. Since ${z_1}^2 - z_2$ is an irreducible polynomial, the ideal I generated by it is a prime ideal and height of I is 1. Thus using Lemma (1.10) we conclude that

$$\dim \mathbb{C}[z_1, z_2]/I = 1.$$

Similarly, we verify the hypotheses of Theorem 2.6 for any hyponormal d- tuple $(T, T^2, \ldots, T^d)$. For this d- tuple, the vanishing ideal I contains the idea $(z_1^2 - z_2, z_1^3 - z_3, \ldots, z_1^d - z_d)$, the ideal generated by the polynomials $z_1^2 - z_2, z_1^3 - z_3, \ldots, z_1^d - z_d$. An argument, as in the previous paragraph shows that it is equal to the vanishing ideal. The kernel of the evaluation map Φ from $\mathbb{C}[z_1, \ldots, z_d]$ to $\mathbb{C}[z_1]$ given by $f(z_1, \ldots, z_d) \to f(z_1, z_1^2, \ldots, z_1^d)$ is I and therefore, $\mathbb{C}[z_1, \ldots, z_d]/I \cong \mathbb{C}[z_1]$. Thus

$$\dim \mathbb{C}[z_1, \ldots, z_d]/I = \dim \mathbb{C}[z_1] = 1.$$

2.2.2 A strengthening of the Douglas-Yan Theorem

Recall that Athavale, in the paper [4], had introduced the notion of the projectively hyponormal operators, although, he called them weakly hyponormal. Among other things, he had proved the following Lemma. We provide a proof which is no different from the proof of Athavale except for minor correction in his proof.

For the proof of the Lemma given below, we will be using the trace norm $\|T\|_1$ of an operator $T \in \mathscr{B}(\mathscr{H})$ in a slightly different form than the one given in Definition 1.1, namely,

$$\|T\|_1 = \sup_{\{f_n\},\{g_n\}} \sum_{n=1}^{\infty} |\langle Tf_n, g_n\rangle|, \tag{2.5}$$

where the supremum is taken over any pair $\{f_n\}$ and $\{g_n\}$ of orthonormal sets in the Hilbert space $\mathscr{H}$, see [40, Proposition 3.6.5.].

Lemma 2.11. *Let $\boldsymbol{T} = (T_1, \dots, T_d)$ be a commuting projectively hyponormal d- tuple of operators on the Hilbert space $\mathscr{H}$. Furthermore, assume that for $1 \le k \le d$, each $[T_k^*, T_k]$ is in trace-class. Then $[T_j^*, T_k]$ is also in trace-class for all $1 \le k, j \le d$.*

Proof. From projective hyponormality of $\boldsymbol{T}$, for any fixed pair of indices j,k and any pair of complex numbers α_j, α_k, we have that $\alpha_j T_j + \alpha_k T_k$ is hyponormal, that is, the commutator $[(\alpha_j T_j + \alpha_k T_k)^*, \alpha_j T_j + \alpha_k T_k]$ is non-negative. Expanding and taking inner product with an arbitrary vector $f \in \mathscr{H}$, we have

$$|\alpha_j|^2\langle f, [T_j^*, T_j]f\rangle + 2\mathrm{Re}\,\bar{\alpha}_j\alpha_k\langle f, [T_j^*, T_k]f\rangle + |\alpha_k|^2\langle f, [T_k^*, T_k]f\rangle \ge 0, \alpha_j, \alpha_k \in \mathbb{C}, f \in \mathscr{H}. \tag{2.6}$$

Since the inequality (2.6) is for all pairs of complex numbers α, β, it follows that $\langle f, [T_j^*, T_k]f\rangle = 0$ if both $\langle f, [T_j^*, T_j]f\rangle = 0$ and $\langle f, [T_k^*, T_k]f\rangle = 0$. On the other hand, suppose that one of $\langle f, [T_k^*, T_k]f\rangle$ or $\langle f, [T_j^*, T_j]f\rangle$ is not equal to 0, say, $\langle f, [T_k^*, T_k]f\rangle \ne 0$. Then choosing $\alpha_j = 1$ and $\alpha_k = \frac{-\overline{\langle f, [T_j^*, T_k]f\rangle}}{\langle f, [T_k^*, T_k]f\rangle}$, the inequality (2.6) gives

$$\langle f, [T_j^*, T_j]f\rangle - 2\frac{|\langle f, [T_j^*, T_k]f\rangle|^2}{\langle f, [T_k^*, T_k]f\rangle} + \frac{|\langle f, [T_j^*, T_k]f\rangle|^2}{\langle f, [T_k^*, T_k]f\rangle} \ge 0,\ f \in \mathscr{H}.$$

Or, equivalently,

$$|\langle f, [T_j^*, T_k]f\rangle|^2 \le \langle f, [T_j^*, T_j]f\rangle\langle f, [T_k^*, T_k]f\rangle,\ f \in \mathscr{H}. \tag{2.7}$$

For any bounded operator A on a Hilbert space $\mathscr{H}$, by the polarization identity, we have

$$\begin{aligned}4\langle f, Ag\rangle = &\langle f+g, A(f+g)\rangle - \langle f-g, A(f-g)\rangle \\ &- i\langle f+ig, A(f+ig)\rangle + i\langle f-ig, A(f-ig)\rangle, \quad f,g \in \mathscr{H},\end{aligned}$$

and hence

$$\begin{aligned}4|\langle f, Ag\rangle| \le\ &|\langle f+g, A(f+g)\rangle| + |\langle f-g, A(f-g)\rangle| \\ &+ |\langle f+ig, A(f+ig)\rangle| + |\langle f-ig, A(f-ig)\rangle| \end{aligned} \tag{2.8}$$

For all $f, g \in \mathscr{H}$, putting $A = [T_j^*, T_k]$ in the inequality (2.8), we get

$$
\begin{aligned}
&4|\langle f, [T_j^*, T_k]g\rangle| \\
&\le |\langle f+g, [T_j^*, T_k](f+g)\rangle| + |\langle f-g, [T_j^*, T_k](f-g)\rangle| \\
&+ |\langle f+ig, [T_j^*, T_k](f+ig)\rangle| + |\langle f-ig, [T_j^*, T_k](f-ig)\rangle| \\
&\le (\langle f+g, [T_k^*, T_k](f+g)\rangle\langle f+g, [T_j^*, T_j](f+g)\rangle)^{\frac{1}{2}} \\
&+ (\langle f-g, [T_k^*, T_k](f-g)\rangle\langle f-g, [T_j^*, T_j](f-g)\rangle)^{\frac{1}{2}} \\
&+ (\langle f+ig, [T_k^*, T_k](f+ig)\rangle\langle f+ig, [T_j^*, T_j](f+ig)\rangle)^{\frac{1}{2}} \\
&+ (\langle f-ig, [T_k^*, T_k](f-ig)\rangle\langle f-ig, [T_j^*, T_j](f-ig)\rangle)^{\frac{1}{2}} \\
&\le \frac{1}{2}\big(\langle f+g, [T_k^*, T_k](f+g)\rangle + \langle f+g, [T_j^*, T_j](f+g)\rangle + \langle f-g, [T_k^*, T_k](f-g)\rangle + \\
&\langle f-g, [T_j^*, T_j](f-g)\rangle + \langle f+ig, [T_k^*, T_k](f+ig)\rangle + \langle f+ig, [T_j^*, T_j](f+ig)\rangle + \\
&\langle f-ig, [T_k^*, T_k](f-ig)\rangle + \langle f-ig, [T_j^*, T_j](f-ig)\rangle\big) \\
&\le 2\big(\langle f, [T_k^*, T_k]f\rangle + \langle g, [T_k^*, T_k]g\rangle + \langle f, [T_j^*, T_j]f\rangle + \langle g, [T_j^*, T_j]g\rangle\big).
\end{aligned}
$$

Therefore, using Equation (2.8), we have

$$
\big|\langle f, [T_j^*, T_k]g\rangle\big| \le \frac{1}{2}\big(\langle f, [T_k^*, T_k]f\rangle + \langle g, [T_k^*, T_k]g\rangle + \langle f, [T_j^*, T_j]f\rangle + \langle g, [T_j^*, T_j]g\rangle\big).
$$

For any pair of orthonormal sets $\{f_n\}$ and $\{g_n\}$,

$$
\begin{aligned}
&\sum \big|\langle f_n, [T_j^*, T_k]g_n\rangle\big| \\
&\le \frac{1}{2}\Big(\sum_n \langle f_n, [T_k^*, T_k]f_n\rangle + \sum_n \langle g_n, [T_k^*, T_k]g_n\rangle + \\
&\sum_n \langle f_n, [T_j^*, T_j]f_n\rangle + \sum_n \langle g_n, [T_j^*, T_j]g_n\rangle\Big) \\
&\le \frac{1}{2}\big(2\big\|[T_k^*, T_k]\big\|_1 + 2\big\|[T_j^*, T_j]\big\|_1\big) \\
&= \big\|[T_k^*, T_k]\big\|_1 + \big\|[T_j^*, T_j]\big\|_1.
\end{aligned}
$$

Taking supremum over all possible orthonormal sets we get,

$$
\begin{aligned}
\big\|[T_j^*, T_k]\big\|_1 = \big\|[T_k^*, T_j]\big\|_1 &= \sup_{\{f_n\}\{g_n\}} \sum_n |\langle [T_k^*, T_j]f_n, g_n\rangle| \\
&= \sup_{\{f_n\}\{g_n\}} \sum_n |\langle f_n, [T_j^*, T_k]g_n\rangle| \le \big\|[T_k^*, T_k]\big\|_1 + \big\|[T_j^*, T_j]\big\|_1 < \infty.
\end{aligned}
$$

Thus $[T_j^*, T_k]$ is in trace class. □

Reviewer pointed out that in the proof of Lemma 2.11 the scalar matrix $((\langle [T_j^*, T_i] f, f\rangle))$ is positive semi-definite, Hence by the Cauchy-Schwarz inequality, one gets the inequality 2.7.

The following estimate for the trace norm of $[T_j^*, T_i]$ assuming that $(T_1, \ldots T_d)$ is projectively hyponormal follows directly from Lemma 2.11 and the Berger-Shaw theorem.

Theorem 2.12. *Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a projectively hyponormal d- tuple of operators. Suppose that each T_i, $1 \le i \le d$, is m_i- polynomially cyclic. Then the operators $[T_j^*, T_i]$ are in trace class. Moreover, we have*

$$\big\| [T_j^*, T_i] \big\|_1 \le \begin{cases} \frac{m_i}{\pi} \nu(\sigma(T_i)) & \text{if } i = j \\ \frac{m_i}{\pi} \nu(\sigma(T_i)) + \frac{m_j}{\pi} \nu(\sigma(T_j)) & \text{if } i \neq j, \end{cases}$$

where $\nu(\sigma(T_i))$ is the Lebesgue measure of the spectrum of T_i.

Proposition 4 of [4] gives an estimate (a) for the trace of the operators $[T_j^*, T_i]$, $1 \le i, j \le d$ assuming only projective hyponormality of $(T_1, \ldots, T_d)$ and (b) for the Hilbert-Schmidt norm of the same set of operators under the stronger assumption of hyponormality. We are unable to verify these assertions with the exact assumptions made in [4]. However, Theorem 2.12 together with Theorem 2.13 stated below, we believe, should replace the corresponding parts of Proposition 4 in [4].

Theorem 2.13. *Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a hyponormal d- tuple of operators. Suppose that each T_i, $1 \le i \le d$, is m_i- polynomially cyclic. Then the operators $[T_j^*, T_i]$ are in trace class. Moreover, we have*

$$\big\| [T_j^*, T_i] \big\|_1 \le \begin{cases} \frac{m_i}{\pi} \nu(\sigma(T_i)) & \text{if } i = j \\ \big(\frac{m_i}{\pi} \nu(\sigma(T_i))\big)^{\frac{1}{2}} \big(\frac{m_j}{\pi} \nu(\sigma(T_j))\big)^{\frac{1}{2}} & \text{if } i \neq j, \end{cases}$$

where $\nu(\sigma(T)$ is the Lebesgue measure of the spectrum of T.

Proof. Hyponormality of the d- tuple $\boldsymbol{T}$ gives for $x, y \in \mathscr{H}$,

$$\big| \langle [T_j^*, T_i] x, y \rangle \big| \le \langle [T_i^*, T_i] x, x \rangle^{\frac{1}{2}} \langle [T_j^*, T_j] y, y \rangle^{\frac{1}{2}}.$$

For any pair of orthonormal sets $\{f_n\}$ and $\{g_n\}$,

$$\begin{aligned} \sum_n \big| \langle [T_j^*, T_i] f_n, g_n \rangle \big| &\le \sum_n \langle [T_i^*, T_i] f_n, f_n \rangle^{\frac{1}{2}} \langle [T_j^*, T_j] g_n, g_n \rangle^{\frac{1}{2}} \\ &\le \big(\sum_n \langle [T_i^*, T_i] f_n, f_n \rangle \big)^{\frac{1}{2}} \big(\sum_n \langle [T_j^*, T_j] g_n, g_n \rangle \big)^{\frac{1}{2}} \\ &\le \big\| [T_i^*, T_i] \big\|_1^{\frac{1}{2}} \big\| [T_j^*, T_j] \big\|_1^{\frac{1}{2}}. \end{aligned}$$

Now the conclusion follows from Equation (2.5) and the Berger-Shaw theorem. □

Lemma 2.11 and the Lemma proved below help to prove the Douglas-Yan theorem with the weaker hypothesis of projective hyponormality.

Lemma 2.14. *Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a projectively hyponormal commuting d- tuple of operators on a Hilbert space $\mathscr{H}$ and a_{ij}, $1 \le i \le m$ and $1 \le j \le d$ be a set of md complex scalars. Suppose $g_i(\boldsymbol{T}) = \sum_{j=1}^{d} a_{ij} T_j, \quad i = 1, \ldots, m,$. Then $(g_1(\boldsymbol{T}), \ldots, g_m(\boldsymbol{T}))$ is also projectively hyponormal on $\mathscr{H}$.*

Proof. $\boldsymbol{T}$ is projectively hyponormal implies the operator $\alpha_1 T_1 + \alpha_2 T_2 + \ldots + \alpha_d T_d$ is hyponormal for all $\alpha_i \in \mathbb{C}$. There exists $b_i \in \mathbb{C}$, $1 \le i \le d$, such that $\sum_{j=1}^{m} \lambda_j g_j(\boldsymbol{T}) = \sum_{i=1}^{d} b_i T_i$. Therefore $(g_1(\boldsymbol{T}), \ldots, g_m(\boldsymbol{T}))$ is also projectively hyponormal. □

The following Theorem is a restatement of [18, Theorem 2] with the weaker hypothesis of projective hyponormality. The proof given below is similar to the one in [18]. As in the original proof, the main idea in the proof below is to find an appropriate linear combination of the operators T_i, $1 \le i \le d$ such that each of them is individually polynomially cyclic. This is ensured by using the crucial assumption that the Krull dimension of the module is 1.

Theorem 2.15. *Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a projectively hyponormal commuting d- tuple of operators on a Hilbert space $\mathscr{H}$ such that $\boldsymbol{T}$ is m- cyclic. Assume that $\mathbb{C}[z_1, \ldots, z_d]/I$ has Krull dimension 1, where I is the vanishing ideal of $\boldsymbol{T}$, then $[T_j^*, T_i]$ is in trace class for all $1 \le i, j \le d$.*

Proof. By hypothesis $\mathbb{C}[z_1, \ldots, z_d]/I$ has Krull dimension 1 and the Noether normalization theorem applies to it. Consequently there exist linear polynomials $g_1, \ldots, g_n$ such that $\mathbb{C}[z_1, \ldots, z_d]$ is integral over $\mathbb{C}[I, g_j]$, $j = 1, \ldots, n$, and for $i = 1, \ldots, d$, $z_i = \sum_{j=1}^{n} a_{ij} g_j$, $a_{ij} \in \mathbb{C}$. Thus

$$T_i = \sum_{j=1}^{n} a_{ij} g_j(\boldsymbol{T}),\ i = 1, \ldots, d,$$

and

$$\begin{aligned}[T_j^*, T_i] &= \Big[(\sum_{k=1}^{n} a_{jk} g_k(\boldsymbol{T}))^*, \sum_{l=1}^{n} a_{il} g_l(\boldsymbol{T}) \Big] \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} \overline{a}_{il} a_{jk} \big[g_k(\boldsymbol{T})^*, g_l(\boldsymbol{T}) \big].\end{aligned}$$

To complete the proof, it is therefore enough to prove that $[(g_k(\boldsymbol{T}))^*, g_l(\boldsymbol{T})], 1 \le k, l \le n$, is in trace class.

It follows from Lemma 2.14 that $\big(g_1(\boldsymbol{T}), \ldots, g_n(\boldsymbol{T})\big)$ is a projectively hyponormal n- tuple. Hence for $i = 1, \ldots, n, g_i(\boldsymbol{T})$ is a hyponormal operator. Since $\boldsymbol{T}$ is m- cyclic there exists $\xi[m] := \{v_1, \ldots, v_m\} \subseteq \mathscr{H}$ such that $\big\{p(\boldsymbol{T})v | v \in \xi[m] \text{ and } p \in \mathbb{C}[z_1, \ldots, z_d]\big\}$ is dense in $\mathscr{H}$. For a fixed but arbitrary i, $1 \le i \le d$, $\mathbb{C}[z_1,\dots,z_d]$ is an integral module over $\mathbb{C}[I,g_i]$, therefore $\mathbb{C}[z_1,\dots,z_d]$ is finitely generated over $\mathbb{C}[I,g_i]$. Thus for any polynomial p in $\mathbb{C}[z_1,\dots,z_d]$, there is a finite subset $\{f_1,\dots,f_t\} \subseteq \mathbb{C}[z_1,\dots,z_d]$, depending on g_i, and $q_1,\dots,q_t$ in $\mathbb{C}[I,g_i]$, depending on p, such that $p = q_1 f_1 + \dots q_t f_t$. By hypothesis I is the vanishing ideal for $\boldsymbol{T}$. Hence $q(\boldsymbol{T})$ is in $\mathbb{C}[g_i(\boldsymbol{T})]$ for any $q \in \mathbb{C}[I,g_i]$. For $v_1,\dots,v_m$ in $\xi[m]$, define $v_{kl} = f_l(\boldsymbol{T})v_k$, $l = 1,\dots,t$. Now, for a fixed but arbitrary $v_k \in \xi[m]$, we have

$$p(\boldsymbol{T})v_k = \sum_{l=1}^{t} q_l(\boldsymbol{T}) f_l(\boldsymbol{T}) v_k = \sum_{l=1}^{t} q_l(\boldsymbol{T}) v_{lk},\ p \in \mathbb{C}[z_1,\dots,z_d].$$

This proves that $\{p(\boldsymbol{T})v | v \in \xi[m] \text{ and } p \in \mathbb{C}[z_1,\dots,z_d]\}$ is subset of

$$\{q(g_i(\boldsymbol{T}))v | v \in \{v_{lk}, l = 1,\dots t, k = 1,\dots,m\} \text{ and } q \in \mathbb{C}[z]\}.$$

Thus $g_i(\boldsymbol{T})$ is finitely polynomially cyclic. By applying Berger-Shaw theorem to $g_i(\boldsymbol{T}), i = 1,\dots,n$, we conclude that $[g_i(\boldsymbol{T})^*, g_i(\boldsymbol{T})]$ is in trace-class. Finally, it follows from Lemma 2.11 that $[(g_k(\boldsymbol{T}))^*, g_l(\boldsymbol{T})]$, $1 \le k,l \le n$, is in trace class. □

In the proof of the following theorem, we consider those polynomials p in $\mathbb{C}[z_1,\dots,z_d]$ with the property that none of the partial derivatives $\partial_k p$, $1 \le k \le d$, is the zero polynomial. We call such a polynomial *pure*.

Suppose (T_1,T_2) is any pair of commuting operators and $p(T_1,T_2) = 0$. If p is not pure, then p is of the form: (a) $p(z_1,z_2) = \sum_{k=1}^{m} a_k z_1^k$, or (b) $p(z_1,z_2) = \sum_{k=1}^{n} b_k z_2^k$. In either case, the spectrum of T_1 or T_2 is finite. Now, if T_1 and T_2 are pure hyponormal operators, then the spectrum of neither of these can be discrete hence cannot be finite (see [41, Cor. 2]). This contradiction shows that for any commuting pair T_1, T_2 of pure hyponormal operators, if p is not a pure polynomial, then $p(T_1,T_2)$ cannot be zero.

Theorem 2.16. *Let $\boldsymbol{T} = (T_1,T_2)$ be a (pure) projectively hyponormal pair of commuting operators on the Hilbert space $\mathscr{H}$ such that $\boldsymbol{T}$ is m- polynomially cyclic. Furthermore, assume that there exists a polynomial $p \in \mathbb{C}[z_1,z_2]$ such that $p(T_1,T_2) = 0$, then $[T_j^*, T_i]$ is in trace class for all $1 \le i,j \le 2$.*

Proof. The discussion preceding the Theorem shows that any polynomial p for which $p(T_1,T_2) = 0$ must be pure. Let degree of z_2 in p be k. For any polynomial $q \in \mathbb{C}[z_1,z_2]$ using the Division algorithm in $(\mathbb{C}[z_1])[z_2]$ we get

$$q(z_1,z_2) = q_1(z_1,z_2)p(z_1,z_2) + r(z_1,z_2)$$

where degree of z_2 in r is less than k. Since $p(T_1,T_2) = 0$ it follows that $q(T_1,T_2) = r(T_1,T_2)$. We conclude that T_1 is mk- polynomially cyclic with the cyclic set $\{T_2^i v : v \in \xi_{\boldsymbol{T}}[m], 0 \le i \le k\}$, where $\xi_{\boldsymbol{T}}[m]$ is the cyclic set for $\boldsymbol{T}$. Now projective hyponomality of the pair $\boldsymbol{T}$ implies that T_1 is hyponormal. Hence, by the Berger-Shaw theorem, $[T_1^*, T_1]$ is in trace-class. Similarly, since T_2 is also polynomially cyclic, one can prove $[T_2^*, T_2]$ is in trace-class. Finally, from Lemma 2.11 we conclude that for $1 \leq i, j \leq 2$, $[T_j^*, T_i]$ is in trace-class. □

Remark 2.17. It is possible to construct a large family of Hilbert modules, using the Theorem we have just proved, where the Douglas-Yan theorem applies with the slightly weaker hypothesis of projective hyponormality:

Suppose that $\mathscr{H}$ is a Hilbert module over the polynomial ring $\mathbb{C}[z_1, z_2]$. For any commuting pair of operators $T_1, T_2 \in \mathscr{B}(\mathscr{H})$, define the the module multiplication by the rule $(p, h) \to p(T_1, T_2)h$, $p \in \mathbb{C}[z_1, z_2]$, $h \in \mathscr{H}$. Suppose $\boldsymbol{T} = (T_1, T_2)$ satisfies the hypotheses of Theorem 2.16. Now, we show that the Krull dimension of $\mathbb{C}[z_1, z_2]/I$, where I is the vanishing ideal of (T_1, T_2), is 1. To verify this claim, note that the only possibilities for the Krull dimension of $\mathbb{C}[z_1, z_2]/I$ are 1 or 0. But if the Krull dimension is 0, then I must be a maximal ideal. Since T_i, $i = 1, 2$ is a pure hyponormal operator, it follows that no maximal ideal can be the vanishing ideal of (T_1, T_2). Hence the $\dim \mathbb{C}[z_1, z_2]/I = 1$.

Chapter 3

Determinant Operator and Generalized Commutator

Let $\boldsymbol{T}$ be a d - tuple of weighted shift operators with (bounded) weight sequence $w_{\boldsymbol{\alpha}}^{(i)}$, that is, $T_i \boldsymbol{x}_{\boldsymbol{\alpha}} = w_{\boldsymbol{\alpha}}^{(i)} \boldsymbol{x}_{\boldsymbol{\alpha}+\epsilon_i}$, $i = 1, \ldots d$, where $\{\boldsymbol{x}_{\boldsymbol{\alpha}}\}$ is an orthonormal basis in the Hilbert space $\ell^2(\mathbb{N}_0^d)$. The d - tuple $\boldsymbol{T}$ is commuting if and only if, for the corresponding weight sequence $w_{\boldsymbol{\alpha}}^{(i)}$ we have the equality

$$w_{\boldsymbol{\alpha}}^{(i)} w_{\boldsymbol{\alpha}+\epsilon_i}^{(j)} = w_{\boldsymbol{\alpha}}^{(j)} w_{\boldsymbol{\alpha}+\epsilon_j}^{(i)},\ 1 \le i, j \le d,\ \boldsymbol{\alpha} \in \mathbb{N}_0^d. \tag{3.1}$$

In this chapter, the d - tuple $\boldsymbol{T}$ of weighted shift operators that we discuss are commuting and hence the equality of Equation (3.1) for the weight sequence $w_{\boldsymbol{\alpha}}^{(i)}$ is assumed throughout. A joint weighted shift would always mean a d - tuple of commuting weighted shift operators as above.

Definition 3.1 ([11], Definition 2.3)**.** The d- tuple of joint weighted shift operator $\boldsymbol{T}$ is said to be spherical if the weights $w_{\boldsymbol{\alpha}}^{(i)}$ of $\boldsymbol{T}$ admit a factorization of the form

$$w_{\boldsymbol{\alpha}}^{(i)} = \delta_k \sqrt{\tfrac{\alpha_i + 1}{|\boldsymbol{\alpha}| + d}},\ k = |\boldsymbol{\alpha}| \in \mathbb{N},\ 1 \le i \le d,$$

for some sequence δ_k of positive real numbers. The operator T_δ is the weighted shift defined by the weight sequence $\{\delta_k\}_{k \in \mathbb{N}}$.

If $\boldsymbol{T}$ is a spherical d - tuple, then $\|T_i\| = \sup\{w_{\boldsymbol{\alpha}}^{(i)} : \boldsymbol{\alpha} \in \mathbb{N}_0^d\} = \sup\{\delta_k : k \in \mathbb{N}_0\}$. It follows that the spherical d - tuple is bounded if and only if the weight sequence δ_k is bounded. One of the main results of [11] is that the hyponormality (resp. subnormality) of the d- tuple of joint weighted shift operators $\boldsymbol{T}$ is equivalent to the hyponormality (resp. joint subnormality) of the weighted shift operator T_δ. In particular, the d- tuple is hyponormal if and only if the weight sequence $\{\delta_{|\boldsymbol{\alpha}|}\}$ is increasing.

The commuting d - tuple $\boldsymbol{M}^{(\lambda)} = (M_1^{(\lambda)}, \ldots, M_d^{(\lambda)})$ of multiplication by the co-ordinate functions on the weighted Bergman spaces $H^{(\lambda)}(\mathbb{B}_d)$, $\lambda > 0$, is unitarily equivalent to a commuting d - tuple of weighted shift operator with weights $w_{\boldsymbol{\alpha}}^{(i)}(\lambda) := \sqrt{\frac{\alpha_i+1}{|\boldsymbol{\alpha}|+\lambda}}$, $1 \le i \le d$. For a fixed $\lambda > 0$, we have the factorization

$$w_{\boldsymbol{\alpha}}^{(i)}(\lambda) = \sqrt{\frac{|\boldsymbol{\alpha}|+d}{|\boldsymbol{\alpha}|+\lambda}}\sqrt{\frac{\alpha_i+1}{|\boldsymbol{\alpha}|+d}},$$

therefore the d- tuple $\boldsymbol{M}^{(\lambda)}$ is spherical and $\delta_k = \sqrt{\frac{k+d}{k+\lambda}}$, $k = |\boldsymbol{\alpha}|$ in this case.

It is known that the d - tuple $\boldsymbol{M}^{(\lambda)}$ is hyponormal if and only if $\lambda \ge d$, see [11, pp. 605]. From the criterion given in [11, Corollary 4.5 and 4.6], it follows that the commutators $[M_j^{(\lambda)^*}, M_i^{(\lambda)}]$, $\lambda > 0$, $1 \le i, j \le d$, are in Schatten p - class if and only if $p > d$. This verifies the Arveson-Douglas conjecture for $I = \{0\}$ in the weighted Bergman spaces $H^{(\lambda)}(\mathbb{B}_d)$, $\lambda > 0$. Consequently, even if $\lambda \ge d$, the commutators $[M_j^{(\lambda)^*}, M_i^{(\lambda)}]$ are not in the trace class. In this chapter, for a commuting d - tuple of operators $\boldsymbol{T}$, we introduce the determinant operator $\mathrm{dEt}\big([\![\boldsymbol{M}^{(\lambda)^*}, \boldsymbol{M}^{(\lambda)}]\!]\big)$. For the weighted Bergman spaces, we verify that the operator $\mathrm{dEt}\big([\![\boldsymbol{M}^{(\lambda)^*}, \boldsymbol{M}^{(\lambda)}]\!]\big)$ is in trace class. We also define a class $BS_{m,\vartheta}(\Omega)$ of commuting d - tuple of operators $\boldsymbol{T}$ with the property that $\mathrm{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$ is in trace class.

3.1 Determinant Operator

For $1 \le i, j \le d$, let $B_{ij} : \mathscr{H} \to \mathscr{H}$ be a bounded linear operator on the complex separable Hilbert space $\mathscr{H}$. Consequently, $\boldsymbol{B} := \big((B_{ij})\big)$ defines a bounded linear operator from the Hilbert space $\mathscr{H} \otimes \ell_2(d)$ to itself. The *determinant* $\mathrm{dEt}(\boldsymbol{B})$ is the operator given by the formula:

$$\mathrm{dEt}(\boldsymbol{B}) := \sum_{\sigma,\tau \in \mathfrak{S}_d} \mathrm{Sgn}(\sigma) B_{\tau(1),\sigma(\tau(1))} B_{\tau(2),\sigma(\tau(2))}, \ldots, B_{\tau(d),\sigma(\tau(d))}.$$

Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a commuting d- tuple of operators. The determinant of the $d \times d$ block operator $[\![\boldsymbol{T}^*, \boldsymbol{T}]\!] = \big(([T_j^*, T_i])\big)$ is then obtained by setting $B_{ij} = [T_j^*, T_i]$. For instance, if $\boldsymbol{T}$ is the pair (T_1, T_2), then

$$\begin{aligned}\mathrm{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big) &= T_1^* T_1 T_2^* T_2 + T_2^* T_2 T_1^* T_1 + T_1 T_1^* T_2 T_2^* + T_2 T_2^* T_1 T_1^* \\ &\quad - T_1^* T_2 T_2^* T_1 - T_2^* T_1 T_1^* T_2 - T_1 T_2^* T_2 T_1^* - T_2 T_1^* T_1 T_2^*. \end{aligned} \tag{3.2}$$

Remark 3.2. Here are some remarks on the determinant operator.

(i) The map $\mathrm{dEt} : \mathscr{B}(\mathscr{H})^d \times \cdots \times \mathscr{B}(\mathscr{H})^d \mapsto \mathscr{B}(\mathscr{H})$ is defined in analogy with the usual definition of the determinant, namely, $\det : \mathbb{C}^d \times \cdots \times \mathbb{C}^d \mapsto \mathbb{C}$, that is, dEt is a multi-linear alternating map. It is not clear if such a map is uniquely determined (up to a scalar multiple).

(ii) The determinant of a positive matrix is positive. However, if $\boldsymbol{B} := (\!(B_{ij})\!)$ is a positive $d \times d$ block operator, then the determinant operator $\mathrm{dEt}(\boldsymbol{B})$ need not be positive. For example let $\boldsymbol{B}$ be the 2×2 block operator with $B_{ij} = E_{ij}$, where E_{ij} is the 2×2 matrix with 1 at the (i, j) entry and 0 everywhere else. The block matrix $\boldsymbol{B}$ is self-adjoint and positive. But $\mathrm{dEt}(\boldsymbol{B}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is not positive.

(iii) In the particular case of $[\![\boldsymbol{T}^*, \boldsymbol{T}]\!]$, there is another competing definition of an alternating multi-linear map, namely, the generalized commutator introduced by Helton and Howe in [24], which we shall discuss in Subsection 3.3.

(iv) By Putnam-Fuglede Theorem, for d - tuple of commuting normal operators $\boldsymbol{N}$,

$$\mathrm{dEt}\big([\![\boldsymbol{N}^*, \boldsymbol{N}]\!]\big) = 0.$$

Reproduced below is a remark made by the Reviewer.

Remark. *If $\boldsymbol{T}$ is a doubly commuting d - tuple of bounded linear operators on $\mathscr{H}$, then*

$$dEt\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big) = d!\,[T_1^*, T_1]\ldots[T_d^*, T_d].$$

In particular, if $[T_1^*, T_1]$ is compact, then $dEt\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$ is compact.

3.2 Determinant Operator Associated to Different Classes of Operators

3.2.1 Hardy Space over $\mathbb{D}^2$

In this subsection we will discuss the determinant operator associated to the pair of multiplication operators (M_{z_1}, M_{z_2}) on the Hardy space $H^2(\mathbb{D}^2)$ over the bidisc $\mathbb{D}^2$ determined by the orthonormal basis $\{z_1^m z_2^n : m, n \geq 0\}$. The space $H^2(\mathbb{D}^2)$ is isometrically isomorphic to $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$ via the map $L : z_1^m z_2^n \mapsto z_1^m \otimes z_2^n$. Extend the map L by linearity and note that it is well-defined and isometric. It is evidently surjective, hence unitary. The unitary L intertwines the pair of operators (M_{z_1}, M_{z_2}) with the pair $(M_z \otimes I, I \otimes M_z)$, where M_z is the multiplication operator on $H^2(\mathbb{D})$. We will let $\boldsymbol{M}$ denote either of these two pairs without causing any ambiguity since the meaning would be clear from the context.

Let P be the orthogonal projection onto the subspace generated by the constant function in $H^2(\mathbb{D})$. Clearly, $M_z^* M_z = I$ and $M_z M_z^* = I - P$. Therefore, we can write down the commutators of $\boldsymbol{M} := (M_z \otimes I, I \otimes M_z)$:

$$[(M_z \otimes I)^*, (M_z \otimes I)] = P \otimes I,\ [(I \otimes M_z)^*, (I \otimes M_z)] = I \otimes P$$

and

$$[(I \otimes M_z)^*, (M_z \otimes I)] = 0 = [(M_z \otimes I)^*, (I \otimes M_z)].$$

Thus the commutator

$$\begin{aligned} [\![\boldsymbol{M}^*, \boldsymbol{M}]\!] &:= \begin{pmatrix} [(M_z \otimes I)^*, (M_z \otimes I)] & [(I \otimes M_z)^*, (M_z \otimes I)] \\ [(M_z \otimes I)^*, (I \otimes M_z)] & [(I \otimes M_z)^*, (I \otimes M_z)] \end{pmatrix} \\ &= \begin{pmatrix} P \otimes I & 0 \\ 0 & I \otimes P \end{pmatrix} \geq 0. \end{aligned} \tag{3.3}$$

This proves that $\boldsymbol{M}$ is hyponormal. In this example, we have

$$\operatorname{dEt}\big([\![\boldsymbol{M}^*, \boldsymbol{M}]\!]\big) = (P \otimes I)(I \otimes P) + (P \otimes I)(I \otimes P) = 2(P \otimes P).$$

Although, none of the non-zero commutators are compact, we see that $\operatorname{dEt}\big([\![\boldsymbol{M}^*, \boldsymbol{M}]\!]\big)$ is positive with $\operatorname{trace}\big(\operatorname{dEt}\big([\![\boldsymbol{M}^*, \boldsymbol{M}]\!]\big)\big) = 2$.

3.2.2 Hardy Space over Symmetrized Bidisk

The usual Hardy space $H^2(\mathbb{D}^2)$ is a module over the polynomial ring $\mathbb{C}[z_1, z_2]$ equipped with the module multiplication m_p given by the point-wise multiplication, namely, $m_p(f) = pf$, $p \in \mathbb{C}[z_1, z_2]$, $f \in H^2(\mathbb{D}^2)$. Obviously, there are several other possibilities for the module multiplication. In this subsection, we consider a different module multiplication, which up to unitary equivalence, is isomorphic to the Hardy module on the symmetrized bidisc. For this, first consider the commuting pair of operators $\boldsymbol{T} = (T_1, T_2)$:

$$T_1 = M_z \otimes I + I \otimes M_z \text{ and } T_2 = M_z \otimes M_z$$

acting on the Hardy space $H^2(\mathbb{D}^2)$. We have

$$[T_1^*, T_1] = P \otimes I + I \otimes P,\ [T_2^*, T_2] = I \otimes P + P \otimes P^{\perp}.$$

Similarly,

$$[T_2^*, T_1] = P \otimes M_z^* + M_z^* \otimes P,\ [T_1^*, T_2] = P \otimes M_z + M_z \otimes P.$$

The operator matrix associated to this pair (T_1, T_2) is

$$\begin{aligned} \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix} &= \begin{pmatrix} P \otimes I + I \otimes P & P \otimes M_z^* + M_z^* \otimes P \\ P \otimes M_z + M_z \otimes P & I \otimes P + P \otimes P^{\perp} \end{pmatrix} \\ &= \begin{pmatrix} P \otimes I & I \otimes P \\ P \otimes M_z & M_z \otimes P \end{pmatrix} \begin{pmatrix} P \otimes I & P \otimes M_z^* \\ I \otimes P & M_z^* \otimes P \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & P \otimes P \end{pmatrix} \\ &= \begin{pmatrix} P \otimes I & I \otimes P \\ P \otimes M_z & M_z \otimes P \end{pmatrix} \begin{pmatrix} P \otimes I & I \otimes P \\ P \otimes M_z & M_z \otimes P \end{pmatrix}^* + \begin{pmatrix} 0 & 0 \\ 0 & P \otimes P \end{pmatrix} \geq 0. \end{aligned}$$

Thus it follows that $\boldsymbol{T}$ is hyponormal. A simple computation gives

$$\operatorname{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big) = 2P \otimes P - PM_z^* \otimes M_z P - M_z P \otimes PM_z^*.$$

We note that the vector $1 \otimes 1$ is an eigenvector of $\operatorname{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$ with eigenvalue 2 while the vector $z \otimes 1 + 1 \otimes z$ is an eigenvector of $\operatorname{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$ with eigenvalue -1. Therefore the operator $\operatorname{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$ acting on $H^2(\mathbb{D}^2)$, is not nonnegative definite.

But if we restrict $\operatorname{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$ to the subspace $H^2_{\text{anti}}(\mathbb{D}^2)$ consisting of those functions in $H^2(\mathbb{D}^2)$ that are anti-symmetric, then it is nonnegative definite. Let $[\![2]\!]$ be the set of all pairs $\boldsymbol{p} = (p_1, p_2)$ such that $p_1 > p_2 \geq 0$, $p_1, p_2 \in \mathbb{N}_0$. Define

$$e_{\boldsymbol{p}}(\boldsymbol{z}) := \frac{z^{p_1} \otimes z^{p_2} - z^{p_2} \otimes z^{p_1}}{\sqrt{2}}.$$

Then $\{e_{\boldsymbol{p}}(\boldsymbol{z}) : \boldsymbol{p} \in [\![2]\!]\}$ is an orthonormal basis for the subspace $H^2_{\text{anti}}(\mathbb{D}^2)$, see [34]. It is also shown in that paper that $H^2_{\text{anti}}(\mathbb{D}^2)$ is module isomorphic to the Hardy module $H^2(G_2)$ on the symmetrized bidisc: $G_2 := \{(z_1 + z_2, z_1 z_2) : |z_1|, |z_2| < 1\}$. In other words, the multiplication by $p(T_1, T_2)$ on the Hardy space $H^2(\mathbb{D}^2)$ is unitarily equivalent to the multiplication by the pair of the coordinate functions on the Hardy space $H^2(G_2)$ of the symmetrized bidisc G_2. A direct and easy computation given below, using the orthonormal basis $\{e_{\boldsymbol{p}}\}$, shows that the operator $\operatorname{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$ is nonnegative definite and is in trace class:

$$\big\langle \operatorname{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big) e_{\boldsymbol{p}}, e_{\boldsymbol{p}} \big\rangle = \begin{cases} 1 & \text{if } \boldsymbol{p} = (1,0) \\ 0 & \text{otherwise}. \end{cases}$$

Remark. The pair of multiplication operators on $H^2(\mathbb{D}^2)$ and also on $H^2(G_2)$ are subnormal.

3.3 Generalized Commutator

Given any d- tuple of operators $\boldsymbol{A}$, not necessarily commuting, it is not clear what represents the degree of noncommutativity among these operators. For two operators, the answer is clear. Helton and Howe proposed the following notion of a generalized commutator that has proved to be quite useful, (see [24, Section A, p. 272]).

Definition 3.3 (Helton-Howe)**.** Let $\boldsymbol{A} = (A_1, \ldots, A_d)$ be a d- tuple of bounded operators. The generalized commutator $\operatorname{GC}(\boldsymbol{A})$ is defined to be the sum

$$\sum_{\sigma \in \mathfrak{S}_d} \operatorname{Sgn}(\sigma) A_{\sigma(1)} A_{\sigma(2)}, \ldots, A_{\sigma(d)}.$$

We adapt the definition of Helton and Howe slightly to the case of a commuting tuple of operators $\boldsymbol{T}$ as follows. Let $\boldsymbol{T} = (T_1, \dots, T_d)$ be a d- tuple of operators. Let $A_1 = T_1^*, A_2 = T_1, \dots, A_{2d-1} = T_d^*, A_{2d} = T_d$. The generalized commutator $\mathrm{GC}(\boldsymbol{T}^*, \boldsymbol{T})$ is defined to be the sum

$$\mathrm{GC}(\boldsymbol{T}^*, \boldsymbol{T}) := \sum_{\sigma \in \mathfrak{S}_{2d}} \mathrm{Sgn}(\sigma) A_{\sigma(1)} A_{\sigma(2)}, \dots, A_{\sigma(2d)}. \tag{3.4}$$

For a pair of commuting operators $\boldsymbol{T} = (T_1, T_2)$,

$$\begin{aligned}\mathrm{GC}(\boldsymbol{T}^*, \boldsymbol{T}) = T_1^* T_1 T_2^* T_2 + T_2^* T_2 T_1^* T_1 + T_1 T_1^* T_2 T_2^* + T_2 T_2^* T_1 T_1^* \\ - T_1^* T_2 T_2^* T_1 - T_2^* T_1 T_1^* T_2 - T_1 T_2^* T_2 T_1^* - T_2 T_1^* T_1 T_2^*.\end{aligned}$$

Thus from the Equation 3.2 it follows that for any pair of commuting operators $\mathrm{GC}(\boldsymbol{T}^*, \boldsymbol{T}) = \mathrm{dEt}\left(\left[\!\left[\boldsymbol{T}^*, \boldsymbol{T}\right]\!\right]\right)$. We now show that the $\mathrm{dEt}\left(\left[\!\left[\boldsymbol{T}^*, \boldsymbol{T}\right]\!\right]\right)$ and $\mathrm{GC}(\boldsymbol{T}^*, \boldsymbol{T})$ coincide for any commuting tuple $\boldsymbol{T}$. We emphasize that the equality need not hold unless $\boldsymbol{T}$ is a d - tuple of commuting operators. In this case, working witth the $\mathrm{dEt}\left(\left[\!\left[\boldsymbol{T}^*, \boldsymbol{T}\right]\!\right]\right)$ has some advantages over $\mathrm{GC}(\boldsymbol{T}^*, \boldsymbol{T})$ since a number of terms in $\mathrm{GC}(\boldsymbol{T}^*, \boldsymbol{T})$ cancel, in case $\boldsymbol{T}$ is d - tuple of commuting operators, and it equals the less formidable expression for $\mathrm{dEt}\left(\left[\!\left[\boldsymbol{T}^*, \boldsymbol{T}\right]\!\right]\right)$.

Proposition 3.4. *For any d-tuple $\boldsymbol{T}$ of commuting operators, the determinant*

$$\mathrm{dEt}\left(\left[\!\left[\boldsymbol{T}^*, \boldsymbol{T}\right]\!\right]\right) = GC(\boldsymbol{T}^*, \boldsymbol{T}).$$

Proof. By definition, we have

$$\begin{aligned}\mathrm{dEt}\left(\left[\!\left[\boldsymbol{T}^*, \boldsymbol{T}\right]\!\right]\right) &= \sum_{\tau, \sigma \in \mathfrak{S}_d} \mathrm{Sgn}(\sigma) \prod_{i=1}^{d} B_{\tau(i)\sigma(\tau(i))} \\ &= \sum_{\tau, \eta \in \mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta) \prod_{i=1}^{d} B_{\tau(i)\eta(i)},\end{aligned}$$

where $B_{ij} = [T_j^*, T_i]$ and $\eta = \sigma\tau$.

Fix a commuting tuple of operators $\boldsymbol{T}$. Suppose one of the terms in $\mathrm{GC}(\boldsymbol{T}^*, \boldsymbol{T})$ has a string of the form PT_iT_jQ, where P and Q are products of operators taken from the remaining set of $(2d-2)$ operators: $(T_1^*, \dots, T_d^*, T_1, \dots, \hat{T}_i, \dots, \hat{T}_j, \dots, T_d)$. (Here i, j are from $\{1, 2, \dots, d\}$ and $\hat{T}$ means that it is not included in the set.) Then there must be a second term in $\mathrm{GC}(\boldsymbol{T}^*, \boldsymbol{T})$ of the form PT_jT_iQ with the opposite sign. However these have to cancel since $T_iT_j = T_jT_i$. A similar argument applies to strings of the form $RT_i^*T_j^*S$. Thus the only terms that survive are those in which a T_i must be followed by a T_j^* and a T_j^* must be followed by a T_i.

There are two sets of terms in $G(\boldsymbol{T}^*,\boldsymbol{T})$, one set which begins with a T^* and another set which begins with a T. Indeed, we have

$$\begin{aligned}\mathrm{GC}(\boldsymbol{T}^*,\boldsymbol{T}) = &\sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta)\, T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)} \dots T^*_{\eta(d)} T_{\tau(d)} \\ &+ (-1)^d \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta)\, T_{\tau(1)} T^*_{\eta(1)} T_{\tau(2)} \dots T_{\tau(d)} T^*_{\eta(d)}.\end{aligned} \tag{3.5}$$

The terms starting with a T^*, which is the first sum in $\mathrm{GC}(\boldsymbol{T}^*,\boldsymbol{T})$ simplifies:

$$\begin{aligned}&\sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta)\, T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)} \dots T^*_{\eta(d)} T_{\tau(d)} \\ &= \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta) \big[T^*_{\eta(1)}, T_{\tau(1)}\big] T^*_{\eta(2)} \dots T^*_{\eta(d)} T_{\tau(d)} \\ &+ \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta)\, T_{\tau(1)} T^*_{\eta(1)} T^*_{\eta(2)} \dots T^*_{\eta(d)} T_{\tau(d)}.\end{aligned}$$

If $d \geq 1$ the second sum on the right is zero since there are two terms containing the string $T^*_{\eta(1)} T^*_{\eta(2)}$ with opposite signs. Repeating this process (note that the vanishing argument does not apply at the last stage), we get

$$\begin{aligned}&\sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta)\, T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)} \dots T^*_{\eta(d)} T_{\tau(d)} \\ &\quad= \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta) \big[T^*_{\eta(1)}, T_{\tau(1)}\big]\big[T^*_{\eta(2)}, T_{\tau(2)}\big] \dots \big[T^*_{\eta(d)}, T_{\tau(d)}\big] \\ &+ \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta) \big[T^*_{\eta(1)}, T_{\tau(1)}\big]\big[T^*_{\eta(2)}, T_{\tau(2)}\big] \dots \big[T^*_{\eta(d-1)}, T_{\tau(d-1)}\big] T_{\tau(d)} T^*_{\eta(d)}.\end{aligned} \tag{3.6}$$

The terms starting with a T, which is the second sum in $\mathrm{GC}(\boldsymbol{T}^*,\boldsymbol{T})$, using the equality

$$T_{\tau(i)} T^*_{\eta(i)} = -[T^*_{\eta(i)}, T_{\tau(i)}] + T^*_{\eta(i)} T_{\tau(i)},$$

simplifies:

$$\begin{aligned}&\sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta)\, T_{\tau(1)} T^*_{\eta(1)} T_{\tau(2)} \dots T_{\tau(d)} T^*_{\eta(d)} \\ &= -\sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta) \big[T^*_{\eta(1)}, T_{\tau(1)}\big] T_{\tau(2)} \dots T_{\tau(d)} T^*_{\eta(d)} \\ &+ \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta)\, T^*_{\eta(1)} T_{\tau(1)} T_{\tau(2)} \dots T_{\tau(d)} T^*_{\eta(d)}.\end{aligned}$$

If $d \geq 1$ the second sum on the right is zero since there is a string with $T_{\tau(1)} T_{\tau(2)}$. Repeating this process $d-1$ times, we get

$$\begin{aligned}
&(-1)^d \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta)\, T_{\tau(1)} T^*_{\eta(1)} T_{\tau(2)} \dots T_{\tau(d)} T^*_{\eta(d)} \\
&= (-1)^d(-1)^{d-1} \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta)\big[T^*_{\eta(1)}, T_{\tau(1)}\big]\big[T^*_{\eta(2)}, T_{\tau(2)}\big]\dots\big[T^*_{\eta(d-1)}, T_{\tau(d-1)}\big] T_{\tau(d)} T^*_{\eta(d)}.
\end{aligned} \tag{3.7}$$

Adding the two sums on the right hand side of the equation (3.6) and the one on the right hand side of the equation (3.7), we get

$$\begin{aligned}
\mathrm{GC}(\boldsymbol{T}^*, \boldsymbol{T}) &= \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta)[T^*_{\eta(1)}, T_{\tau(1)}][T^*_{\eta(2)}, T_{\tau(2)}]\dots[T^*_{\eta(d)}, T_{\tau(d)}] \\
&= \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta) B_{\tau(1)\eta(1)} B_{\tau(2)\eta(2)} \dots B_{\tau(d)\eta(d)} \\
&= \mathrm{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)
\end{aligned}$$

completing the verification that $GC(\boldsymbol{T}^*, \boldsymbol{T}) = \mathrm{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$ for a commuting tuple $\boldsymbol{T}$. □

This proof was made up with substantial help from Dr. Cherian Varughese.

3.4 The class $BS_{m,\vartheta}(\Omega)$

Let $\Omega \subset \mathbb{C}^d$ be a bounded domain and let $\mathscr{H} \subset \mathrm{Hol}(\Omega)$ be a Hilbert space. A commuting tuple of bounded linear operators $\boldsymbol{T} = (T_1, \dots, T_d)$ defines a module multiplication on $\mathscr{H}$ over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$ via the map

$$\mathfrak{p}_{\boldsymbol{T}}(h) = p(\boldsymbol{T})h,\ p \in \mathbb{C}[\boldsymbol{z}],\ h \in \mathscr{H}.$$

Evidently, $p \to \mathfrak{p}_{\boldsymbol{T}}$ is an algebra homomorphism.

Definition 3.5. Let $\boldsymbol{\xi}[n]$ denote a set of linearly independent vectors $\xi_1, \dots, \xi_n$ in $\mathscr{H}$. For a commuting tuple of operators $\boldsymbol{T} = (T_1, \dots, T_d)$, we say that $\boldsymbol{\xi}[k]$ is cyclic for $\boldsymbol{T}$ if the linear span of the vectors

$$\Big\{T_1^{i_1} T_2^{i_2} \dots T_d^{i_d} v |\ v \in \boldsymbol{\xi}[k] \text{ and } i_1, i_2, \dots, i_d \geq 0\Big\}$$

is dense in $\mathscr{H}$. The commuting tuple $\boldsymbol{T}$ is said to be *m-polynomially cyclic*, where

$$m = \min\{k : \boldsymbol{\xi}[k] \text{ is cyclic for } \boldsymbol{T}\}.$$

The set $\boldsymbol{\xi}_{\boldsymbol{T}}[m]$ is then said to be m-cyclic for $\boldsymbol{T}$.

Remark 3.6. 1. In this thesis, we assume that the spectrum of the d-tuple of operators $\boldsymbol{T}$ is *polynomially convex*. Consequently polynomially cyclic, as opposed to rationally cyclic, would be the natural hypothesis for us. Therefore, we write *"m-cyclic"* instead of m-polynomially cyclic throughout the thesis.

2. A second consequence of the polynomial convexity is that $\boldsymbol{T}$ is m- cyclic if and only if the subspace

$$\left\{f(\boldsymbol{T})v \mid v \in \boldsymbol{\xi}[m] \text{ and } f \in \mathrm{Hol}(\sigma(\boldsymbol{T}))\right\}$$

is dense in $\mathscr{H}$. Here, $\mathrm{Hol}(\sigma(\boldsymbol{T}))$ is the algebra of all functions which are holomorphic in some open neighbourhood of the closed set $\sigma(\boldsymbol{T})$. The operator $f(\boldsymbol{T})$ is then defined using the usual holomorphic functional calculus.

Definition 3.7. For a m-cyclic d-tuple $\boldsymbol{T}$, let

$$\mathscr{H}_N := \bigvee\left\{T_1^{i_1}T_2^{i_2}\dots T_d^{i_d}v \mid v \in \boldsymbol{\xi}[m] \text{ and } 0 \le i_1 + i_2 + \dots i_d \le N\right\}$$

and P_N be the orthogonal projection onto $\mathscr{H}_N$.

We list below some of the basic properties of the projection P_N that will be used in the proof of the main theorem.

Lemma 3.8. *For a m-cyclic d-tuple of operators $\boldsymbol{T}$, we have P_N increasing strongly to I and $rank(P_N^{\perp}T_jP_N) \le m\binom{N+d-1}{d-1}$, $j = 1, \dots, d$.*

Proof. Evidently, $\mathscr{H}_N \subseteq \mathscr{H}_{N+1}$ and hence the projections $\{P_N\}$ are increasing. By hypothesis, $\boldsymbol{T}$ is m-cyclic, therefore by definition, the linear span of $\{\mathscr{H}_N : N \in \mathbb{N}_0\}$ is dense in $\mathscr{H}$. The number of vectors from $\mathscr{H}_N$ that are pushed out of it by the operator T_j provides a reasonable upper bound on the rank of the operator $P_N^{\perp}T_jP_N$. Such vectors can only be a subset of the subspace $\mathscr{H}_N \ominus \mathscr{H}_{N-1}$. Clearly, the dimension of this subspace is the same as the dimension of the space of homogeneous polynomials of degree N in d-variables tensored with $\mathbb{C}^m$, which is $m\binom{N+d-1}{d-1}$. Therefore, $\mathrm{rank}(P_N^{\perp}T_jP_N) \le m\binom{N+d-1}{d-1}$. □

We recall a well-known inequality between the trace norm and the operator norm of a finite rank bounded operator.

Lemma 3.9. *If $F \in \mathscr{B}(\mathscr{H})$ is of finite rank, then $\|F\|_1 \le (rank F)\|F\|$.*

Proof. Since F is a finite rank operator, choosing an arbitrary but fixed $\{\varphi_1, \dots, \varphi_n\}$ orthonormal basis for the range of F, for any $x \in \mathscr{H}$, we have

$$Fx = \sum_{k=1}^{n} \langle x, v_k\rangle \varphi_k \tag{3.8}$$

for some set of n vectors $\{v_1,\ldots,v_n\}$ in $\mathscr{H}$. To see this, observe that $Fx=\sum_{k=1}^n\langle Fx,\varphi_k\rangle\varphi_k$. For any fixed k, $1\le k\le n$, the map $x\mapsto\langle Fx,\varphi_k\rangle$ is a bounded linear functional because F is a bounded operator. Therefore, there is a $v_k\in\mathscr{H}$ such that $\langle Fx,\varphi_k\rangle=\langle x,v_k\rangle$. (Here, the vector v_k depends on the operator F.) For any pair of vectors $x,y\in\mathscr{H}$, we see that

$$\begin{aligned}\langle Fx,y\rangle&=\Big\langle\sum_{k=1}^n\langle x,v_k\rangle\varphi_k,y\Big\rangle\\&=\sum_{i=1}^n\langle x,v_k\rangle\langle\varphi_k,y\rangle\\&=\Big\langle x,\Big(\sum_{k=1}^n\langle\varphi_k,y\rangle v_k\Big)\Big\rangle.\end{aligned}$$

Therefore, we have proved that $F^*y=\left(\sum_{k=1}^n\langle\varphi_k,y\rangle v_k\right)$. It follows that

$$\begin{aligned}F^*Fx&=F^*\Big(\sum_{k=1}^n\langle x,v_k\rangle\varphi_k\Big)\\&=\sum_{j=1}^n\Big\langle\varphi_j,\sum_{i=1}^n\langle x,v_i\rangle\varphi_i\Big\rangle v_j\\&=\sum_{j=1}^n\langle x,v_j\rangle v_j\end{aligned}$$

Therefore, setting V to be the linear span of the vectors $\{v_1,\ldots,v_n\}$, we have $F^*F(V)\subseteq V$. The restriction of F^*F to $V^\perp$ is zero. Hence $F^*F=(F^*F)_{|V}\oplus 0$. Since V is finite dimensional and for any finite dimensional positive operator A, we have the inequality $\|A\|_1\le(\text{rank A})\,\|A\|$, we conclude that

$$\|F\|_1=\left\|(F^*F)^{1/2}\right\|_1\le(\operatorname{rank}F)\|F\|$$

completing the proof of the theorem. □

We next define a class $BS_{m,\vartheta}(\Omega)$ of d - tuples of commuting operators. The rest of this chapter is devoted to showing that if $\boldsymbol{T}$ is in $BS_{m,\vartheta}(\Omega)$, then $\operatorname{trace}\big(\operatorname{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)\big)$ is finite. Moreover, if $\boldsymbol{T}$ is in $BS_{m,\vartheta}(\Omega)$, then

$$\operatorname{trace}\big(\operatorname{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)\big)\le m\,\vartheta\,d!\prod_{i=1}^d\|T_i\|^2.$$

Definition 3.10. Fix a bounded domain $\Omega\subset\mathbb{C}^d$ such that $\overline{\Omega}$ is polynomially convex. A m-cyclic commuting d - tuple of operators with $\sigma(\boldsymbol{T})=\overline{\Omega}$ is said to be in the class $BS_{m,\vartheta}(\Omega)$, if

(i) $P_NT_jP_N^\perp=0$, $j=1,\ldots,d$.

(ii) $\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$ is non-negative definite.

(iii) For a fixed but arbitrary τ in the permutation group $\mathfrak{S}_d$ of d symbols, there exists $\vartheta\in\mathbb{N}$, independent of N, such that

$$\|P_N\big(\sum_{\eta\in\mathfrak{S}_d}\mathrm{Sgn}(\eta)\,T^*_{\eta(1)}T_{\tau(1)}T^*_{\eta(2)}\dots T^*_{\eta(d)}\big)P_N^{\perp}T_{\tau(d)}P_N\|\le\vartheta\begin{pmatrix}N+d-1\\ d-1\end{pmatrix}^{-1}\prod_{i=1}^{d}\|T_i\|^2.$$

Remark 3.11. (a) For a single operator T on a Hilbert space $\mathscr{H}$, condition (iii) of Definition 3.10 reduces to

$$\|P_NT^*P_N^{\perp}TP_N\|\le\vartheta\|T\|^2,$$

which is true with $\vartheta=1$. It follows that a m-cyclic hyponormal operator T with $\sigma(T)=\overline{\Omega}$ is in the class $BS_{m,1}(\Omega)$, if $P_NTP_N^{\perp}=0$.

(b) If for each $\tau\in\mathfrak{S}_d$, there exists a unitary operator U_τ on the Hilbert space such that

$$U_\tau T_{\tau(i)}U_\tau^*=T_i,\ 1\le i\le d,$$

then it is enough to check condition (iii) for identity permutation, that is,

$$\|P_N\big(\sum_{\eta\in\mathfrak{S}_d}\mathrm{Sgn}(\eta)\,T^*_{\eta(1)}T_1T^*_{\eta(2)}\dots T_{d-1}T^*_{\eta(d)}\big)P_N^{\perp}T_dP_N\|\le\vartheta\begin{pmatrix}N+d-1\\ d-1\end{pmatrix}^{-1}\prod_{i=1}^{d}\|T_i\|^2$$

implies all the other inequalities, one for each τ of (iii).

To see this, pick $\tau_0\in\mathfrak{S}_d$ such that $\eta=\tau\cdot\tau_0$. With this choice of τ_0, we have $\mathrm{Sgn}\,\eta=\mathrm{Sgn}\,\tau\,\mathrm{Sgn}\,\tau_0$. It now follows that

$$\begin{aligned}&P_N\big(\sum_{\eta\in\mathfrak{S}_d}\mathrm{Sgn}(\eta)\,T^*_{\eta(1)}T_{\tau(1)}T^*_{\eta(2)}\dots T_{\tau(d-1)}T^*_{\eta(d)}\big)P_N^{\perp}T_{\tau(d)}P_N\\&=P_NU_\tau\big(\mathrm{Sgn}(\tau)\sum_{\hat\eta\in\mathfrak{S}_d}\mathrm{Sgn}(\hat\eta)\,T^*_{\hat\eta(1)}T_1T^*_{\hat\eta(2)}\dots T_{d-1}T^*_{\hat\eta(d)}\big)U_\tau^*P_N^{\perp}U_\tau T_dU_\tau^*P_N\\&=\mathrm{Sgn}(\tau)U_\tau P_N\big(\sum_{\hat\eta\in\mathfrak{S}_d}\mathrm{Sgn}(\hat\eta)\,T^*_{\hat\eta(1)}T_1T^*_{\hat\eta(2)}\dots T_{d-1}T^*_{\hat\eta(d)}\big)P_N^{\perp}T_dP_NU_\tau^*.\end{aligned}$$

(c) There exists unitary representation U of the symmetric group $\mathfrak{S}_d$ and commuting d -tuples of operators $\boldsymbol{T}$ in $BS_{1,1}(\mathbb{B}_d)$ with the property

$$U_\tau T_{\tau(i)}U_\tau^*=T_i,\ \tau\in\mathfrak{S}_n,\ 1\le i\le d.$$

Explicit examples are given in the following section and in [11].

3.5 Examples of operators in the class $BS_{m,\vartheta}(\Omega)$

We consider two sets of examples, the first is based on the Euclidean ball $\mathbb{B}_d \subset \mathbb{C}^d$ while the second set of examples comes from considering the ball $\mathbb{B}_{2,1} := \{(z_1, z_2) : |z_1|^2 + |z_2| < 1\} \subseteq \mathbb{C}^2$.

3.5.1 The case of the Euclidean ball $BS_{1,1}(\mathbb{B}_d)$

In the following examples, we have taken $\Omega = \mathbb{B}_d$. Let $H^{(\lambda)}(\mathbb{B}_d)$ be the weighted Bergman spaces of the unit Euclidean ball $\mathbb{B}_d$. In particular, $\lambda = d$ is the Hardy space $H^2(\mathbb{B}_d)$. These spaces are determined by the orthonormal set of vectors:

$$\{c_{\boldsymbol{\alpha}}^{(\lambda)} z_1^{\alpha_1} \cdots z_d^{\alpha_d} : \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d\},$$

where $\mathbb{N}_0$ is the set of non-negative integers and $c_{\boldsymbol{\alpha}}^{(\lambda)} = \frac{(\lambda)_{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!}$. Here

$$(\lambda)_n := \lambda(\lambda+1)\cdots(\lambda+n-1)$$

is the Pochhammer symbol and $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_d$. Let $\boldsymbol{M}^{(\lambda)} = (M_1^{(\lambda)}, \ldots, M_d^{(\lambda)})$ be the d-tuple of multiplication by the coordinate functions on $H^{(\lambda)}(\mathbb{B}_d)$. These are examples of *spherical operators* [5, 11] described below.

Let $\mathscr{U}(d)$ be the group of unitary linear transformations on $\mathbb{C}^d$, let $\boldsymbol{T}$ be a commuting d-tuple of bounded linear operators on $\mathscr{H}$ and finally, let $\mathscr{U}(\mathscr{H})$ be the group of unitary linear transformations on $\mathscr{H}$. Clearly, the group $\mathscr{U}(d)$ acts on any commuting d-tuple of operators $\boldsymbol{T}$, namely,

$$U \cdot \boldsymbol{T} := \Big(\sum_{j=1}^d U_{1j} T_j, \ldots, \sum_{j=1}^d U_{dj} T_j\Big),\ U = \big(\!\big(U_{ij}\big)\!\big) \in \mathscr{U}(d). \tag{3.9}$$

The d-tuple $\boldsymbol{T}$ is said to be *spherical* if there is a map $\Gamma : \mathscr{U}(d) \to \mathscr{U}(\mathscr{H})$ such that

$$\Gamma_U \boldsymbol{T} \Gamma_U^* := (\Gamma_U T_1 \Gamma_U^*, \ldots, \Gamma_U T_d \Gamma_U^*) = U \cdot \boldsymbol{T} \text{ for all } U \in \mathscr{U}(d). \tag{3.10}$$

The set of vectors

$$e_{\boldsymbol{\alpha}}(\boldsymbol{z}) = \sqrt{\tfrac{(d)_{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!}}\, z_1^{\alpha_1} \cdots z_d^{\alpha_d},\ \boldsymbol{\alpha} \in \mathbb{N}_0^d, \boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_d!$$

is an orthonormal basis of $H^2(\mathbb{B}_d)$. The d- tuple $\boldsymbol{S}$ of multiplication operators by the coordinate functions and its adjoint $\boldsymbol{S}^*$ on the Hardy space $H^2(\mathbb{B}_d)$ are commuting tuples of weighted shift operators:

$$S_i e_{\boldsymbol{\alpha}} = \sqrt{\tfrac{\alpha_i + 1}{|\boldsymbol{\alpha}| + d}}\, e_{\alpha + \epsilon_i},\ S_i^* e_{\boldsymbol{\alpha}} = \begin{cases} \sqrt{\tfrac{\alpha_i}{|\boldsymbol{\alpha}| + d - 1}}\, e_{\boldsymbol{\alpha} - \epsilon_i} & \text{if } \alpha_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the operator $\boldsymbol{S}$ is the same as the commuting tuple $\boldsymbol{M}^{(d)}$. However, it is convenient to use a different notation for this particular d-tuple as will be apparent soon. The basic properties of commuting tuples of weighted shifts, also called joint weighted shifts, are in [26].

The proof of the two main results of this Section, appearing below, are given for the case of $d = 2$. The proof, in general, is obtained inductively starting from this case. The details are in the Appendix.

Theorem 3.12. *For the d-tuple $\boldsymbol{S}$ of multiplication by the coordinate functions on the Hardy space $H^2(\mathbb{B}_d)$, the operator $dEt\big([\![\boldsymbol{S}^*,\boldsymbol{S}]\!]\big)$ is non-negative definite and trace$\big(dEt\big([\![\boldsymbol{S}^*,\boldsymbol{S}]\!]\big)\big) = 1$.*

Proof. ($d = 2$): In this particular case, we have

$$S_i(e_{\boldsymbol{\alpha}}) = w_{\boldsymbol{\alpha}}^{(i)} e_{\boldsymbol{\alpha}+\epsilon_i}, \text{ where } w_{\boldsymbol{\alpha}}^{(i)} = \sqrt{\frac{\alpha_i+1}{|\boldsymbol{\alpha}|+2}},\ i = 1,2,$$

relative to the orthonormal basis $\{e_{\boldsymbol{\alpha}}\}$ of the Hardy space $H^2(\mathbb{B}_2)$ and

$$S_i^*(e_{\boldsymbol{\alpha}}) = \begin{cases} w_{\boldsymbol{\alpha}-\epsilon_i}^{(i)} e_{\boldsymbol{\alpha}-\epsilon_i}, & \text{for } \alpha_i > 0,\\ 0 & \text{otherwise.}\end{cases}$$

A simple computation gives

$$\begin{aligned}
[S_i^*,S_i](e_{\boldsymbol{\alpha}}) &= \{(w_{\boldsymbol{\alpha}}^{(i)})^2 - (w_{\boldsymbol{\alpha}-\epsilon_i}^{(i)})^2\}e_{\boldsymbol{\alpha}}\\
[S_i^*,S_j](e_{\boldsymbol{\alpha}}) &= \{(w_{\boldsymbol{\alpha}-\epsilon_i+\epsilon_j}^{(i)})(w_{\boldsymbol{\alpha}}^{(j)}) - (w_{\boldsymbol{\alpha}-\epsilon_i}^{(j)})(w_{\boldsymbol{\alpha}-\epsilon_i}^{(i)})\}e_{\boldsymbol{\alpha}-\epsilon_i+\epsilon_j}.
\end{aligned}$$

So the action of the determinant operator on the basis vectors e_α is given by the formula:

$$\begin{aligned}
&\mathrm{dEt}([\![\boldsymbol{S}^*,\boldsymbol{S}]\!])(e_{\boldsymbol{\alpha}})\\
&= \big([S_1^*,S_1][S_2^*,S_2] + [S_2^*,S_2][S_1^*,S_1] - [S_1^*,S_2][S_2^*,S_1] - [S_2^*,S_1][S_1^*,S_2]\big)(e_{\boldsymbol{\alpha}})\\
&= \Big\{\{(w_{\boldsymbol{\alpha}}^{(1)})^2 - (w_{\boldsymbol{\alpha}-\epsilon_1}^{(1)})^2\}\{(w_{\boldsymbol{\alpha}}^{(2)})^2 - (w_{\boldsymbol{\alpha}-\epsilon_2}^{(2)})^2\}\\
&+ \{(w_{\boldsymbol{\alpha}}^{(1)})^2 - (w_{\boldsymbol{\alpha}-\epsilon_1}^{(1)})^2\}\{(w_{\boldsymbol{\alpha}}^{(2)})^2 - (w_{\boldsymbol{\alpha}-\epsilon_2}^{(2)})^2\}\\
&- \{(w_{\boldsymbol{\alpha}}^{(2)})(w_{\boldsymbol{\alpha}-\epsilon_1+\epsilon_2}^{(1)}) - (w_{\boldsymbol{\alpha}-\epsilon_1}^{(1)})(w_{\boldsymbol{\alpha}-\epsilon_1}^{(2)})\}\{(w_{\boldsymbol{\alpha}-\epsilon_1+\epsilon_2}^{(1)})(w_{\boldsymbol{\alpha}}^{(2)}) - (w_{\boldsymbol{\alpha}-\epsilon_1}^{(2)})(w_{\boldsymbol{\alpha}-\epsilon_1}^{(1)})\}\\
&-\{(w_{\boldsymbol{\alpha}}^{(1)})(w_{\boldsymbol{\alpha}-\epsilon_2+\epsilon_1}^{(2)}) - (w_{\boldsymbol{\alpha}-\epsilon_2}^{(2)})(w_{\boldsymbol{\alpha}-\epsilon_2}^{(1)})\}\{(w_{\boldsymbol{\alpha}-\epsilon_2+\epsilon_1}^{(2)})(w_{\boldsymbol{\alpha}}^{(1)}) - (w_{\boldsymbol{\alpha}-\epsilon_2}^{(1)})(w_{\boldsymbol{\alpha}-\epsilon_2}^{(2)})\}\Big\}(e_{\boldsymbol{\alpha}})\\
&= \Big(\frac{1}{(|\boldsymbol{\alpha}|+2)} - \frac{|\boldsymbol{\alpha}|}{(|\boldsymbol{\alpha}|+1)^2}\Big)e_{\boldsymbol{\alpha}}.
\end{aligned}$$

Since $\big(\frac{1}{(|\boldsymbol{\alpha}|+2)} - \frac{|\boldsymbol{\alpha}|}{(|\boldsymbol{\alpha}|+1)^2}\big) \geq 0$, it follows that $\mathrm{dEt}\big([\![\boldsymbol{S}^*,\boldsymbol{S}]\!]\big)$ is non-negative definite. We now compute the trace of the determinant operator:

$$\begin{aligned}\operatorname{trace}\big(\mathrm{dEt}\big([\![\boldsymbol{S}^*,\boldsymbol{S}]\!]\big)\big) &= \sum_{\alpha\in\mathbb{N}_0^2}\big\langle \mathrm{dEt}\big([\![\boldsymbol{S}^*,\boldsymbol{S}]\!]\big)e_{\boldsymbol{\alpha}}, e_{\boldsymbol{\alpha}}\big\rangle\\ &= \sum_{k=0}^{\infty}\sum_{\substack{\alpha_1,\alpha_2\\|\boldsymbol{\alpha}|=k}}\Big(\frac{1}{(|\boldsymbol{\alpha}|+2)}-\frac{|\boldsymbol{\alpha}|}{(|\boldsymbol{\alpha}|+1)^2}\Big)\\ &= \sum_{k=0}^{\infty}\Big(\frac{k+1}{k+2}-\frac{k}{k+1}\Big)\\ &= 1.\end{aligned}$$

This completes the proof. □

For $\boldsymbol{a}\in\mathbb{C}^d$ and $r>0$, let $\mathbb{B}[\boldsymbol{a},r]$ be the ball $\{\boldsymbol{z}\in\mathbb{C}^d : \|\boldsymbol{z}-\boldsymbol{a}\|_2<r\}$. We let $\mathbb{B}[r]$ denote the ball of radius r centred at 0. Finally, $\mathbb{B}_d$ is the unit ball in $\mathbb{C}^d$.

Theorem 3.13. *Let $\boldsymbol{T}$ be a d- tuple of spherical joint weighted shift operators and T_δ be the one variable weighted shift corresponding to $\boldsymbol{T}$. If T_δ is hyponormal, then $\boldsymbol{T}$ is in $BS_{1,1}(\mathbb{B}[r])$, where $\mathbb{B}[r]=\{\boldsymbol{z}\in\mathbb{C}^d : \|\boldsymbol{z}\|_2<r\}$, $r>0$.*

Proof. ($d=2$): Clearly, the d- tuple $\boldsymbol{T}$ is 1- cyclic. The joint spectrum of $\boldsymbol{T}$ is $\mathbb{B}[r]$ for some $r>0$, where the radius r of the ball $\mathbb{B}[r]$ is determined by $\{\delta_{|\boldsymbol{\alpha}|}\}$ (cf. [11, Proposition 3.2]). A calculation similar to the one in the proof of the previous theorem gives the equality:

$$\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)(x_{\boldsymbol{\alpha}}) = \Big(\frac{\delta^4_{|\boldsymbol{\alpha}|}}{(|\boldsymbol{\alpha}|+2)}-\frac{|\boldsymbol{\alpha}|\delta^4_{|\boldsymbol{\alpha}|-1}}{(|\boldsymbol{\alpha}|+1)^2}\Big)x_{\boldsymbol{\alpha}}.$$

Since $\delta_{|\boldsymbol{\alpha}|}$ is an increasing sequence, it follows that $\Big(\frac{\delta^4_{|\boldsymbol{\alpha}|}}{(|\boldsymbol{\alpha}|+2)}-\frac{|\boldsymbol{\alpha}|\delta^4_{|\boldsymbol{\alpha}|-1}}{(|\boldsymbol{\alpha}|+1)^2}\Big)\geq\frac{|\boldsymbol{\alpha}|\delta^4_{|\boldsymbol{\alpha}|-1}}{(|\boldsymbol{\alpha}|+2)(|\boldsymbol{\alpha}|+1)^2}\geq 0$. Hence $\mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$ is non-negative definite. Now it only remains to check the norm estimates in the Definition 3.10. A straightforward computation shows that

$$\Big(\sum_{\eta\in\mathfrak{S}_2}\mathrm{Sgn}(\eta)\,T^*_{\eta(1)}T_{\tau(1)}T^*_{\eta(2)}\Big)x_{\boldsymbol{\alpha}} = \mathrm{Sgn}(\tau)\delta^3_{|\boldsymbol{\alpha}|}\sqrt{\frac{\alpha_{\tau(2)}}{|\boldsymbol{\alpha}|+1}}(|\boldsymbol{\alpha}|+1)^{-1}x_{\boldsymbol{\alpha}-\epsilon_{\tau(2)}}.$$

Consequently, we have

$$\begin{aligned}\|P_N\Big(\sum_{\eta\in\mathfrak{S}_2}\mathrm{Sgn}(\eta)\,T^*_{\eta(1)}T_{\tau(1)}T^*_{\eta(2)}\Big)P_N^{\perp}\| &\leq \max_{|\boldsymbol{\alpha}|=N+1}\Big\{\delta^3_{|\alpha|}\sqrt{\frac{\alpha_{\tau(2)}}{|\boldsymbol{\alpha}|+1}}(|\boldsymbol{\alpha}|+1)^{-1}\Big\}\\ &= \frac{\delta^3_{N+1}}{N+2}\\ &\leq \frac{1}{N+2}\|T_{\tau(1)}\|^2\|T_{\tau(2)}\|.\end{aligned}$$

Thus

$$\|P_N\big(\sum_{\eta\in\mathfrak{S}_2}\mathrm{Sgn}(\eta)T^*_{\eta(1)}T_{\tau(1)}T^*_{\eta(2)}\big)P_N^{\perp}T_{\tau(2)}P_N\| \le \frac{1}{N+2}\|T_{\tau(1)}\|^2\|T_{\tau(2)}\|^2.$$

Since $\boldsymbol{T}$ is a d - tuple of joint weighted shifts, by definition, $P_N T_j P_N^{\perp} = 0$, $i = 1,\dots,d$. □

Corollary 3.14. *Suppose $\delta_n \uparrow 1$ then $trace\big(dEt([\![\boldsymbol{T}^*,\boldsymbol{T}]\!])\big) = 1$.*

Proof. The string of equalities

$$\begin{aligned}
\mathrm{trace}\big(\mathrm{dEt}([\![\boldsymbol{T}^*,\boldsymbol{T}]\!])\big) &= \sum_{\alpha\in\mathbb{N}^d}\big\langle \mathrm{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)x_{\boldsymbol{\alpha}}, x_{\boldsymbol{\alpha}}\big\rangle\\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^d}(d-1)!\Big(\frac{\delta^{2d}_{|\alpha|}}{(|\boldsymbol{\alpha}|+d)^{(d-1)}} - \frac{\delta^{2d}_{|\alpha|-1}|\boldsymbol{\alpha}|}{(|\boldsymbol{\alpha}|+d-1)^{d}}\Big)\\
&= \sum_{k=0}^{\infty}\sum_{\substack{\alpha_1,\dots,\alpha_d\\ |\boldsymbol{\alpha}|=k}}(d-1)!\Big(\frac{\delta^{2d}_{|\alpha|}}{(|\boldsymbol{\alpha}|+d)^{(d-1)}} - \frac{\delta^{2d}_{|\alpha|-1}|\boldsymbol{\alpha}|}{(|\boldsymbol{\alpha}|+d-1)^{d}}\Big)\\
&= \sum_{k=0}^{\infty}\Big(\delta_k^{2d}\frac{(k+d-1)(k+d-2)\dots(k+1)}{(k+d)^{(d-1)}} - \delta_{k-1}^{2d}\frac{(k+d-2)\dots(k+1)k}{(k+d-1)^{d-1}}\Big)\\
&= \lim_{k\to\infty}\delta_k^{2d}\frac{(k+d-1)(k+d-2)\dots(k+1)}{(k+d)^{(d-1)}} = 1.
\end{aligned}$$

where second equality follows from Equation (6.7), verifies the claim. □

3.5.2 The case of an ellipsoid $BS_{1,2}(\mathbb{B}_{2,1})$

For $p,q\in\mathbb{N}$, let $\mathbb{B}_{p,q} = \{\boldsymbol{z}\in\mathbb{C}^2 : |z_1|^p + |z_2|^q < 1\}$. These are examples of pseudo convex Reinhardt domains in $\mathbb{C}^2$. The usual Euclidean ball $\mathbb{B}_2$ is obtained by taking $p = q = 2$, i.e., $\mathbb{B}_{2,2} = \mathbb{B}_2$.

The pair $(z_1,z_2)\in\mathbb{C}^2$ is in $\mathbb{B}_{2,1}$ if and only if $r_1^2 + r_2 < 1$, where $r_k := |z_k|$, $k = 1,2$. The volume measure ν restricted to $\mathbb{B}_{2,1}$ is of the form $d\nu(\boldsymbol{z}) = r_1 r_2 dr_1 dr_2 d\theta_1 d\theta_2$, $z_k = r_k\exp(i\theta_k)$, $k = 1,2$ and set

$$d\mu_\lambda(z) := (1 - r_1^2 - r_2)^{\lambda-4} r_1 r_2 dr_1 dr_2 d\theta_1 d\theta_2. \tag{3.11}$$

The measure $d\mu_\lambda$ defines an inner product on the space $\mathbb{C}[\boldsymbol{z}]$ of polynomials in two variables by integration over $\mathbb{B}_{2,1}$:

$$\langle p,q\rangle_\lambda := \int_{\mathbb{B}_{2,1}} p\bar{q}\,d\mu_\lambda.$$

Let $\mathbb{A}^{(\lambda)}(\mathbb{B}_{2,1})$ denote the Hilbert space obtained by taking the completion of the inner product space $\big(\mathbb{C}[\boldsymbol{z}],\langle\cdot,\cdot\rangle_\lambda\big)$. The Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{B}_{2,1})$ is non-zero if and only if $\lambda > 3$. This follows from the norm computation below. For any multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, we have

$$\begin{aligned}\|\boldsymbol{z}^{\boldsymbol{\alpha}}\|_\lambda^2 =& \int_{\Omega_{2,1}} |\boldsymbol{z}^{\boldsymbol{\alpha}}|^2 d\mu_\lambda \\ =&(2\pi)^2 \int_{r_1=0}^{1} \int_{r_2=0}^{1-r_1^2} r_1^{2\alpha_1+1} r_2^{2\alpha_2+1} (1 - r_1^2 - r_2)^{\lambda-4} dr_1 dr_2 \\ =&2(\pi)^2 B(2\alpha_2+2, \lambda-3) B(\alpha_1+1, 2\alpha_2+\lambda-1) \\ =&2(\pi)^2 \Gamma(\lambda-3) \frac{\Gamma(\alpha_1+1)\Gamma(2\alpha_2+2)}{\Gamma(2\alpha_2+\alpha_1+\lambda)}.\end{aligned}$$

Integrating first, with respect to the measure $d\theta_1 d\theta_2$, we see that $\{\boldsymbol{z}^{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathbb{N}_0^2\}$ is an orthogonal set of vectors relative to the inner product $\langle \cdot, \cdot \rangle_\mu$ and hence the set of vectors $\{\phi_{\boldsymbol{\alpha}} := \frac{\boldsymbol{z}^{\boldsymbol{\alpha}}}{\|\boldsymbol{z}^{\boldsymbol{\alpha}}\|_\lambda} : \boldsymbol{\alpha} \in \mathbb{N}_0^2\}$ is a complete orthonormal set in the Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{B}_{2,1})$. Now, it is easy to see that the multiplication operators M_{z_i}, $i = 1,2$ on the Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{B}_{2,1})$ are weighted shifts relative to this orthonormal basis, that is, $M_{z_i}(\phi_{\boldsymbol{\alpha}}) = w_{\boldsymbol{\alpha}}^{(i)} \phi_{\boldsymbol{\alpha}+\epsilon_i}$, where the weights are given explicitly by the formulae:

$$w_{\boldsymbol{\alpha}}^{(1)} = \sqrt{\frac{\alpha_i+1}{\alpha_1+2\alpha_2+\lambda}} \text{ and } w_{\boldsymbol{\alpha}}^{(2)} = \sqrt{\frac{(2\alpha_2+2)(2\alpha_2+3)}{(\alpha_1+2\alpha_2+\lambda)(\alpha_1+2\alpha_2+\lambda+1)}}.$$

Since $\operatorname{Sup}\{w_{\boldsymbol{\alpha}}^{(i)}\} = 1$, it follows that $\|M_{z_i}\| = 1$, $i = 1,2$.

Theorem 3.15. *Let $\boldsymbol{M} = (M_{z_1}, M_{z_2})$ be the pair of multiplication operators on $\mathbb{A}^{(\lambda)}(\mathbb{B}_{2,1})$ by the co-ordinate functions. If $\lambda \geq 4$, then $\boldsymbol{M}$ is in $BS_{1,2}(\mathbb{B}_{2,1})$.*

Proof. Since $\boldsymbol{M}$ is a pair of joint weighted shifts, by definition, $P_N M_j P_N^\perp = 0$, $i = 1,2$. The commuting pair $\boldsymbol{M}$ is 1-cyclic and the Taylor joint spectrum $\sigma(\boldsymbol{M}) = \overline{\mathbb{B}}_{2,1}$, see [15]. The following computation verifies the estimate (iii) of the Definition 3.10.

$$\begin{aligned}&\big(M_{z_1}^* M_{z_1} M_{z_2}^* - M_{z_2}^* M_{z_1} M_{z_1}^*\big)\phi_{\boldsymbol{\alpha}} \\ &= \Big(\frac{(2\alpha_1+2\alpha_2+\lambda-1)}{(\alpha_1+2\alpha_2+\lambda-2)(\alpha_1+2\alpha_2+\lambda-1)} \sqrt{\frac{(2\alpha_2)(2\alpha_2+1)}{(\alpha_1+2\alpha_2+\lambda-2)(\alpha_1+2\alpha_2+\lambda-1)}}\Big)\phi_{\boldsymbol{\alpha}-\epsilon_2}.\end{aligned}$$

Thus $\|P_N\big(M_{z_1}^* M_{z_1} M_{z_2}^* - M_{z_2}^* M_{z_1} M_{z_1}^*\big)P_N^\perp\| \leq \frac{2}{N+1}\|M_{z_1}\|^2\|M_{z_2}\|$ and therefore

$$\|P_N\big(M_{z_1}^* M_{z_1} M_{z_2}^* - M_{z_2}^* M_{z_1} M_{z_1}^*\big)P_N^\perp M_{z_2} P_N\| \leq \frac{2}{N+1}\|M_{z_1}\|^2\|M_{z_2}\|^2.$$

$$\begin{aligned}&\big(M_{z_2}^* M_{z_2} M_{z_1}^* - M_{z_1}^* M_{z_2} M_{z_2}^*\big)\phi_{\boldsymbol{\alpha}} \\ &\qquad = \Big(\frac{(2\alpha_2+1)(\alpha_1+\lambda-2) + 2(\alpha_2+1)(\alpha_1+2\alpha_2+\lambda-2)}{(\alpha_1+2\alpha_2+\lambda-2)(\alpha_1+2\alpha_2+\lambda)} \\ &\qquad\qquad \frac{2}{(\alpha_1+2\alpha_2+\lambda-1)} \sqrt{\frac{\alpha_1}{\alpha_1+2\alpha_2+\lambda-1}}\Big)\phi_{\boldsymbol{\alpha}-\epsilon_1}.\end{aligned}$$

Consequently, we have

$$\left\| P_N\left(M_{z_2}^* M_{z_2} M_{z_1}^* - M_{z_1}^* M_{z_2} M_{z_2}^*\right) P_N^{\perp} \right\| \leq \frac{2}{N+1} \|M_{z_2}\|^2 \|M_{z_1}\|$$

and

$$\left\| P_N\left(M_{z_2}^* M_{z_2} M_{z_1}^* - M_{z_1}^* M_{z_2} M_{z_2}^*\right) P_N^{\perp} M_{z_1} P_N \right\| \leq \frac{2}{N+1} \|M_{z_2}\|^2 \|M_{z_1}\|^2.$$

To complete the proof, we only need to verify that the operator $\mathrm{dEt}\left(\left[\!\left[\boldsymbol{M}^*, \boldsymbol{M}\right]\!\right]\right)$ is non-negative definite. Evaluating on the orthonormal basis $\{\phi_{\boldsymbol{\alpha}}\}$, we see that $\mathrm{dEt}\left(\left[\!\left[\boldsymbol{M}^*, \boldsymbol{M}\right]\!\right]\right)\phi_{\boldsymbol{\alpha}} = \chi_{\boldsymbol{\alpha}}\phi_{\boldsymbol{\alpha}}$, where

$$\begin{aligned}\chi_{\boldsymbol{\alpha}} = &-\frac{2\alpha_2(2\alpha_2+1)(2\alpha_1+2\alpha_2+\lambda-1)}{(\alpha_1+2\alpha_2+\lambda-2)^2(\alpha_1+2\alpha_2+\lambda-1)^2} + \frac{(2\alpha_2+2)(2\alpha_2+3)(2\alpha_1+2\alpha_2+\lambda+1)}{(\alpha_1+2\alpha_2+\lambda)^2(\alpha_1+2\alpha_2+\lambda+1)^2}\\ &+ \frac{2(\alpha_1+1)((2\alpha_2+1)(\alpha_1+\lambda-1)+2(\alpha_2+1)(\alpha_1+2\alpha_2+\lambda-1))}{(\alpha_1+2\alpha_2+\lambda-1)(\alpha_1+2\alpha_2+\lambda)^2(\alpha_1+2\alpha_2+\lambda+1)}\\ &- \frac{2\alpha_1((2\alpha_2+1)(\alpha_1+\lambda-2)+2(\alpha_2+1)(\alpha_1+2\alpha_2+\lambda-2))}{(\alpha_1+2\alpha_2+\lambda-2)(\alpha_1+2\alpha_2+\lambda-1)^2(\alpha_1+2\alpha_2+\lambda)}.\end{aligned}$$

Gathering these terms over a common denominator and simplifying we find that $\chi_{\boldsymbol{\alpha}}$ is a fraction with a positive denominator and the numerator is 4 times the expression given below.

$$\begin{aligned}&\alpha_1^4(4\alpha_2\lambda-10\alpha_2+3\lambda-9)+2\alpha_1\left(8\alpha_2^3\left(12\lambda^2-30\lambda+1\right)+\alpha_2^2\left(48\lambda^3-96\lambda^2-84\lambda+98\right)\right)\\ &+2\alpha_1^3\left(8\alpha_2^2(2\lambda-5)+2\alpha_2\left(4\lambda^2-6\lambda-7\right)+3\left(2\lambda^2-7\lambda+6\right)\right)+\alpha_1^2\left(4\alpha_2^2\left(24\lambda^2-54\lambda-5\right)\right)\\ &+\alpha_1^2\left(48\alpha_2^3(2\lambda-5)+4\alpha_2\left(6\lambda^3-3\lambda^2-35\lambda+32\right)+18\lambda^3-72\lambda^2+93\lambda-33\right)\\ &+2\alpha_1\left(32\alpha_2^4(2\lambda-5)+\alpha_2\left(8\lambda^4+4\lambda^3-98\lambda^2+128\lambda-39\right)+3\left(2\lambda^4-9\lambda^3+13\lambda^2-5\lambda-2\right)\right)+\\ &+16\alpha_2^4\left(8\lambda^2-21\lambda+2\right)+16\alpha_2^3\left(6\lambda^3-15\lambda^2-4\lambda+9\right)+4\alpha_2^2\left(8\lambda^4-14\lambda^3-37\lambda^2+61\lambda-14\right)\\ &+32\alpha_2^5(2\lambda-5)+2\alpha_2\left(2\lambda^5+3\lambda^4-42\lambda^3+64\lambda^2-14\lambda-13\right)+3(\lambda+1)\left(\lambda^2-3\lambda+2\right)^2\end{aligned}$$

It is then not hard to verify that the coefficients of α_1^i, $1 \leq i \leq 4$ and that of α_2^j, $1 \leq j \leq 5$ and the constant term are all positive if $\lambda \geq 4$ completing the proof. □

Remark 3.16. Although, unlike the case of the Euclidean ball $\mathbb{B}_2$, we are not able to explicitly compute the trace of the operator $\mathrm{dEt}\left(\left[\!\left[\boldsymbol{T}^*, \boldsymbol{T}\right]\!\right]\right)$ for the weighted Bergman spaces $\mathbb{A}^{(\lambda)}(\mathbb{B}_{2,1})$, extensive numerical computations show that it is approximately equal to $\frac{2}{3}$, which is the $\frac{2}{\pi^2}$ times the volume of the ellipsoid $\mathbb{B}_{2,1}$.

3.6 Trace estimate of the determinant operators

In this Section, for a commuting d-tuple of operators $\boldsymbol{T}$ in the class $\mathrm{BS}_{m,\vartheta}(\Omega)$, we obtain an estimate for the trace of the operator $\mathrm{dEt}\left(\left[\!\left[\boldsymbol{T}^*, \boldsymbol{T}\right]\!\right]\right)$ in Theorem 3.18. We make the standing assumption that the spectrum of the d - tuple of operators $\boldsymbol{T}$ is polynomially convex and consider only finitely polynomially cyclic d - tuples $\boldsymbol{T}$.

Lemma 3.17. *Suppose the d -tuple $\boldsymbol{T}$ is m- cyclic and $P_N T_j P_N^{\perp} = 0$, $1 \le j \le d$. Then*

$$\begin{aligned}&\left|trace\left(P_N dEt\left([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\right) P_N\right)\right| \\ &\qquad \le m\tbinom{N+d-1}{d-1}\Big(\sum_{\tau\in\mathfrak{S}_d}\big\|\big(\sum_{\eta\in\mathfrak{S}_d} Sgn(\eta) P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)} P_N^{\perp} T_{\tau(d)} P_N\big)\big\|\Big).\end{aligned}$$

Proof. For a d-tuple of commuting operators $\boldsymbol{T}$, by Proposition 3.4 and using Equation (3.5), we infer that the determinant

$$\begin{aligned}\mathrm{dEt}\left([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\right) &= \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta) T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)} T_{\tau(d)} + \\ &\qquad (-1)^d \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta) T_{\tau(1)} T^*_{\eta(1)} T_{\tau(2)}\dots T_{\tau(d)} T^*_{\eta(d)} \\ &= \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta) T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)} T_{\tau(d)} + \\ &\qquad - \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta) T_{\tau(d)} T^*_{\eta(1)} T_{\tau(1)}\dots T_{\tau(d-1)} T^*_{\eta(d)} \qquad (3.12) \\ &= \sum_{\tau,\eta\in\mathfrak{S}_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta)\left[T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)}, T_{\tau(d)}\right].\end{aligned}$$

The second equality in Equation (3.12) is obtained by replacing the permutation τ in the second sum by the permutation $\tau' = \tau\circ(1,\dots,d)$, where $(1,\dots,d)$ is the permutation taking $1\to 2$, $\dots$, $(d-1)\to d$ and $d\to 1$. Since the sums are over all permutations, this does not change it. But the sign of τ' differs from that of τ by $(-1)^{d-1}$. Therefore, it follows that

$$P_N \,\mathrm{dEt}\left([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\right) P_N = \sum_{\tau,\eta\in S_d} \mathrm{Sgn}(\tau)\mathrm{Sgn}(\eta) P_N\left[T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)}, T_{\tau(d)}\right] P_N.$$

Also, for $\tau\in\mathfrak{S}_d$,

$$\begin{aligned}&P_N\left[T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)}, T_{\tau(d)}\right] P_N \\ &\quad = P_N\left(T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)} T_{\tau(d)} - T_{\tau(d)} T^*_{\eta(1)} T_{\tau(1)}\dots T_{\tau(d-1)} T_{\eta(d)}\right) P_N \\ &= P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)} (P_N + P_N^{\perp}) T_{\tau(d)} P_N - P_N T_{\tau(d)} (P_N + P_N^{\perp}) T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)} P_N \\ &\quad = P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)} P_N^{\perp} T_{\tau(d)} P_N + \left[P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)} P_N, P_N T_{\tau(d)} P_N\right] \\ &\qquad - P_N T_{\tau(d)} P_N^{\perp} T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)} P_N \\ &\quad = P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)} P_N^{\perp} T_{\tau(d)} P_N + \left[P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)}\dots T^*_{\eta(d)} P_N, P_N T_{\tau(d)} P_N\right],\end{aligned}$$

where, the validity of the last equality follows from the assumption that $P_N T_j P_N^{\perp} = 0$, $j = 1,2,\dots,d$. If A, B are any two operators with one in trace class and the other bounded, then trace (AB) = trace (BA). A bounded operator of finite rank is trace class. Therefore, the commutator of any two bounded operators of finite rank must be 0. Hence

$$\operatorname{trace}\Big(\big[P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)} \dots T^*_{\eta(d)} P_N, P_N T_{\tau(d)} P_N\big]\Big) = 0.$$

Putting these together we obtain the equality below

$$\begin{aligned}&\operatorname{trace}\big(P_N \operatorname{dEt}\big(\big[\!\big[\boldsymbol{T}^*, \boldsymbol{T}\big]\!\big]\big) P_N\big)\\ &\qquad = \sum_{\tau\in\mathfrak{S}_d} \operatorname{Sgn}(\tau)\operatorname{trace}\Big(\sum_{\eta\in\mathfrak{S}_d} \operatorname{Sgn}(\eta) P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)} \dots T^*_{\eta(d)} P_N^{\perp} T_{\tau(d)} P_N\Big).\end{aligned}$$

By Lemma 3.8 we have the inequalities:

$$\begin{aligned}&\operatorname{rank}\Big(\big(\sum_{\eta\in\mathfrak{S}_d} \operatorname{Sgn}(\eta) P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)} T^*_{\eta(d)} P_N^{\perp}\big)\big(P_N^{\perp} T_{\tau(d)} P_N\big)\Big)\\ &\qquad \le \operatorname{rank}(P_N^{\perp} T_{\tau(d)} P_N) \le m\tbinom{N+d-1}{d-1}. \qquad (3.13)\end{aligned}$$

This implies

$$\begin{aligned}&\big|\operatorname{trace}\big(P_N \operatorname{dEt}\big(\big[\!\big[\boldsymbol{T}^*, \boldsymbol{T}\big]\!\big]\big) P_N\big)\big|\\ &\le \sum_{\tau\in\mathfrak{S}_d} \Big|\operatorname{trace}\Big(\sum_{\eta\in\mathfrak{S}_d} \operatorname{Sgn}(\eta) P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)} \dots T^*_{\eta(d)} P_N^{\perp} T_{\tau(d)} P_N\Big)\Big|\\ &\le \sum_{\tau\in\mathfrak{S}_d} \Big\{\Big\|\Big(\sum_{\eta\in\mathfrak{S}_d} \operatorname{Sgn}(\eta) P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)} \dots T^*_{\eta(d)} P_N^{\perp} T_{\tau(d)} P_N\Big)\Big\|\\ &\qquad \operatorname{rank}\Big(\sum_{\eta\in\mathfrak{S}_d} \operatorname{Sgn}(\eta) P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)} \dots T^*_{\eta(d)} P_N^{\perp} T_{\tau(d)} P_N\Big)\Big\}\\ &\le \sum_{\tau\in\mathfrak{S}_d} \Big\|\Big(\sum_{\eta\in\mathfrak{S}_d} \operatorname{Sgn}(\eta) P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)} \dots T^*_{\eta(d)} P_N^{\perp} T_{\tau(d)} P_N\Big)\Big\| \operatorname{rank}\big(P_N^{\perp} T_{\tau(d)} P_N\big)\\ &\le m\tbinom{N+d-1}{d-1}\Big(\sum_{\tau\in\mathfrak{S}_d} \Big\|\Big(\sum_{\eta\in\mathfrak{S}_d} \operatorname{Sgn}(\eta) P_N T^*_{\eta(1)} T_{\tau(1)} T^*_{\eta(2)} \dots T^*_{\eta(d)} P_N^{\perp} T_{\tau(d)} P_N\Big)\Big\|\Big).\end{aligned}$$

The two penultimate inequalities follow from the inequality 3.13. □

The following Theorem shows that the operator $\operatorname{dEt}\big(\big[\!\big[\boldsymbol{T}^*, \boldsymbol{T}\big]\!\big]\big)$ is in trace class whenever $\boldsymbol{T}$ is in $BS_{m,\vartheta}(\Omega)$.

Theorem 3.18. *Let $\boldsymbol{T} = (T_1, \dots, T_d)$ be a commuting tuple of operators on a Hilbert space $\mathscr{H}$ such that $\boldsymbol{T}$ is in the class $BS_{m,\vartheta}(\Omega)$. Then the determinant operator $dEt\big(\big[\!\big[\boldsymbol{T}^*, \boldsymbol{T}\big]\!\big]\big)$ is in trace-class and*

$$\mathit{trace}\Big(\mathit{dEt}\big(\big[\!\big[\boldsymbol{T}^*, \boldsymbol{T}\big]\!\big]\big)\Big) \le m\,\vartheta\, d! \prod_{i=1}^{d} \|T_i\|^2.$$

Proof. By hypothesis,

$$\Big\|P_N\Big(\sum_{\eta\in\mathfrak{S}_d}\operatorname{Sgn}(\eta)\,T^*_{\eta(1)}T_{\tau(1)}T^*_{\eta(2)}\dots T_{\tau(d-1)}T^*_{\eta(d)}\Big)P_N^{\perp}T_{\tau(d)}P_N\Big\|\leq\frac{\vartheta}{\binom{N+d-1}{d-1}}\prod_{i=1}^{d}\|T_i\|^2.$$

Thus combining this inequality with the one from Lemma 3.17, we have

$$\big|\operatorname{trace}\big(P_N\operatorname{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)P_N\big)\big|\leq m\,\vartheta\,d!\prod_{i=1}^{d}\|T_i\|^2.$$

Since $\operatorname{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$ is non-negative definite by assumption and $P_N\uparrow I$, we obtain the inequality

$$\operatorname{trace}\big(\operatorname{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)\big)\leq m\,\vartheta\,d!\prod_{i=1}^{d}\|T_i\|^2$$

completing the proof. □

From Theorems 7.1 and 7.2 of [24], it follows that the trace of the generalized commutator $\operatorname{GC}(\boldsymbol{T}^*,\boldsymbol{T})$ of a class of analytic Toeplitz operators is bounded above by 1. In particular, the explicit formula given in Theorem 7.2 (a) of [24] shows that equality is achieved for the tuple of multiplication by the coordinate functions. Of course, the same is true of the determinant operator $\operatorname{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$. In the example of weighted Bergman spaces over the Euclidean ball $\mathbb{B}_{1,1}$, we have been able to compute the trace of the operator $\operatorname{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)$ and shown that $\operatorname{trace}\big(\operatorname{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)\big)=\frac{md!}{\pi^d}\nu(\mathbb{B}_{1,1})$. Also, for the generalized ellipsoid $\mathbb{B}_{1,2}$, we have numerical evidence for such an equality. Taking all of this into account, we make the following conjecture.

Conjecture 3.19. Suppose that $\boldsymbol{T}=(T_1,\dots,T_d)$ is a commuting tuple of operators on a Hilbert space $\mathscr{H}$ in the class $BS_{m,\vartheta}(\Omega)$. Then

$$\operatorname{trace}\big(\operatorname{dEt}\big([\![\boldsymbol{T}^*,\boldsymbol{T}]\!]\big)\big)\leq\frac{md!}{\pi^d}\nu(\overline{\Omega}),$$

where ν is the Lebesgue measure.

3.7 The tensor product model

For $i=1,2$, let $\boldsymbol{T}^{(i)}=(T_1^{(i)},\dots,T_{d_i}^{(i)})$ be a d_i- tuple of commuting bounded operators. Set

$$\begin{aligned}(\boldsymbol{T}^{(1)}\#\boldsymbol{T}^{(2)}) &:= (\boldsymbol{T}^{(1)}\otimes\boldsymbol{I},\boldsymbol{I}\otimes\boldsymbol{T}^{(2)})\\ &= (T_1^{(1)}\otimes I,\dots,T_{d_1}^{(1)}\otimes I, I\otimes T_1^{(2)},\dots I\otimes T_{d_2}^{(2)})\end{aligned}$$

This definition clearly extends to d_i- tuples of commuting operators, $i=1,\dots,n$.

Lemma 3.20. *The spectrum* $\sigma\big(\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)}\big)$ *of the operator* $\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)}$ *is* $\sigma\big(\boldsymbol{T}^{(1)}\big) \times \sigma\big(\boldsymbol{T}^{(2)}\big)$. *Moreover, if the* d_i*- tuples* $\boldsymbol{T}^{(i)}$, $i = 1,2$, *are* m_i*- cyclic, then the operator* $\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)}$ *is* m*- cyclic, where* $m \le m_1 m_2$.

Proof. The joint spectrum of $\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)}$ is explicitly given in [10, Theorem 2.2]. If $\xi_{\boldsymbol{T}^{(i)}}[m_i]$, $i = 1,2$, is the cyclic set for the d_i- tuple $\boldsymbol{T}^{(i)}$, then the cyclic set of the operator $\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)}$ is clearly contained in the set of vectors

$$\big\{x \otimes y \mid x \in \xi_{\boldsymbol{T}^{(1)}}[m_1] \text{ and } y \in \xi_{\boldsymbol{T}^{(2)}}[m_2]\big\}.$$

Thus the claim that $m \le m_1 m_2$ is verified. □

We now obtain a trace inequality for the operator $\mathrm{dEt}\big([\![(\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})^*, (\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})]\!]\big)$. A similar inequality can be proved for $T^{(1)} \# \cdots \# T^{(n)}$.

Theorem 3.21. *Assume that* $\boldsymbol{T}^{(i)}$ *is in the class* $BS_{m_i,1}(\Omega_i)$, $i = 1,2$. *Then the determinant operator*

$$dEt\big([\![(\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})^*, (\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})]\!]\big)$$

is non-negative definite and

$$trace\Big(dEt\big([\![(\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})^*, (\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})]\!]\big)\Big) \le 2 d_1! d_2! m_1 m_2 \prod_{i=1}^{d_1} \| T_i^{(1)} \|^2 \prod_{i=1}^{d_2} \| T_i^{(2)} \|^2.$$

Proof. It is easy to see that

$$[\![(\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})^*, (\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})]\!] = \begin{pmatrix} [\![(\boldsymbol{T}^{(1)})^*, \boldsymbol{T}^{(1)}]\!] \otimes \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \otimes [\![(\boldsymbol{T}^{(2)})^*, \boldsymbol{T}^{(2)}]\!] \end{pmatrix}.$$

Thus

$$\mathrm{dEt}\big([\![(\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})^*, (\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})]\!]\big) = 2\, \mathrm{dEt}\big([\![\boldsymbol{T}^{(1)*}, \boldsymbol{T}^{(1)}]\!]\big) \otimes \mathrm{dEt}\big([\![\boldsymbol{T}^{(2)*}, \boldsymbol{T}^{(2)}]\!]\big).$$

Since for $i = 1,2$, $\boldsymbol{T}^{(i)}$ is in the class $BS_{m_i,1}(\Omega_i)$, $\mathrm{dEt}\big([\![(\boldsymbol{T}^{(i)})^*, \boldsymbol{T}^{(i)}]\!]\big)$ is non-negative definite and

$$\mathrm{trace}\big(\mathrm{dEt}\big([\![(\boldsymbol{T}^{(i)})^*, \boldsymbol{T}^{(i)}]\!]\big)\big) \le d_i! m_i \prod_{j=1}^{d_i} \| T_j^{(i)} \|^2.$$

Hence, $\mathrm{dEt}\big([\![(\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})^*, (\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})]\!]\big)$ is non-negative definite and

$$\begin{aligned}\mathrm{trace}\big(\mathrm{dEt}\big([\![(\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})^*, (\boldsymbol{T}^{(1)} \# \boldsymbol{T}^{(2)})]\!]\big)\big) &= 2\, \mathrm{trace}\big(\mathrm{dEt}\big([\![(\boldsymbol{T}^{(1)})^*, \boldsymbol{T}^{(1)}]\!]\big)\big) \mathrm{trace}\big(\mathrm{dEt}\big([\![(\boldsymbol{T}^{(2)})^*, \boldsymbol{T}^{(2)}]\!]\big)\big) \\ &\le 2 d_1! m_1 \prod_{i=1}^{d_1} \| T_i^{(1)} \|^2 \cdot d_2! m_2 \prod_{i=1}^{d_2} \| T_i^{(2)} \|^2 \\ &= 2 d_1! d_2! m_1 m_2 \prod_{i=1}^{d_1} \| T_i^{(1)} \|^2 \prod_{i=1}^{d_2} \| T_i^{(2)} \|^2.\end{aligned}$$

This completes the proof. □

Remark 3.22. 1. Let $\boldsymbol{T}^{(i)}$, $i=1,\ldots,n$ be a set of n commuting d_i- tuple of operators. A similar proof, as given above, shows that if $T^{(i)} \in \mathrm{BS}_{m_i,1}(\Omega_i)$, then

$$\mathrm{dEt}\big(\big[\!\big[(\boldsymbol{T}^{(1)}\#\cdots\#\boldsymbol{T}^{(n)})^*,(\boldsymbol{T}^{(1)}\#\cdots\#\boldsymbol{T}^{(n)})\big]\!\big]\big)$$

is non-negative definite and

$$\mathrm{trace}\Big(\mathrm{dEt}\big(\big[\!\big[(\boldsymbol{T}^{(1)}\#\cdots\#\boldsymbol{T}^{(n)})^*,(\boldsymbol{T}^{(1)}\#\cdots\#\boldsymbol{T}^{(n)})\big]\!\big]\big)\Big) \le n!\,d_1!\cdots d_n!\,m_1\cdots m_n\prod_{i=1}^{n}\|\boldsymbol{T}^{(i)}\|^2,$$

where $\|\boldsymbol{T}^{(i)}\|^2=\prod_{j=1}^{d_i}\|T_j^{(i)}\|^2$.

2. If $d_i=1$, $i=1,\ldots,n$, then $(\boldsymbol{T}^{(1)}\#\cdots\#\boldsymbol{T}^{(n)})$ is of the form $(T_1\otimes I\cdots\otimes I,\ldots I\otimes\cdots\otimes T_n)$. Now, we can apply the Berger-Shaw inequality to each of the operators T_i, $1\le i\le n$, to conclude

$$\mathrm{trace}\Big(\mathrm{dEt}\big(\big[\!\big[(\boldsymbol{T}^{(1)}\#\cdots\#\boldsymbol{T}^{(n)})^*,(\boldsymbol{T}^{(1)}\#\cdots\#\boldsymbol{T}^{(n)})\big]\!\big]\big)\Big) \le n!\,m_1\cdots m_n\frac{\nu(\Omega_1\times\cdots\times\Omega_n)}{\pi^n}.$$

Let $\boldsymbol{M}=(M_1,\ldots M_d)$ be the d- tuple of multiplication by the coordinate functions on the Hardy space $H^2(\mathbb{D}^d)$. Clearly, $\boldsymbol{M}=M\#\cdots\#M$ where M is the multiplication operator on $H^2(\mathbb{D})$.

Corollary 3.23. *For the d- tuple $\boldsymbol{M}=M\#\cdots\#M$ on the Hardy space $H^2(\mathbb{D}^d)$, we have that the operator $dEt\big(\big[\!\big[\boldsymbol{M}^*,\boldsymbol{M}\big]\!\big]\big)$ is non-negative definite and*

$$trace\Big(dEt\big(\big[\!\big[\boldsymbol{M}^*,\boldsymbol{M}\big]\!\big]\big)\Big)=trace\Big(dEt\big(\big[\!\big[(M\#\cdots\#M)^*,(M\#\cdots\#M)\big]\!\big]\big)\Big)\le d!.$$

Remark 3.24. The inequality of the Corollary 3.23 is actually an equality which follows from an easy direct computation. This shows that the inequality obtained in Theorem 3.21 is sharp.

Chapter 4

$\mathbb{K}$-Homogeneous Operators on Bounded Symmetric Domains

Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a commuting d- tuple of bounded linear operators acting on a complex separable Hilbert space $\mathscr{H}$. Also, let $D_{\boldsymbol{T}} : \mathscr{H} \to \mathscr{H} \oplus \cdots \oplus \mathscr{H}$ be the operator

$$D_{\boldsymbol{T}} h := (T_1 h, \ldots, T_d h), \qquad h \in \mathscr{H}.$$

We note that $\ker D_T = \cap_{i=1}^d \ker T_i$ is the *joint kernel* and $\sigma_p(\boldsymbol{T}) = \{\boldsymbol{w} \in \mathbb{C}^d : \ker D_{\boldsymbol{T}-\boldsymbol{w}I} \neq \boldsymbol{0}\}$ is the *joint point spectrum* of the d- tuple $\boldsymbol{T} = (T_1, \ldots, T_d)$.

The group $\mathbb{K}$ acts on Ω by the rule

$$k \cdot \boldsymbol{z} := \big(k_1(\boldsymbol{z}), \ldots, k_d(\boldsymbol{z})\big), \qquad k \in \mathbb{K} \text{ and } \boldsymbol{z} \in \Omega.$$

Note that $k_1(\boldsymbol{z}), \ldots, k_d(\boldsymbol{z})$ are linear polynomials. Thus $k \in \mathbb{K}$ acts on any commuting d- tuple of bounded linear operators $\boldsymbol{T} = (T_1, \ldots, T_d)$, defined on complex separable Hilbert space $\mathscr{H}$, naturally, via the map

$$k \cdot \boldsymbol{T} := \big(k_1(T_1, \ldots, T_d), \ldots, k_d(T_1, \ldots, T_d)\big).$$

Definition 4.1. A d- tuple $\boldsymbol{T} = (T_1, \ldots, T_d)$ of commuting bounded linear operators on $\mathscr{H}$ is said to be $\mathbb{K}$-homogeneous if for all k in $\mathbb{K}$ the operators $\boldsymbol{T}$ and $k \cdot \boldsymbol{T}$ are unitarily equivalent, that is, for all k in $\mathbb{K}$ there exists a unitary operator $\Gamma(k)$ on $\mathscr{H}$ such that

$$T_j \Gamma(k) = \Gamma(k) k_j(T_1, \ldots, T_d), \qquad j = 1, 2, \ldots, d. \tag{4.1}$$

For brevity, we will write

$$\boldsymbol{T}\Gamma(k) = \Gamma(k)(k \cdot \boldsymbol{T}).$$

Definition 4.2. A commuting d- tuple of $\mathbb{K}$-homogeneous operators $\boldsymbol{T}$ possessing the following properties

(i) $\dim\ker D_{\boldsymbol{T}^*} = 1$,

(ii) any non-zero vector e in $\ker D_{\boldsymbol{T}^*}$ is cyclic for $\boldsymbol{T}$,

(iii) $\Omega \subseteq \sigma_p(\boldsymbol{T}^*)$

is said to be in the class $\mathscr{A}\mathbb{K}(\Omega)$.

4.1 Model for operators in $\mathscr{A}\mathbb{K}(\Omega)$

We begin this section by providing a well known family of examples, namely, the d- tuple of multiplication by the coordinate functions on the weighted Bergman spaces, which belongs to the class $\mathscr{A}\mathbb{K}(\Omega)$.

Recall that the *Wallach set* $\mathscr{W}(\Omega)$ of a classical bounded symmetric domain Ω is of the form $\mathscr{W}_d(\Omega) \cup \mathscr{W}_c(\Omega)$, where

$$\mathscr{W}_d(\Omega) := \left\{0, \frac{a}{2}, \ldots, \frac{a}{2}(r-1)\right\},\ \mathscr{W}_c(\Omega) := \left(\frac{a}{2}(r-1), \infty\right),$$

see [22]. For $\lambda > 0$ consider the function $K^{(\lambda)} : \Omega \times \Omega \to \mathbb{C}$ given by the formula

$$K^{(\lambda)}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\underline{s}} (\lambda)_{\underline{s}} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w}), \qquad \boldsymbol{z}, \boldsymbol{w} \in \Omega,$$

where $(\lambda)_{\underline{s}}$ is the generalized Pochhammer symbol

$$(\lambda)_{\underline{s}} := \prod_{j=1}^{r} \left(\lambda - \frac{a}{2}(j-1)\right)_{s_j} = \prod_{j=1}^{r} \prod_{l=1}^{s_j} \left(\lambda - \frac{a}{2}(j-1) + l - 1\right).$$

The function $K^{(\lambda)}$ is non-negative definite if and only if λ is in the Wallach set $\mathscr{W}(\Omega)$. Let $\mathscr{H}^{(\lambda)}$ denote the Hilbert space determined by the non-negative definite kernel $K^{(\lambda)}$, $\lambda \in \mathscr{W}(\Omega)$. If $\lambda = \frac{d}{r}$ and $\lambda = \frac{a}{2}(r-1) + \frac{d}{r}$, then the Hilbert spaces $\mathscr{H}^{(\lambda)}$ coincide with the Hardy space $H^2(S)$ over the *Shilov boundary* S of Ω and the classical Bergman space $\mathbb{A}^2(\Omega)$ respectively. For this reason with a slight abuse of language, the Hilbert spaces $\mathscr{H}^{(\lambda)}$, $\lambda \in \mathscr{W}(\Omega)$, are called weighted Bergman spaces.

For $\lambda > \frac{a}{2}(r-1)$, the multiplication d- tuple $\boldsymbol{M}^{(\lambda)} = (M_1^{(\lambda)}, \ldots, M_d^{(\lambda)})$ on $\mathscr{H}^{(\lambda)}$ is bounded and homogeneous (cf. [6], [2]). One can also verify that $\boldsymbol{M}^{(\lambda)}$ is in $\mathscr{A}\mathbb{K}(\Omega)$. Replacing $(\lambda)_{\underline{s}}$ by any arbitrary positive number $a_{\underline{s}}$ with some boundedness condition, we get a large class of operator tuples in $\mathscr{A}\mathbb{K}(\Omega)$ and we prove that they are the all.

To facilitate the study of $\mathbb{K}$-homogeneous operators, we recall the following result from [1] describing all the $\mathbb{K}$-invariant kernels on Ω.

Proposition 4.3 (Proposition 3.4, [1]). *For any $\mathbb{K}$-invariant semi-inner product $\langle\cdot,\cdot\rangle$ on the the space of polynomials $\mathscr{P}$, the following statements hold:*

(i) *$\mathscr{P}_{\underline{s}}$ is orthogonal to $\mathscr{P}_{\underline{s}'}$ whenever $\underline{s} \neq \underline{s}'$.*

(ii) *There exists a constant $b_{\underline{s}} \geq 0$ associated to each $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$ such that*

$$\langle p, q\rangle = b_{\underline{s}}\langle p, q\rangle_{\mathscr{F}}, \qquad \text{for all } p, q \in \mathscr{P}_{\underline{s}}.$$

(iii) *$b_{\underline{s}} > 0$ for all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$ if and only if $\langle\cdot,\cdot\rangle$ is an inner product.*

(iv) *If the evaluation map at each point of Ω is continuous on $(\mathscr{P}, \langle\cdot,\cdot\rangle)$, then the completion $\mathscr{H}$ of $(\mathscr{P}, \langle\cdot,\cdot\rangle)$ is a reproducing kernel Hilbert space. Moreover the kernel $K(\boldsymbol{z}, \boldsymbol{w})$ is of the form*

$$K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r} b_{\underline{s}}^{-1} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w}),$$

where convergence is both pointwise and uniformly on compact subsets of $\Omega \times \Omega$ and in norm.

The following result is a generalization of [11, Lemma 2.10] which is necessary for the proof of Theorem 4.5 giving a model for commuting d- tuple of operators in the class $\mathscr{A}\mathbb{K}(\Omega)$.

Lemma 4.4. *Let $\boldsymbol{T} = (T_1, \ldots, T_d)$ be a $\mathbb{K}$-homogeneous d- tuple of commuting operators on $\mathscr{H}$. Suppose that $\ker D_{\boldsymbol{T}^*}$ is one-dimensional and is spanned by a vector $e \in \mathscr{H}$ which is cyclic for $\boldsymbol{T}$. Then there exists a sequence $\{a_{\underline{s}}\}_{\underline{s} \in \overrightarrow{\mathbb{N}}_0^r}$ of non-negative real numbers such that for any polynomial $p \in \mathscr{P}$,*

$$||p(\boldsymbol{T})e||_{\mathscr{H}}^2 = \sum_{k=0}^{deg\, p} \sum_{|\underline{s}|=k} a_{\underline{s}} \|p_{\underline{s}}\|_{\mathscr{F}}^2, \tag{4.2}$$

where deg p is the degree of p and

$$p = \sum_{k=0}^{deg\, p} \sum_{|\underline{s}|=k} p_{\underline{s}}$$

is the Peter-Weyl decomposition.

Proof. Since $\boldsymbol{T}$ is $\mathbb{K}$-homogeneous, for each $k \in \mathbb{K}$, there exists a unitary operator $\Gamma(k)$ on $\mathscr{H}$ such that

$$T_j\Gamma(k) = \Gamma(k)k_j(\boldsymbol{T}), \qquad j = 1, \ldots, d.$$

Hence $T_j^*\Gamma(k) = \Gamma(k)k_j(\boldsymbol{T})^*$, $j = 1, \ldots, d$. Since $k_j(\boldsymbol{T})$ is a linear combination of $T_1, \ldots, T_d$ and $e \in \ker D_{\boldsymbol{T}^*}$, it follows that $\Gamma(k)e$ belongs to $\ker D_{\boldsymbol{T}^*}$ for all $k \in \mathbb{K}$.

Furthermore, since $\ker D_{\boldsymbol{T}^*}$ is one dimensional and spanned by e, we obtain that $\Gamma(k)e = \eta(k)e$ for some $\eta(k)$ such that $|\eta(k)| = 1$. We now define a semi-inner product on $\mathscr{P}_{\underline{s}}$ for all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$ by the formula

$$\langle p_{\underline{s}}, q_{\underline{s}} \rangle_{\mathscr{P}_{\underline{s}}} := \langle p_{\underline{s}}(\boldsymbol{T})e, q_{\underline{s}}(\boldsymbol{T})e \rangle_{\mathscr{H}}, \qquad p_{\underline{s}}, q_{\underline{s}} \in \mathscr{P}_{\underline{s}}.$$

Now for any $k \in \mathbb{K}$ we have

$$\begin{aligned}
\langle p_{\underline{s}}(k\cdot \boldsymbol{z}), q_{\underline{s}}(k\cdot \boldsymbol{z}) \rangle_{\mathscr{P}_{\underline{s}}} &= \langle p_{\underline{s}}(k\cdot \boldsymbol{T})e, q_{\underline{s}}(k\cdot \boldsymbol{T})e \rangle_{\mathscr{H}} \\
&= \langle \Gamma(k)^* p_{\underline{s}}(\boldsymbol{T})\Gamma(k)e, \Gamma(k)^* q_{\underline{s}}(\boldsymbol{T})\Gamma(k)e \rangle_{\mathscr{H}} \\
&= \langle p_{\underline{s}}(\boldsymbol{T})\Gamma(k)e, q_{\underline{s}}(\boldsymbol{T})\Gamma(k)e \rangle_{\mathscr{H}} \\
&= \langle p_{\underline{s}}(\boldsymbol{T})\eta(k)e, q_{\underline{s}}(\boldsymbol{T})\eta(k)e \rangle_{\mathscr{H}} \\
&= |\eta(k)|^2 \langle p_{\underline{s}}(\boldsymbol{T})e, q_{\underline{s}}(\boldsymbol{T})e \rangle_{\mathscr{H}} \\
&= \langle p_{\underline{s}}(\boldsymbol{T})e, q_{\underline{s}}(\boldsymbol{T})e \rangle_{\mathscr{H}} \\
&= \langle p_{\underline{s}}, q_{\underline{s}} \rangle_{\mathscr{P}_{\underline{s}}}.
\end{aligned}$$

So $\langle \cdot, \cdot \rangle_{\mathscr{P}_{\underline{s}}}$ is a $\mathbb{K}$-invariant semi-inner product on $\mathscr{P}_{\underline{s}}$ for each $\underline{s}$. Therefore, on $\mathscr{P}$,

$$\langle p, q \rangle := \sum_{k=0}^{\ell} \sum_{|\underline{s}|=k} \langle p_{\underline{s}}, q_{\underline{s}} \rangle_{\mathscr{P}_{\underline{s}}},$$

where p and q have the Peter-Weyl decomposition $\sum_{k=0}^{\deg p} \sum_{|\underline{s}|=k} p_{\underline{s}}$ and $\sum_{k=0}^{\deg q} \sum_{|\underline{s}|=k} q_{\underline{s}}$ respectively and $\ell = \min\{\deg p, \deg q\}$, defines a $\mathbb{K}$-invariant semi-inner product. Thus by Proposition 4.3, there exists a sequence of non-negative real numbers $a_{\underline{s}}$ such that

$$\langle p, q \rangle = \sum_{k=0}^{\ell} \sum_{|\underline{s}|=k} a_{\underline{s}} \langle p_{\underline{s}}, q_{\underline{s}} \rangle_{\mathscr{F}}.$$

This completes the proof. □

For all the classical bounded symmetric domains, it can be easily verified that $\Omega = \{\boldsymbol{w} \in \mathbb{C}^d : \overline{\boldsymbol{w}} \in \Omega\}$. Consequently, in the following theorem, the Hilbert space that we construct consists of holomorphic functions on Ω rather than $\{\boldsymbol{w} \in \mathbb{C}^d : \overline{\boldsymbol{w}} \in \Omega\}$. The next result provides an analytic model for any d- tuple of operators $\boldsymbol{T}$ in $\mathscr{A}\mathbb{K}(\Omega)$.

Theorem 4.5. *If $\boldsymbol{T}$ is a d- tuple of operators in $\mathscr{A}\mathbb{K}(\Omega)$, then $\boldsymbol{T}$ is unitarily equivalent to a d- tuple $\boldsymbol{M} = (M_1, \ldots, M_d)$ of multiplication by the coordinate functions $z_1, \ldots, z_d$ on a reproducing kernel Hilbert space H_K of holomorphic functions defined on Ω with $K(\boldsymbol{z}, \boldsymbol{w}) = \sum a_{\underline{s}}^{-1} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w})$ for all $\boldsymbol{z}, \boldsymbol{w} \in \Omega$, for some choice of positive real numbers $a_{\underline{s}}$ with $a_{\underline{0}} = 1$.*

Proof. Since $\Omega \subseteq \sigma_p(\boldsymbol{T}^*)$, for each $\boldsymbol{w} \in \Omega$ there exists a non-zero vector $x \in \mathscr{H}$, such that $T_j^* x = \bar{w}_j x$ for all $j = 1, \ldots, d$. Thus for any polynomial $p \in \mathscr{P}$, we have $p(\boldsymbol{T}^*)x = p(\overline{\boldsymbol{w}})x$. Let $e \in ker D_{\boldsymbol{T}^*}$ be a cyclic vector for $\boldsymbol{T}$ of norm 1. Then

$$p(\boldsymbol{w})\langle e, x\rangle_{\mathscr{H}} = \langle e, \overline{p(\boldsymbol{w})} x\rangle_{\mathscr{H}} = \langle e, \bar{p}(\boldsymbol{T}^*)x\rangle_{\mathscr{H}} = \langle p(\boldsymbol{T})e, x\rangle_{\mathscr{H}}, \tag{4.3}$$

where $\bar{p}(\boldsymbol{z}) = \overline{p(\bar{\boldsymbol{z}})}$, $z \in \Omega$. Since $x \neq 0$ and e is cyclic for $\boldsymbol{T}$, we get $\langle e, x\rangle_{\mathscr{H}} \neq 0$ and

$$|p(\boldsymbol{w})| \leq \frac{\|p(\boldsymbol{T})e\|_{\mathscr{H}} \|x\|_{\mathscr{H}}}{|\langle e, x\rangle_{\mathscr{H}}|}. \tag{4.4}$$

Thus it follows that evaluation at $\boldsymbol{w} \in \Omega$ is bounded and therefore, the semi-inner product defined by the rule $\langle p, q\rangle_{\mathscr{P}_{\underline{s}}} = \langle p(\boldsymbol{T})e, q(\boldsymbol{T})e\rangle_{\mathscr{H}}$ is an inner product on each $\mathscr{P}_{\underline{s}}$. This gives rise to an inner product $\langle\cdot,\cdot\rangle$ on the space of polynomials $\mathscr{P}$. The sequence $\{a_{\underline{s}}\}_{\underline{s}\in\vec{\mathbb{N}}_0^r}$ of Lemma 4.4, using Proposition 4.3(iii), is now evidently positive. Moreover, since $\|e\| = 1$, it follows from (4.2) that $a_{\underline{0}} = 1$. Thus, by Proposition 4.3(iv), the completion of $(\mathscr{P}, \langle\cdot,\cdot\rangle)$, say H_K, is a reproducing kernel Hilbert space, where

$$K(\boldsymbol{z}, \boldsymbol{w}) = \sum a_{\underline{s}}^{-1} K_{\underline{s}}(\boldsymbol{z}, \boldsymbol{w}), \qquad \boldsymbol{z}, \boldsymbol{w} \in \Omega. \tag{4.5}$$

Clearly, the map $p \mapsto p(\boldsymbol{T})e$ extends to a unitary from H_K to $\mathscr{H}$, which intertwines $\boldsymbol{T}$ with the multiplication d- tuple $\boldsymbol{M} = (M_1, \ldots, M_d)$ on H_K. □

If $\boldsymbol{T}$ is a $\mathbb{K}$-homogeneous d- tuple of operators, then, in general, the map $k \mapsto \Gamma(k)$ of (4.1) need not be a homomorphism. The next proposition assures that if $\boldsymbol{T}$ is in the class $\mathscr{A}\mathbb{K}(\Omega)$, then there exists a choice of $\Gamma(k)$ for which the map $k \mapsto \Gamma(k)$ is a homomorphism.

Proposition 4.6. *If $\boldsymbol{T}$ is a d-tuple of operators in $\mathscr{A}\mathbb{K}(\Omega)$, then there exists a unitary representation $\Gamma : \mathbb{K} \to \mathscr{U}(\mathscr{H})$ such that*

$$\boldsymbol{T}\Gamma(k) = \Gamma(k)(k \cdot \boldsymbol{T}).$$

Proof. By Theorem 4.5, $\boldsymbol{T}$ is unitarily equivalent to the d- tuple $\boldsymbol{M} = (M_1, \ldots, M_d)$ of multiplication operators on a reproducing kernel Hilbert space H_K of holomorphic functions defined on Ω with a kernel $K(\boldsymbol{z}, \boldsymbol{w})$ which is $\mathbb{K}$-invariant. Clearly, the map Γ on H_K given by $\Gamma(k)(f) = f \circ k^{-1}(\cdot)$ is a unitary representation of $\mathbb{K}$ satisfying the intertwining condition.

□

Remark 4.7. Since $\mathbb{K}$ is a subgroup of the group $\mathscr{U}(d)$ of unitary linear transformations on $\mathbb{C}^d$, every spherical d- tuple $\boldsymbol{T} = (T_1, \ldots, T_d)$ is $\mathbb{K}$-homogeneous. Conversely, a $\mathbb{K}$-homogeneous d- tuple of Theorem 4.5 is spherical if and only if $a_{\underline{s}} = a_{\underline{s}'}$ for all $\underline{s}, \underline{s}' \in \vec{\mathbb{N}}_0^r$ with $|\underline{s}| = |\underline{s}'|$.

Remark 4.8. We also point out that, by the spectral mapping theorem, the Taylor joint spectrum $\sigma(\boldsymbol{T})$ of a $\mathbb{K}$-homogeneous operator $\boldsymbol{T}$ is $\mathbb{K}$-invariant, that is, if $\boldsymbol{w}$ belongs to $\sigma(\boldsymbol{T})$, then $k \cdot \boldsymbol{w}$ also belongs to $\sigma(\boldsymbol{T})$ for all $k \in \mathbb{K}$.

4.2 Boundedness of the multiplication tuple

Let $K^{(a)} : \Omega\times\Omega \to \mathbb{C}$ denote the kernel function given by the formula $K^{(a)}(\boldsymbol{z},\boldsymbol{w}) = \sum_{\underline{s}} a_{\underline{s}} K_{\underline{s}}(\boldsymbol{z},\boldsymbol{w})$, $\boldsymbol{z},\boldsymbol{w} \in \Omega$, for some choice of positive real numbers $a_{\underline{s}}$ for which the series is convergent. The positivity of the sequence $a_{\underline{s}}$ ensures that $K^{(a)}$ is a positive definite kernel. Thus it determines a unique Hilbert space $\mathscr{H}^{(a)} \subseteq \mathrm{Hol}(\Omega)$ with the reproducing property: $\langle f, K^{(a)}(\cdot,\boldsymbol{w})\rangle = f(\boldsymbol{w})$, $f \in \mathscr{H}^{(a)}$, $\boldsymbol{w} \in \Omega$. It follows from Proposition 4.3 that the polynomial ring $\mathscr{P}$ is dense in $\mathscr{H}^{(a)}$ and $\mathscr{P}_{\underline{s}}$ is orthogonal to $\mathscr{P}_{\underline{s}'}$ whenever $\underline{s} \neq \underline{s}'$, that is, $\mathscr{H}^{(a)} = \oplus_{\underline{s}\in\vec{\mathbb{N}}_0^r}\mathscr{P}_{\underline{s}}$. In this section, we discuss the boundedness of the d- tuple $\boldsymbol{M}^{(a)} := (M_1^{(a)},\dots,M_d^{(a)})$ of multiplication by the coordinate functions $z_1,\dots,z_d$ on $\mathscr{H}^{(a)}$. We begin with the following basic lemma, which is surely known to the experts, but we provide a proof for the sake of completeness.

Lemma 4.9. *For the bounded d- tuple of multiplication operators $\boldsymbol{M}^{(a)} := (M_1^{(a)},\dots,M_d^{(a)})$, the operators $\sum_{i=1}^d M_i^{(a)*} M_i^{(a)}$ and $\sum_{i=1}^d M_i^{(a)} M_i^{(a)*}$, acting on $\mathscr{H}^{(a)}$, are block diagonal with respect to the decomposition $\oplus_{\underline{s}\in\vec{\mathbb{N}}_0^r}\mathscr{P}_{\underline{s}}$, where each block is a non-negative scalar multiple of the identity operator.*

Proof. It is enough to give the proof for the operator $\sum_{i=1}^d M_i^{(a)*} M_i^{(a)}$ since the proof for the operator $\sum_{i=1}^d M_i^{(a)} M_i^{(a)*}$ follows exactly in the same way. First, note that $\Gamma(k)^* M_i^{(a)}\Gamma(k) = M^{(a)}_{z_i\circ k^{-1}}$ for $k \in \mathbb{K}$. Let $e_1,\dots,e_d$ be the standard basis vectors in $\mathbb{C}^d$. Note that $M^{(a)}_{z_i\circ k^{-1}} = \sum_{j=1}^d \langle k^{-1}e_j, e_i\rangle M_j^{(a)}$. In consequence, we have

$$\begin{aligned}
\Gamma(k)^*\left(\sum_{i=1}^d M_i^{(a)*} M_i^{(a)}\right)\Gamma(k) &= \sum_{i=1}^d \Gamma(k)^* M_i^{(a)*}\Gamma(k)\Gamma(k)^* M_i^{(a)}\Gamma(k)\\
&= \sum_{i=1}^d {M^{(a)}_{z_i\circ k^{-1}}}^* M^{(a)}_{z_i\circ k^{-1}}\\
&= \sum_{i=1}^d \sum_{p,q=1}^d \langle e_i, k^{-1}e_p\rangle\langle k^{-1}e_q, e_i\rangle M_p^{(a)*} M_q^{(a)}\\
&= \sum_{p,q=1}^d \langle k^{-1}e_q, k^{-1}e_p\rangle M_p^{(a)*} M_q^{(a)}\\
&= \sum_{i=1}^d M_i^{(a)*} M_i^{(a)}.
\end{aligned}$$

Here the last equality follows from the fact that the subgroup $\mathbb{K}$ is contained in the group $\mathscr{U}(d)$ of unitary linear transformations on $\mathbb{C}^d$. Since $\{\mathscr{P}_{\underline{s}}\}_{\underline{s}\in\vec{\mathbb{N}}_0^r}$ are $\mathbb{K}$-irreducible, mutually $\mathbb{K}$-inequivalent subspaces of $\mathscr{H}^{(a)}$ and $\mathscr{H}^{(a)} = \oplus_{\underline{s}\in\vec{\mathbb{N}}_0^r}\mathscr{P}_{\underline{s}}$, the conclusion follows from Schur's lemma. □

For any $\underline{s}$ in $\overrightarrow{\mathbb{N}}_0^r$, let $I^+(\underline{s})$ and $I^-(\underline{s})$ denote the sets given by

$$I^+(\underline{s}) := \{j : 1 \le j \le r,\ \underline{s} + \epsilon_j \in \overrightarrow{\mathbb{N}}_0^r\},$$
$$I^-(\underline{s}) := \{j : 1 \le j \le r,\ \underline{s} - \epsilon_j \in \overrightarrow{\mathbb{N}}_0^r\}.$$

Further, in the remaining portion of this paper, we set

$$c_{\underline{s}}(j) = \prod_{k \neq j} \frac{s_j - s_k + \frac{a}{2}(k - j + 1)}{s_j - s_k + \frac{a}{2}(k - j)},\ j = 1, \dots, r,$$

and

$$c'_{\underline{s}}(j) = \prod_{k \neq j} \frac{s_j - s_k + \frac{a}{2}(k - j - 1)}{s_j - s_k + \frac{a}{2}(k - j)},\ j = 1, \dots, r.$$

If $j \in I^+(\underline{s})$, then it is easy to see that $c_{\underline{s}}(j) > 0$. Otherwise, $c_{\underline{s}}(j) = 0$. Similarly, if $j \in I^-(\underline{s})$, then $c'_{\underline{s}}(j) > 0$. Otherwise, $c'_{\underline{s}}(j) = 0$.

The following lemma describing the operator $\sum_{i=1}^d M_i^{(a)} M_i^{(a)*}$ on the Hilbert space $\mathscr{H}^{(a)}$ was obtained for weighted Bergman spaces by Arazy and Zhang [2, Proposition 4.4]. Although the proof for this much larger class of operators is very similar to the original proof [2, Proposition 4.4], we recall it for completeness.

Lemma 4.10. *For $f \in \mathscr{P}_{\underline{s}}$, we have $\sum_{i=1}^d M_i^{(a)} M_i^{(a)*} f = \tau(\underline{s}) f$, where*

$$\tau(\underline{s}) = \begin{cases} \sum_{j \in I^-(\underline{s})} \frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} \frac{\frac{a}{2}(r-j)+s_j}{b+\frac{a}{2}(r-j)+s_j} c'_{\underline{s}}(j) & \text{if } \underline{s} \neq 0, \\ 0 & \text{if } \underline{s} = 0. \end{cases}$$

Proof. Let $f \in \mathscr{P}_{\underline{s}}$ be arbitrary. It follows from Lemma 4.9 that $\sum_{i=1}^d M_i^{(a)} M_i^{(a)*} f = \tau(\underline{s}) f$ for some complex number $\tau(\underline{s})$. In order to find $\tau(\underline{s})$, first note that

$$\begin{aligned} \tau(\underline{s}) f(\boldsymbol{w}) &= \langle \big(\sum_{i=1}^d M_i^{(a)} M_i^{(a)*}\big) f, K^{(a)}(\cdot, \boldsymbol{w}) \rangle \\ &= \langle f, \big(\sum_{i=1}^d M_i^{(a)} M_i^{(a)*}\big) K^{(a)}(\cdot, \boldsymbol{w}) \rangle \\ &= \langle f, \langle \cdot, \boldsymbol{w} \rangle K^{(a)}(\cdot, \boldsymbol{w}) \rangle. \end{aligned}$$

Since for any signature $\underline{s}'$, $\langle \boldsymbol{z}, \boldsymbol{w} \rangle K_{\underline{s}'}(\boldsymbol{z}, \boldsymbol{w})$ is a $\mathbb{K}$-invariant kernel, it follows from [2, Lemma 4.2 and Lemma 4.3] that

$$\langle \boldsymbol{z}, \boldsymbol{w} \rangle K_{\underline{s}'}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{j \in I^+(\underline{s})} \beta_{\underline{s}'}(j) K_{\underline{s}'+\epsilon_j}(\boldsymbol{z}, \boldsymbol{w}) \tag{4.6}$$

for some constants $\beta_{\underline{s}'}(j)$. Thus

$$\begin{aligned}\langle \boldsymbol{z}, \boldsymbol{w}\rangle K^{(a)}(\boldsymbol{z}, \boldsymbol{w}) &= \sum_{\underline{s}'} a_{\underline{s}'} \sum_{j\in I^+(\underline{s}')} \beta_{\underline{s}'}(j) K_{\underline{s}'+\epsilon_j}(\boldsymbol{z}, \boldsymbol{w})\\ &= \sum_{\underline{s}'} \Big(\sum_{j\in I^-(\underline{s}')} \beta_{\underline{s}'-\epsilon_j}(j) a_{\underline{s}'-\epsilon_j}\Big) K_{\underline{s}'}(\boldsymbol{z}, \boldsymbol{w}).\end{aligned}$$

Since $f \in \mathscr{P}_{\underline{s}}$, we obtain

$$\begin{aligned}\tau(\underline{s}) f(\boldsymbol{w}) &= \big\langle f, \langle \cdot, \boldsymbol{w}\rangle K^{(a)}(\cdot, \boldsymbol{w})\big\rangle\\ &= \Big(\frac{1}{a_{\underline{s}}} \sum_{j\in I^-(\underline{s})} \beta_{\underline{s}-\epsilon_j}(j) a_{\underline{s}-\epsilon_j}\Big) \big\langle f, a_{\underline{s}} K_{\underline{s}}(\cdot, \boldsymbol{w})\big\rangle\\ &= \Big(\frac{1}{a_{\underline{s}}} \sum_{j\in I^-(\underline{s})} \beta_{\underline{s}-\epsilon_j}(j) a_{\underline{s}-\epsilon_j}\Big) f(\boldsymbol{w}).\end{aligned}$$

Hence $\tau(\underline{s}) = \sum_{j\in I^-(\underline{s})} \frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} \beta_{\underline{s}-\epsilon_j}(j)$. Also it follows from the proof of [2, Proposition 4.4] that

$$\begin{aligned}\beta_{\underline{s}-\epsilon_j}(j) &= \Big(\frac{a}{2}(r-j) + s_j\Big) \prod_{l\neq j} \frac{s_j - s_l + \frac{a}{2}(l-j-1)}{s_j - s_l + \frac{a}{2}(l-j)}\\ &= \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} \frac{\frac{a}{2}(r-j) + s_j}{b + \frac{a}{2}(r-j) + s_j} c'_{\underline{s}}(j).\end{aligned}$$

This completes the proof. □

The following lemma, while of independent interest, is useful in obtaining a criterion for the boundedness of $\boldsymbol{M}^{(a)}$ and in many other proofs.

Lemma 4.11. *For any fixed but arbitrary $\underline{s} \in \vec{\mathbb{N}}_0^r$, we have*

$$\sum_{j=1}^r c'_{\underline{s}}(j) = \sum_{j=1}^r c_{\underline{s}}(j) = r.$$

Proof. Evidently, we have

$$\begin{aligned}\sum_{j=1}^r c'_{\underline{s}}(j) &= \sum_{j=1}^r \prod_{k\neq j} \frac{s_j - s_k + \frac{a}{2}(k-j-1)}{s_j - s_k + \frac{a}{2}(k-j)}\\ &= \sum_{j=1}^r \prod_{k\neq j} \Big(1 - \frac{\frac{a}{2}}{s_j - s_k + \frac{a}{2}(k-j)}\Big)\\ &= \sum_{j=1}^r \prod_{k\neq j} \Big(1 - \frac{\frac{a}{2}}{(s_j - \frac{a}{2}j) - (s_k - \frac{a}{2}k)}\Big).\end{aligned}$$

Setting $s'_j = \frac{s_j - \frac{a}{2} j}{\frac{a}{2}}$, we see that $s'_1 > s'_2 > \cdots > s'_r$, and

$$\sum_{j=1}^r c'_{\underline{s}}(j) = \sum_{j=1}^r \prod_{k\neq j} \left(1 - \frac{1}{s'_j - s'_k}\right) = r + \sum_{j=1}^r \sum_{\substack{A\subseteq\{1,\ldots,j-1,j+1,\ldots,r\}\\ A\neq\phi}} (-1)^{|A|} \prod_{k\in A} \frac{1}{s'_j - s'_k}$$

$$= r + \sum_{\substack{A\subseteq\{1,\ldots,r\}\\ |A|\geq 2}} (-1)^{|A|-1} \sum_{j\in A} \prod_{\substack{k\in A\\ k\neq j}} \frac{1}{s'_j - s'_k},$$

where for any finite set A, $|A|$ denotes the cardinality of A. Now, by [6, Corollary 2.3], it follows that $\sum_{j\in A}\prod_{\substack{k\in A\\ k\neq j}} \frac{1}{s'_j - s'_k} = 0$ for all $A \subseteq \{1,\ldots,r\}$ with $|A| \geq 2$. Therefore, $\sum_{j=1}^r c'_{\underline{s}}(j) = r$. The proof of the other part follows exactly in the same way. □

Theorem 4.12. *The commuting d- tuple $\boldsymbol{M}^{(a)} = (M_1^{(a)},\ldots,M_d^{(a)})$ of multiplication operators on $\mathscr{H}^{(a)}$ is bounded if and only if*

$$A := \sup\left\{ \frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} : \underline{s},\ \underline{s}-\epsilon_j \in \overrightarrow{\mathbb{N}}_0^r,\ j = 1,\ldots,r \right\}$$

is finite.

Proof. First assume that A is finite. Then we show that $\tau(\underline{s})$ is bounded for all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$:

$$\begin{aligned}
\tau(\underline{s}) &= \sum_{j\in I^-(\underline{s})} \frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} \frac{\frac{a}{2}(r-j)+s_j}{b+\frac{a}{2}(r-j)+s_j} c'_{\underline{s}}(j)\\
&\leq A \sum_{j=1}^r \frac{\frac{a}{2}(r-j)+s_j}{b+\frac{a}{2}(r-j)+s_j} c'_{\underline{s}}(j)\\
&\leq A \sum_{j=1}^r c'_{\underline{s}}(j)\\
&= Ar.
\end{aligned}$$

for any $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$. Here, the last equality follows from Lemma 4.11. It follows form the proof of Lemma 4.10 that $\langle z,w\rangle K^{(a)}(z,w) = \sum_{\underline{s}} \tau(\underline{s}) a_{\underline{s}} K_{\underline{s}}(z,w)$. Consequently, $(\sup\tau(\underline{s}) - \langle z,w\rangle)K^{(a)}(z,w)$ is non-negative definite. Hence by [6, Lemma 3.1] it follow that the d-tuple $\boldsymbol{M}^{(a)} = (M_1^{(a)},\ldots,M_d^{(a)})$ on $\mathscr{H}^{(a)}$ is bounded.

To prove the other direction, assume that $\boldsymbol{M}^{(a)}$ is bounded. This implies that $\sum_{i=1}^d M_i^{(a)} M_i^{(a)*}$ is bounded and therefore $\tau(\underline{s})$ is bounded, that is, $\tau(\underline{s}) \leq B$ for some positive real number B and for all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$. Thus

$$\frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} \frac{\frac{a}{2}(r-j)+s_j}{b+\frac{a}{2}(r-j)+s_j} c'_{\underline{s}}(j) \leq \tau(\underline{s}) \leq B, \qquad j \in I^-(\underline{s}).$$

Now, note that if $j \in I^-(\underline{s})$, then

$$\begin{aligned}\frac{1}{c'_{\underline{s}}(j)} &= \prod_{k\neq j} \frac{s_j - s_k + \frac{a}{2}(k-j)}{s_j - s_k + \frac{a}{2}(k-j-1)} \\ &= \prod_{k<j} \frac{s_j - s_k + \frac{a}{2}(k-j)}{s_j - s_k + \frac{a}{2}(k-j-1)} \prod_{k>j} \frac{s_j - s_k + \frac{a}{2}(k-j)}{s_j - s_k + \frac{a}{2}(k-j-1)} \\ &\leq \prod_{k>j} \frac{s_j - s_k + \frac{a}{2}(k-j)}{s_j - s_k + \frac{a}{2}(k-j-1)} \\ &\leq \prod_{k>j} \frac{s_j - s_k + \frac{a}{2}(k-j)}{s_j - s_k} \\ &\leq \prod_{k>j} (1 + \frac{\frac{a}{2}(k-j)}{s_j - s_k}) \\ &\leq (1 + \frac{a}{2}(r-1))^r. \end{aligned} \tag{4.7}$$

Here the third inequality holds since $\frac{s_j - s_k + \frac{a}{2}(k-j)}{s_j - s_k + \frac{a}{2}(k-j-1)} \leq 1$ for $k < j$. Now, it follows that

$$\frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} \leq \frac{B}{c'_{\underline{s}}(j)} \frac{b + \frac{a}{2}(r-j) + s_j}{\frac{a}{2}(r-j) + s_j} \leq B(1 + \frac{a}{2}(r-1))^r (1+b).$$

This completes the proof. □

Corollary 4.13. *The multiplication d- tuple $\boldsymbol{M}^{(\lambda)}$ on the weighted Bergman space $\mathscr{H}^{(\lambda)}$ is bounded if $\lambda > \frac{a}{2}(r-1)$.*

Proof. If $\lambda > \frac{a}{2}(r-1)$, then

$$\frac{(\lambda)_{\underline{s}-\epsilon_j}}{(\lambda)_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} = \frac{\frac{d}{r} - \frac{a}{2}(j-1) + s_j - 1}{\lambda - \frac{a}{2}(j-1) + s_j - 1} \leq \max\left\{1, \frac{1+b}{\lambda - \frac{a}{2}(r-1)}\right\}.$$

Therefore, from Theorem 4.12, it follows that $\boldsymbol{M}^{(\lambda)}$ is bounded. □

4.2.1 Cowen-Douglas Class

Having determined (a) the condition for boundedness of the operator $\boldsymbol{M}^{(a)}$, (b) noting that each $\boldsymbol{w}$ in Ω is a joint eigenvalue for the multiplication d- tuple $\boldsymbol{M}^{(a)*}$ and finally since the constant vector 1 is cyclic for $\boldsymbol{M}^{(a)}$, it is natural to investigate the question of which of these are in the Cowen-Douglas class $\mathrm{B}_1(\Omega)$, see [12], [13] for the definition of this very important class of operators. As shown in [19, pp. 285], the cyclicity implies that the dimension of the joint eigenspace at each $\boldsymbol{w}$ in Ω is 1. Thus to determine the membership in the Cowen-Douglas class in a neighbourhood of the origin contained in Ω, we only need to find when $\operatorname{ran} D_{\boldsymbol{M}^{(a)*}}$ is closed. The following theorem provides the precise condition for this.

Theorem 4.14. *For a multiplication d- tuple $\boldsymbol{M}^{(a)} = (M_1^{(a)}, \dots, M_d^{(a)})$ on $\mathscr{H}^{(a)}$, $\operatorname{ran} D_{\boldsymbol{M}^{(a)*}}$ is closed if and only if*

$$B := \inf\left\{ \sum_{j \in I^-(\underline{s})} \frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} : \underline{s} \in \overrightarrow{\mathbb{N}}_0^r \right\}$$

is positive.

Proof. It is elementary to see that $\operatorname{ran} D_{\boldsymbol{M}^{(a)*}}$ is closed if and only if $\sum_{i=1}^d M_i^{(a)} M_i^{(a)*}$ is bounded below on $(\ker D_{\boldsymbol{M}^{(a)*}})^{\perp}$. Also, for the d- tuple $\boldsymbol{M}^{(a)}$ on $\mathscr{H}^{(a)}$, we have $\ker D_{\boldsymbol{M}^{(a)*}} = \mathscr{P}_0$, the space of constant functions. Therefore, in view of Lemma 4.10, it suffices to show that B is non-zero positive if and only if $\inf\{\tau(\underline{s}) : \underline{s} \neq 0,\ \underline{s} \in \overrightarrow{\mathbb{N}}_0^r\}$ is non-zero positive. Suppose that B is a non-zero positive number. Now, for any non-zero $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$, we have

$$\begin{aligned}
\tau(\underline{s}) &= \sum_{j \in I^-(\underline{s})} \frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} \frac{\frac{a}{2}(r-j)+s_j}{b+\frac{a}{2}(r-j)+s_j} c'_{\underline{s}}(j) \\
&\geq \frac{1}{b+1} \sum_{j \in I^-(\underline{s})} \frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} c'_{\underline{s}}(j) \\
&\geq \frac{1}{b+1} \sum_{j \in I^-(\underline{s})} \frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} \frac{1}{(1+\frac{a}{2}(r-1))^r} \\
&\geq \frac{B}{(b+1)(1+\frac{a}{2}(r-1))^r}.
\end{aligned}$$

Here the third inequality follows from (4.7).

Conversely, assume that $\inf\{\tau(\underline{s}) : \underline{s} \neq 0,\ \underline{s} \in \overrightarrow{\mathbb{N}}_0^r\}$ is a non-zero positive number, say C. Thus for each non-zero $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$,

$$\sum_{j \in I^-(\underline{s})} \frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} \frac{\frac{a}{2}(r-j)+s_j}{b+\frac{a}{2}(r-j)+s_j} c'_s(j) \geq C. \tag{4.8}$$

Hence, noting that $c'_s(j) \leq r$ by Lemma 4.11 and $\frac{\frac{a}{2}(r-j)+s_j}{b+\frac{a}{2}(r-j)} \leq 1$, it follows that

$$\sum_{j \in I^-(\underline{s})} \frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} \geq \frac{C}{r}.$$

□

Corollary 4.15. *The range of $D_{\boldsymbol{M}^{(\lambda)*}}$ is closed if $\lambda > \frac{a}{2}(r-1)$.*

Proof. Suppose $\lambda = \frac{a}{2}(r-1)+\epsilon$ for some $\epsilon > 0$. Then

$$\sum_{j\in I^-(\underline{s})} \frac{(\lambda)_{\underline{s}-\epsilon_j}}{(\lambda)_{\underline{s}}} \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}-\epsilon_j}} = \sum_{j\in I^-(\underline{s})} \frac{b+\frac{a}{2}(r-j)+s_j}{\frac{a}{2}(r-j)+s_j+\epsilon-1}$$

which is always bounded below by 1 if $\epsilon \le b+1$. On the other hand, for $\epsilon \ge b+1$, it is bounded below by $\frac{1}{\epsilon}$. Hence, by Theorem 4.14, $\operatorname{ran} D_{\boldsymbol{M}^{(\lambda)*}}$ is closed. □

Now, we wish to show that the adjoint $\boldsymbol{M}^{(\lambda)^*}$ of the d- tuple of multiplication operators on $\mathscr{H}^{(\lambda)}$ is in the Cowen-Douglas class $\mathrm{B}_1(\Omega)$ for $\lambda > \frac{a}{2}(r-1)$.

Recall that the left essential spectrum $\pi_e^{\ell,0}(\boldsymbol{T})$ of a commuting d- tuple of operators $\boldsymbol{T}$ is defined to be the complement of the set of all $\boldsymbol{w} \in \mathbb{C}^d$ with the property:

1. $\dim\ker D_{(\boldsymbol{T}-\boldsymbol{w}I)}$ is finite,
2. $\operatorname{ran} D_{(\boldsymbol{T}-\boldsymbol{w}I)}$ is closed.

If $0 \notin \pi_e^{\ell,0}(\boldsymbol{T})$, then the d- tuple $\boldsymbol{T}$ is said to be left semi-Fredholm.

The essential ingredient of the proof of the following theorem is based on the spectral mapping property of the left essential spectrum, which appears in [21] and was pointed out to G. Misra by J. Eschmeier during a conversation at University of Saarbrucken in February 2014.

Theorem 4.16. *The adjoint $\boldsymbol{M}^{(\lambda)^*}$ of the multiplication d- tuple on $\mathscr{H}^{(\lambda)}$ is in the Cowen-Douglas class $B_1(\Omega)$ whenever $\lambda > \frac{a}{2}(r-1)$.*

Proof. Since polynomials are dense in the Hilbert space $\mathscr{H}^{(\lambda)}$, it follows that $\dim\ker D_{\boldsymbol{M}^{(\lambda)*}}$ is 1. By Corollary 4.15, we also have that ran $D_{\boldsymbol{M}^{(\lambda)*}}$ is closed. Therefore, $D_{\boldsymbol{M}^{(\lambda)*}}$ is left semi-Fredholm and hence there is an $\varepsilon > 0$ such that for $\boldsymbol{w} \in \Omega$ with $\sum_{i=1}^d |w_i|^2 < \varepsilon$, the operators $D_{(\boldsymbol{M}-\boldsymbol{w}I)^*}$ are left semi-Fredholm. Thus $\boldsymbol{M}^{(\lambda)^*}$ is in the Cowen-Douglas class $\mathrm{B}_1(\Omega_\varepsilon)$, where $\Omega_\varepsilon = \{\boldsymbol{w} \in \Omega : \sum_{i=1}^d |w_i|^2 < \varepsilon\}$. Note that the operator $\boldsymbol{M}^{(\nu)}$ is homogeneous (see [2]). This together with the spectral mapping property of the left essential spectrum shows that $\boldsymbol{M}^{(\nu)^*}$ is actually in $\mathrm{B}_1(\Omega)$.

To complete the proof, first note that if $\boldsymbol{w} \in \Omega$ is any fixed but arbitrary point, then there exists a biholomorphic automorphism φ of Ω with the property: $\varphi(0) = \boldsymbol{w}$. We have seen that $0 \notin \pi_e^{\ell,0}(\boldsymbol{M}^{(\lambda)^*})$. An analytic spectral mapping property for the left essential spectrum is ensured by [21, Corollary 2.6.9]. It follows that

$$\boldsymbol{w} = \varphi(0) \notin \varphi(\pi_e^{\ell,0}(\boldsymbol{M}^{(\lambda)^*})) = \pi_e^{\ell,0}(\varphi(\boldsymbol{M}^{(\lambda)^*})) = \pi_e^{\ell,0}(\boldsymbol{M}^{(\lambda)^*}).$$

Here the last equality follows from the homogeneity assumption. □

4.3 Unitary equivalence and Similarity

In this section, we study the question of unitary equivalence and similarity of two commuting d- tuple of operators in the class $\mathscr{A}\mathbb{K}(\Omega)$. In particular, when $\mathbb{K}$ is the unit circle $\mathbb{T}$, these results were obtained by Shields in [39] and the case when $\mathbb{K}$ is $\mathscr{U}(d)$, similarity result was obtained in [29, Lemma 2.2].

By Theorem 4.5, any d- tuple of operators $\boldsymbol{T}$ in $\mathscr{A}\mathbb{K}(\Omega)$ is unitarily equivalent to $\boldsymbol{M}^{(a)}$ consisting of multiplication operators by the coordinate functions $z_1,\ldots,z_d$ on the reproducing kernel Hilbert space $\mathscr{H}^{(a)}$ with the reproducing kernel $K^{(a)}(\boldsymbol{z},\boldsymbol{w}) = \sum_{\underline{s}} a_{\underline{s}} K_{\underline{s}}(\boldsymbol{z},\boldsymbol{w})$, where $a_{\underline{s}} > 0$ with $a_{\underline{0}} = 1$. Thus we assume, without loss of generality, that $\boldsymbol{T} \sim_u \boldsymbol{M}^{(a)}$.

Theorem 4.17. *Let $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ be two operator tuples in $\mathscr{A}\mathbb{K}(\Omega)$. Suppose that $\boldsymbol{T}_1 \sim_u \boldsymbol{M}^{(a)}$ and $\boldsymbol{T}_2 \sim_u \boldsymbol{M}^{(b)}$.*

Then the following statements are equivalent.

(i) *$\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are unitarily equivalent.*

(ii) *$a_{\underline{s}} = b_{\underline{s}}$ for all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$.*

(iii) $K^{(a)} = K^{(b)}$.

Proof. It is easy to see that (ii) and (iii) are equivalent. It is obvious that (iii) implies (i). Therefore it remains to verify that (i) implies (iii). Assume that the d- tuples $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are unitarily equivalent. Then so are the operators $\boldsymbol{M}^{(a)}$ and $\boldsymbol{M}^{(b)}$. By [14, Theorem 3.7], there exists a holomorphic function g on Ω such that

$$K^{(a)}(\boldsymbol{z},\boldsymbol{w}) = g(\boldsymbol{z})K^{(b)}(\boldsymbol{z},\boldsymbol{w})\overline{g(\boldsymbol{w})}, \qquad \boldsymbol{z},\boldsymbol{w} \in \Omega.$$

In particular, $K^{(a)}(\boldsymbol{z},0) = g(\boldsymbol{z})K^{(b)}(\boldsymbol{z},0)\overline{g(0)}$, $\boldsymbol{z} \in \Omega$. Therefore, $a_{\underline{0}} = b_{\underline{0}} g(\boldsymbol{z})\overline{g(0)}$, and consequently, $g(z)\overline{g(0)} = 1$ since $a_{\underline{0}} = b_{\underline{0}} = 1$. Hence $K^{(a)} = K^{(b)}$. □

Recall that two commuting d- tuples $\boldsymbol{A} = (A_1,\ldots,A_d)$ and $\boldsymbol{B} = (B_1,\ldots,B_d)$, defined on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, are said to be similar if there exists an invertible operator $X : \mathscr{H}_1 \to \mathscr{H}_2$ such that $XA_i = B_iX$ for all $i = 1,\ldots,d$. For a non-negative integer n, as before, $\mathscr{P}_n$ denotes the space of homogeneous polynomials of degree n in d variables. For two non-negative definite kernels K and $\tilde{K}$, we write $K \preceq \tilde{K}$ if $\tilde{K} - K$ is a non-negative definite kernel.

Theorem 4.18. *Let $\Omega \subseteq \mathbb{C}^d$ be any bounded domain (not necessarily symmetric), and let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two reproducing kernel Hilbert spaces determined by two positive definite kernels K_1 and K_2 respectively. Suppose that*

(i) *the space of polynomials $\mathscr{P}$ is dense in both $\mathscr{H}_1$ and $\mathscr{H}_2$,*

(ii) *$\mathscr{P}_n$ is orthogonal to $\mathscr{P}_m$ if $m \neq n$ in $\mathscr{H}_1$ and $\mathscr{H}_2$,*

(iii) *for each $i = 1,2$, the d- tuple $\boldsymbol{M}^{(i)} = (M_1^{(i)}, \dots, M_d^{(i)})$ of multiplication operators by the coordinate functions $z_1, \dots, z_d$ on $\mathscr{H}_i$ is bounded.*

Then the following statements are equivalent.

(i) *$\boldsymbol{M}^{(1)}$ and $\boldsymbol{M}^{(2)}$ are similar.*

(ii) *There exist constants $\alpha, \beta > 0$ such that*

$$\alpha \|p\|_{\mathscr{H}_1} \leq \|p\|_{\mathscr{H}_2} \leq \beta \|p\|_{\mathscr{H}_1}, \qquad p \in \mathscr{P}. \tag{4.9}$$

(iii) *$\mathscr{H}_1 = \mathscr{H}_2$ (as sets).*

(iv) *There exist constants $\alpha, \beta > 0$ such that*

$$\alpha K_1 \preceq K_2 \preceq \beta K_1.$$

Proof. The equivalence of (iii) and (iv) follows from the standard theory of reproducing kernel Hilbert spaces (cf. [3], [38, Theorem 6.25]). Let $f \in \mathscr{H}_1$. Since polynomials are dense in $\mathscr{H}_1$ there exists a sequence of polynomials $\{p_n\}$ which converges to f in $\mathscr{H}_1$. Hence $\{p_n\}$ is Cauchy in $\mathscr{H}_1$ and therefore by (ii) it is also Cauchy in $\mathscr{H}_2$. Thus $\{p_n\}$ converges to some g in $\mathscr{H}_2$. Since Evaluations are continuous in both $\mathscr{H}_1$ and $\mathscr{H}_2$, we see that $f(w) = \lim p_n(w) = g(w)$, $w \in \Omega$. Hence $\mathscr{H}_1 \subseteq \mathscr{H}_2$. Similarly we can show that $\mathscr{H}_2 \subseteq \mathscr{H}_1$. Hence (ii) implies (iii). If $\mathscr{H}_1 = \mathscr{H}_2$ (as sets), then the identity operator from $\mathscr{H}_1$ to $\mathscr{H}_2$ is a bounded invertible operator which intertwines the multiplication d- tuples $\boldsymbol{M}^{(1)}$ and $\boldsymbol{M}^{(2)}$, and consequently, (iii) implies (i). Now, to complete the proof, it remains to show that (i) implies (ii).

Suppose that $\boldsymbol{M}^{(1)}$ and $\boldsymbol{M}^{(2)}$ are similar. Then there exists an invertible operator $X : \mathscr{H}_1 \to \mathscr{H}_2$ such that

$$X M_j^{(1)} = M_j^{(2)} X, \qquad j = 1, \dots, d. \tag{4.10}$$

Since the subspaces $\mathscr{P}_n$, $n \geq 0$, are mutually orthogonal, it suffices to show that (4.9) is satisfied for all $p \in \mathscr{P}_n$ and for some $\alpha, \beta > 0$ (which is independent of n). Fix a polynomial p in $\mathscr{P}_n$. Clearly, it follows from (4.10) that

$$X M_p^{(1)} = M_p^{(2)} X, \tag{4.11}$$

where $M_p^{(i)}$ is the operator of multiplication by the polynomial p on $\mathscr{H}_i$ for $i = 1,2$.

Let $\big(X_{r,s}\big)_{r,s=0}^{\infty}$ be the matrix representation of X with respect to $\oplus_{n=0}^{\infty}\mathscr{P}_n$, that is, $X_{r,s} = P_{\mathscr{P}_r} X_{|\mathscr{P}_s}$. Similarly, let $M_p^{(i)} = \big((M_p^{(i)})_{r,s}\big)_{r,s=0}^{\infty}$ be the matrix representation of $M_p^{(i)}$, $i = 1,2$. Since $M_p^{(i)}$ maps $\mathscr{P}_s$ into $\mathscr{P}_{s+n}$, $i = 1,2$, it clear that

$$(M_p^{(i)})_{r,s} = \begin{cases} (M_p^{(i)})_{|\mathscr{P}_s}, & \text{if } r = s+n \\ 0, & \text{otherwise.} \end{cases} \tag{4.12}$$

Therefore it follows from (4.11) that

$$X_{r,s+n}(M_p^{(1)})_{s+n,s} = \begin{cases} (M_p^{(2)})_{r,r-n}X_{r-n,s}, & \text{if } r-n \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{4.13}$$

Choosing $r = n$ and $s = 0$, we see that

$$(M_p^{(2)})_{n,0}X_{0,0} = X_{n,n}(M_p^{(1)})_{n,0}. \tag{4.14}$$

Therefore

$$(M_p^{(1)})_{n,0}^{*}X_{n,n}^{*}X_{n,n}(M_p^{(1)})_{n,0} = X_{0,0}^{*}(M_p^{(2)})_{n,0}^{*}(M_p^{(2)})_{n,0}X_{0,0}. \tag{4.15}$$

Since $\|X_{n,n}\| \leq \|X\|$, we have

$$X_{n,n}^{*}X_{n,n} \leq \|X\|^2 I.$$

Hence from (4.15) we obtain

$$X_{0,0}^{*}(M_p^{(2)})_{n,0}^{*}(M_p^{(2)})_{n,0}X_{0,0} \leq \|X\|^2(M_p^{(1)})_{n,0}^{*}(M_p^{(1)})_{n,0}. \tag{4.16}$$

Note that $X_{0,0}$ is a linear map from $\mathscr{P}_0$ to $\mathscr{P}_0$ and $\dim\mathscr{P}_0 = 1$. Hence $X_{0,0}1 = \eta 1$ for some $\eta \in \mathbb{C}$. Also, taking p to be the polynomial z_j, $1 \leq j \leq d$, and $r = 0$ in (4.13) we see that

$$X_{0,s+1}(M_{z_j}^{(1)})_{s+1,s} = 0, \qquad \text{for all } s \geq 0.$$

Since this is true for all $j = 1,\ldots,d$, it follows that $X_{0,s+1} = 0$ for all $s \geq 0$. Moreover, since X is invertible we must have $X_{0,0} \neq 0$. Otherwise, $X_{0,s} = 0$ for all $s \geq 0$, implying that $\mathscr{P}_0$ is orthogonal to the range of X, which is a contradiction. Hence $X_{0,0} \neq 0$, and consequently $\eta \neq 0$. Therefore, from (4.16), we obtain

$$\langle (M_p^{(2)})_{n,0}X_{0,0}1, (M_p^{(2)})_{n,0}X_{0,0}1\rangle \leq \|X\|^2\langle (M_p^{(1)})_{n,0}1, (M_p^{(1)})_{n,0}1\rangle.$$

Consequently,

$$|\eta|^2\|p\|_{\mathscr{H}_2}^2 \leq \|X\|^2\|p\|_{\mathscr{H}_1}^2. \tag{4.17}$$

To finish the proof, note that (4.10) implies

$$X^{-1}M_j^{(2)} = M_j^{(1)}X^{-1},\ j = 1,\ldots,d.$$

Hence repeating the arguments used to establish (4.17) we obtain that

$$|\zeta|^2\|p\|^2_{\mathscr{H}_1} \le \|X^{-1}\|^2\|p\|^2_{\mathscr{H}_2},$$

where $(X^{-1})_{0,0}1 = \zeta 1$, $\zeta \neq 0$. This completes the proof. □

Remark 4.19. In the proof given above, we have shown that $X_{0,s} = 0$ for all $s \ge 1$. But using (4.13), it can be easily verified that $X_{r,s} = 0$ for all $s > r$, that is, X is lower triangular with respect to the decomposition $\oplus_{n=0}^{\infty}\mathscr{P}_n$. Consequently, $\zeta = \frac{1}{\eta}$.

Theorem 4.20. *Let $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ be two operator tuples in $\mathscr{A}\mathbb{K}(\Omega)$. Suppose that $\boldsymbol{T}_1 \sim_u \boldsymbol{M}^{(a)}$ and $\boldsymbol{T}_2 \sim_u \boldsymbol{M}^{(b)}$. Then the following statements are equivalent.*

(i) *$\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are similar.*

(ii) *There exist constants $\alpha, \beta > 0$ such that*

$$\alpha\|p\|_{\mathscr{H}^{(a)}} \le \|p\|_{\mathscr{H}^{(b)}} \le \beta\|p\|_{\mathscr{H}^{(a)}}, \qquad p \in \mathscr{P}. \tag{4.18}$$

(iii) *$\mathscr{H}^{(a)} = \mathscr{H}^{(b)}$ (as sets).*

(iv) *There exist constants $\alpha, \beta > 0$ such that*

$$\alpha K^{(a)} \preceq K^{(b)} \preceq \beta K^{(a)}.$$

(v) *there exist constants $\alpha, \beta > 0$ such that*

$$\alpha a_{\underline{s}} \le b_{\underline{s}} \le \beta a_{\underline{s}}, \qquad \underline{s} \in \overrightarrow{\mathbb{N}}_0^r.$$

Proof. The equivalence of (i), (ii), (iii) and (iv) follows easily from Theorem 4.18. Assume that (ii) holds. Then (v) is easily verified by choosing any polynomial p in $\mathscr{P}_{\underline{s}}$ and using $\|p\|^2_{\mathscr{H}^{(a)}} = \frac{\|p\|^2_{\mathscr{F}}}{a_{\underline{s}}}$ and $\|p\|^2_{\mathscr{H}^{(b)}} = \frac{\|p\|^2_{\mathscr{F}}}{b_{\underline{s}}}$ in (4.9). Also, it is trivial to see that (v) implies (iv). □

Corollary 4.21. *Let $\lambda_1, \lambda_2 > \frac{a}{2}(r-1)$. Then the d- tuple of multiplication operators $\boldsymbol{M}^{(\lambda_1)}$ on $\mathscr{H}^{(\lambda_1)}$ and $\boldsymbol{M}^{(\lambda_2)}$ on $\mathscr{H}^{(\lambda_2)}$ are similar if and only if $\lambda_1 = \lambda_2$.*

Proof. Suppose that $\boldsymbol{M}^{(\lambda_1)}$ and $\boldsymbol{M}^{(\lambda_2)}$ are similar. Then, by Theorem 4.20, there exist constants $\alpha, \beta > 0$ such that $\alpha(\lambda_1)_{\underline{s}} \le (\lambda_2)_{\underline{s}} \le \beta(\lambda_1)_{\underline{s}}$ for all $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$. Take $\underline{s} = (s_1, 0, \ldots, 0)$, $s_1 \in \mathbb{N}_0$. By the properties of the Gamma function we have $\frac{(\lambda_1)_{\underline{s}}}{(\lambda_2)_{\underline{s}}} = \frac{(\lambda_1)_{s_1}}{(\lambda_2)_{s_1}} \sim {s_1}^{\lambda_1 - \lambda_2}$. Hence $\lambda_1 = \lambda_2$. The other implication is trivial. □

Chapter 5

Computation of $\sum M_i^* M_i$

We have studied the commuting tuple $\boldsymbol{M}^{(a)}$ of multiplication by the coordinate functions on the Hilbert space $\mathscr{H}^{(a)}$. In particular, we have determined when they are bounded, when they belong to the Cowen-Douglas class and when they are mutually unitarily equivalent, respectively, similar. As we have pointed out earlier, the boundedness of the commuting tuple $\boldsymbol{M}^{(a)}$ follows form the explicit formula for $\sum M_i^{(a)} M_i^{(a)*}$. Surprisingly, no formula was worked out for the operator $\sum M_i^{(a)*} M_i^{(a)}$. If nothing else, it should have been done for the sake of completeness. Of course, the boundedness of the commuting d- tuple $\boldsymbol{M}^{(a)}$ would follow from this computation as easily as from the other one. However, the initial motivation for computing this lies in the equality for a commuting pair of operators $\boldsymbol{T} = (T_1, T_2)$.

$$\mathrm{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big) = \Big(\sum_{i=1}^{2} T_i^* T_i\Big)^2 + \Big(\sum_{i=1}^{2} T_i T_i^*\Big)^2 - \sum_{j=1}^{2} T_j \Big(\sum_{i=1}^{2} T_i^* T_i\Big) T_j^* - \sum_{j=1}^{2} T_j^* \Big(\sum_{i=1}^{2} T_i T_i^*\Big) T_j.$$

It is likely that a similar formula connecting $\mathrm{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$ and the two sums $\sum T_i T_i^*$ and $\sum T_i^* T_i$ exist for any commuting d- tuple of operators $\boldsymbol{T} = (T_1, \ldots, T_d)$, $d > 2$.

The results of this Chapter are in two parts. In the first part, we obtain an explicit formula for the operator $\sum M_i^{(\frac{d}{r})^*} M_i^{(\frac{d}{r})}$ on the Hardy space of the Shilov boundary of any classical bounded symmetric domain Ω of rank r, using this, in the second part, a formula for this operator acting on the Hilbert space $\mathscr{H}^{(a)}$ is given as long as the rank of Ω is 2. Based on these computations, a conjecture was made for the actual form of the operator for a domain of any rank. H. Upmeier has actually verified this conjecture to be true. It follows, as a corollary, that all the commutators of the multiplication by the coordinate functions on $\mathscr{H}^{(\lambda)}$ are compact if and only if rank of Ω is 1.

5.1 Multiplication operators on $H^2(S_\Omega)$

The Hilbert space $\mathscr{H}^{(\lambda)}$, $\lambda = \frac{d}{r}$ is the Hardy space of the Shilov boundary of the domain Ω. It will be useful to first determine the form of the operator $\sum M_i^* M_i$ on the Shilov boundary. This computation is particularly simple for two reasons: First, the inner product in the Hardy space is given by integration against a quasi-invariant measure on S_Ω and second, the action of the operator $\sum M_i^* M_i$ on functions from $H^2(S_\Omega)$ is explicit.

In this section, for each one of the classical bounded symmetric domains Ω, we compute the operator $\sum_{i=1}^d M_i^* M_i$ on the Hardy space $H^2(S_\Omega)$ of the Shilov boundary S_Ω. This is easily done since the function $\sum_{i=1}^d |z_i|^2 = r$ on the Shilov boundary S_Ω. This was also noted in a paper of A. Athavale [5], although, his answers are different since the realization of the classical bounded symmetric domains that he uses are different from the Harish-Chandra realization used here.

5.1.1 Multiplication operators on $H^2(S_\Omega)$ for type-I domains

$I_{n,m}$: The domains Ω of type I consists of matrices of order $n \times m$, $n \le m$, of norm strictly less than 1. The rank of the domain Ω is n. It is known that the Shilov boundary S_Ω consists of maximal partial isometries in $n \times m$ complex matrices. For a positive integer n, let $H_n^2(S_\Omega)$ denote the direct sum of n copies of $H^2(S_\Omega)$. Let $\mathbb{M} : H_m^2(S_\Omega) \to H_n^2(S_\Omega)$ be the block operator $\left(M_{ij}\right)_{i=1,j=1}^{n,m}$, where M_{ij} is the multiplication operator by the coordinate function z_{ij} on the Hardy space $H^2(S_\Omega)$ over S_Ω. Let $\mathbb{M}^t$ denote the block operator $\left(M_{ji}\right)_{j=1,i=1}^{m,n} : H_n^2(S_\Omega) \to H_m^2(S_\Omega)$ and $\mathbb{M}^{t*}$ be the adjoint of the operator $\mathbb{M}^t$.

Theorem 5.1. *The operator $\mathbb{M}^{t*}\mathbb{M}^t : H_n^2(S_\Omega) \to H_n^2(S_\Omega)$ equals the identity operator I_n on $H_n^2(S_\Omega)$.*

Proof. Since the Shilov boundary S_Ω of Ω is the set of all $z \in M_{n,m}$ such that $zz^* = I$, it follows that $\sum_{j=1}^m z_{ij}\bar{z}_{jk} = \delta_{ik}$ for all $i,k = 1,\dots,n$, where $z = (z_{ij})_{i=1,j=1}^{n,m} \in S_\Omega$. Any $f \in H_n^2(S_\Omega)$ is of the form $\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$ for some choice of $f_1,\dots,f_n \in H^2(S_\Omega)$. For such a $f \in H_n^2(S_\Omega)$, we have

$$\begin{aligned}
\|\mathbb{M}^t f\|^2 &= \sum_{j=1}^m \left\| \sum_{i=1}^n z_{ij} f_i \right\|^2 = \sum_{j=1}^m \sum_{i,k=1}^n \langle z_{ij} f, z_{kj} f_k \rangle \\
&= \sum_{i,k=1}^n \sum_{j=1}^m \int_{S_\Omega} z_{ij} f_i \bar{z}_{kj} \bar{f}_k \, d\sigma = \sum_{i,k=1}^n \int_{S_\Omega} \left(\sum_{j=1}^m z_{ij}\bar{z}_{kj} \right) f_i \bar{f}_k \, d\sigma \\
&= \sum_{i,k=1}^n \int_{S_\Omega} \delta_{ik} f_i \bar{f}_k \, d\sigma = \sum_{i=1}^n \int_{S_\Omega} |f_i|^2 \, d\sigma = \|f\|^2.
\end{aligned}$$

completing the proof. □

The following corollary is an immediate consequence.

Corollary 5.2. *If M_{ij} is the multiplication operator by the coordinate function z_{ij} on $H^2(S_\Omega)$ then $\sum_{i=1,j=1}^{n,m} M_{ij}^* M_{ij} = nI$ on $H^2(S_\Omega)$.*

5.1.2 Multiplication operators on $H^2(S_\Omega)$ for type-II domains

II_n : The domains Ω of Type-II consist of symmetric complex matrices $\boldsymbol{z}$ of order n with $\|\boldsymbol{z}\| < 1$. In this case, the rank of the domain Ω is n. The Shilov boundary S_Ω consists of symmetric complex matrices $\boldsymbol{z}$ of order n such that $\boldsymbol{z}^*\boldsymbol{z} = I$. Pick one of these domains of dimension $\frac{n(n+1)}{2}$. It is convenient to put $\frac{n(n+1)}{2}$ variables in the form of a symmetric matrix, where the inner product is given by $\operatorname{tr}(AB^*)$. Now, in the space of these symmetric matrices of size n, the matrices E_{ii}, $i = 1,\dots,n$ together with $\frac{E_{ij}+E_{ji}}{\sqrt{2}}$, $1 \le i \ne j \le n$, form an orthonormal basis. Consequently, the coordinates of this domain are of the form

$$\tilde{\boldsymbol{z}} := \big(z_{11}, \sqrt{2}z_{12}, \dots, \sqrt{2}z_{1n}, z_{22}, \dots, \sqrt{2}z_{2n}, \dots, z_{n-1n-1}, \sqrt{2}z_{n-1n}, z_{nn}\big),$$

see [25, pp. 130]. Thus the coordinate $\tilde{z}_{ii} = z_{ii}$ and for $(i,j) : j > i$, $\tilde{z}_{ij} = \sqrt{2}z_{ij}$.

Theorem 5.3. *If M_{ij} is the multiplication operator by the coordinate function $\tilde{z}_{ij}$ on $H^2(S_\Omega)$ then $\sum_{i\le j} M_{ij}^* M_{ij} = nI$ on $H^2(S_\Omega)$.*

Proof. Since $\operatorname{trace}(\boldsymbol{z}^*\boldsymbol{z}) = \operatorname{trace}(I) = n$ it follows that $\sum_{i\le j} |\tilde{z}_{ij}|^2 = \sum_{i,j=1}^n |z_{ij}|^2 = n$. Thus

$$\begin{aligned}
\big\langle \big(\sum_{i\le j} M_{ij}^* M_{ij}\big) f, f \big\rangle &= \sum_{i\le j} \big\| M_{ij} f \big\|^2 \\
&= \sum_{i\le j} \int_{S_\Omega} |\tilde{z}_{ij} f(\boldsymbol{z})|^2 d\sigma \\
&= \sum_{i,j=1}^n \int_{S_\Omega} |z_{ij} f(\boldsymbol{z})|^2 d\sigma \\
&= \int_{S_\Omega} \sum_{i,j=1}^n |z_{ij}|^2 |f(\boldsymbol{z})|^2 d\sigma \\
&= n \int_{S_\Omega} |f(\boldsymbol{z})|^2 d\sigma = n\|f\|^2.
\end{aligned}$$

This completes the proof. □

5.1.3 Multiplication operators on $H^2(S_\Omega)$ for type-III domains

III_n: The domains Ω of Type-III consist of anti-symmetric complex matrices $\boldsymbol{z}$ of order n with $\|\boldsymbol{z}\| < 1$. In this case, the rank of the domain Ω is $[\frac{n}{2}]$. If n is an even number the Shilov boundary S_Ω consists of anti-symmetric complex matrices $\boldsymbol{z}$ of order n such that $\boldsymbol{z}^*\boldsymbol{z} = I$. If n is odd then the Shilov boundary S_Ω consists of all matrices of the form UDU^*, where U is an arbitrary unitary matrix [25] and

$$D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus (0).$$

Coordinates of this domain are of the form

$$z_{12}, z_{13}, \ldots, z_{1n}, z_{23} \ldots z_{2n}, \ldots, z_{n-1n},$$

see [25, pp.138]

Theorem 5.4. *If M_{ij} is the multiplication operator by the coordinate function z_{ij} on $H^2(S_\Omega)$ then $\sum_{i<j} M_{ij}^* M_{ij} = [\frac{n}{2}]I$ on $H^2(S_\Omega)$.*

Proof. If $\boldsymbol{z}$ is in S_Ω, then trace $(\boldsymbol{z}^*\boldsymbol{z}) = 2[\frac{n}{2}]$. Thus $\sum_{i<j} |z_{ij}|^2 = [\frac{n}{2}]$. The remaining portion of the proof is the same as the proof of Theorem 5.3. □

5.1.4 Multiplication operators on $H^2(S_\Omega)$ for type-IV domains

Type-IV domains, which are also known as the Lie balls, consist of all $\boldsymbol{z} \in \mathbb{C}^d$ $(d \geq 5)$ such that $1 + \frac{1}{4}|\boldsymbol{z}^t\boldsymbol{z}|^2 > \overline{\boldsymbol{z}}^t\boldsymbol{z}$ and $\overline{\boldsymbol{z}}^t\boldsymbol{z} < 2$, where $\overline{\boldsymbol{z}}^t$ is the complex conjugate of the transpose $\boldsymbol{z}^t$. These domains have rank 2 independent of the dimension d. In this case the Shilov boundary $S_\Omega \subseteq \mathbb{S}_d(0, \sqrt{2})$, where $\mathbb{S}_d(0, \sqrt{2})$ is the sphere of radius $\sqrt{2}$ centred at 0 in $\mathbb{C}^d$, see [36, pp. 190 and Theorem X.4.6].

Theorem 5.5. *If M_i is the multiplication operator by the coordinate function z_i on $H^2(S_\Omega)$ then $\sum_{i=1}^d M_i^* M_i = 2I$ on $H^2(S_\Omega)$.*

Proof. Since the Shilov boundary S_Ω is a subset of $\mathbb{S}(0, \sqrt{2})$, it follows that $\sum_{i=1}^d |z_i|^2 = 2$. Hence $\sum_{i=1}^d M_i^* M_i = 2I$ on $H^2(S_\Omega)$. □

The discussion of this section is summarized in the Theorem stated below.

Theorem 5.6. *Let $\boldsymbol{M}^{(\frac{d}{r})} = (M_1^{(\frac{d}{r})}, \ldots, M_d^{(\frac{d}{r})})$ be the d- tuple of multiplication operators by the coordinate functions $z_1, \ldots, z_d$ on the Hardy space $H^2(S_\Omega)$. Then*

$$\sum_{i=1}^{d} M_i^{(\frac{d}{r})^*} M_i^{(\frac{d}{r})} = rI. \tag{5.1}$$

5.2 $\sum_i^d M_i^* M_i$ on $\mathscr{H}^{(a)}$

Let $\boldsymbol{M}^{(a)} := (M_1^{(a)}, \ldots, M_d^{(a)})$ be the d-tuple of multiplication by the coordinate functions $z_1, \ldots, z_d$ on $\mathscr{H}^{(a)}$. In this section, we wish to compute the operator $\boldsymbol{M}^{(a)*}\boldsymbol{M}^{(a)} := \sum_{i=1}^d M_i^{(a)*} M_i^{(a)}$ on the Hilbert space $\mathscr{H}^{(a)}$.

By Lemma 4.9, note that $\boldsymbol{M}^{(a)*}\boldsymbol{M}^{(a)}$ is a block diagonal operator with respect to the decomposition $\oplus \mathscr{P}_{\underline{s}}$, where each block is a non-negative scalar multiple of the identity, that is, $\boldsymbol{M}^{(a)*}\boldsymbol{M}^{(a)} p = \delta(\underline{s}) p,\ p \in \mathscr{P}_{\underline{s}}$ for some non-negative real number $\delta(\underline{s})$. Therefore, in order to compute the operator $\boldsymbol{M}^{(a)*}\boldsymbol{M}^{(a)}$, it is sufficient to find the constants $\delta(\underline{s})$ for all $\underline{s}$ in $\overrightarrow{\mathbb{N}}_0^r$. Unfortunately, we are only able to find $\delta(\underline{s})$ when $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$ and $|I^+(\underline{s})| \le 2$. In particular, we have the complete answer in case the rank $r = 2$.

The following lemma gives a description of the operator $M_i^{(a)*}$ on $\mathscr{H}^{(a)}$. In case of weighted Bergman spaces, it is described in [44, Lemma 4.12.19].

Lemma 5.7. *If $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$ and p is a polynomial in $\mathscr{P}_{\underline{s}}$, then*

$$M_i^{(a)*} p = \sum_{j \in I^-(\underline{s})} \frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} (\partial_i p)_{\underline{s}-\epsilon_j},$$

where ∂_i denotes the partial derivative with respect to the variable z_i.

Proof. By [44, Theorem 4.11.86], we have that $z_i \mathscr{P}_{\underline{s}}$ is contained in $\oplus_{j \in I^+(\underline{s})} \mathscr{P}_{\underline{s}+\epsilon_j}$. Thus, for any polynomial p in $\mathscr{P}_{\underline{s}}$, it follows that $M_i^* p$ belongs to $\oplus_{j \in I^-(\underline{s})} \mathscr{P}_{\underline{s}-\epsilon_j}$. Now for $j \in I^-(\underline{s})$ and $q \in \mathscr{P}_{\underline{s}-\epsilon_j}$, we have

$$\begin{aligned}
\langle M_i^* p, q \rangle_{\mathscr{H}^{(a)}} = \langle p, z_i q \rangle_{\mathscr{H}^{(a)}} &= \langle p, (z_i q)_{\underline{s}} \rangle_{\mathscr{H}^{(a)}} \\
&= \frac{1}{a_{\underline{s}}} \langle p, (z_i q)_{\underline{s}} \rangle_{\mathscr{F}} \\
&= \frac{1}{a_{\underline{s}}} \langle p, z_i q \rangle_{\mathscr{F}} \\
&= \frac{1}{a_{\underline{s}}} \langle \partial_i p, q \rangle_{\mathscr{F}} \\
&= \frac{1}{a_{\underline{s}}} \langle (\partial_i p)_{\underline{s}-\epsilon_j}, q \rangle_{\mathscr{F}} \\
&= \frac{a_{\underline{s}-\epsilon_j}}{a_{\underline{s}}} \langle (\partial_i p)_{\underline{s}-\epsilon_j}, q \rangle_{\mathscr{H}^{(a)}}.
\end{aligned}$$

Here the equality $\langle p, z_i q \rangle_{\mathscr{F}} = \langle \partial_i p, q \rangle_{\mathscr{F}}$ follows from [44, Proposition 4.11.36]. This completes the proof. □

The following theorem describes the operator $\boldsymbol{M}^{(a)*}\boldsymbol{M}^{(a)}$ on some subspace of $\mathscr{H}^{(a)}$.

Theorem 5.8. *Let $\underline{s} \in \overrightarrow{\mathbb{N}}_0^r$ such that $|I^+(\underline{s})| \le 2$. Then $\boldsymbol{M}^{(a)*}\boldsymbol{M}^{(a)}p = \delta(\underline{s})p$, $p \in \mathscr{P}_{\underline{s}}$, where*

$$\delta(\underline{s}) = \sum_{j \in I^+(\underline{s})} \frac{a_{\underline{s}}}{a_{\underline{s}+\epsilon_j}} \frac{(\frac{d}{r})_{\underline{s}+\epsilon_j}}{(\frac{d}{r})_{\underline{s}}} c_{\underline{s}}(j). \tag{5.2}$$

Proof. First note that, for $p \in \mathscr{P}_{\underline{s}}$, we have

$$\begin{aligned}\sum_{i=1}^d M_i^{(a)*} M_i^{(a)} p = \sum_{i=1}^d M_i^{(a)*}(z_i p) &= \sum_{i=1}^d \Big(M_i^{(a)*}\Big(\sum_{j \in I^+(\underline{s})} (z_i p)_{\underline{s}+\epsilon_j}\Big)\Big)_{\underline{s}} \\ &= \sum_{i=1}^d \Big(\sum_{j \in I^+(\underline{s})} \frac{a_{\underline{s}}}{a_{\underline{s}+\epsilon_j}} \partial_i\big((z_i p)_{\underline{s}+\epsilon_j}\big)\Big)_{\underline{s}} \\ &= \sum_{j \in I^+(\underline{s})} \frac{a_{\underline{s}}}{a_{\underline{s}+\epsilon_j}} \sum_{i=1}^d \Big(\partial_i\big((z_i p)_{\underline{s}+\epsilon_j}\big)\Big)_{\underline{s}},\end{aligned}$$

where the third equality follows from Lemma 5.7. Let Q_j be the linear map on $\mathscr{P}_{\underline{s}}$ given by

$$Q_j(p) = \begin{cases} \sum_{i=1}^d \Big(\partial_i\big((z_i p)_{\underline{s}+\epsilon_j}\big)\Big)_{\underline{s}}, & \text{if } j \in I^+(\underline{s}) \\ 0, & \text{otherwise.}\end{cases}$$

Then clearly,

$$\delta(\underline{s})p = \sum_{j \in I^+(\underline{s})} \frac{a_{\underline{s}}}{a_{\underline{s}+\epsilon_j}} Q_j(p). \tag{5.3}$$

Note that, for $p \in \mathscr{P}_{\underline{s}}$, Q_j satisfies the following:

$$\begin{aligned}\sum_{j \in I^+(\underline{s})} Q_j(p) = \sum_{i=1}^d \sum_{j \in I^+(\underline{s})} \Big(\partial_i\big((z_i p)_{\underline{s}+\epsilon_j}\big)\Big)_{\underline{s}} &= \sum_{i=1}^d \Big(\partial_i\Big(\sum_{j \in I^+(\underline{s})} (z_i p)_{\underline{s}+\epsilon_j}\Big)\Big)_{\underline{s}} \\ &= \sum_{i=1}^d \Big(\partial_i(z_i p)\Big)_{\underline{s}} \\ &= dp + \sum_{i=1}^d \Big(z_i \partial_i p\Big)_{\underline{s}}.\end{aligned}$$

Therefore, by Euler's formula, we obtain

$$\sum_{j \in I^+(\underline{s})} Q_j(p) = (d + |\underline{s}|)p. \tag{5.4}$$

Now, assume that $|I^+(\underline{s})| = 1$. Then $\underline{s}$ is necessarily of the form $(s_1, 0, \ldots, 0)$ and $I^+(\underline{s}) = \{1\}$. Thus it follows easily from (5.3) and (5.4) that $\delta(\underline{s}) = \frac{a_{\underline{s}}}{a_{\underline{s}+\epsilon_1}} r(\frac{d}{r} + s_1)$.

To complete the proof, assume that $|I^+(\underline{s})| = 2$. Then $I^+(\underline{s}) = \{1, k\}$, where $2 \le k \le r$. Note that by (5.3) and Theorem 5.6, we have

$$\frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}+\epsilon_1}} Q_1(p) + \frac{(\frac{d}{r})_{\underline{s}}}{(\frac{d}{r})_{\underline{s}+\epsilon_k}} Q_k(p) = rp. \tag{5.5}$$

By solving equations (5.4) and (5.5), it is easily verified that

$$Q_1(p) = \frac{(k-1)(\frac{d}{r}+s_1)(s_1 - s_k + \frac{ar}{2})}{(s_1 - s_k + \frac{a}{2}(k-1))} p,$$

and

$$Q_k(p) = \frac{(r-k+1)(\frac{d}{r} - \frac{a}{2}(k-1) + s_k)(s_1 - s_k)}{(s_1 - s_k + \frac{a}{2}(k-1))} p.$$

Now, the proof is completed by (5.3). □

As an immediate consequence of Theorem 5.8, we obtain the following corollary giving the complete form of the operator $\boldsymbol{M}^{(a)*}\boldsymbol{M}^{(a)}$ on $\mathscr{H}^{(a)}$ when the domain Ω is of rank 2.

Corollary 5.9. *Let Ω be an irreducible bounded symmetric domain of rank 2. Then, for any polynomial p in $\mathscr{P}_{\underline{s}}$, $\boldsymbol{M}^{(a)*}\boldsymbol{M}^{(a)} p = \delta(\underline{s}) p$, where*

$$\delta(\underline{s}) = \sum_{j \in I^+(\underline{s})} \frac{a_{\underline{s}}}{a_{\underline{s}+\epsilon_j}} \frac{(\frac{d}{r})_{\underline{s}+\epsilon_j}}{(\frac{d}{r})_{\underline{s}}} c_{\underline{s}}(j).$$

As a consequence of Theorem 5.8, we also obtain the following corollary about the essential normality of the multiplication operators by the coordinate functions on the weighted Bergman spaces.

Corollary 5.10. *Let $\lambda > \frac{a}{2}(r-1)$ and $\boldsymbol{M}^{(\lambda)} = (M_1^{(\lambda)}, \ldots, M_d^{(\lambda)})$ be the d- tuple of multiplication operators on $\mathscr{H}^{(\lambda)}$. Then the operator $M_i^{(\lambda)}$ is essentially normal, that is, the commutator $M_i^{(\lambda)*} M_i^{(\lambda)} - M_i^{(\lambda)} M_i^{(\lambda)*}$ is compact for all $i = 1, \ldots, d$ if and only if $r = 1$.*

Proof. If $r = 1$, then by a direct computation it is easily verified that each $M_i^{(\lambda)}$ is essentially normal. For the converse part, first set $\underline{\boldsymbol{l}}$ to be the signature $(l, 0, \ldots, 0)$, where l is a positive integer. Then, by Lemma 4.10 and Theorem 5.8, we see that,

$$\sum_{i=1}^d (M_i^{(\lambda)*} M_i^{(\lambda)} - M_i^{(\lambda)} M_i^{(\lambda)*}) p = \eta(\underline{\boldsymbol{l}}) p, \ p \in \mathscr{P}_{\underline{\boldsymbol{l}}},$$

where

$$\eta(\underline{\boldsymbol{l}}) = \frac{(\frac{d}{r}+l)(l+\frac{ar}{2})}{(\lambda+l)(l+\frac{ar}{2})} + \frac{l(r-1)(\frac{d}{r}-\frac{a}{2})}{(\lambda-\frac{a}{2})(l+\frac{a}{2})} - \frac{l}{\lambda+l-1}. \tag{5.6}$$

Suppose that each $M_i^{(\lambda)}$ is essentially normal. Then the operator $\sum_{i=1}^d (M_i^{(\lambda)*} M_i^{(\lambda)} - M_i^{(\lambda)} M_i^{(\lambda)*})$ is compact. Hence $\eta(\underline{\boldsymbol{l}})$ must converge to 0 as $l \to \infty$. Thus, from (5.6), we obtain that $\frac{(r-1)(\frac{d}{r}-\frac{a}{2})}{\lambda-\frac{a}{2}} = 0$. Finally, since $\frac{d}{r} = 1 + \frac{a}{2}(r-1) + b$, we conclude that $r = 1$. □

We finish this section with the following conjecture on the description of the operator $\boldsymbol{M}^{(a)*}\boldsymbol{M}^{(a)}$ on the Hilbert space $\mathscr{H}^{(a)}$ when the domain Ω is of arbitrary rank.

Conjecture 5.11. Let Ω be an irreducible bounded symmetric domain of rank r. Then, for any polynomial p in $\mathscr{P}_{\underline{s}}$, $\boldsymbol{M}^{(a)*}\boldsymbol{M}^{(a)} p = \delta(\underline{s}) p$ on the Hilbert space $\mathscr{H}^{(a)}$, where

$$\delta(\underline{s}) = \sum_{j \in I^+(\underline{s})} \frac{a_{\underline{s}}}{a_{\underline{s}+\epsilon_j}} \frac{(\frac{d}{r})_{\underline{s}+\epsilon_j}}{(\frac{d}{r})_{\underline{s}}} c_{\underline{s}}(j). \tag{5.7}$$

Chapter 6

Appendix

Proof of Theorem 3.12: The constant function $\mathbf{1}$ is a cyclic vector for the d- tuple $\boldsymbol{S}$. Also, $\sigma(\boldsymbol{S}) = \overline{\mathbb{B}}_d$, see [11]. To show that $\mathrm{dEt}\big([\![\boldsymbol{S}^*, \boldsymbol{S}]\!]\big)$ is non-negative definite, we claim that

$$\Big(\sum_{\eta\in\mathfrak{S}_d} \mathrm{sgn}(\eta) S^*_{\eta(1)} S_{\tau(1)} \dots S_{\tau(d-1)} S^*_{\eta(d)}\Big) e_{\boldsymbol{\alpha}} = \mathrm{sgn}(\tau) \sqrt{\frac{\alpha_{\tau(d)}}{|\boldsymbol{\alpha}| + d - 1}} (|\boldsymbol{\alpha}| + d - 1)^{-(d-1)} e_{\boldsymbol{\alpha} - \epsilon_{\tau(d)}}, \tag{6.1}$$

for each τ in $\mathfrak{S}_d$. Here we have assumed $\alpha_{\tau(d)} > 0$ without loss of generality.

For any fixed but arbitrary $\tau \in \mathfrak{S}_k$ and an arbitrary k- tuple $(i_1, \dots, i_k)$, $k \le d$, with $1 \le i_1 < i_2 < \cdots < i_k \le d$, let "$P_\tau(i_1, \dots, i_k)$" be the induction hypothesis, namely, the statement

$$\Big(\sum_{\eta\in\mathfrak{S}_{d-1}} \mathrm{sgn}(\eta) S^*_{i_{\eta(1)}} S_{i_{\tau(1)}} \dots S_{i_{\tau(k-1)}} S^*_{i_{\eta(k)}}\Big) e_{\boldsymbol{\alpha}} = \mathrm{sgn}(\tau) \sqrt{\frac{\alpha_{i_{\tau(k)}}}{|\boldsymbol{\alpha}| + d - 1}} (|\boldsymbol{\alpha}| + d - 1)^{-(k-1)} e_{\boldsymbol{\alpha} - \epsilon_{i_{\tau(k)}}}. \tag{6.2}$$

We see that the equality of Equation (6.1) is the same as the equality asserted in the statement $P_\tau(i_1, \dots, i_d)$ with $k = d$ and $\tau = \mathrm{id}$, the identity permutation on d elements.

Thus, a proof of the equality (6.1) follows from showing that the validity of all the equalities in $P_\tau(i_1, \dots, i_k)$, $\tau \in \mathfrak{S}_k$, $k < d$, implies the validity of all the equalities in $P_\tau(i_1, \dots, i_{k+1})$ $\tau \in \mathfrak{S}_{k+1}$.

To establish this, first note that if $k = 1$, then $P_\tau(i_1)$, $\tau \in S_1$ is the equalty $(\alpha_i > 0)$: $S^*_{i_1} e_\alpha = \sqrt{\frac{\alpha_{i_1}}{|\boldsymbol{\alpha}| + d - 1}}\, e_{\alpha - \epsilon_{i_1}}$, which is clearly valid. Now if $k = 2$, we note that

$$\Big(\sum_{\eta\in\mathfrak{S}_2} \mathrm{sgn}(\eta) S^*_{i_{\eta(1)}} S_{i_{\tau(1)}} S^*_{i_{\eta(2)}}\Big) e_{\boldsymbol{\alpha}} = \mathrm{sgn}(\tau) \sqrt{\frac{\alpha_{i_{\tau(2)}}}{|\boldsymbol{\alpha}| + d - 1}} (|\boldsymbol{\alpha}| + d - 1)^{-1} e_{\boldsymbol{\alpha} - \epsilon_{i_{\tau(2)}}}$$

for any pair i_1, i_2 with $1 \le i_1 < i_2 \le d$ and any fixed but arbitrary $\tau \in \mathfrak{S}_2$. This establishes the validity of $P_\tau(i_1, i_2)$, $\tau \in \mathfrak{S}_2$.

More generally, assume that the equalities in $P_\tau(i_1, \dots, i_{k-1})$ are valid for each tuple $(i_1, \dots, i_{k-1})$ with $1 \le i_1 < i_2 \cdots < i_{k-1} \le d$, and any fixed but arbitrary $\tau \in S_{k-1}$. To show that the equality in $P_\tau(i_1, \dots, i_k)$ is valid for each fixed but arbitrary $(i_1, \dots, i_k)$ and $\tau \in \mathfrak{S}_k$, it is enough to verify it with $\tau = id$. For this, we split the left hand side of $P_{id}(i_1, \dots, i_k)$ into several sums fixing $\eta(k) = j$, $j = 1, \dots, k$, in each one of these sums, that is,

$$\begin{aligned}
&\Big(\sum_{\eta\in\mathfrak{S}_k} \operatorname{sgn}(\eta) S^*_{i_{\eta(1)}} S_{i_1} S^*_{i_{\eta(2)}} \dots S^*_{i_{\eta(k-1)}} S_{i_{k-1}} S^*_{i_{\eta(k)}}\Big) e_{\boldsymbol{\alpha}} = \\
&\quad \Big(\sum_{\eta\in\mathfrak{S}_k, \eta(k)=k} \operatorname{sgn}(\eta) S^*_{i_{\eta(1)}} S_{i_1} S^*_{i_{\eta(2)}} \dots S^*_{i_{\eta(k-1)}} S_{i_{k-1}} S^*_{i_k}\Big) e_{\boldsymbol{\alpha}} \\
&\quad + \Big(\sum_{\eta\in\mathfrak{S}_k, \eta(k)=k-1} \operatorname{sgn}(\eta) S^*_{i_{\eta(1)}} S_{i_1} S^*_{i_{\eta(2)}} \dots S^*_{i_{\eta(k-1)}} S_{i_{k-1}} S^*_{i_{k-1}}\Big) e_{\boldsymbol{\alpha}} \\
&\quad + \Big(\sum_{\eta\in\mathfrak{S}_k, \eta(k)=k-2} \operatorname{sgn}(\eta) S^*_{i_{\eta(1)}} S_{i_1} S^*_{i_{\eta(2)}} \dots S^*_{i_{\eta(k-1)}} S_{i_{k-1}} S^*_{i_{k-2}}\Big) e_{\boldsymbol{\alpha}} \\
&\quad + \dots + \Big(\sum_{\eta\in\mathfrak{S}_k, \eta(k)=1} \operatorname{sgn}(\eta) S^*_{i_{\eta(1)}} S_{i_1} S^*_{i_{\eta(2)}} \dots S^*_{i_{\eta(k-1)}} S_{i_{k-1}} S^*_{i_1}\Big) e_{\boldsymbol{\alpha}}.
\end{aligned} \tag{6.3}$$

Pick a fixed but arbitrary sum in $P_{id}(i_1, \dots, i_k)$ with $\eta(k) = j$, $j = 1, \dots, k-2$. We claim that these sums vanish. Each one of these sums is of the form

$$\sum_{\eta\in\mathfrak{S}_k, \eta(k)=j} \operatorname{sgn}(\eta) S^*_{i_{\eta(1)}} S_{i_1} S^*_{i_{\eta(2)}} \dots S_{i_{j-1}} S^*_{i_{\eta(j)}} S_{i_j} S^*_{i_{\eta(j+1)}} S_{i_{j+1}} S^*_{i_{\eta(j+2)}} \dots S^*_{i_{\eta(k-1)}} S_{i_{k-1}} S^*_{i_j}. \tag{6.4}$$

For a fixed $\eta \in \mathfrak{S}_k$, let $\eta_\sigma \in \mathfrak{S}_k$ be the permutation:

$$\eta_\sigma(i) = \begin{cases} \eta(i) & i \notin \{j, j+1\}, \\ \eta(j+1) & \text{if } i = j, \\ \eta(j) & \text{if } i = j+1. \end{cases}$$

The sign of η_σ is opposite of the sign of η and these occur in pairs. Also, $S^*_i S_l S^*_p = S^*_p S_l S^*_i$ for any choice of $((i, l, p)$ with $i \neq l \neq p$. Clearly, $\eta(j) \neq j \neq \eta(j+1)$ by choice. Putting these observations together, we conclude that the sum (6.4) vanishes.

Now, we examine the two nonzero sums that remain on the right hand side of (6.3). The first of these two sums, namely,

$$\begin{aligned}
&\Big(\sum_{\eta\in\mathfrak{S}_k} \operatorname{sgn}(\eta) S^*_{i_{\eta(1)}} S_{i_1} S^*_{i_{\eta(2)}} \dots S^*_{i_{\eta(k-1)}} S_{i_{k-1}} S^*_{i_k}\Big) e_{\boldsymbol{\alpha}} \\
&\quad = \Big(\sum_{\sigma\in\mathfrak{S}_{k-1}} \operatorname{sgn}(\sigma) S^*_{i_{\sigma(1)}} S_{i_1} S^*_{i_{\sigma(2)}} \dots S^*_{i_{\sigma(k-1)}}\Big) S_{i_{k-1}} S^*_{i_k} e_{\boldsymbol{\alpha}}
\end{aligned}$$

Applying the equality in $P_{id}(i_1,\ldots,i_{k-1})$ to the vector $S_{i_{k-1}}S^*_{i_k}e_{\boldsymbol\alpha}$, we have

$$\Big(\sum_{\eta\in\mathfrak{S}_k}\operatorname{sgn}(\eta)S^*_{i_{\eta(1)}}S_{i_1}S^*_{i_{\eta(2)}}\ldots S^*_{i_{\eta(k-1)}}S_{i_{k-1}}S^*_{i_k}\Big)e_{\boldsymbol\alpha}=\sqrt{\frac{\alpha_{i_k}}{|\boldsymbol\alpha|+d-1}}\frac{\alpha_{i_{k-1}}+1}{(|\boldsymbol\alpha|+d-1)^{(k-1)}}e_{\boldsymbol\alpha-\epsilon_{i_k}}. \tag{6.5}$$

The second sum

$$\Big(\sum_{\eta\in\mathfrak{S}_k}\operatorname{sgn}(\eta)S^*_{i_{\eta(1)}}S_{i_1}S^*_{i_{\eta(2)}}\ldots S^*_{i_{\eta(k-1)}}S_{i_{k-1}}S^*_{i_{k-1}}\Big)e_{\boldsymbol\alpha}$$
$$=-\Big(\sum_{\sigma\in\mathfrak{S}_{k-1}}\operatorname{sgn}(\sigma)S^*_{i_{\sigma(1)}}S_{i_1}S^*_{i_{\sigma(2)}}\ldots S^*_{i_{\sigma(k)}}\Big)S_{i_{k-1}}S^*_{i_{k-1}}e_{\boldsymbol\alpha}.$$

The verification of this equality is an immediate consequence of the following observations.

1. Each permutation $\eta\in\hat{\mathfrak{S}}_k:=\{\eta\in\mathfrak{S}_k:\eta(k)=k-1\}$ is of the form $\hat\sigma(k,k-1)$, where $\hat\sigma$ is a bijection from $\{1,\ldots k-1\}$ to $\{1,\ldots,k-2,k\}$ and $(k,k-1)$ is the cycle taking k to $k-1$.

2. The bijections $\hat\sigma$ are in one to one correspondence with permutations $\mathfrak{P}_{k-1}$ of the set $\{1,\ldots,k-2,k\}$.

3. Since $\eta=\sigma(k,k-1)$ for some $\sigma\in\mathfrak{P}_k$, it follows that $\operatorname{sgn}(\eta)=-\operatorname{sgn}(\sigma)$.

4. $\mathfrak{S}_{k-1}$ is isomorphic to $\mathfrak{P}_{k-1}$.

Now, as before, applying the equality in $P_{id}(i_1,\ldots,i_{k-2},i_k)$ to the vector $S_{i_{k-1}}S^*_{i_{k-1}}e_\alpha$, we have

$$\Big(\sum_{\eta\in S_k}\operatorname{sgn}(\eta)S^*_{i_{\eta(1)}}S_{i_1}S^*_{i_{\eta(2)}}\ldots S^*_{i_{\eta(k-1)}}S_{i_{k-1}}S^*_{i_{k-1}}\Big)e_{\boldsymbol\alpha}=-\sqrt{\frac{\alpha_{i_k}}{|\boldsymbol\alpha|+d-1}}\frac{\alpha_{i_{k-1}}}{(|\boldsymbol\alpha|+d-1)^{(k-1)}}e_{\boldsymbol\alpha-\epsilon_{i_k}}. \tag{6.6}$$

Combining (6.5) and (6.6), we obtain

$$\Big(\sum_{\eta\in\mathfrak{S}_k}\operatorname{sgn}(\eta)S^*_{i_{\eta(1)}}S_{i_1}S^*_{i_{\eta(2)}}\ldots S^*_{i_{\eta(k-1)}}S_{i_{k-1}}S^*_{i_{\eta(k)}}\Big)e_{\boldsymbol\alpha}=\sqrt{\frac{\alpha_{i_k}}{|\boldsymbol\alpha|+d-1}}(|\boldsymbol\alpha|+d-1)^{-(k-1)}e_{\boldsymbol\alpha-\epsilon_{i_k}},$$

which verifies $P_\tau(i_1,\ldots i_k)$ for $\tau=id$. The verification of (6.1) for an arbitrary choice of $\tau\in\mathfrak{S}_k$ is similar. Recall the expression of determinant operator in (3.12)

$$\operatorname{dEt}\big([\![\boldsymbol S^*,\boldsymbol S]\!]\big)=\sum_{\tau,\eta\in\mathfrak{S}_d}\operatorname{sgn}(\tau)Sgn(\eta)S^*_{\eta(1)}S_{\tau(1)}S^*_{\eta(2)}\ldots S^*_{\eta(d)}S_{\tau(d)}+$$
$$-\sum_{\tau,\eta\in\mathfrak{S}_d}\operatorname{sgn}(\tau)\operatorname{sgn}(\eta)S_{\tau(d)}S^*_{\eta(1)}S_{\tau(1)}\ldots S_{\tau(d-1)}S^*_{\eta(d)}.$$

Now using the equality in (6.1) we get,

$$\sum_{\tau,\eta\in\mathfrak{S}_d}\operatorname{sgn}(\tau)\operatorname{sgn}(\eta)S^*_{\eta(1)}S_{\tau(1)}S^*_{\eta(2)}\ldots S^*_{\eta(d)}S_{\tau(d)}e_{\boldsymbol\alpha}=\frac{(d-1)!}{(|\boldsymbol\alpha|+d)^{(d-1)}}e_{\boldsymbol\alpha},$$

and

$$\sum_{\tau,\eta\in\mathfrak{S}_d} \operatorname{sgn}(\tau)\operatorname{sgn}(\eta) S_{\tau(d)} S^*_{\eta(1)} S_{\tau(1)} \dots S_{\tau(d-1)} S^*_{\eta(d)} e_{\boldsymbol{\alpha}} = \frac{(d-1)!|\boldsymbol{\alpha}|}{(|\boldsymbol{\alpha}|+d-1)^d} e_{\boldsymbol{\alpha}}.$$

Thus combining the above equalities we get

$$\operatorname{dEt}\big([\![\boldsymbol{S}^*, \boldsymbol{S}]\!]\big) e_{\boldsymbol{\alpha}} = \Big(\frac{(d-1)!}{(|\boldsymbol{\alpha}|+d)^{(d-1)}} - \frac{(d-1)!|\boldsymbol{\alpha}|}{(|\boldsymbol{\alpha}|+d-1)^d}\Big) e_{\boldsymbol{\alpha}}.$$

It is easy to see $\operatorname{dEt}\big([\![\boldsymbol{S}^*, \boldsymbol{S}]\!]\big)$ is non-negative definite since

$$\frac{1}{(|\boldsymbol{\alpha}|+d)^{(d-1)}} - \frac{|\boldsymbol{\alpha}|}{(|\boldsymbol{\alpha}|+d-1)^d} \geq 0.$$

Now,

$$\begin{aligned}
\operatorname{trace}\big(\operatorname{dEt}\big([\![\boldsymbol{S}^*, \boldsymbol{S}]\!]\big)\big) &= \sum_{\alpha\in\mathbb{N}_0^d} \big\langle \operatorname{dEt}\big([\![\boldsymbol{S}^*, \boldsymbol{S}]\!]\big) e_{\boldsymbol{\alpha}}, e_{\boldsymbol{\alpha}} \big\rangle \\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^d} (d-1)!\Big(\frac{1}{(|\boldsymbol{\alpha}|+d)^{(d-1)}} - \frac{|\boldsymbol{\alpha}|}{(|\boldsymbol{\alpha}|+d-1)^d}\Big) \\
&= \sum_{k=0}^{\infty} \sum_{\substack{\alpha_1,\dots,\alpha_d \\ |\boldsymbol{\alpha}|=k}} (d-1)!\Big(\frac{1}{(|\boldsymbol{\alpha}|+d)^{(d-1)}} - \frac{|\boldsymbol{\alpha}|}{(|\boldsymbol{\alpha}|+d-1)^d}\Big) \\
&= \sum_{k=0}^{\infty} \Big(\frac{(k+d-1)(k+d-2)\dots(k+1)}{(k+d)^{(d-1)}} - \frac{(k+d-2)\dots(k+1)k}{(k+d-1)^{d-1}}\Big) \\
&= 1.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 3.13: As in the case of two variables, the d- tuple $\boldsymbol{T}$ is 1- cyclic, $P_N T_j P_N^{\perp} = 0$, $1 \leq j \leq d$, and its spectrum is of the form $\mathbb{B}[r]$ for some $r > 0$ depending on $\{\delta_{|\boldsymbol{\alpha}|}\}$.

A calculation similar to the one in the proof of Theorem 3.12 given above shows that

$$\operatorname{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big) x_{\boldsymbol{\alpha}} = \Big(\frac{(d-1)!\delta^{2d}_{|\boldsymbol{\alpha}|}}{(|\boldsymbol{\alpha}|+d)^{(d-1)}} - \frac{(d-1)!|\boldsymbol{\alpha}|\delta^{2d}_{|\boldsymbol{\alpha}|-1}}{(|\boldsymbol{\alpha}|+d-1)^d}\Big) x_{\boldsymbol{\alpha}}. \tag{6.7}$$

Since $\delta_{|\boldsymbol{\alpha}|}$ is an increasing sequence, it follows

$$\Big(\frac{(d-1)!\delta^{2d}_{|\boldsymbol{\alpha}|}}{(|\boldsymbol{\alpha}|+d)^{(d-1)}} - \frac{(d-1)!|\boldsymbol{\alpha}|\delta^{2d}_{|\boldsymbol{\alpha}|-1}}{(|\boldsymbol{\alpha}|+d-1)^d}\Big) \geq (d-1)!\delta^{2d}_{|\boldsymbol{\alpha}|-1}\Big(\frac{1}{(|\boldsymbol{\alpha}|+d)^{(d-1)}} - \frac{|\boldsymbol{\alpha}|}{(|\boldsymbol{\alpha}|+d-1)^d}\Big) \geq 0.$$

Hence $\operatorname{dEt}\big([\![\boldsymbol{T}^*, \boldsymbol{T}]\!]\big)$ is non-negative definite.

To complete the proof, we need to verify the norm estimate (iii) of Definition 3.10. For this, taking τ to be the identity permutation in (6.1), we obtain the equality

$$\begin{aligned}\Big(\sum_{\eta\in\mathfrak{S}_d}\operatorname{sgn}(\eta)\,T^*_{\eta(1)}T_1T^*_{\eta(2)}\dots T_{d-1}T^*_{\eta(d)}\Big)x_{\boldsymbol{\alpha}}\\ =\delta^{2d-1}_{|\boldsymbol{\alpha}|}\sqrt{\frac{\alpha_d}{|\boldsymbol{\alpha}|+d-1}}\,(|\boldsymbol{\alpha}|+d-1)^{-(d-1)}x_{\boldsymbol{\alpha}-\epsilon_d}.\end{aligned}\tag{6.8}$$

Clearly, we have

$$\begin{aligned}\|P_N\Big(\sum_{\eta\in\mathfrak{S}_d}\operatorname{sgn}(\eta)\,T^*_{\eta(1)}T_1T^*_{\eta(2)}\dots T_{d-1}T^*_{\eta(d)}\Big)P_N^{\perp}\|\\ \le\binom{N+d-1}{d-1}^{-1}\prod_{\{i:\,i\neq d\}}\|T_i\|^2\|T_d\|.\end{aligned}\tag{6.9}$$

Consequently,

$$\|P_N\Big(\sum_{\eta\in\mathfrak{S}_d}\operatorname{sgn}(\eta)\,T^*_{\eta(1)}T_1T^*_{\eta(2)}\dots T_{d-1}T^*_{\eta(d)}\Big)P_N^{\perp}T_dP_N\|\le\binom{N+d-1}{d-1}^{-1}\prod_{i=1}^{d}\|T_i\|^2.\tag{6.10}$$

It follows from Remark 3.11(b) that the inequality in Equation (6.10) remains unchanged when we replace the identity permutation by any other permutation from $\mathfrak{S}_d$. Therefore, the d- tuple $\boldsymbol{T}$ is in the class $BS_{1,1}(\mathbb{B}[r])$ and the proof is complete. □

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