Infinitely Divisible Metrics, Curvature Inequalities and Curvature Formulae

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DINESH KUMAR KESHARI

Department of Mathematics
Indian Institute of Science
Bangalore - 560012

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Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Prof. Gadadhar Misra at the Department of Mathematics, Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

Dinesh Kumar Keshari
S. R. No. 6910-110-071-05486

Indian Institute of Science,
Bangalore,
July, 2012.

Prof. Gadadhar Misra
(Research advisor)

DEDICATED TO MY GRANDMOTHERS

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Abstract

The curvature of a contraction T in the Cowen-Douglas class is bounded above by the curvature of the backward shift operator. However, in general, an operator satisfying the curvature inequality need not be contractive. In this thesis, we characterize a slightly smaller class of contractions using a stronger form of the curvature inequality. Along the way, we find conditions on the metric of the holomorphic Hermitian vector bundle E corresponding to the operator T in the Cowen-Douglas class which ensures negative definiteness of the curvature function. We obtain a generalization for commuting tuples of operators in the Cowen-Douglas class.

Secondly, we obtain an explicit formula for the curvature of the jet bundle of the Hermitian holomorphic bundle E_f on a planar domain Ω. Here E_f is assumed to be a pull-back of the tautological bundle on $\mathcal{G}r(n,\mathcal{H})$ by a nondegenerate holomorphic map $f:\Omega\to\mathcal{G}r(n,\mathcal{H})$. Clearly, finding relationships amongs the complex geometric invariants inherent in the short exact sequence

$$0\to\mathcal{J}_k(E_f)\to\mathcal{J}_{k+1}(E_f)\to\mathcal{J}_{k+1}(E_f)/\mathcal{J}_k(E_f)\to 0$$

is an important problem, where $\mathcal{J}_k(E_f)$ represents the k-th order jet bundle. It is known that the Chern classes of these bundles must satisfy

$$c(\mathcal{J}_{k+1}(E_f))=c(\mathcal{J}_k(E_f))\,c(\mathcal{J}_{k+1}(E_f)/\mathcal{J}_k(E_f)).$$

We obtain a refinement of this formula:

$$\big(\mathrm{trace}\otimes\mathrm{Id}_{n\times n}\big)(\mathcal{K}_{\mathcal{J}_k(E_f)})-\big(\mathrm{trace}\otimes\mathrm{Id}_{n\times n}\big)(\mathcal{K}_{\mathcal{J}_{k-1}(E_f)})=\mathcal{K}_{\mathcal{J}_k(E_f)/\mathcal{J}_{k-1}(E_f)}(z).$$

Contents

Chapter 1

Introduction

Let $\mathcal{H}$ be a complex Hilbert space. Given a bounded linear operator T on $\mathcal{H}$, it is natural to ask if there exists a canonical model (modulo unitary equivalence) for T and obtain a complete set of unitary invariants. In general, it is not possible to find a solution to this problem. However, if T is a normal operator, the spectral theorem provides both a model (multiplication operator) and a complete set of invariants (spectrum, spectral multiplicity function and spectral measure). For a contraction T, Sz-Nagy and Foias model theory provides a canonical model as well as a complete set of invariants. In a very influential paper [7], Cowen and Douglas introduced the class $\mathcal{B}_n(\Omega)$ of operators, where Ω is a domain in $\mathbb{C}$. They showed that an operator T in $\mathcal{B}_n(\Omega)$ determines a Hermitian holomorphic vector bundle E_T on Ω and that the equivalence classes of T and of E_T are in one to one correspondence. Exploiting this correspondence and using techniques from complex geometry, they obtained a complete set of invariants for operators in $\mathcal{B}_n(\Omega)$. They also showed that an operator in $\mathcal{B}_n(\Omega)$ can be realized as the adjoint of a multiplication operator on a Hilbert space consisting of holomorphic functions on Ω and possessing a reproducing kernel. This provides a model for operators in the class $\mathcal{B}_n(\Omega)$. This latter description was elaborated and studied in detail by Curto and Salinas to determine when two operators in the class $\mathcal{B}_n(\Omega)$ are unitarily equivalent (cf. [9]).

For a normal operator, via spectral theory, one attempts to synthesize the operator from elementary operators. For example, a normal operator on a finite dimensional space can be written as the orthogonal direct sum of scalar operators on eigenspaces, where the scalars are just the eigenvalues of the operator, which together with multiplicities determine the operator up to unitary equivalence. On infinite dimensional Hilbert spaces, the results are essentially the same. In this case, the direct sum is replaced by a continuous direct sum or direct integral. For an arbitrary operator this approach fails spectacularly. Consider the following example.

Let $U_+ : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the shift operator defined by

$$U_+(\alpha_0, \alpha_1, \alpha_2, \ldots) = (0, \alpha_0, \alpha_1, \alpha_2, \ldots)$$

for $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ in $l^2(\mathbb{N})$ and let U_+^* denote the adjoint of U_+ defined by

$$U_+^*(\alpha_0, \alpha_1, \alpha_2, \ldots) = (\alpha_1, \alpha_2, \alpha_3, \ldots).$$

Since

$$U_+^*(1, \lambda, \lambda^2, \ldots) = \lambda(1, \lambda, \lambda^2, \ldots),$$

where $(1, \lambda, \lambda^2, \ldots)$ is in $l^2(\mathbb{N})$ for $|\lambda| < 1$, it follows that the spectrum of U_+^* contains $\mathbb{D}$. We can not write $l^2(\mathbb{N}) = \mathbf{M} + \mathbf{N}$, where $\mathbf{M}$ and $\mathbf{N}$ are invariant non zero proper subspaces for U_+^* [17, theorem 2.2.1, page 43]. Therefore, the conventional spectral theory is not of much use in studying U_+^*. Cowen and Douglas initiated, in their foundational paper [7], a systematic study of a class of operators which includes the operator U_+^* and many other operators possessing an open set of eigenvalues.

Definition 1.1. For a domain $\Omega \subseteq \mathbb{C}$ and $n \in \mathbb{N}$, the class $\mathcal{B}_n(\Omega)$ consists of those operators T whose spectrum $\sigma(T)$ is contained in Ω and

(1) ran $(T - w) = \mathcal{H}$ for w in Ω;

(2) span $\{\ker(T - w) : w \in \Omega\}$ is dense in $\mathcal{H}$;

(3) dim $\ker(T - w) = n$ for w in Ω.

It was shown in [7, proposition 1.11] that the eigenspaces for each T in $\mathcal{B}_n(\Omega)$ form a rank n Hermitian holomorphic vector bundle E_T over Ω, that is,

$$E_T := \{(w, x) \in \Omega \times \mathcal{H} : x \in \ker(T - w)\},\ \pi(w, x) = w$$

and there exist a holomorphic frame $w \to \gamma(w) := (\gamma_1(w), \ldots, \gamma_n(w))$ with $\ker(T - w) = \operatorname{span}\{\gamma_i(w) : 1 \le i \le n\}$. The Hermitian structure at w is the one that $\ker(T - w)$ inherits as a subspace of the Hilbert space $\mathcal{H}$. The metric of the vector bundle E_T at w is $h(w) = (\!(\langle \gamma_j(w), \gamma_i(w) \rangle)\!)_{i,j=1}^n$. The curvature $\mathcal{K}_T$ of the bundle E_T is given by the following formula [21, proposition 2.2, pp. 79]

$$\mathcal{K}_T(w) = \frac{\partial}{\partial \bar{w}}\big(h^{-1}(w)\frac{\partial}{\partial w}h(w)\big)\, d\bar{w} \wedge dw.$$

It was also shown in [7] that the equivalence class of the Hermitian holomorphic bundle E_T and the unitary equivalence class of the operator T determine each other.

Theorem 1.2. *[7, theorem 1.14] The operators T and $\widetilde{T}$ in $\mathcal{B}_n(\Omega)$ are unitarily equivalent if and only if the corresponding Hermitian holomorphic vector bundles E_T and $E_{\widetilde{T}}$ are equivalent.*

The curvature of a vector bundle E transforms according to the rule, [21, pp. 72] $\mathcal{K}(fg)w = (g^{-1}\mathcal{K}(f)g)w$, $w \in \Omega_0$, where $f = (e_1, \dots, e_n)$ is a frame for E over an open subset $\Omega_0 \subset \Omega$ and $g : \Omega_0 \to GL(n, \mathbb{C})$ is a change of frame. In the case when the rank n of the vector bundle E is strictly greater than 1, the curvature of E depends on the choice of a frame. Thus the curvature $\mathcal{K}$ cannot be an invariant for the vector bundle E. However, the eigenvalues of $\mathcal{K}$ are invariants for the bundle E. The complete set of invariants given in [7, Definition 2.17 and Theorem 3.17] involve the curvature and the covariant derivatives

$$\mathcal{K}_{z^i\bar{z}^j}\ 0 \leq i \leq j \leq i+j \leq n,\ (i,j \neq (0,n),(n,0)),$$

where rank of $E = n$. The curvature

$$\mathcal{K}(w) = \frac{\partial^2}{\partial w \partial \bar{w}} \log \parallel \gamma(w) \parallel^2 \quad d\bar{w} \wedge dw,$$

of the line bundle E, defined with respect to a non-zero holomorphic section γ of E, is a complete invariant. The definition of the curvature is independent of the choice of the section γ: If γ_0 is another holomorphic section of E, then $\gamma_0 = \phi\gamma$ for some holomorphic function ϕ on some open subset Ω_0 of Ω. The harmonicity of $\log|\phi|$ completes the verification. Hence Theorem 1.2 for the line bundle has the form

Theorem 1.3. *[8, pp. 4] Operators $T, \tilde{T}$ in $\mathcal{B}_1(\Omega)$ are unitarily equivalent if and only if $\mathcal{K}_T(w) = \mathcal{K}_{\tilde{T}}(w)$ for all w in Ω.*

An operator T in the class $B_1(\Omega)$, as is well-known (cf. [7, pp. 194]), is unitarily equivalent to the adjoint M^* of the multiplication operator M by the co-ordinate function on some Hilbert space $\mathcal{H}_K$ of holomorphic functions on $\Omega^* := \{z \in \mathbb{C} : \bar{z} \in \Omega\}$ possessing a reproducing kernel K.

The kernel K is a complex valued function defined on $\Omega^* \times \Omega^*$ which is holomorphic in the first variable and anti-holomorphic in the second. In consequence, the map $\bar{w} \to K(\cdot, w)$, $w \in \Omega^*$, is holomorphic on Ω. We have $K(z,w) = \overline{K(w,z)}$ making it Hermitian. It is positive definite in the sense that the $n \times n$ matrix

$$\big(\!\big(\, K(w_i, w_j)\big)\!\big)_{i,j=1}^n$$

is positive definite for every subset $\{w_1, \dots, w_n\}$ of Ω^*, $n \in \mathbb{N}$. Finally, the kernel K reproduces the value of functions in $\mathcal{H}_K$, that is, for any fixed $w \in \Omega^*$, the holomorphic function $K(\cdot, w)$ belongs to $\mathcal{H}_K$ and

$$f(w) = \langle f, K(\cdot, w)\rangle,\ f \in \mathcal{H}_K,\ w \in \Omega^*.$$

The correspondence between the operator T in $\mathcal{B}_1(\Omega)$ and the operator M^* on the Hilbert space of holomorphic functions is easy to describe [7, pp. 194]). Let γ be a non-zero holomorphic section of E_T (for a bounded domain in $\mathbb{C}$, a global section exists by Grauert's Theorem) for the operator T acting on the Hilbert space $\mathcal{H}$. Consider the map $\Gamma_\gamma : \mathcal{H} \to \mathcal{O}(\Omega^*)$, where $\mathcal{O}(\Omega^*)$ is the space of holomorphic functions on Ω^*, defined by $\Gamma_\gamma(x)(z) = \langle x, \gamma(\bar{z})\rangle$, $z \in \Omega^*$. Transplant the inner product from $\mathcal{H}$ on the range of Γ_γ. The map Γ_γ is now unitary from $\mathcal{H}$ onto $\operatorname{ran}\Gamma_\gamma = \mathcal{H}_\gamma$. Define K_γ to be the function $K_\gamma(z,w) = \Gamma_\gamma\big(\gamma(\bar{w})\big)(z) = \langle \gamma(\bar{w}), \gamma(\bar{z})\rangle$, $z, w \in \Omega^*$. Set $(K_\gamma)_w(\cdot) := K_\gamma(\cdot, w)$. Thus $(K_\gamma)_w$ is the function $\Gamma_\gamma\big(\gamma(\bar{w})\big)$. It is then easily verified that K_γ has the reproducing property, that is,

$$\begin{aligned}
\langle \Gamma_\gamma(x)(\cdot), K_\gamma(\cdot, w)\rangle_{\operatorname{ran}\Gamma_\gamma} &= \langle \Gamma_\gamma(x)(\cdot), \Gamma_\gamma(\gamma(\bar{w}))(\cdot)\rangle_{\operatorname{ran}\Gamma_\gamma} \\
&= \langle x, \gamma(\bar{w})\rangle_{\mathcal{H}} \\
&= \Gamma_\gamma(x)(w),\ x \in \mathcal{H},\ w \in \Omega^*.
\end{aligned}$$

It follows that $\|(K_\gamma)_w(\cdot)\|^2 = K_\gamma(w,w)$. Also, $(K_\gamma)_w(\cdot)$ is an eigenvector for the operator $\Gamma_\gamma\, T\, \Gamma_\gamma^*$ with eigenvalue $\bar{w}$ in Ω;

$$\begin{aligned}
\Gamma_\gamma\, T\, \Gamma_\gamma^*((K_\gamma)_w(\cdot)) &= \Gamma_\gamma\, T\, \Gamma_\gamma^*\big(\Gamma_\gamma(\gamma(\bar{w}))\big) \\
&= \Gamma_\gamma\, T\, \gamma(\bar{w}) \\
&= \Gamma_\gamma\, \bar{w}\, \gamma(\bar{w}) \\
&= \bar{w}\,(K_\gamma)_w(\cdot),\ w \in \Omega^*.
\end{aligned}$$

Since the linear span of the vectors $\{(K_\gamma)_w : w \in \Omega^*\}$ is dense in $\mathcal{H}_\gamma$, it follows that $\Gamma_\gamma\, T\, \Gamma_\gamma^*$ is the adjoint M^* of the multiplication operator M on $\mathcal{H}_\gamma$. We therefore assume, without loss of generality, that an operator T in $\mathcal{B}_1(\Omega)$ can be viewed as the adjoint M^* of the multiplication operator M on some Hilbert space $\mathcal{H}_\gamma$ of holomorphic functions on Ω^* possessing a reproducing kernel K.

More generally, an operator $T \in \mathcal{B}_n(\Omega)$ can be realized as the adjoint of the multiplication operator on a reproducing kernel Hilbert space of holomorphic $\mathbb{C}^n$-valued functions on Ω^*. Let E_T be the Hermitian holomorphic vector bundle over Ω corresponding to T. Let $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a holomorphic frame for E_T.

Define the map $\Gamma_\gamma : \mathcal{H} \to \mathcal{O}(\Omega^*, \mathbb{C}^n)$ as follows

$$\Gamma_\gamma(x)(z) = \big(\langle x, \gamma_1(\bar{z})\rangle, \ldots, \langle x, \gamma_n(\bar{z})\rangle\big)^{\mathrm{tr}} \qquad z \in \Omega^*,\ x \in \mathcal{H},$$

where $\mathcal{O}(\Omega^*, \mathbb{C}^n)$ is the space of holomorphic functions defined on Ω^* which take values in $\mathbb{C}^n$. It is easy to see that the map Γ_γ is an injective map. Transplant the inner product from $\mathcal{H}$ on the range of Γ_γ. The map Γ_γ is now unitary from $\mathcal{H}$ onto $\mathcal{H}_\gamma := \operatorname{ran}\Gamma_\gamma$. Define K_γ to be the function on $\Omega^* \times \Omega^*$ taking values in the $n \times n$ matrices $\mathcal{M}_n(\mathbb{C})$:

$$K_\gamma(z,w) = \big(\big(\langle \gamma_j(\bar{w}), \gamma_i(\bar{z})\rangle\big)\big)_{i,j=1}^n$$

for $z, w \in \Omega^*$. Set $(K_\gamma)_w(\cdot) = K_\gamma(\cdot, w)$. It is then easily verified that K has the reproducing property, that is,

$$\begin{aligned}
\langle \Gamma_\gamma(x)(\cdot), (K_\gamma)_w(\cdot)\eta\rangle_{\operatorname{ran}\Gamma_\gamma} &= \langle \Gamma_\gamma(x)(\cdot), \sum_{i=1}^n \Gamma_\gamma(\gamma_i(\bar{w}))(\cdot)\eta_i\rangle_{\operatorname{ran}\Gamma_\gamma} \\
&= \sum_{i=1}^n \bar{\eta}_i \langle \Gamma_\gamma(x)(\cdot), \Gamma_\gamma(\gamma_i(\bar{w}))(\cdot)\rangle_{\operatorname{ran}\Gamma_\gamma} \\
&= \sum_{i=1}^n \langle x, \gamma_i(\bar{w})\rangle_{\mathcal{H}}\, \bar{\eta}_i \\
&= \langle \Gamma_\gamma(x)(w), \eta\rangle_{\mathbb{C}^n}, \quad x \in \mathcal{H}, \eta \in \mathbb{C}^n, w \in \Omega^*.
\end{aligned}$$

Now consider,

$$\begin{aligned}
\Gamma_\gamma(T^*x)(w) &= (\langle T^*x, \gamma_1(\bar{w})\rangle, \ldots, \langle T^*x, \gamma_n(\bar{w})\rangle)^{\mathrm{tr}} \\
&= \big(\langle x, T\gamma_1(\bar{w})\rangle, \ldots, \langle x, T\gamma_n(\bar{w})\rangle\big)^{\mathrm{tr}} \\
&= \big(\langle x, \bar{w}\gamma_1(\bar{w})\rangle, \ldots, \langle x, \bar{w}\gamma_n(\bar{w})\rangle\big)^{\mathrm{tr}} \\
&= w\big(\langle x, \gamma_1(\bar{w})\rangle, \ldots, \langle x, \gamma_n(\bar{w})\rangle\big)^{\mathrm{tr}} \\
&= w\Gamma_\gamma(x)(w) \\
&= (M(\Gamma_\gamma(x)))(w).
\end{aligned}$$

Hence

$$\Gamma_\gamma T^* = M\Gamma_\gamma.$$

Also, $(K_\gamma)_w(\cdot)\eta$ is an eigenvector for the operator $M^* = \Gamma_\gamma T \Gamma_\gamma^*$ with eigenvalue $\bar{w}$ in Ω;

$$\begin{aligned} M^*(K_\gamma)_w(\cdot)\eta &= \Gamma_\gamma T\Gamma_\gamma^*\big(\sum_{i=1}^n (\Gamma_\gamma(\gamma_i(\bar{w}))(\cdot)\eta_i)\big) \\ &= \Gamma_\gamma\big(\sum_{i=1}^n (T(\gamma_i(\bar{w}))\eta_i)\big)(\cdot) \\ &= \Gamma_\gamma\big(\sum_{i=1}^n (\bar{w}(\gamma_i(\bar{w}))\eta_i)\big)(\cdot) \\ &= \bar{w}\Gamma_\gamma\big(\sum_{i=1}^n \gamma_i(\bar{w})\eta_i\big)(\cdot) \\ &= \bar{w}\sum_{i=1}^n \Gamma_\gamma(\gamma_i(\bar{w}))(\cdot)\eta_i \\ &= \bar{w}(K_\gamma)_w(\cdot)\eta. \end{aligned}$$

A remark in [16] relates the trace of the curvature of vector bundle E_T to the Hilbert-Schmidt norm of second fundamental form of E_T (viewed as a sub bundle of the trivial bundle $\Omega \times \mathcal{H}$) as follows. Let $P : \Omega \to \mathcal{L}(\mathcal{H})$ be the map:

$$P(\lambda) = P_{\ker(\mathrm{T}-\lambda)}, \quad \lambda \in \Omega,$$

where $P_{\ker(\mathrm{T}-\lambda)}$ denotes the orthogonal projection from $\mathcal{H}$ to $\ker(\mathrm{T}-\lambda)$. Treating E_T as a sub bundle of $\Omega \times \mathcal{H}$, they note that $-\frac{\partial}{\partial\lambda}P(\lambda)$ is the second fundamental form and

$$\text{trace } \mathcal{K}_{\mathrm{T}}(\lambda) = -\|\tfrac{\partial}{\partial\lambda}\mathrm{P}(\lambda)\|_{\mathcal{HS}},$$

where $\|\cdot\|_{\mathcal{HS}}$ denotes the Hilbert-Schmidt norm.

The results of the thesis are in two parts which we briefly describe below.

It is shown in [18] that the curvature $\mathcal{K}_{S^*}$ of the backward shift operator dominates the curvature $\mathcal{K}_T$ if T is a contraction in $\mathcal{B}_1(\Omega)$. It is natural to ask if the converse is valid. However, it is easy to construct an example (see Chapter 3) of a non-contraction which satisfies the curvature inequality. However, since $\mathcal{K}_T(w,w)$ is real analytic, polarization gives a Hermitian function $\mathcal{K}_T(z,w)$. Thus it is natural to study the stronger inequality: For any subset $\{z_1,\ldots,z_n\}$ of $\mathbb{D}$,

$$\big(\big(\mathcal{K}_{S^*}(z_i,z_j) - \mathcal{K}_T(z_i,z_j)\big)\big)_{i,j=1}^n$$

is positive definite. One of the main results in this thesis says that the curvature inequality is equivalent to T being a contraction in a stronger sense than the usual. In the first part of the thesis, these results have been proved using the familiar notion of infinite divisibility. Extension to more general domains Ω in $\mathbb{C}^m$ and several applications have been given.

In the second part, we obtain an explicit formula for the curvature of the jet bundle of the Hermitian holomorphic bundle E_f on a planar domain Ω. Here E_f is assumed to be a pull-back of the tautological bundle on $\mathcal{G}r(n,\mathcal{H})$ by a nondegenerate holomorphic map $f:\Omega \to \mathcal{G}r(n,\mathcal{H})$ as in Definition 5.14. Clearly, finding relationships among the complex geometric invariants inherent in the short exact sequence

$$0 \to \mathcal{J}_k(E_f) \to \mathcal{J}_{k+1}(E_f) \to \mathcal{J}_{k+1}(E_f)/\mathcal{J}_k(E_f) \to 0 \tag{1.1}$$

is an important problem, where $\mathcal{J}_k(E_f)$ represents the k-th order jet bundle. In the paper [5], it is shown that the Chern classes of these bundles must satisfy

$$c(\mathcal{J}_{k+1}(E_f)) = c(\mathcal{J}_k(E_f))\, c(\mathcal{J}_{k+1}(E_f)/\mathcal{J}_k(E_f)).$$

We obtain a refinement of this formula.

We now give some of the details.

Let T be an operator in $\mathcal{B}_1(\mathbb{D})$, where $\mathbb{D}$ is the open unit disc. The following proposition was proved in [18].

Proposition 1.4. *If T is a contractive operator in $\mathcal{B}_1(\mathbb{D})$, then $\mathcal{K}_T(w) \le \mathcal{K}_{S^*}(w)$, $w \in \mathbb{D}$, where S^* is the backward shift operator.*

However if $\mathcal{K}_T(w) \le \mathcal{K}_{S^*}(w)$ for all w in $\mathbb{D}$, then it does not necessarily follow that T is a contraction!

If K is a positive definite kernel on a planar domain, then by an application of the Cauchy-Schwarz inequality, we see that $\frac{\partial^2}{\partial w \partial \bar{w}} \log K(w,w)$ is positive. In general the real analytic function $\frac{\partial^2}{\partial z \partial \bar{w}} \log K(z,w)$ obtained by polarizing $\frac{\partial^2}{\partial w \partial \bar{w}} \log K(w,w)$ need not be a positive definite function. We provide an example of a positive definite kernel K, in Chapter 3, for which $\frac{\partial^2}{\partial z \partial \bar{w}} \log K(z,w)$ is not positive definite. Our main Theorem gives a necessary and sufficient condition for the positive definiteness of $\frac{\partial^2}{\partial z \partial \bar{w}} \log K(z,w)$.

Definition 1.5. A positive definite kernel K is said to be *infinitely divisible* if for all $t > 0$, the kernel K^t is also positive definite.

Definition 1.6. Let G be a real analytic function of $w, \bar{w}$ for w in some open connected subset Ω of $\mathbb{C}^m$. Polarizing G, we obtain a new function $\tilde{G}$ defined on $\Omega \times \Omega$ which is holomorphic in the first variable and anti-holomorphic in the second and restricts to G on the diagonal set $\{(w,w) : w \in \Omega\}$, that is, $\tilde{G}(w,w) = G(w,w)$, $w \in \Omega$. If the function $\tilde{G}$ is positive definite, that is, the $n \times n$ matrix $\big(\!\!\big(\tilde{G}(w_i,w_j)\big)\!\!\big)$ is positive definite for all finite subsets $\{w_1,\ldots,w_m\}$ of Ω, then we say that G is a positive definite function on Ω.

Theorem 1.7. *Let Ω be a domain in $\mathbb{C}$ and let K be a positive, real analytic function on $\Omega \times \Omega$. If K is infinitely divisible then there exists a domain $\Omega_0 \subset \Omega$ such that negative of the curvature $\frac{\partial^2}{\partial w\, \partial \bar{w}} \log K$ is a positive definite function on Ω_0. Conversely, if $\hat{K}$ is a real analytic function on Ω and the function $\frac{\partial^2}{\partial w\, \partial \bar{w}} \log \hat{K}$ is positive definite on Ω, then there exists a neighborhood $\Omega_0 \subseteq \Omega$ of w_0, for every point $w_0 \in \Omega$, and an infinitely divisible kernel K on $\Omega_0 \times \Omega_0$ such that $K(w,w) = \hat{K}(w,w)$ for all $w \in \Omega_0$.*

Definition 1.8. If K is a non negative definite kernel such that $(1-z\bar{w})K(z,w)$ is infinitely divisible then we say that M^* on $\mathcal{H}_K$ is an infinitely divisible contraction.

The following Corollary completes the study of curvature inequalities begun in [18].

Corollary 1.9. *Let K be a positive definite kernel on the open unit disc. Assume that the adjoint M^* of the multiplication operator M on the reproducing kernel Hilbert space $(\mathcal{H}, K)$ belongs to $\mathcal{B}_1(\mathbb{D})$. The function $\frac{\partial^2}{\partial z\, \partial \bar{w}} \log \big((1 - z\bar{w})K(z,w)\big)$ is positive definite if and only if the multiplication operator M is an infinitely divisible contraction.*

Definition 1.10. Let $\mathcal{H}$ be a Hilbert space and let $\boldsymbol{T} = (T_1, \ldots, T_m)$ be a commuting tuple of bounded linear operators on $\mathcal{H}$. We say that $\boldsymbol{T}$ is a row contraction if $\sum_{i=1}^m T_i T_i^* \leq I_{\mathcal{H}}$.

Let $\mathbb{B}^m$ be the unit ball in $\mathbb{C}^m$ and $\boldsymbol{M} = (M_1, \ldots, M_m)$ be the m-tuple of (co-ordinate) multiplication operators on a reproducing kernel Hilbert space with reproducing kernel K, which is assumed to be bounded. Then $\boldsymbol{M}$ is a row contraction if and only if $(1 - \langle z, w\rangle)K(z,w)$ is positive definite.

Definition 1.11. Let K be positive definite kernel on $\mathbb{B}^m$ and $\boldsymbol{M} = (M_1, \ldots, M_m)$ be the m-tuple of (co-ordinate) multiplication operators on a reproducing kernel Hilbert space with reproducing kernel K. Then we say that $\boldsymbol{M}$ is an infinitely divisible row contraction if $(1 - \langle z, w\rangle)K(z,w)$ is an infinitely divisible kernel.

In Chapter 4, a multi-variate analogue of Theorem 1.7 is given. The following corollary is an immediate consequence.

Corollary 1.12. *Let K be a positive definite kernel on the open unit ball $\mathbb{B}^m \subset \mathbb{C}^m$. Assume that the adjoint $\boldsymbol{M}^* = (M_1^*, \ldots, M_m^*)$ of the tuple of multiplication operators $\boldsymbol{M} = (M_1, \ldots, M_m)$ on the reproducing kernel Hilbert space $(\mathcal{H}, K)$ belongs to $\boldsymbol{\mathcal{B}}_1(\mathbb{B}^m)$. The function $\Big(\!\!\Big(\frac{\partial^2}{\partial w_i\, \partial \bar{w}_j} \log(1 - \langle w, w\rangle)K(w,w)\Big)\!\!\Big)_{i,j=1}^m$, $w \in \mathbb{B}^n$, is positive definite if and only if the operator $\boldsymbol{M}$ is an infinitely divisible row contraction.*

Several other applications are given for domain like the polydisc.

In Chapter 5, we compute the curvature of the jet bundle obtained from a Hermitian holomorphic line bundle $\mathcal{L}_f$ in closed form. Here $\mathcal{L}_f$ is the line bundle on a planar domain

Ω which is the pull-back of the tautological bundle $S(n,\mathcal{H})$ on $\mathcal{G}r(1,\mathcal{H})$ by a holomorphic nondegenerate map $f:\Omega\to\mathcal{G}r(1,\mathcal{H})$.

The following Theorem is an immediate consequence of this curvature formula for the jet bundle.

Theorem 1.13. *Let $\mathcal{L}_f$ and $\mathcal{L}_{\tilde{f}}$ be Hermitian holomorphic line bundles. Let $\mathcal{J}_k(\mathcal{L}_f)$ and $\mathcal{J}_k(\mathcal{L}_{\tilde{f}})$ be corresponding jet bundles of rank $k+1$. The two jet bundles $\mathcal{J}_k(\mathcal{L}_f)$ and $\mathcal{J}_k(\mathcal{L}_{\tilde{f}})$ are locally equivalent as Hermitian holomorphic vector bundles if and only if the two line bundles $\mathcal{L}_f$ and $\mathcal{L}_{\tilde{f}}$ are locally equivalent as Hermitian holomorphic vector bundles.*

Also, the curvature of the determinant bundle of the jet bundle $\mathcal{J}_k(\mathcal{L}_f)$ corresponding to the line bundle $\mathcal{L}_f$ is explicitly obtained.

Proposition 1.14. *The curvature of the determinant bundle* $\det\mathcal{J}_k(\mathcal{L}_f)$ *is given by the following formula*

$$\mathcal{K}_{\det\mathcal{J}_k(\mathcal{L}_f)}(z)=\frac{(\det\mathcal{J}_{k-1}h)(z)(\det\mathcal{J}_{k+1}h)(z)}{(\det\mathcal{J}_kh)^2(z)}\ d\bar{z}\wedge dz.$$

The following Corollary is an immediate consequence of this formula.

Corollary 1.15. *Let $\mathcal{L}_f$ and $\mathcal{L}_{\tilde{f}}$ be Hermitian holomorphic line bundles over a domain $\Omega\subset\mathbb{C}$. The following statements are equivalent:*

(1) $\det\mathcal{J}_k(\mathcal{L}_f)$ *is locally equivalent to* $\det\mathcal{J}_k(\mathcal{L}_{\tilde{f}})$ *and* $\det\mathcal{J}_{k+1}(\mathcal{L}_f)$ *is locally equivalent to* $\det\mathcal{J}_{k+1}(\mathcal{L}_{\tilde{f}})$*, for some $k\in\mathbb{N}$*

(2) $\mathcal{L}_f$ is locally equivalent to $\mathcal{L}_{\tilde{f}}$.

We now describe a formula for a Hermitian holomorphic vector bundle E_f on a bounded domain $\Omega\subset\mathbb{C}^m$.

Let $\{s_1,\cdots,s_n\}$ be a local holomorphic frame for E_f. Set

$$\tau_p^j(z)=s_1(z)\wedge\cdots\wedge s_n(z)\wedge\frac{\partial s_p}{\partial z_j}(z),\qquad 1\le p\le n,\ 1\le j\le m.$$

and

$$h_{ij}(z)=\Big(\big(\langle\tau_p^i(z),\tau_q^j(z)\rangle\big)\Big)_{p,q=1}^n,\quad 1\le i,j\le m,\ z\in\Omega.$$

With this notation, the curvature $\mathcal{K}_{E_f}$ of the vector bundle E_f may be expressed as

$$\mathcal{K}_{E_f}(z)=(\det h(z))^{-1}h^{-1}(z)\sum_{i,j=1}^m h_{ij}(z)d\bar{z}_j\wedge dz_i,$$

where h is the metric $h(z) = ((\langle s_j(z), s_i(z)\rangle))$. Applying this formula to domains in $\mathbb{C}$, we obtain a natural generalization for the curvature formula of the jet bundle $\mathcal{J}_k(E_f)$, where the rank of the Hermitian holomorphic vector bundle E_f is assumed to be n. This closed form for the curvature of $\mathcal{J}_k(E_f)$ gives to a very interesting relationship involving the jet bundles $\mathcal{J}_k(E_f),\ k = 1, 2, \ldots$ and their quotients:

$$\big(\mathrm{trace} \otimes \mathrm{Id}_{n\times n}\big)(\mathcal{K}_{\mathcal{J}_k(E_f)}) - \big(\mathrm{trace} \otimes \mathrm{Id}_{n\times n}\big)(\mathcal{K}_{\mathcal{J}_{k-1}(E_f)}) = \mathcal{K}_{\mathcal{J}_k(E_f)/\mathcal{J}_{k-1}(E_f)}(z).$$

involving the short exact sequence 1.1.

Chapter 2

Preliminaries

2.1 Reproducing Kernel

Let Ω be a bounded, connected, open subset of $\mathbb{C}^m$ and $\mathcal{M}_n(\mathbb{C})$ be the set of all $n \times n$ matrices over $\mathbb{C}$.

Definition 2.1. A non negative definite function $K : \Omega \times \Omega \to \mathcal{M}_n(\mathbb{C})$ which is holomorphic in the first variable and antiholomorphic in the second variable is said to be a reproducing kernel on Ω if it satisfies the positivity condition:

$$\sum_{i,j=1}^{q} \langle K(w^{(i)}, w^{(j)})\zeta_j, \zeta_i\rangle_{\mathbb{C}^n} \geq 0,\ w^{(1)}, \ldots, w^{(q)} \in \Omega,\ \zeta_1, \ldots, \zeta_q \in \mathbb{C}^n,\ q \geq 1. \tag{2.1}$$

Given a non negative definite kernel K, consider the linear span $\mathcal{H}^0$ of all vector from the set

$$S := \{K(\cdot, w)\zeta, w \in \Omega, \zeta \in \mathbb{C}^n\}.$$

Define the inner product on $\mathcal{H}^0$ as follows,

$$\Big\langle \sum_{i=1}^{p} K(\cdot, w^{(i)})\zeta_i, \sum_{i=1}^{p} K(\cdot, w^{(i)})\zeta_i \Big\rangle = \sum_{i,j=1}^{p} \langle K(w^{(i)}, w^{(j)})\zeta_j, \zeta_i\rangle_{\mathbb{C}^n}. \tag{2.2}$$

The completion $\mathcal{H}$ of the inner product space $\mathcal{H}^0$ is a Hilbert space of holomorphic functions on Ω. It can be seen easily that

$$\langle f(w), \zeta\rangle_{\mathbb{C}^n} = \langle f, K(\cdot, w)\zeta\rangle_{\mathcal{H}},\ w \in \Omega,\ \zeta \in \mathbb{C}^n,\ f \in \mathcal{H}. \tag{2.3}$$

Remark 2.2. In the above definition, we have assumed that the function K defines a non negative definite sesquilinear form. It then follows that K is positive definite. This is a consequence of the Cauchy-Schwarz inequality: For $f \in \mathcal{H}, \zeta \in \mathbb{C}^n$ and $w \in \Omega$, we have

$$\begin{aligned} |\langle f(w), \zeta \rangle_{\mathbb{C}^n}| &= |\langle f, K(., w)\zeta \rangle_{\mathcal{H}}| \\ &\leq ||f||_{\mathcal{H}} \langle K(w,w)\zeta, \zeta \rangle_{\mathbb{C}^n}. \end{aligned}$$

Thus if $||f||_{\mathcal{H}} = 0$ then $f = 0$.

Conversely, If $\mathcal{H}$ is a Hilbert space of holomorphic functions defined on Ω taking values in $\mathbb{C}^n$ and the evaluation function e_w is bounded for each $w \in \Omega$, then there exists a function $e_w^* : \mathbb{C}^n \to \mathcal{H}$ such that $\langle e_w(f), \zeta \rangle = \langle f, e_w^*(\zeta) \rangle$, for all $f \in \mathcal{H}$ and $\zeta \in \mathbb{C}^n$. Clearly, $f \perp \operatorname{ran} e_w^*$ if and only if $\langle e_w f, \zeta \rangle = \langle f, e_w^*(\zeta) \rangle = 0$ for every $\zeta \in \mathbb{C}^n$. Hence $f \perp \operatorname{ran} e_w^*$ for all $w \in \Omega$ if and only if $f = 0$. Hence $\mathcal{H}$ is generated by the subspaces $e_w^*(\mathbb{C}^n)$. Therefore the linear space

$$\widetilde{\mathcal{H}} := \left\{ \sum_{j=1}^{r} e_{w_j}^*(\zeta_j) | w_j \in \Omega, \zeta_j \in \mathbb{C}^n, r \in \mathbb{N} \right\}$$

is dense in $\mathcal{H}$. For $f \in \widetilde{\mathcal{H}}$,

$$\begin{aligned} ||f||^2 &= \left\langle \sum_{j=1}^{r} e_{w_j}^*(\zeta_j), \sum_{j=1}^{r} e_{w_j}^*(\zeta_j) \right\rangle \\ &= \sum_{j,k=1}^{r} \left\langle e_{w_k} e_{w_j}^*(\zeta_j), \zeta_k \right\rangle. \end{aligned}$$

Since $||f||^2 \geq 0$, it follows that the function $K(z,w) = e_z e_w^*$ is non negative definite as in (2.1). The function K has the reproducing property

$$\begin{aligned} \langle f, K(., w)\zeta \rangle_{\mathcal{H}} &= \langle f, e_w^*(\zeta) \rangle_{\mathcal{H}} \\ &= \langle e_w(f), \zeta \rangle_{\mathbb{C}^n} \\ &= \langle f(w), \zeta \rangle_{\mathbb{C}^n}. \end{aligned}$$

The reproducing property (2.3) implies that K is uniquely determined.

Definition 2.3. A Hilbert space of holomorphic functions on some bounded domain $\Omega \subset \mathbb{C}^m$ will be called a reproducing kernel Hilbert space if the evaluation e_w at w is bounded for w in some open subset of Ω.

If K is a reproducing kernel for some Hilbert space $\mathcal{H}$, then

$$\mathcal{H} = \overline{\operatorname{span}}\{K(\cdot, w)\zeta : w \in \Omega, \zeta \in \mathbb{C}^n\}.$$

We can give an alternative description of a reproducing kernel K in terms of an orthonormal basis $\{e_k : k \geq 0\}$ of the Hilbert space $\mathcal{H}$ as follows; We think of $e_k(w) \in \mathbb{C}^n$ as a column vector for a fixed $w \in \Omega$ and let $e_k(w)^*$ be the row vector $(\overline{e_k^1(w)}, \ldots, \overline{e_k^n(w)})$. We see that

$$\begin{aligned}
\langle K(z,w)\zeta, \eta\rangle &= \langle K(.,w)\zeta, K(.,z)\eta\rangle \\
&= \left\langle \sum_{j=0}^{\infty}\langle K(.,w)\zeta, e_j\rangle e_j, \sum_{k=0}^{\infty}\langle K(.,z)\eta, e_k\rangle e_k \right\rangle \\
&= \sum_{k=0}^{\infty}\langle K(.,w)\zeta, e_k\rangle \overline{\langle K(.,z)\eta, e_k\rangle} \\
&= \sum_{k=0}^{\infty}\overline{\langle e_k(w), \zeta\rangle}\langle e_k(z), \eta\rangle \\
&= \sum_{k=0}^{\infty}\langle e_k(z)e_k(w)^*\zeta, \eta\rangle
\end{aligned}$$

for any pair of vectors $\zeta, \eta \in \mathbb{C}^n$. Therefore, we have the following very useful representation for the reproducing kernel K;

$$K(z,w) = \sum_{k=0}^{\infty} e_k(z)e_k(w)^*, \tag{2.4}$$

where $\{e_k : k \geq 0\}$ is any orthonormal basis in $\mathcal{H}$.

Definition 2.4. A non negative definite kernel K is said to be normalized at $w_0 \in \Omega$ if there exist a neighborhood Ω_0 of w_0 in Ω such that $K(z, w_0) = 1$ for all $z \in \Omega_0$.

A detailed discussion of reproducing kernels is given in [3].

2.2 The Cowen-Douglas Class

Let $\mathcal{H}$ be a separable Hilbert space and $T_i : \mathcal{H} \to \mathcal{H}, 1 \leq i \leq m$, be bounded linear operators such that $T_iT_j = T_jT_i, 1 \leq i, j \leq m$. Let $\boldsymbol{T} = (T_1, \ldots, T_m)$ denote the m-tuple of operators. We associate with the m-tuple $\boldsymbol{T}$ a bounded linear operator $D_{\boldsymbol{T}} : \mathcal{H} \to \mathcal{H} \oplus \ldots \oplus \mathcal{H}$ defined by $D_{\boldsymbol{T}}(x) = (T_1x, \ldots, T_mx), x \in \mathcal{H}$. Let Ω be a domain (open and connected set) in $\mathbb{C}^m$. For $w = (w_1, \ldots, w_m)$ in Ω, let $\boldsymbol{T} - w$ denote the operator tuple $(T_1 - w_1, \ldots, T_m - w_m)$. Let n be a positive integer.

Definition 2.5. The m-tuple $\boldsymbol{T}$ is said to be in the Cowen-Douglas class $\boldsymbol{\mathcal{B}}_n(\Omega)$ if

(1) ran $D_{\boldsymbol{T}-w}$ is closed for all $w \in \Omega$;

(2) $\text{span}\{\ker\ D_{\boldsymbol{T}-w} : w \in \Omega\}$ is dense in $\mathcal{H}$; and

(3) $\dim\ \ker D_{\boldsymbol{T}-w} = n$ for all $w \in \Omega$.

For $\boldsymbol{T}$ in $\boldsymbol{\mathcal{B}}_n(\Omega)$ let $(E_{\boldsymbol{T}}, \pi)$ denote the sub-bundle of the trivial bundle $\Omega \times \mathcal{H}$ defined by

$$E_{\boldsymbol{T}} = \{(w, x) \in \Omega \times \mathcal{H} : x \in \ker D_{\boldsymbol{T}-w}\} \text{ and } \pi(w, x) = w.$$

To show that $E_{\boldsymbol{T}}$ is actually a holomorphic vector bundle over Ω we need to show that locally in a neighborhood of each point $w_0 \in \Omega$ there exist holomorphic $\mathcal{H}$- valued functions $\gamma_1(w), \dots, \gamma_n(w)$, whose values span $\ker D_{\boldsymbol{T}-w}$. This is given in [8, pp. 16]. Since $\pi^{-1}(w) = (E_{\boldsymbol{T}})_w = \ker D_{\boldsymbol{T}-w}$ is a subspace of $\mathcal{H}$, the Hermitian structure on $E_{\boldsymbol{T}}$ comes from $\mathcal{H}$. Hence $E_{\boldsymbol{T}}$ is a Hermitian holomorphic vector bundle.

Theorem 2.6. *Two commuting tuples of operators $\boldsymbol{T}$ and $\widetilde{\boldsymbol{T}}$ in $\boldsymbol{\mathcal{B}}_n(\Omega)$ are unitarily equivalent if and only if the vector bundles $E_{\boldsymbol{T}}$ and $E_{\widetilde{\boldsymbol{T}}}$ are equivalent as Hermitian holomorphic vector bundles over some open subset Ω_0 of $\Omega \subset \mathbb{C}^m$.*

When Ω is an open subset of $\mathbb{C}$ and $T, \widetilde{T} \in \mathcal{B}_n(\Omega)$, Theorem 2.6 is proved in [7, Theorem 1.14].

For a domain Ω in $\mathbb{C}^m$, it is noted in [8, pp. 16] that theorem 2.6 is valid. In general, for a vector bundle $E_{\boldsymbol{T}}$ of rank n, $n \geq 1$, the curvature of $E_{\boldsymbol{T}}$, along with certain covariant derivatives of the curvature, form a complete set of invariants for the operator $\boldsymbol{T}$ (cf. [7] and [8]). For line bundles, however, the curvature forms a complete invariant. In this case, theorem 2.6 amounts to saying that two operators $\boldsymbol{T}, \widetilde{\boldsymbol{T}}$ in $\boldsymbol{\mathcal{B}}_1(\Omega)$ are unitarily equivalent if and only if the curvatures of the corresponding line bundles $E_{\boldsymbol{T}}$ and $E_{\widetilde{\boldsymbol{T}}}$ are equal on some open subset of Ω.

Every commuting m-tuple of operators $\boldsymbol{T} = (T_1, \dots, T_m) \in \boldsymbol{\mathcal{B}}_n(\Omega)$ can be realized as the adjoint of an m-tuple of multiplication operators by coordinate functions on a Hilbert space of holomorphic functions on an open set $\Omega^* = \{w \in \mathbb{C}^m : \bar{w} \in \Omega\}$. We choose a holomorphic frame $\{\gamma_1, \dots, \gamma_n\}$ on some open subset Ω_0 of Ω. The map $\Gamma : \Omega_0 \to \mathcal{L}(\mathbb{C}^n, \mathcal{H})$ defined by

$$\Gamma(w)\zeta = \sum_{i=1}^{n} \zeta_i \gamma_i(w),\ \zeta = (\zeta_1, \dots, \zeta_n)$$

is holomorphic. Let $\mathcal{O}(\Omega_0^*, \mathbb{C}^n)$ be the set of all holomorphic $\mathbb{C}^n$-valued functions on Ω_0^*. Define the map $U_\Gamma : \mathcal{H} \to \mathcal{O}(\Omega_0^*, \mathbb{C}^n)$ by

$$(U_\Gamma x)(w) = \Gamma(w)^*(x), \quad x \in \mathcal{H}, \quad w \in \Omega_0.$$

It is easy to see that U_Γ is linear and injective. Let $\mathcal{H}_\Gamma = \operatorname{ran} U_\Gamma$ and define the sesquilinear form $\langle\ ,\ \rangle_\Gamma$ on $\mathcal{H}_\Gamma$ by

$$\langle U_\Gamma x, U_\Gamma y\rangle_\Gamma = \langle x, y\rangle, \quad x, y \in \mathcal{H}.$$

It is shown in [9, Remark 2.6] that

(1) $U_\Gamma T_i = M_i^* U_\Gamma$, where $(M_i f)(z) = z_i f(z), z = (z_1, \ldots, z_m) \in \Omega$.

(2) $K_\Gamma(z, w) = \Gamma(\bar{z})^* \Gamma(\bar{w}), z, w \in \Omega_0^*$ is a reproducing kernel for the Hilbert space $\mathcal{H}_\Gamma$.

(3) There exists $w_0 \in \Omega_0^*$ such that $K_\Gamma(z, w_0) = I$ for all $z \in \Omega_0^*$.

Conversely, by imposing certain conditions on the kernel $K : \Omega \times \Omega \to \mathcal{M}_n(\mathbb{C})$ we can ensure the boundedness of each of the multiplication operators $M_1, \ldots, M_m$ on the associated reproducing kernel Hilbert space. One may impose additional conditions on K to ensure that $\boldsymbol{M}^* = (M_1^*, \ldots, M_m^*)$ is in $\boldsymbol{\mathcal{B}}_n(\Omega^*)$ by following [9].

Let $\mathcal{H}_K$ be the reproducing kernel Hilbert space of holomorphic functions with reproducing kernel K defined on Ω. Let the multiplication operators $M_i : \mathcal{H}_K \to \mathcal{H}_K, 1 \leq i \leq m$, be bounded linear operators. Let $\zeta, \eta \in \mathbb{C}^n$ and for fixed $\widetilde{w} \in \Omega$, $1 \leq i \leq m$,

$$\begin{aligned}
\langle K(\cdot, w)\zeta, M_i^* K(\cdot, \widetilde{w})\eta\rangle &= \langle M_i K(\cdot, w)\zeta, K(\cdot, \widetilde{w})\eta\rangle \\
&= \langle z_i K(\cdot, w)\zeta, K(\cdot, \widetilde{w})\eta\rangle \\
&= \langle \widetilde{w}_i K(\widetilde{w}, w)\zeta, \eta\rangle \\
&= \langle K(\cdot, w)\zeta, \overline{\widetilde{w}_i} K(\cdot, \widetilde{w})\eta\rangle.
\end{aligned}$$

Hence

$$M_i^* K(\cdot, \widetilde{w})\eta = \overline{\widetilde{w}_i} K(\cdot, \widetilde{w})\eta. \tag{2.5}$$

Let $\boldsymbol{M} = (M_1, \ldots, M_m)$ be the commuting m-tuple of multiplication operators and let $\boldsymbol{M}^*$ be the $(M_1^*, \ldots, M_m^*)$. It then follows from 2.5 that the eigenspace of $\boldsymbol{M}^*$ at $\widetilde{w} \in \Omega^*$ contains the n- dimensional subspace $\operatorname{ran} K(\cdot, \widetilde{w})$.

Suppose $\boldsymbol{M}^*$ is in $\boldsymbol{\mathcal{B}}_n(\Omega^*)$ and $K(w, w)$ is invertible for every $w \in \Omega$. For fixed $w_0 \in \Omega$ there exists a neighborhood Δ_0 of w_0 such that $K(z, w_0)$ is invertible for all $z \in \Delta_0$. Let K_{res} be the restriction of K on $\Delta_0 \times \Delta_0$. Define a kernel function K_0 on Δ_0 by

$$K_0(z, w) = \phi(z) K(z, w) \phi(w)^*, \quad z, w \in \Delta_0 \tag{2.6}$$

where $\phi(z) = K_{\text{res}}(w_0, w_0)^{1/2} K_{\text{res}}(z, w_0)^{-1}$. Clearly K_0 is normalized at w_0. Let $\boldsymbol{M}_0$ be the m-tuple of multiplication operators on $\mathcal{H}_{K_0}$. It is not hard to establish the unitary equivalence of the two m-tuples $\boldsymbol{M}$ and $\boldsymbol{M}_0$ (cf. [9, Lemma 3.9 and Remark 3.8]). First the restriction map res : $f \mapsto f_{|\text{res}}$, which restricts a function in $\mathcal{H}_K$ to Δ_0 is a unitary map intertwining the m-tuple $\boldsymbol{M}$ on $\mathcal{H}_K$ and the m-tuple $\boldsymbol{M}$ on $(\mathcal{H}_K)_{\text{res}} = $ ran res. The Hilbert space $(\mathcal{H}_K)_{\text{res}}$ is the reproducing kernel Hilbert space with reproducing kernel K_{res}. Second, suppose that the m-tuples $\boldsymbol{M}$ defined on two different reproducing kernel Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are in $\boldsymbol{\mathcal{B}}_n(\Omega)$ and $X : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator intertwining these two operator tuples. Then X must map the joint kernel of one tuple in to the other, that is, $XK_1(\cdot, w)\eta = K_2(\cdot, w)\phi(w)\eta,\ \eta \in \mathbb{C}^n$, for some function $\phi : \Omega \to \mathcal{M}_n(\mathbb{C})$. Assuming that the kernel functions K_1 and K_2 are holomorphic in the first variable and anti-holomorphic in the second variable, it follows, as in [9, pp. 472], that ϕ is anti-holomorphic. An easy calculation shows that X^* is the multiplication operator $M_{\bar{\phi}^{\text{tr}}}$. If the two operator tuples are unitarily equivalent then there exists an unitary operator U intertwining them. Hence U^* must be of the form M_ψ for some holomorphic function ψ. Also, the operator U must map the joint kernel of $(\boldsymbol{M} - w)^*$ acting on $\mathcal{H}_1$ isometrically onto the joint kernel of $(\boldsymbol{M} - w)^*$ acting on $\mathcal{H}_2$ for all $w \in \Omega$. The unitarity of U is equivalent to the relation $K_1(\cdot, w)\eta = U^*K_2(\cdot, w)\overline{\psi(w)}^{\text{tr}}\eta$ for all $w \in \Omega$ and $\eta \in \mathbb{C}^n$. It then follows that

$$K_1(z, w) = \psi(z)K_2(z, w)\overline{\psi(w)}^{\text{tr}} \tag{2.7}$$

where $\psi : \Delta_0 \to \mathcal{GL}(\mathbb{C}^n)$ is some holomorphic function. Here $\mathcal{GL}(\mathbb{C}^n)$ denotes the group of all invertible linear transformation on $\mathbb{C}^n$.

Conversely, if two kernels are related as in equation 2.7, then the corresponding tuples of operators are unitarily equivalent since

$$M_i^* K(\cdot, w)\eta = \overline{w}_i K(\cdot, w)\eta,\ w \in \Omega,\ \eta \in \mathbb{C}^n,$$

where $(M_i f)(z) = z_i f(z),\ f \in \mathcal{H}_K$ for $1 \leq i \leq m$.

2.3 A Co-ordinate Free Approach to the Operators in the Cowen-Douglas Class

A slightly different description of these ideas is given in [14], where the vector bundle of Cowen-Douglas appears as an antiholomorphic vector bundle instead of a holomorphic one. We reproduce, closely following [14, pp. 5339], the correspondence between an operator in the Cowen-Douglas class and its realization as the adjoint of a multiplication operator.

2.3.1 Vector Bundles

Let N be a complex manifold and let $E \xrightarrow{\pi} N$ be complex vector bundle over N of rank n. We shall assume for our discussion that E is a trivial vector bundle, that is, there exists a holomorphic function $E \to N \times \mathbb{C}^n$ such that $\phi(v) = (z, \phi_z(v))$ for $v \in E_z = \pi^{-1}(z)$ with $\phi_z : E_z \to \mathbb{C}^n$ linear. Let E_z^* be the complex anti-linear dual of E_z for $z \in M$. We write $[u, v]$ for $u(v), u \in {E_z}^*, v \in E_z$. We consider $\mathbb{C}^n$ to be equipped with its natural inner product and identify it with its own anti-linear dual (so $\xi \in \mathbb{C}^n$ is identified with the anti-linear map $\eta \mapsto \langle \xi, \eta \rangle_{\mathbb{C}^n}$). Then $\phi_z^* : \mathbb{C}^n \to E_z^*$ is the dual of the map $\phi_z : E_z \to \mathbb{C}^n$. We set $\psi_z = \phi_z^{*-1}$ and $\psi(u) = (z, \psi_z(u))$ for $u \in E_z^*$. This makes E^* into a complex vector bundle with trivialization ψ. If E is a holomorphic vector bundle then E^* is an anti holomorphic vector bundle and vice-versa.

2.3.2 Reproducing Kernels

Let $\mathcal{H}$ be a Hilbert space whose elements are sections of a vector bundle $E \to M$ and suppose the evaluation maps $e_z : \mathcal{H} \to E_z$ are continuous for all $z \in M$. Then setting $K_z = e_z^*$, we have

$$[u, f(z)] = [u, e_z(f)] = \langle K_z u, f \rangle_{\mathcal{H}}, \quad u \in E_z^*, f \in \mathcal{H}. \tag{2.8}$$

For all $w \in M$, $K_w u$ is in $\mathcal{H}$ and is linear in u. So we can write $K_w(z)u = e_z(K_w u) = e_z e_w^*(u)$. We also write $K(z, w) = K_w(z) = e_z e_w^*$ which is a linear map $E_w^* \to E_z$, and is called the reproducing kernel of $\mathcal{H}$, (2.8) is the reproducing property.

Clearly, $K(z, w)^* = K(w, z)$. We have the positivity $\sum_{j,k=1}^p [u_k, K(z_k, z_j)u_j] \geq 0$ for any $z_1, \ldots, z_p$ in M and $u_1, \ldots, u_p \in E_z^*$ which is nothing but the inequality

$$\sum_{j,k=1}^p \langle e_{z_k}^* u_k, e_{z_j}^* u_j \rangle_{\mathcal{H}} \geq 0.$$

conversely, a K with these properties is always the reproducing kernel of a Hilbert space of sections of E (cf. [4]).

Suppose each e_z is non-singular, that is, its range is the whole of E_z. Then $K_z = e_z^*$ is an embedding of E_z^* into $\mathcal{H}$. Postulating that this embedding is an isometry we obtain a canonical Hermitian structure on E^*. Using (2.8) we can write for the norm on E^*,

$$\|u\|_z^2 = \|K_z u\|_{\mathcal{H}}^2 = [u, K(z, z)u], \quad u \in E_z^*.$$

The vector bundle E has the dual Hermitian structure. For $v \in E_z$ we have

$$\|v\|_z^2 = [K(z, z)^{-1}v, v].$$

It follow that

$$|[u,v]|^2 \le [K(z,z)^{-1}v,v][u,K(z,z)u]$$

for all u and v. Since $K(z,w)$ is bijective by hypothesis, any $v \in E_z$ can be written as $v = K(z,z)u'$ with $u' \in E_z^*$ and the inequality to be proved is equivalent to

$$|[u,K(z,z)u']|^2 \le [u',K(z,z)u'][u,K(z,z)u].$$

But this is just the Cauchy-Schwarz inequality.

When E is a holomorphic vector bundle, $K(z,w)$ depends on z holomorphically and on w anti-holomorphically. Hence $K(z,w)$ is completely determined by $K(z,z)$. It follows that $K(z,w)$ is completely determined by the canonical Hermitian structure of E(or E^*).

In the last paragraphs, we had a Hilbert space $\mathcal{H}$ of sections of E and (under the assumption that each e_z is surjective) we associated to it a family of embeddings of E_z^*, the fibre of E^*, into $\mathcal{H}$. This procedure can be reversed. Suppose now that E is a vector bundle and fibres E_z^* of E^* form a smooth family of subspaces of some Hilbert space H which together span H, that is, E^* is an anti-holomorphic sub-bundle of the trivial bundle $M \times H$. We write $\iota_z : E_z^* \to H$ for the (identity) embeddings. We define $\tilde{f}(z) = \iota_z^* f$ for $f \in H, z \in M$. If we denote by $\mathcal{H}$ the Hilbert space of all $\tilde{f}$, where $f \in H$, with norm $\|\tilde{f}\| = \|f\|$, each e_z is continuous, so we have a reproducing kernel Hilbert space. The reproducing kernel is determined by $K_z u = \widetilde{\iota_z u}$.

2.3.3 Operators in the Cowen-Douglas Class

Given a domain $\Omega \subset \mathbb{C}$, we say an operator T on a Hilbert space H is in $\mathcal{B}_n(\Omega)$ if $\bar{z}$ is an eigenvalue of T, the range of the operator $T - \bar{z}$ is closed, and the corresponding eigenspaces F_z are of constant dimension n for every $z \in \Omega$. It is proved in [7] that the spaces F_z determine an anti-holomorphic Hermitian vector bundle $F \subset \Omega \times H$. (In [7] the eigenvalues z are assumed to be in Ω so, F is a holomorphic vector bundle.) We write, for $z \in \Omega, \iota_z : F_z \to H$ for the identity embedding. Hence, $E = F^*$ is a holomorphic vector bundle.

To the element f of H there corresponds the section $\tilde{f}$ of E (defined by $\tilde{f}(z) = \iota_z^* f$) and these sections form a Hilbert space $\mathcal{H}$ isomorphic with H and having the reproducing kernel determined by $K_z u = \widetilde{\iota_z u}$.

Under this isomorphism, the operator on $\mathcal{H}$ corresponding to T is M^*, where M is the multiplication operator $M\tilde{f}(z) = z\tilde{f}(z)$. In fact (cf. [7]) for any $u \in E_z^*$,

$$[u, \widetilde{T^* f}(z)] = \langle \iota_z u, T^* f\rangle = \langle T\iota_z u, f\rangle = \bar{z}\langle \iota_z u, f\rangle = [u, z\tilde{f}(z)] = [u, M\tilde{f}(z)].$$

Chapter 3

Infinitely Divisible Metrics and Curvature Inequalities - Planar Case

In this section we consider operators in the Cowen-Douglas class $\mathcal{B}_1(\Omega)$, where Ω is a planar domain. Let $T \in \mathcal{B}_1(\Omega)$. Fix $w \in \Omega$ and let γ be a holomorphic section of the line bundle E_T. From [7, Lemma 1.22], it follows that the vectors $\gamma(w)$ and $\frac{\partial}{\partial w}\gamma(w)$ from a basis of $\ker(T-w)^2$. Let $N_T(w) = T|_{\ker(T-w)^2}$ and $\{\gamma_1(w), \gamma_2(w)\}$ be the basis obtained by applying Gram-Schmidt ortho-normalization to the vectors $\gamma(w)$ and $\frac{\partial}{\partial w}\gamma(w)$. The linear transformation $N_T(w)$ with respect to the basis $\{\gamma_1(w), \gamma_2(w)\}$ has the matrix representation

$$N_T(w) = \begin{pmatrix} w & h_T(w) \\ 0 & w \end{pmatrix},$$

where $h_T(w) = \big(-\mathcal{K}_T(w)\big)^{-\frac{1}{2}}$.

The curvature $\mathcal{K}_T(w)$ of an operator T in $\mathcal{B}_1(\Omega)$ is negative. To see this, recall that the curvature may also be expressed (cf. [7, pp. 195]) in the form

$$\mathcal{K}_T(w) = -\frac{\|\gamma(w)\|^2\|\gamma'(w)\|^2 - |\langle\gamma'(w), \gamma(w)\rangle|^2}{\|\gamma(w)\|^4} \tag{3.1}$$

Applying the Cauchy-Schwarz inequality, we see that the numerator is positive.

Let $\{e_0, e_1\}$ be an orthonormal set of vectors. Suppose N is a nilpotent linear transformation defined by the rule

$$e_1 \to a\, e_0,\ e_0 \to 0,\ \ a \in \mathbb{C}.$$

Then $|a|$ determines the unitary equivalence class of N.

The localization $N_T(w) - wI_2 = \left(\begin{smallmatrix} 0 & h_T(w) \\ 0 & 0 \end{smallmatrix}\right)$ of the operator T in $\mathcal{B}_1(\Omega)$ is nilpotent. Now, $h_T(w) > 0$ since we have shown that the curvature $\mathcal{K}_T(w)$ is negative. Hence the curvature $\mathcal{K}_T(w)$ is an invariant for the operator T. The non-trivial converse of this statement follows from Theorem 2.6. Thus the operators T and $\widetilde{T}$ in $\mathcal{B}_1(\Omega)$ are unitarily equivalent if and only if $N_T(w)$ is unitarily equivalent to $N_{\widetilde{T}}(w)$ for every w in Ω.

Note that if $T \in \mathcal{B}_1(\mathbb{D})$ is a contraction, that is, $\|T\| \leq 1$, then $N_T(w)$ is a contraction for each $w \in \mathbb{D}$. Observe that $\left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right)$ is a contraction if and only if $|a| \leq 1$ and $|c|^2 \leq (1-|a|^2)(1-|b|^2)$. Thus $\|N_T(w)\| \leq 1$ if and only if $\mathcal{K}_T(w) \leq -\frac{1}{(1-|w|^2)^2}$, $w \in \mathbb{D}$. The adjoint S^* of the unilateral shift operator S is in $\mathcal{B}_1(\mathbb{D})$. It is easy to see that $\gamma_{S^*}(w) = (1, w, \ldots, w^n, \ldots) \in \ell^2_+$, $w \in \mathbb{D}$, is a holomorphic section for the corresponding Hermitian holomorphic line bundle E_{S^*}. The norm $\|\gamma_{S^*}(w)\|^2$ of the section γ_{S^*} is $(1-|w|^2)^{-1}$ and hence the curvature $\mathcal{K}_{S^*}(w)$ of the operator S^* is given by the formula $-\frac{1}{(1-|w|^2)^2}$, $w \in \mathbb{D}$. We have therefore proved:

Proposition 3.1. *If T is a contractive operator in $\mathcal{B}_1(\mathbb{D})$, then the curvature of T is bounded above by the curvature of the backward shift operator S^*.*

We think of the operator S^* as an extremal operator within the class of contractions in $\mathcal{B}_1(\mathbb{D})$. This is a special case of the curvature inequality proved in [18]. The curvature inequality is equivalent to contractivity of the operators $N_T(w)$, $w \in \mathbb{D}$, while the contractivity of the operator T is global in nature. So, it is natural to expect that the validity of the inequality $\mathcal{K}_T(w) \leq -\frac{1}{(1-|w|^2)^2}$, $w \in \mathbb{D}$, need not force T to be a contraction. Indeed, there exists an operator T, $\|T\| > 1$, in $\mathcal{B}_1(\mathbb{D})$ with $\mathcal{K}_T(w) \leq \mathcal{K}_{S^*}(w)$. We provide such an example here.

Remark 3.2. The main point of this note is to investigate additional conditions on the curvature, apart from the inequality we have discussed above, which will ensure contractivity. We give an alternative proof the curvature inequality. A stronger inequality becomes apparent from this proof. It is this stronger inequality which, as we will show below, admits a converse.

The contractivity of the adjoint M^* of the multiplication operator M on some reproducing kernel Hilbert space $\mathcal{H}_K$ is equivalent to the requirement that $K^\ddagger(z,w) := (1-z\bar{w})K(z,w)$ is positive definite on $\mathbb{D}$ (cf. [1, Corollary 2.37] and [12, Lemma 1]). Suppose that the operator M^* is in $\mathcal{B}_1(\mathbb{D})$. Here is a second proof of the curvature inequality:

We have

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log K(w,w) = \frac{\partial^2}{\partial w \partial \bar{w}} \log \frac{1}{(1-|w|^2)} + \frac{\partial^2}{\partial w \partial \bar{w}} \log K^\ddagger(w,w),\ w \in \mathbb{D},$$

which we rewrite as

$$\mathcal{K}_{M^*}(w) = \mathcal{K}_{S^*}(w) - \frac{\partial^2}{\partial w \partial \bar{w}} \log K^\ddagger(w,w), w \in \mathbb{D}.$$

Recalling that $\frac{\partial^2}{\partial w \partial \bar{w}} \log K^{\ddagger}(w,w)$ must be positive (see (3.1)) as long as $K^{\ddagger}$ is positive definite, we conclude that

$$\mathcal{K}_{M^*}(w) \leq \mathcal{K}_{S^*}(w),\ \ w \in \mathbb{D}.$$

The fibre at $\bar{w}$ of the holomorphic bundle E_{M^*} for M^* in $\mathcal{B}_1(\Omega)$ is the one-dimensional kernel of the operator $M^* - \bar{w}$ spanned by $K_w(\cdot)$, $w \in \Omega^*$. In general, there is no obvious way to define an inner product between the two vectors $K_w(\cdot)$ and $(\frac{\partial}{\partial \bar{w}} K_w)(\cdot)$. However since these vectors belong to the same Hilbert space (cf. [9, Lemma 4.3]), in our special case, there is a natural inner product defined between them. This ensures, via the Cauchy-Schwarz inequality, the negativity of the curvature $\mathcal{K}_T$. The reproducing kernel function K of the Hilbert space $\mathcal{H}_K$ encodes the mutual inner products of the vectors $\{K_w(\cdot) : w \in \Omega^*\}$. The Cauchy-Schwarz inequality, in turn, is just the positivity of the Gramian of the two vectors $K_w(\cdot)$ and $(\frac{\partial}{\partial \bar{w}} K_w)(\cdot)$, $w \in \Omega^*$. The positive definiteness of K is a much stronger positivity requirement involving all the derivatives of the holomorphic section $K_w(\cdot)$ defined on Ω^*. We exploit this to show that the function $-\big(\frac{\partial^2}{\partial z \partial \bar{w}} \log K\big)(z,w)$ obtained by polarizing the curvature $-\big(\frac{\partial^2}{\partial w \partial \bar{w}} \log K\big)(w,w)$ is actually negative definite not just negative, whenever K^t is assumed to be positive definite for all $t > 0$.

We now construct an example of an operator which is not contractive but its curvature is dominated by the curvature of the backward shift. Expanding the function $K(z,w) = \frac{8+8z\bar{w}-z^2\bar{w}^2}{1-z\bar{w}}$ in $z\bar{w}$, we see that it has the form $8 + 16z\bar{w} + 15\frac{z^2\bar{w}^2}{1-z\bar{w}}$. Therefore, it defines a positive definite kernel on the unit disk $\mathbb{D}$. The monomials $\frac{z^n}{\|z^n\|}$ (with $\|1\|^2 = \frac{1}{8}$, $\|z\|^2 = \frac{1}{16}$ and $\|z^n\|^2 = \frac{1}{15}$ for $n \geq 2$) form an orthonormal basis in the corresponding Hilbert space $\mathcal{H}_K$. The multiplication operator M maps $\frac{z^n}{\|z^n\|}$ to $\frac{\|z^{n+1}\|}{\|z^n\|} \frac{z^{n+1}}{\|z^{n+1}\|}$. Hence it corresponds to a weighted shift operator W with the weight sequence $\{\sqrt{\frac{1}{2}}, \sqrt{\frac{16}{15}}, 1, 1, \ldots\}$. Evidently, it is not a contraction. (This is the same as saying that the function $K^{\ddagger}(z,w) = 8 + 8z\bar{w} - z^2\bar{w}^2$ is not positive definite.) The operator W is similar to the forward shift S. Since the class $\mathcal{B}_1(\mathbb{D})$ is invariant under similarity and $S \in \mathcal{B}_1(\mathbb{D})$, it follows that W is in it as well. However,

$$-\frac{\partial^2}{\partial w \partial \bar{w}} \log K^{\ddagger}(w,w) = -\frac{8(8 - 4|w|^2 - |w|^4)}{(8 + 8|w|^2 - |w|^4)^2},\ \ w \in \mathbb{D},$$

is negative for $|w| < 1$. Hence we have shown that $\mathcal{K}_{M^*}(w) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log K(w,w) \leq \mathcal{K}_{S^*}(w)$, $w \in \mathbb{D}$, although M^* is not a contraction.

This is not an isolated example, it is easy to modify this example to produce a family of examples parameterized by a real parameter.

3.1 Infinite Divisibility and Curvature Inequality

Starting with a positive definite kernel K on a bounded domain Ω in $\mathbb{C}$, it is possible to construct several new positive definite kernel functions. For instance, if K is positive definite then the kernel K^n, $n \in \mathbb{N}$, is also positive definite. Indeed, a positive definite kernel K is said to be *infinitely divisible* if, for all $t > 0$, the kernel K^t is also positive definite. The following Lemma shows that if K is positive definite then the kernel $\big(\frac{\partial^2}{\partial z\,\partial\bar{w}}K\big)(z,w)$ is positive definite as well.

Lemma 3.3. *For any bounded domain Ω in $\mathbb{C}$, if K defines a positive definite kernel on Ω then $\big(\frac{\partial^2}{\partial z\,\partial\bar{w}}K\big)(z,w)$ is also positive definite.*

First Proof. Without loss of generality, assume that 0 is in Ω and let

$$K(z,w) = \sum_{m,n}^{\infty} a_{mn} z^m \bar{w}^n$$

be the power series expansion of K around 0. It is shown in [9, Lemma 4.1 and 4.3] that the positivity of the kernel K is equivalent to the positivity of the matrix of Taylor co-efficients of K at 0, namely,

$$H_n(0;K) := \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0n} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

for each $n \in \mathbb{Z}_+$. The function $\frac{\partial^2}{\partial z\,\partial\bar{w}}K(z,w)$ admits the expansion

$$\sum_{m,n=0}^{\infty} (m+1)(n+1)a_{(m+1)(n+1)} z^m \bar{w}^n.$$

Therefore,

$$H_{n-1}(0;\tfrac{\partial^2}{\partial z\,\partial\bar{w}}K) = \begin{pmatrix} a_{11} & 2a_{12} & \cdots & na_{1n} \\ 2a_{21} & 4a_{22} & \cdots & 2na_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ na_{n1} & 2na_{n2} & \cdots & n^2 a_{nn} \end{pmatrix}.$$

Clearly, we have

$$\begin{pmatrix} 0_{1\times 1} & 0_{1\times n} \\ 0_{n\times 1} & H_{n-1}(0;\frac{\partial^2}{\partial z\,\partial\bar{w}}K) \end{pmatrix} = D\big(H_n(0;K)\big)D,$$

where $D : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is the linear map which is diagonal and is determined by the sequence $\{0, 1, \ldots, k, \ldots, n\}$. It therefore follows that $H_n(0; \frac{\partial^2}{\partial z\, \partial \bar{w}} K)$ is positive definite for all $n \in \mathbb{N}$. Consequently, $\frac{\partial^2}{\partial z\, \partial \bar{w}} K$ is a positive definite kernel.

This completes the proof.

Second Proof. Let $K^{\#}(z, w) = \big(\frac{\partial^2}{\partial z\, \partial \bar{w}} K\big)(z, w)$. Let $w_1, \ldots, w_l$ be l points in Ω and $\alpha_1, \ldots, \alpha_l$ be scalars in $\mathbb{C}$. We have

$$\begin{aligned}
\sum_{i,j=1}^{l} \alpha_i K^{\#}(w_i, w_j) \bar{\alpha}_j &= \sum_{i,j=1}^{l} \alpha_i \bar{\alpha}_j \langle \tfrac{\partial}{\partial \bar{w}} K_{w_j}, \tfrac{\partial}{\partial \bar{w}} K_{w_i} \rangle_{\mathcal{H}_K} \\
&= || \sum_{i=1}^{l} \bar{\alpha}_i \tfrac{\partial}{\partial \bar{w}} K_{w_i} ||^2_{\mathcal{H}_K} \\
&\geq 0
\end{aligned}$$

This completes the proof. □

Definition 3.4. Let G be a real analytic function of $w, \bar{w}$ for w in some open connected subset Ω of $\mathbb{C}^m$. Polarizing G, we obtain a new function $\tilde{G}$ defined on $\Omega \times \Omega$ which is holomorphic in the first variable and anti-holomorphic in the second and restricts to G on the diagonal set $\{(w, w) : w \in \Omega\}$, that is, $\tilde{G}(w, w) = G(w, w)$, $w \in \Omega$. If the function $\tilde{G}$ is positive definite, that is, the $n \times n$ matrix $\big(\!\big(\tilde{G}(w_i, w_j)\big)\!\big)$ is positive definite for all finite subsets $\{w_1, \ldots, w_m\}$ of Ω, then we say that G is a positive definite function on Ω.

The curvature $\mathcal{K}$ of a line bundle is a real analytic function. We have shown that $-\mathcal{K}(w)$, $w \in \Omega$, is positive. However, the following example shows that $-\mathcal{K}$ need not be positive definite!

Example 3.5. Let $K(z, w) = 1 + \sum_{i=1}^{\infty} a_i z^i \bar{w}^i$ be a positive definite kernel on the unit disc $\mathbb{D}$. The kernel K then admits a power series expansion on some small neighborhood of 0. Consequently, we have

$$\begin{aligned}
\log K(z, w) &= \log(1 + \sum_{i=1}^{\infty} a_i z^i \bar{w}^i) \\
&= \sum_{i=1}^{\infty} a_i z^i \bar{w}^i - \frac{(\sum_{i=1}^{\infty} a_i z^i \bar{w}^i)^2}{2} + \frac{(\sum_{i=1}^{\infty} a_i z^i \bar{w}^i)^3}{3} - \cdots \\
&= a_1 z \bar{w} + (a_2 - \tfrac{a_1^2}{2}) z^2 \bar{w}^2 + (a_3 - a_1 a_2 + \tfrac{a_1^3}{3}) z^3 \bar{w}^3 + \ldots
\end{aligned}$$

It follows that

$$\big(\tfrac{\partial^2}{\partial z\, \partial \bar{w}} \log K\big)(z, w) = a_1 + 4(a_2 - \tfrac{a_1^2}{2}) z \bar{w} + 9(a_3 - a_1 a_2 + \tfrac{a_1^3}{3}) z^2 \bar{w}^2 + \ldots$$

Thus if we choose $0 < a_i$, $i \in \mathbb{N}$, such that $a_2 < \frac{a_1^2}{2}$, then from [9, Lemma 4.1 and 4.3], it follows that $\frac{\partial^2}{\partial z \partial \bar{w}} \log K$ is not positive definite.

In particular, taking K to be the function $1 + z\bar{w} + \frac{1}{4}z^2\bar{w}^2 + \sum_{i=3}^{\infty} z^i \bar{w}^i$, we see that

$$K^t(z,w) = 1 + tz\bar{w} + \frac{t(2t-1)}{4} z^2\bar{w}^2 + \cdots$$

is not positive definite for $t < \frac{1}{2}$.

It is therefore natural to ask if assuming that K is infinitely divisible is both necessary and sufficient for positive definiteness of the curvature function $-\mathcal{K}$. The following Theorem provides an affirmative answer.

For the proof of the following Theorem, it will be useful to recall the notion of conditional positive definiteness.

Definition 3.6. Let Ω be domain in $\mathbb{C}^m$. A complex valued function L on $\Omega \times \Omega$ which is holomorphic in the first variable and antiholomorphic in the second variable is called a Hermitian kernel if $L(z,w) = \overline{L(w,z)}$ for all $z, w \in \Omega$. A Hermitian kernel is said to be conditionally positive definite if, for any positive integer n and any choice of elements $w_1, \ldots, w_n$ in Ω and complex scalars $\alpha_1, \ldots, \alpha_n$ with $\sum_{i=1}^n \alpha_i = 0$, the inequality

$$\sum_{i,j=1}^n \alpha_i \bar{\alpha}_j L(w_i, w_j) \geq 0$$

holds.

Theorem 3.7. *Let Ω be a domain in $\mathbb{C}$ and let K be a positive, real analytic function on $\Omega \times \Omega$. If K is infinitely divisible then there exists a domain $\Omega_0 \subset \Omega$ such that negative of the curvature $\frac{\partial^2}{\partial w \partial \bar{w}} \log K$ is a positive definite function on Ω_0. Conversely, if $\hat{K}$ is a real analytic function on Ω and the function $\frac{\partial^2}{\partial w \partial \bar{w}} \log \hat{K}$ is positive definite on Ω, then there exists a neighborhood $\Omega_0 \subseteq \Omega$ of w_0, for every point $w_0 \in \Omega$, and an infinitely divisible kernel K on $\Omega_0 \times \Omega_0$ such that $K(w,w) = \hat{K}(w,w)$ for all $w \in \Omega_0$.*

Proof. For each $t > 0$, assume that K^t is positive definite on Ω. This is the same as the positive definiteness of $\exp(t \log K)$, $t > 0$. Clearly $t^{-1}(\exp(t\log K) - 1)$ is conditionally positive definite. By letting t tends to 0, it follows that $\log K$ is conditionally positive definite. Hence at an arbitrary point in Ω, in particular at w_0, the kernel

$$L_{w_0}(z,w) = \log K(z,w) - \log K(z,w_0) - \log K(w_0,w) + \log K(w_0,w_0)$$

is positive definite. This is essentially the Lemma 1.7 in [19]. From Lemma 3.3, it follows that $\frac{\partial^2}{\partial w \partial \bar{w}} L_{w_0}$ is positive definite on Ω. Note that that there exist a neighborhood $\Omega_0 \subseteq \Omega$ of

w_0 such that $\log K(z, w_0)$ is holomorphic on Ω_0. Hence from the equation above, negative of the curvature $\frac{\partial^2}{\partial w\, \partial \bar{w}} \log K$ is positive definite on Ω_0. This proves the Theorem in the forward direction.

For the other direction, without loss of generality, assume that $w_0 = 0$. The function $\mathcal{K}$ defined on some open neighborhood $U \times U$ of $(0,0)$ obtained by polarizing the real analytic function $\frac{\partial^2}{\partial w\, \partial \bar{w}} \log \hat{K}$ is holomorphic in the first variable and anti-holomorphic in the second. It is positive definite on it by hypothesis. Let $\mathcal{K}(z,w) = \sum_{m,n}^{\infty} a_{mn} z^m \bar{w}^n$ be the power series expansion of $\mathcal{K}$ on $U \times U$. The function

$$\widetilde{\mathcal{K}}(z,w) := \sum_{m,n=0}^{\infty} \frac{a_{mn}}{(m+1)(n+1)} z^{m+1} \bar{w}^{n+1}$$

is convergent on $U \times U$. Then

$$H_n(0; \widetilde{\mathcal{K}}) := \begin{pmatrix} a_{00} & \frac{a_{01}}{2} & \frac{a_{02}}{3} & \cdots & \frac{a_{0n}}{(n+1)} \\ \frac{a_{10}}{2} & \frac{a_{11}}{4} & \frac{a_{12}}{6} & \cdots & \frac{a_{1n}}{2(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n0}}{(n+1)} & \frac{a_{n1}}{2(n+1)} & \frac{a_{n2}}{3(n+1)} & \cdots & \frac{a_{nn}}{(n+1)^2} \end{pmatrix}.$$

Just as in the proof of Lemma 3.3, this time, setting $D : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ to be the linear map which is diagonal with the diagonal sequence $\{1, 2, \ldots, n+1\}$, we find that

$$H_n(0; \widetilde{\mathcal{K}}) = D^{-1}\big(H_n(0; \mathcal{K})\big) D^{-1}.$$

Appealing to [9, Lemma 4.1 and 4.3], as before, we conclude that $\widetilde{\mathcal{K}}$ is a positive definite kernel on $U \times U$. We also have

$$\frac{\partial^2}{\partial w\, \partial \bar{w}} (\log \hat{K} - \widetilde{\mathcal{K}})(w,w) = 0, \;\; w \in U.$$

Therefore, $(\log \hat{K} - \widetilde{\mathcal{K}})(w,w)$ is a real harmonic function on U and hence there exists a holomorphic function ϕ such that

$$\log \hat{K}(w,w) - \widetilde{\mathcal{K}}(w,w) = (\Re \phi)(w) := \frac{\phi(w) + \overline{\phi(w)}}{2}$$

and thus

$$\hat{K}(w,w) = \exp(\tfrac{\phi(w)}{2}) \exp(\widetilde{\mathcal{K}}(w,w)) \exp(\tfrac{\overline{\phi(w)}}{2}), \;\; w \in U.$$

Let $K : \Omega \times \Omega \to \mathbb{C}$ be the function defined by the rule

$$K(z,w) = \exp(\tfrac{\phi(z)}{2}) \exp(\widetilde{\mathcal{K}}(z,w)) \exp(\tfrac{\overline{\phi(w)}}{2}), \;\; z, w \in U.$$

For $t > 0$, we then have

$$K^t(z,w) = \exp(t\tfrac{\phi(z)}{2})\exp(t\widetilde{\mathcal{K}}(z,w))\exp(t\overline{\tfrac{\phi(w)}{2}}),\ \ z,w \in U.$$

By construction $K(w,w) = \hat{K}(w,w)$, $w \in U$. Since $\widetilde{\mathcal{K}}$ is a positive definite kernel as shown above, it follows from [19, Lemma 1.6] that $\exp(t\widetilde{\mathcal{K}})$ is a positive definite kernel and therefore K^t is a positive definite kernel on U for $t > 0$ completing the proof of the converse. ☐

Remark 3.8. If K is a non negative definite kernel such that $(1 - z\bar{w})K(z,w)$ is infinitely divisible then we say that M^* on $\mathcal{H}_K$ is an infinitely divisible contraction.

We now give an example to show that a contraction need not be infinitely divisible. Take

$$\begin{aligned} K(z,w) &= (1 - z\bar{w})^{-1}\big(1 + z\bar{w} + \tfrac{1}{4}z^2\bar{w}^2 + \sum_{i=1}^{3} z^i\bar{w}^i\big) \\ &= 1 + 2z\bar{w} + \sum_{n=2}^{\infty}(n + \tfrac{1}{4})z^n\bar{w}^n \end{aligned}$$

Clearly K defines a positive definite kernel on $\mathbb{D}$. Since $(1 - z\bar{w})K(z,w)$ is also positive definite, it follows that the adjoint of the multiplication operator M^* on $\mathcal{H}_K$ is contractive. But

$$\big((1 - z\bar{w})K(z,w)\big)^t = 1 + tz\bar{w} + \frac{t(2t-1)}{4}z^2\bar{w}^2 + \cdots$$

is not positive definite for $t < \frac{1}{2}$ as was pointed out earlier. Hence M^* is not an infinitely divisible contraction on $\mathcal{H}_K$.

The following Corollary is a partial converse to the curvature inequality from [18] for operators in the Cowen-Douglas class $B_1(\mathbb{D})$.

Corollary 3.9. *Let K be a positive definite kernel on the open unit disc. Assume that the adjoint M^* of the multiplication operator M on the reproducing kernel Hilbert space $\mathcal{H}_K$ belongs to $\mathcal{B}_1(\mathbb{D})$. The function $\frac{\partial^2}{\partial z\,\partial\bar{w}}\log\big((1 - z\bar{w})K(z,w)\big)$ is positive definite if and only if the multiplication operator M is an infinitely divisible contraction.*

Proof. Theorem 3.7 says that the positive definiteness of

$$\tfrac{\partial^2}{\partial z\,\partial\bar{w}}\log\big((1 - z\bar{w})K(z,w)\big)$$

is equivalent to $\big((1 - z\bar{w})K(z,w)\big)^t$ is positive definite for all $t \geq 0$. Hence the function $\frac{\partial^2}{\partial z\,\partial\bar{w}}\log\big((1 - z\bar{w})K(z,w)\big)$ is positive definite if and only if the multiplication operator M is an infinitely divisible contraction. In particular, M is a contraction ($t = 1$). ☐

Chapter 4

Infinitely Divisible Metrics and Curvature Inequalities - Higher Dimensional Case

We generalize the results of the previous chapter to operators in the Cowen-Douglas class $\boldsymbol{\mathcal{B}}_1(\Omega)$ when Ω is a domain in $\mathbb{C}^m$.

4.1 Negativity of the Curvature in General

In this section, we discuss the Cowen-Douglas class of commuting m-tuples of operators $\boldsymbol{T} = (T_1, \ldots, T_m)$, acting on a separable complex Hilbert space $\mathcal{H}$, for a bounded domain Ω, not necessarily planar, the corresponding Hermitian holomorphic vector bundle

$$E_{\boldsymbol{T}} = \{(w, x) \in \Omega \times \mathcal{H} : x \in \ker\ D_{\boldsymbol{T}-w}\}$$

and the curvature of $E_{\boldsymbol{T}}$ (cf. [8,9]). Here, the operator $D_{\boldsymbol{T}} : \mathcal{H} \to \mathcal{H} \oplus \ldots \oplus \mathcal{H}$ is defined by $D_{\boldsymbol{T}}(x) = (T_1 x, \ldots, T_m x)$, $x \in \mathcal{H}$. For $w = (w_1, \ldots, w_m) \in \Omega$, let $\boldsymbol{T} - w$ denote the operator tuple $(T_1 - w_1, \ldots, T_m - w_m)$. We see that $\ker D_{\boldsymbol{T}-w} = \cap_{j=1}^m \ker(T_j - w_j)$. Recall that the curvature of the holomorphic hermitian line bundle $E_{\boldsymbol{T}}$ is the $(1,1)$ form

$$\mathcal{K}_{\boldsymbol{T}}(w) = -\sum_{i,j=1}^{m} \frac{\partial^2 \log \|\gamma(w)\|^2}{\partial w_i \partial \bar{w}_j} dw_i \wedge d\bar{w}_j,\ w \in \Omega_0,$$

for some open subset $\Omega_0 \subseteq \Omega$ and a non-zero holomorphic section γ of $E_{\boldsymbol{T}}$ defined on Ω_0. Let

$$\mathsf{K}_{\boldsymbol{T}}(w) = \Big(\Big(-\frac{\partial^2 \log \parallel \gamma(w) \parallel^2}{\partial w_i \partial \bar{w}_j}\Big)\Big)_{i,j=1}^m,\ w \in \Omega_0,$$

denote the curvature matrix. In general, for a Hermitian holomorphic vector bundle, there are two well-known notions of positivity due to Nakano and Griffiths (cf. [10, page 338]). These two notions coincide in the case of a line bundle, and one talks of a positive line bundle in an unambiguous manner. The following Proposition shows that the line bundle corresponding to a commuting tuple of operators in $\boldsymbol{\mathcal{B}}_1(\Omega)$ is negative.

Proposition 4.1. *For an operator $\boldsymbol{T}$ in $\boldsymbol{\mathcal{B}}_1(\Omega^*)$, the matrix $\mathsf{K}_{\boldsymbol{T}}(w)$ is negative definite for each $w \in \Omega^*$.*

First Proof. Fix $w_0 \in \Omega$. As before (cf. [9]), it follows that $\boldsymbol{T}$ can be realized as $\boldsymbol{M}^* = (M_1^*, \ldots, M_m^*)$ where M_i is the multiplication operator by the co-ordinate function z_i on the Hilbert space $\mathcal{H}_K$ of holomorphic functions on $\Omega_0 \subseteq \Omega$ possessing a reproducing kernel K with $K(w, w) \neq 0$ for $w \in \Omega_0$. Fix $w_0 \in \Omega_0$. The function

$$K_0(z, w) = K(w_0, w_0)^{\frac{1}{2}} \varphi(z)^{-1} K(z, w) \overline{\varphi(w)^{-1}} K(w_0, w_0)^{\frac{1}{2}}$$

is defined on some open neighborhood $U \times U$ of (w_0, w_0), where U is the open set on which $K(z, w_0)$ is non-zero and $\varphi(z) = K(z, w_0)$ is holomorphic on U. The kernel K_0 is said to be normalized at w_0 (cf. [9]). The operator of multiplication by the holomorphic function φ^{-1} then defines a unitary operator from the Hilbert space $\mathcal{H}_K$ determined by the kernel function K to the Hilbert space $\mathcal{H}_{K_0}$ determined by the normalized kernel function K_0. This unitary operator intertwines the two multiplication operators on $\mathcal{H}_K$ and $\mathcal{H}_{K_0}$. Thus $\mathcal{K}_{\boldsymbol{M}^*}(w_0)$ is equal to the curvature $\mathcal{K}_{\boldsymbol{M}^{(0)*}}(w_0)$ [9, Lemma 3.9], where $\boldsymbol{M}^{(0)}$ is the m-tuple of multiplication operators by the co-ordinate functions on the Hilbert space $\mathcal{H}_{K_0}$. Let

$$K_0(z, w) = \sum_{I,J} a_{IJ} (z - w_0)^I (\bar{w} - \bar{w}_0)^J, \; z, w \in U, \; I, J \in \mathbb{Z}_+^m,$$

be the power series expansion of K_0 around the point (w_0, w_0). Here, as usual z^I is $z_1^{i_1} \ldots z_m^{i_m}, I = (i_1, \ldots, i_m)$. Since $K_0(z, w_0) = 1$, we have that $a_{00} = 1$ and $a_{I0} = 0$ for all I with $|I| > 0$. Similarly, $K_0(w_0, z) = \overline{K_0(z, w_0)}$ shows that $a_{0J} = 0$ for all J with $|J| > 0$. Also note that if

$$K_0(z, w)^{-1} = \sum_{I,J} b_{IJ} (z - w_0)^I (\bar{w} - \bar{w}_0)^J, \; z, w \in U, \; I, J \in \mathbb{Z}_+^m,$$

then $b_{00} = 1$ and $b_{I0} = 0 = b_{0J}$ for all I, J with $|I|, |J| > 0$. Since $\gamma(w) = K_0(\cdot, \bar{w})$, $w \in U^* := \{\bar{z} : z \in U\}$, is a section of the Hermitian holomorphic line bundle $E_{\boldsymbol{M}^{(0)*}}$ over

U^*, we have

$$\begin{aligned}
&\frac{\partial^2 \log \| \gamma(w) \|^2}{\partial w_i \partial \bar{w}_j}\Big|_{w=w_0} \\
&= \frac{\partial}{\partial \bar{w}_j}(K_0(\bar{w},\bar{w})^{-1}\frac{\partial}{\partial w_i}K_0(\bar{w},\bar{w}))\Big|_{w=w_0} \\
&= \frac{\partial}{\partial \bar{w}_j}\big\{(1+\sum_{\substack{|I|\geq 1\\|J|\geq 1}} b_{IJ}(\bar{w}-\bar{w}_0)^I(w-w_0)^J)(\sum_{\substack{|I|\geq 1\\|J|\geq 0}} a_{IJ+\varepsilon_i(J_i+1)}(\bar{w}-\bar{w}_0)^I(w-w_0)^J)\big\}_{|w=w_0} \\
&= a_{\varepsilon_j\varepsilon_i}
\end{aligned}$$

where ε_i is the standard unit vector in $\mathbb{C}^m$ with 1 at the i-th co-ordinate and 0 elsewhere. On the other hand, we have

$$a_{\varepsilon_j\varepsilon_i} = \frac{\partial^2 K_0(\bar{w},\bar{w})}{\partial w_i \partial \bar{w}_j}\Big|_{w=w_0} = \langle \frac{\partial}{\partial w_i}K_0(\cdot,\bar{w}), \frac{\partial}{\partial w_j}K_0(\cdot,\bar{w})\rangle\Big|_{w=w_0}.$$

Thus for any complex constants $\alpha_1,\ldots,\alpha_m$,

$$-\sum_{i,j=1}^m \alpha_i\bar{\alpha}_j \frac{\partial^2 \log \| \gamma(w) \|^2}{\partial w_i \partial \bar{w}_j}\Big|_{w=w_0} = -\|\sum_{i=1}^m \alpha_i \frac{\partial}{\partial w_i}K_0(\cdot,\bar{w})\|^2\Big|_{w=w_0} \leq 0.$$

This completes the proof.

Second Proof. We show that $-\mathsf{K}_{\boldsymbol{T}}(w)$ is the Gramian of a set of n vectors which can be explicitly exhibited. These vectors are

$$e_i(w) = K_w \otimes \frac{\partial}{\partial \bar{w}_i}K_w - \frac{\partial}{\partial \bar{w}_i}K_w \otimes K_w,\ 1 \leq i \leq n,$$

in $\mathcal{H}_K \otimes \mathcal{H}_K$. Then

$$\begin{aligned}
\langle e_i(w), e_j(w)\rangle &= \langle K_w \otimes \frac{\partial}{\partial \bar{w}_i}K_w - \frac{\partial}{\partial \bar{w}_i}K_w \otimes K_w, K_w \otimes \frac{\partial}{\partial \bar{w}_j}K_w - \frac{\partial}{\partial \bar{w}_j}K_w \otimes K_w\rangle \\
&= 2(K(w,w)\frac{\partial^2 K(w,w)}{\partial w_i \partial \bar{w}_j} - \frac{\partial}{\partial w_i}K(w,w)\frac{\partial}{\partial \bar{w}_j}K(w,w)).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial^2 \log \| \gamma(w) \|^2}{\partial w_i \partial \bar{w}_j}\Big|_{w=w_0} &= \frac{K(w,w)\frac{\partial^2 K(w,w)}{\partial w_i \partial \bar{w}_j} - \frac{\partial}{\partial w_i}K(w,w)\frac{\partial}{\partial \bar{w}_j}K(w,w)}{K(w,w)^2}\Big|_{w=w_0} \\
&= \frac{\langle e_i(w_0), e_j(w_0)\rangle}{2K(w_0,w_0)^2}.
\end{aligned}$$

This completes the proof. ∎

Definition 4.2. Let $\mathcal{H}$ be a Hilbert space and let $\mathbf{T} = (T_1, \ldots, T_m)$ be a commuting tuple of bounded linear operators on $\mathcal{H}$. We say that $\mathbf{T}$ is a row contraction if $\sum_{i=1}^m T_i T_i^* \leq I_{\mathcal{H}}$.

The following Lemma is well known, however we provide a proof for completeness.

Lemma 4.3. *Let $\mathbb{B}^m$ be the unit ball in $\mathbb{C}^m$ and $\boldsymbol{M} = (M_1, \ldots, M_m)$ be the m-tuple of multiplication operators on a reproducing kernel Hilbert space with reproducing kernel K. Then $\boldsymbol{M}$ is a row contraction if and only if $(1 - \langle z, w\rangle)K(z, w)$ is positive definite.*

Proof. For $1 \leq i \leq k, k \in \mathbb{N}$, let $\alpha_i \in \mathbb{C}$ and $w^i \in \mathbb{B}^m$, we have

$$
\begin{aligned}
&\Big\langle \big(I_{\mathcal{H}} - \sum_{l=1}^m M_l M_l^*\big) \sum_{i=1}^k \alpha_i K_{w^i}, \sum_{i=1}^k \alpha_i K_{w^i} \Big\rangle \\
= \;& \|\sum_{i=1}^k \alpha_i K_{w^i}\|^2 - \sum_{l=1}^m \Big\langle M_l M_l^* \sum_{i=1}^k \alpha_i K_{w^i}, \sum_{i=j}^k \alpha_j K_{w^j} \Big\rangle \\
= \;& \|\sum_{i=1}^k \alpha_i K_{w^i}\|^2 - \sum_{l=1}^m \Big\langle M_l^* \sum_{i=1}^k \alpha_i K_{w^i}, M_l^* \sum_{i=j}^k \alpha_j K_{w^j} \Big\rangle \\
= \;& \|\sum_{i=1}^k \alpha_i K_{w^i}\|^2 - \sum_{l=1}^m \Big(\sum_{i,j=1}^k \alpha_i \bar{\alpha}_j \langle \bar{w}_l^i K_{w^i}, \bar{w}_l^j K_{w^j} \rangle \Big) \\
= \;& \|\sum_{i=1}^k \alpha_i K_{w^i}\|^2 - \sum_{l=1}^m \Big(\sum_{i,j=1}^k \alpha_i \bar{\alpha}_j \bar{w}_l^i w_l^j K(w^j, w^i) \Big) \\
= \;& \|\sum_{i=1}^k \alpha_i K_{w^i}\|^2 - \sum_{i,j=1}^k \alpha_i \bar{\alpha}_j \Big(\sum_{l=1}^m w_l^j \bar{w}_l^i \Big) K(w^j, w^i) \\
= \;& \sum_{i,j=1}^k \alpha_i \bar{\alpha}_j K(w^j, w^i) - \sum_{i,j=1}^k \alpha_i \bar{\alpha}_j \langle w^j, w^i \rangle K(w^j, w^i) \\
= \;& \sum_{i,j=1}^k \alpha_i \bar{\alpha}_j (1 - \langle w^j, w^i \rangle) K(w^j, w^i).
\end{aligned}
$$

Hence

$$
\sum_{i=1}^m M_i M_i^* \leq I_{\mathcal{H}} \text{ if and only if } \sum_{i,j=1}^k \alpha_i \bar{\alpha}_j (1 - \langle w^j, w^i \rangle) K(w^j, w^i) \geq 0,
$$

which is equivalent to the positive definiteness of the kernel $(1 - \langle z, w\rangle)K(z, w)$. $\square$

Let $\boldsymbol{R}_m^*$ be the adjoint the commuting tuple $(M_1, \ldots, M_m)$ on the Dury-Arveson space H_m^2 which is determined by the reproducing kernel $\frac{1}{1-\langle z,w\rangle}$, $z = (z_1, \ldots, z_m)$, $w = (w_1, \ldots, w_m) \in \mathbb{B}^m$. As in Remark 3.2, using Proposition 4.1 and Lemma 4.3, we obtain a version of curvature inequality for the multi-variate case. It appeared earlier in [12] with a different proof.

Corollary 4.4. *If $\boldsymbol{T} = (T_1, \ldots, T_m)$ is a row contraction in $\boldsymbol{\mathcal{B}}_1(\mathbb{B}^m)$, then $\mathsf{K}_{\boldsymbol{R}_m^*}(w) - \mathsf{K}_{\boldsymbol{T}}(w)$ is positive for each w in the unit ball $\mathbb{B}^m$.*

4.2 Infinitely Divisible Metrics and Curvature Inequalities

Starting with a positive definite kernel K on a bounded domain Ω in $\mathbb{C}^m$, it is possible to construct several new positive definite kernel functions. For instance, if K is positive definite then the kernel K^n, $n \in \mathbb{N}$, is also positive definite. Indeed, a positive definite kernel K is said to be *infinitely divisible* if for all $t > 0$, the kernel K^t is also positive definite. While the Bergman kernel for the Euclidean ball is easily seen to be infinitely divisible, it is not infinitely divisible for the unit ball of the $n \times n$ matrices (with respect to the operator norm). We give the details for $n = 2$ in the final section of this note. The following Lemma shows that if K is positive definite then the matrix valued kernel $\big(\big(\frac{\partial^2}{\partial z_i \partial \bar{w}_j} K\big)(z, w)\big)_{i,j=1}^m$ is positive definite as well.

Lemma 4.5. *For any bounded domain Ω in $\mathbb{C}^m$, if K defines a positive definite kernel on Ω, then $\big(\big(\frac{\partial^2}{\partial z_i \partial \bar{w}_j} K\big)(z, w)\big)_{i,j=1}^m$ is also a positive definite kernel on Ω.*

Proof. Let $\mathsf{K}(z, w) = \big(\big(\frac{\partial^2}{\partial z_i \partial \bar{w}_j} K\big)(z, w)\big)_{i,j=1}^m$. Let $u_1, \ldots, u_n$ be n points in Ω and $\xi_i = (\xi_i(1), \ldots, \xi_i(m))$, $1 \le i \le m$, be vectors in $\mathbb{C}^m$. From [9], it follows that

$$\begin{aligned}
\sum_{i,j=1}^n \langle \mathsf{K}(u_i, u_j)\xi_j, \xi_i\rangle_{\mathbb{C}^m} &= \sum_{i,j=1}^n \sum_{k,l=1}^m \big(\tfrac{\partial^2}{\partial w_k \partial \bar{w}_l} K\big)(u_i, u_j)\xi_j(l)\overline{\xi_i(k)} \\
&= \sum_{i,j=1}^n \sum_{k,l=1}^m \langle \tfrac{\partial}{\partial \bar{w}_l} K_{u_j}, \tfrac{\partial}{\partial \bar{w}_k} K_{u_i}\rangle_{\mathcal{H}_K} \xi_j(l)\overline{\xi_i(k)} \\
&= \| \sum_{i=1}^n \sum_{k=1}^m \xi_i(k) \tfrac{\partial}{\partial \bar{w}_k} K_{u_i} \|_{\mathcal{H}_K}^2 \\
&\ge 0
\end{aligned}$$

This completes the proof. □

The following Lemma encodes a way to extract scalar valued positive definite kernels from matrix valued ones.

Lemma 4.6. *If K is a $n \times n$ matrix valued positive definite kernels on a bounded domain $\Omega \subset \mathbb{C}^m$, then for every $\zeta \in \mathbb{C}^n$, $\langle K(z,w)\zeta, \zeta\rangle_{\mathbb{C}^n}$ is also a positive definite kernel on Ω.*

Proof. Let $K_\zeta(z,w) = \langle K(z,w)\zeta, \zeta\rangle_{\mathbb{C}^n}$. Let $u_1, \dots, u_l$ be l points in Ω and α_i, $1 \le i \le l$, be scalars in $\mathbb{C}$. From [9], it follows that

$$\begin{aligned}\sum_{i,j=1}^{l} \alpha_i K_\zeta(u_i, u_j)\bar{\alpha}_j &= \sum_{i,j=1}^{l} \alpha_i \bar{\alpha}_j \langle K(\cdot, u_j)\zeta, K(\cdot, u_i)\zeta\rangle_{\mathcal{H}_K} \\ &= \|\sum_{j=1}^{l} \bar{\alpha}_j K(\cdot, u_j)\zeta\|^2_{\mathcal{H}_K} \\ &\ge 0\end{aligned}$$

This completes the proof. □

Theorem 4.7. *Let Ω be a domain in $\mathbb{C}^m$ and let K be a positive, real analytic function on $\Omega \times \Omega$. If K is infinitely divisible then there exist a domain $\Omega_0 \subseteq \Omega$ such that negative of the curvature matrix $\big(\big(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log K\big)\big)_{i,j=1}^m$ is a positive definite function on Ω_0. Conversely, if the function $\big(\big(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \hat{K}\big)\big)_{i,j=1}^m$ is positive definite on Ω, then there exists a neighborhood $\Omega_0 \subseteq \Omega$ of w_0 and an infinitely divisible kernel K on $\Omega_0 \times \Omega_0$ such that $K(w,w) = \hat{K}(w,w)$ for all $w \in \Omega_0$.*

Proof. For each $t > 0$, assume that K^t is positive definite on Ω. This is the same as the positive definiteness of $\exp(t \log K)$, $t > 0$. Clearly $t^{-1}(\exp(t \log K) - 1)$ is conditionally positive definite. By letting t tend to 0, it follows that $\log K$ is conditionally positive definite. Hence at an arbitrary point in Ω, in particular at w_0, the kernel

$$L_{w_0}(z,w) = \log K(z,w) - \log K(z,w_0) - \log K(w_0,w) + \log K(w_0,w_0)$$

is positive definite. This is essentially the Lemma 1.7 in [19]. From Lemma 4.5, it follows that the matrix $\big(\big(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} L_{w_0}\big)\big)$ is positive definite on Ω. Note that there exists a neighborhood $\Omega_0 \subseteq \Omega$ of w_0 such that $\log K(z,w_0)$ is holomorphic on Ω_0. Hence from the equation above, negative of the curvature matrix $\big(\big(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log K\big)\big)$ is positive definite on Ω_0. This proves the Theorem in the forward direction.

For the other direction, without loss of generality, assume that $w_0 = 0$. Let $\mathsf{K}(z,w)$ be the function obtained by polarizing the real analytic $m \times m$ matrix valued function

$$\big(\big(\tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \hat{K}(w,w)\big)\big)_{i,j=1}^m$$

defined on some bounded domain Ω in $\mathbb{C}^m$. Suppose that $\log \hat{K}$ has the power series expansion $\sum a_{IJ} z^I \bar{w}^J$, where the sum is over all multi-indices I, J of length m and $z^I =$

$z_1^{i_1}\cdots z_m^{i_m}$, $\bar w^J = \bar w_1^{j_1}\cdots \bar w_m^{j_m}$. Then

$$\mathsf{K}(z,w) = \sum_{I,J} a_{IJ}\big(\big(A_{IJ}(k,\ell) z^{I-\epsilon_k}\bar w^{J-\epsilon_\ell}\big)\big)_{k,\ell=1}^m,$$

where $A_{IJ}(k,\ell) = i_k\, j_\ell,\ 1\le k,\ell\le m$, and the sum is again over all multi-indices I,J of size m. Clearly, A_{IJ} can be written as the product $D(I)\,E_m\,D(J)$, where $D(I)$ and $D(J)$ are the $m\times m$ diagonal matrices with $(i_1,\dots,i_m)$ and $(j_1,\dots,j_m)$ on their diagonals respectively, and E_m is the $m\times m$ matrix all of whose entries are 1.

Let $D(z)$ be the holomorphic function on Ω taking values in the $m\times m$ diagonal matrices which has z_i in the (i,i) position for $z := (z_1,z_2,\dots,z_m)\in\Omega$. If the function K is assumed to be positive definite then

$$\widetilde{\mathsf{K}}(z,w) := D(z)\,\mathsf{K}(z,w)\,D(\bar w) = \sum_{I,J} a_{IJ} D(I)\,E_m\,D(J) z^I \bar w^J$$

is positive definite on Ω_0.

Let $\Lambda(I) = \{k : 1\le k\le m \text{ and } i_k\neq 0\}$. Consider the $m\times m$ matrix $E(I,J)$ defined as below:

$$E(I,J)_{ij} = \begin{cases} 1 & \text{if } i\in\Lambda(I) \text{ and } j\in\Lambda(J),\\ 0 & \text{otherwise.}\end{cases}$$

Note that if $\Lambda(I) = \Lambda(J) = \{1,\dots,m\}$, then $E(I,J) = E_m$. Consider the function on $\Omega_0\times\Omega_0$, defined by

$$\widehat{\widetilde{\mathsf{K}}}(z,w) = \sum_{I,J\neq 0} a_{IJ}\tfrac{E(I,J)}{c(I)c(J)} z^I \bar w^J,$$

where $c(I)$ denotes the cardinality of the set $\Lambda(I)$. We will prove that $\widehat{\widetilde{\mathsf{K}}}$ is a positive definite kernel on Ω_0. To facilitate the proof, we need to fix some notations.

Let δ be a multi-index of size m. Also let $p(\delta) = \prod_{j=1}^m(\delta_j+1)$ which is the number of multi-indices $I\le\delta$, that is, $i_l\le\delta_l$, $1\le l\le m$. As per the notation in [9], given a function L on a domain $U\times U$ which is holomorphic in the first variable and antiholomorphic in the second, let $H_\delta(w_0;L)$ be the $p(\delta)\times p(\delta)$ matrix whose (I,J)-entry is $\frac{\partial^I\bar\partial^J L(w_0,w_0)}{I!J!}$, $0\le I,J\le\delta$. For convenience, one uses the colexicographic order to write down the matrix, that is, $I\le_c J$ if and only if $(i_m<j_m)$ or $(i_m=j_m$ and $i_{m-1}<j_{m-1})$ or $\cdots$ or $(i_m = j_m$ and $\dots i_2 = j_2$ and $i_1<j_1)$ or $I=J$.

Let $D(I)^\sharp$ be the diagonal matrix with the diagonal entry $D(I)^\sharp_{\ell\ell}$ equal to $\frac{1}{i_\ell}$ or 0 according as i_ℓ is non-zero or zero. Using this notation, we have

$$D(I)^\sharp D(I)\,E_m\,D(J)D(J)^\sharp = E(I,J).$$

Let A_δ be the block diagonal matrix, written in the colexicographic ordering, of the form

$$(A_\delta)_{IJ} = \begin{cases} \frac{D(I)^\sharp}{c(I)} & \text{if } I = J (\neq 0) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, in this setup, for any multi-index δ, we have

$$H_\delta(0; \widehat{\widetilde{\mathsf{K}}}) = A_\delta H_\delta(0; \widetilde{\mathsf{K}}) A_\delta^*.$$

Clearly $H_\delta(0; \widehat{\widetilde{\mathsf{K}}})$ is positive definite since $H_\delta(0; \widetilde{\mathsf{K}})$ is, by [9, Lemma 4.1]. Thus from [9, Lemma 4.3], it follows that $\widehat{\widetilde{\mathsf{K}}}$ is a positive definite kernel.

Let K_0 be the scalar function on $\Omega_0 \times \Omega_0$ defined by

$$K_0(z, w) := \sum a_{IJ} z^I \bar{w}^J,$$

where the sum is over all pairs (I, J) excluding those of the form $(I, 0)$ and $(0, J)$. From Lemma 4.6, it follows that the function K_0 is positive definite since it is of the form $\langle \widehat{\widetilde{\mathsf{K}}}(z, w)\mathbf{1}, \mathbf{1}\rangle$, $\mathbf{1} = (1, \ldots, 1)$. It is evident that

$$\Big(\!\Big(\big(\tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} K_0\big)(w, w) \Big)\!\Big) = \mathsf{K}(w, w),$$

that is,

$$\frac{\partial^2}{\partial w_i\, \partial \bar{w}_j} (\log \hat{K} - K_0)(w, w) = 0,\ 1 \leq i, j \leq m,\ w \in \Omega_0.$$

Therefore, $(\log \hat{K} - K_0)(w, w)$ is a real pluriharmonic function on Ω_0 and hence there exists a holomorphic function ϕ such that

$$\log \hat{K}(w, w) - K_0(w, w) = (\Re\phi)(w) := \frac{\phi(w) + \overline{\phi(w)}}{2}.$$

(Alternatively, since $\log \hat{K}$ is real analytic, it follows that

$$\sum_{I,J} a_{IJ} w^I \bar{w}^J \;=\; \sum_{I,J} \bar{a}_{IJ} w^J \bar{w}^I$$

Equating coefficients, we get $a_{IJ} = \bar{a}_{JI}$ for all multi-indices I, J. In particular, we have $a_{I0} = \overline{a_{0I}}$ for all multi-indices I. The power series

$$(a_{00}/2) + \sum_I a_{I0} z^I$$

defines a holomorphic function ψ on Ω_0 such that $\log \hat{K}(w, w) - K_0(w, w) = \psi(w) + \overline{\psi(w)}$.)

Thus

$$\hat{K}(w,w) = \exp(\tfrac{\phi(w)}{2})\exp(K_0(w,w))\exp(\overline{\tfrac{\phi(w)}{2}}),\ w\in\Omega_0.$$

Let $K:\Omega_0\times\Omega_0\to\mathbb{C}$ be the function defined by the rule

$$K(z,w) = \exp(\tfrac{\phi(z)}{2})\exp(K_0(z,w))\exp(\overline{\tfrac{\phi(w)}{2}}).$$

For $t>0$, we then have

$$K^t(z,w) = \exp(t\tfrac{\phi(z)}{2})\exp(tK_0(z,w))\exp(t\overline{\tfrac{\phi(w)}{2}}),\ z,w\in\Omega_0.$$

By construction $K(w,w) = \hat{K}(w,w)$, $w\in\Omega_0$. Since K_0 is a positive definite kernel as shown above, it follows from [19, Lemma 1.6] that $\exp(tK_0)$ is a positive definite kernel for all $t>0$ and therefore K^t is positive definite on Ω_0 for all $t>0$, completing the proof of the converse.

□

4.3 Applications

Let $\boldsymbol{M}^*$ be the adjoint of the commuting tuple of multiplication operators acting on the Hilbert space $\mathcal{H}_K\subseteq\mathcal{O}(\Omega)$. Fix a positive definite kernel $\mathfrak{K}$ on Ω. Let us say that $\boldsymbol{M}$ is infinitely divisible with respect to $\mathfrak{K}$ if $\mathfrak{K}(z,w)^{-1}K(z,w)$ is infinitely divisible in some open subset Ω_0 of Ω. As an immediate application of Theorem 4.7 we obtain :

Theorem 4.8. *Assume that the the adjoint $\boldsymbol{M}^*$ of the multiplication operator $\boldsymbol{M}$ on the reproducing kernel Hilbert space $\mathcal{H}_K$ belongs to $\boldsymbol{\mathcal{B}}_1(\Omega)$. The function*

$$\Big(\big(\tfrac{\partial^2}{\partial w_i\partial\bar{w}_j}\log\big(\mathfrak{K}(w,w)^{-1}K(w,w)\big)\big)\Big)$$

is positive definite, if and only if the multiplication operator $\boldsymbol{M}$ is infinitely divisible with respect to $\mathfrak{K}$.

We say that a commuting tuple of multiplication operators $\boldsymbol{M}$ is an infinitely divisible row contraction if $(1-\langle z,w\rangle)K(z,w)$ is infinitely divisible, that is, $\big((1-\langle z,w\rangle)K(z,w)\big)^t$ is positive definite for all $t>0$.

Recall that $\boldsymbol{R}_m^*$ is the adjoint the commuting tuple $(M_1,\ldots,M_m)$ on the Dury-Arveson space H_m^2 whose reproducing kernel is $(1-\langle z,w\rangle)^{-1}$. The following theorem is a characterization of infinitely divisible row contractions.

Corollary 4.9. *Let K be a positive definite kernel on the Euclidean ball $\mathbb{B}^m$. Assume that the adjoint $\boldsymbol{M}^*$ of the multiplication operator $\boldsymbol{M}$ on the reproducing kernel Hilbert space $\mathcal{H}_K$ belongs to $\boldsymbol{\mathcal{B}}_1(\mathbb{B}^m)$. The function $\big(\big(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \big((1-\langle w, w\rangle)K(w,w)\big)\big)\big)_{i,j=1}^m$, $w \in \mathbb{B}^m$, is positive definite if and only if the multiplication operator $\boldsymbol{M}$ is an infinitely divisible row contraction.*

Proof. We have shown in theorem 4.7 that the positive definiteness of

$$\Big(\!\!\Big(\tfrac{\partial^2}{\partial w_i \partial \bar{w}_j} \log(1-\langle w, w\rangle)K(w,w)\Big)\!\!\Big)_{i,j=1}^m$$

is equivalent to $\big((1-\langle z, w\rangle)K(z,w)\big)^t$ is positive definite for all $t \geq 0$. Hence the function $\big(\big(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \big((1-\langle w, w\rangle)K(w,w)\big)\big)\big)_{i,j=1}^m$ is positive definite if and only if the multiplication operator $\boldsymbol{M}$ is an infinitely divisible row contraction.

□

We give one last example, namely that of the polydisc $\mathbb{D}^m$. In this case, we say a commuting tuple $\boldsymbol{M}$ of multiplication by the co-ordinate functions acting on the Hilbert space $\mathcal{H}_K$ is infinitely divisible if $\big(S^{-1}(z,w)K(z,w)\big)^t$, where $S(z,w) := \prod_{i=1}^m (1-z_i\bar{w}_i)^{-1}$, $z, w \in \mathbb{D}^m$, is positive definite for all $t > 0$ (this amounts to infinite divisibility with respect to the kernel S). Every commuting tuple of contractions $\boldsymbol{M}^*$ need not be infinitely divisible. Let $\boldsymbol{S}_m$ be the commuting m-tuple of the joint shift induced by the commuting tuple of (co-ordinate) multiplication operators defined on the Hardy space $H^2(\mathbb{D}^m)$.

Corollary 4.10. *Let K be a positive definite kernel on the polydisc $\mathbb{D}^m$. Assume that the adjoint $\boldsymbol{M}^*$ of the multiplication operator $\boldsymbol{M}$ on the reproducing kernel Hilbert space $\mathcal{H}_K$ belongs to $\boldsymbol{\mathcal{B}}_1(\mathbb{D}^m)$. The function $\big(\big(\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log \big(S^{-1}(w,w)K(w,w)\big)\big)\big)_{i,j=1}^m$, $w \in \mathbb{D}^m$, is positive definite if and only if the multiplication operator $\boldsymbol{M}$ is an infinitely divisible m-tuple of contractions.*

For a second application of these ideas, assume that K is a positive definite kernel on $\mathbb{D}^m$ with the property:

$$K_i(z,w) = (1-z_i\bar{w}_i)^m K(z,w),\ 1 \leq i \leq m,$$

is infinitely divisible. Then

$$K^m(z,w) = \Big(\prod_{i=1}^m (1-z_i\bar{w}_i)\Big)^{-m} \prod_{i=1}^m K_i(z,w).$$

It now follows that

$$K(z,w) = S(z,w)\Big(\prod_{i=1}^m K_i(z,w)\Big)^{\frac{1}{m}}.$$

Let $\boldsymbol{M}$ be the commuting tuple of multiplication operators on the Hilbert space $\mathcal{H}_K$, which is contractive since K admits the Sz̈ego kernel S as a factor. Clearly, the infinite divisibility of K_i, $1 \leq i \leq m$, implies that $K^0(z,w) = \big(\prod_{i=1}^m K_i(z,w)\big)^{\frac{1}{m}}$ is positive definite. As pointed out in [12], in consequence, for any polynomial p in m - variables,

$$p(M_1, \ldots, M_m) = P_{\mathcal{S}}\, p(S_m)_{|\mathcal{S}},$$

where $\mathcal{S}$ is the invariant subspace of functions in the Hilbert space $H^2_m \otimes \mathcal{H}_{K^0} \subseteq \mathcal{O}(\mathbb{D}^m \times \mathbb{D}^m)$ vanishing on the diagonal. $P_{\mathcal{S}}$ is the projection onto the subspace $\mathcal{S}$. We have therefore proved the following proposition.

Proposition 4.11. *If a commuting tuple in the Cowen-Douglas class $\boldsymbol{\mathcal{B}}_1(\mathbb{D}^m)$ is infinitely divisible with respect to the kernel $S(z,w)^m$, then it admits an isometric dilation to the Hardy space $H^2(\mathbb{D}^m)$.*

A basic question raised in the paper of Cowen and Douglas [7, Section 4] is the determination of nondegenerate holomorphic curves in the Grassmannian. Clearly, a necessary condition for this is the negative definiteness of the curvature matrix function. Thus we have the following Corollary to Theorem 4.7.

Let E be a Hermitian holomorphic vector bundle of rank 1 over a bounded domain $\Omega \subset \mathbb{C}^m$.

Corollary 4.12. *In the following, the implications "(iii) $\Longrightarrow$ (i)" and "(i) $\Longrightarrow$ (ii)" are valid.*

(i) There exists a Hilbert space $\mathcal{H}$ and a holomorphic map $\gamma : \Omega_0 \to \mathcal{H}$, Ω_0 open in Ω, such that E is isomorphic to the pullback, by the holomorphic map $\gamma : \Omega_0 \to \mathcal{G}r(1, \mathcal{H})$, of the tautological bundle defined over $\mathcal{G}r(1, \mathcal{H})$

(ii) The curvature matrix $\mathsf{K}(\mathsf{w}, \mathsf{w})$ is negative definite for $w \in \Omega_0$.

(iii) The Hermitian matrix valued function $\mathsf{K}(\mathsf{z}, \mathsf{w})$ is negative definite on Ω_0.

Moreover, if we assume (iii), then the existence of γ as in (i) follows, where the real analytic function $\langle \gamma(z), \gamma(w) \rangle$ is infinitely divisible.

Here is another amusing application of Theorem 4.7. Let K be the function on the unit ball $\big(\mathbb{C}^{2\times 2}\big)_1$ of 2×2 matrices (with respect to the operator norm), given by the formula $K(Z,W) := \det(I - ZW^*)^{-1}$, $Z, W \in \big(\mathbb{C}^{2\times 2}\big)_1$. It is known (cf. [2, Corollary 4.6]) that K is not infinitely divisible. The kernel K is normalized at 0 by definition. For $\delta = (1,0,0,3)$, the matrix

$$\Big(\Big(\frac{\partial^\alpha \bar{\partial}^\beta \log K(0,0)}{\alpha! \beta!}\Big)\Big)_{0 \leq \alpha, \beta \leq \delta}$$

is diagonal with $\frac{\partial_1\partial_4^3\bar{\partial}_1\bar{\partial}_4^3 \log K(0,0)}{3!3!} = -1 < 0$ (in fact for $|\delta| \leq 3$, the corresponding matrix is diagonal with non-negative entries). Here, $\delta \geq \mu$ if and only if $\delta_i \geq \mu_i$ for all $i \in \{1, \ldots, m\}$ and the matrix is written with respect to the colexicographic ordering. From [9, Lemma 4.1 and 4.3], it follows that $\log K$ is not positive definite. Hence Theorem 4.7 shows that the function $\det(I - ZW^*)^{-t}$ cannot be positive definite for all $t > 0$. Of course, the wallach set for the domain $\left(\mathbb{C}^{2\times 2}\right)_1$ (the Wallach set here is $\{t > 0 : \det(1 - ZW^*)^{-t}, \text{ is nnd}\}$) is known to be $\{1\} \cup \{2 \leq t < \infty\}$. The methods described here do not determine the Wallach set but only help in finding out if it consists of all positive real numbers or not.

Chapter 5

Curvature Calculation for the Jet Bundle

For a domain Ω in $\mathbb{C}$ and an operator T in $\mathcal{B}_n(\Omega)$, Cowen and Douglas construct a Hermitian holomorphic vector bundle E_T over Ω corresponding to T. The Hermitian holomorphic vector bundle E_T is obtained as a pull-back of the tautological bundle $S(n, \mathcal{H})$ defined over $\mathcal{G}r(n, \mathcal{H})$ by a nondegenerate holomorphic map $z \mapsto \ker(T - z),\ z \in \Omega$ as in Definition 5.14. To find the answer to the converse, namely, when a given Hermitian holomorphic vector bundle is a pull-back of the tautological bundle by a nondegenerate holomorphic map, Cowen and Douglas studied the jet bundle in their foundational paper [7, pp. 235]. The computations in this paper for the curvature of the jet bundle are somewhat difficult to comprehend. They have given a set of invariants to determine if two rank n Hermitian holomorphic vector bundle are equivalent. These invariants are complicated and not easy to compute. It is natural to expect that the equivalence of Hermitian holomorphic jet bundles should be easier to characterize. In fact, in the case of the Hermitian holomorphic jet bundle $\mathcal{J}_k(\mathcal{L}_f)$, where the line bundle $\mathcal{L}_f$ is a pull-back of the tautological bundle on $\mathcal{G}r(1, \mathcal{H})$, we have shown that the curvature of the line bundle $\mathcal{L}_f$ completely determines the class of $\mathcal{J}_k(\mathcal{L}_f)$. In general, however, our results are not as complete. Relating the complex geometric invariants inherent in the short exact sequence

$$0 \to E_I \to E \to E_{II} \to 0. \tag{5.1}$$

is an important problem. In the paper [5], it is shown that the Chern classes of these bundles must satisfy

$$c(E) = c(E_I)\, c(E_{II}).$$

Donaldson [11] obtains similar relations involving what are known as secondary invariants. We obtain a refinement, in case $E_I = \mathcal{J}_k(E_f)$ and $E = \mathcal{J}_{k+1}(E_f)$, namely,

$$\big(\text{trace} \otimes \text{Id}_{n\times n}\big)(\mathcal{K}_{\mathcal{J}_k(E_f)}) - \big(\text{trace} \otimes \text{Id}_{n\times n}\big)(\mathcal{K}_{\mathcal{J}_{k-1}(E_f)}) = \mathcal{K}_{\mathcal{J}_k(E_f)/\mathcal{J}_{k-1}(E_f)}.$$

5.1 Basic Definitions and Notation

5.1.1 Notation

Let $E \xrightarrow{\pi} \Omega$ be a C^∞ complex vector bundle of rank n.

1. $\mathcal{A}(\Omega)$ is the sheaf of C^∞ functions on Ω.
2. $\mathcal{E}^p(\Omega)$ is the sheaf of C^∞ complex p-forms over Ω.
3. $\mathcal{E}^{p,q}(\Omega)$ is the sheaf of (p,q)-forms over Ω.
4. $\mathcal{E}(\Omega, E)$ is the sheaf of C^∞ sections of the vector bundle E on Ω.
5. $\mathcal{E}^p(\Omega, E)$ is the sheaf of C^∞ complex p-forms over Ω with values in E.
6. $\mathcal{E}^{p,q}(\Omega, E)$ is the sheaf of (p,q)-forms over Ω with values in E.

We recall, following [21], some basic definitions and results from complex geometry which we will be using repeatedly in this chapter. Let Ω be a bounded domain in $\mathbb{C}^m$.

Definition 5.1. Let E be a holomorphic (resp. C^∞ over $\mathbb{C}$) manifold of dimension $m+n$ and $\pi : E \to \Omega$ be a holomorphic (resp. C^∞) map. Then $\pi : E \to \Omega$ is called a holomorphic (resp. C^∞ over $\mathbb{C}$) vector bundle of rank n if the following conditions are satisfied:

(1) $E_z = \pi^{-1}(z),\ z \in \Omega$, is a $\mathbb{C}$-vector space of dimension n.

(2) For every $z \in \Omega$, there exists a neighborhood U of z in Ω and a biholomorphism (resp. diffeomorphism)

$$\phi : \pi^{-1}(U) \to U \times \mathbb{C}^n$$

such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi"] \arrow[dr, "\pi"'] & U \times \mathbb{C}^n \arrow[d, "Pr_1"] \\ & U \end{tikzcd}$$

commutes and $\phi_{|E_z} : E_z \to \{z\} \times \mathbb{C}^n$ is a vector space isomorphism over $\mathbb{C}$.

Definition 5.2. Let $E \xrightarrow{\pi} \Omega$ be a holomorphic (resp. C^∞ over $\mathbb{C}$) vector bundle of rank n.

(1) A local holomorphic (resp. C^∞) section of $E \xrightarrow{\pi} \Omega$ is a map $s : \Omega_0 \to E$ such that $\pi \circ s = \mathrm{Id}_{\Omega_0}$, on some open subset Ω_0 of Ω.

(2) A local holomorphic (resp. C^∞) frame on $\Omega_0 \subset \Omega$ for $E \xrightarrow{\pi} \Omega$ consists of local holomorphic (resp. C^∞ over $\mathbb{C}$) sections $\{s_1, \dots, s_n\}$ of $E \xrightarrow{\pi} \Omega$ defined on Ω_0 such that $\{s_1(z), \dots, s_n(z)\}$ is a basis for E_z, $\;z \in \Omega_0$.

Definition 5.3. Let $E \xrightarrow{\pi_1} \Omega$ and $F \xrightarrow{\pi_2} \Omega$ be two holomorphic (resp. C^∞) vector bundles over Ω.

(1) A holomorphic (resp. C^∞) map $\Psi : E \to F$ is called a bundle map if $\pi_2 \circ \Psi = \pi_1$ and the restricted map $\Psi_{|E_z} : E_z \to F_{\pi_2(\Psi(z))}$ is linear.

(2) A bundle map Ψ is called an isomorphism if it is a biholomorphism (diffeomorphism).

Remark 5.4. Let $E \xrightarrow{\pi_1} \Omega$ and $F \xrightarrow{\pi_2} \Omega$ be two holomorphic (resp. C^∞ over $\mathbb{C}$) vector bundle of rank n and p respectively. Let $s := \{s_1, \dots, s_n\}$ and $\sigma := \{\sigma_1, \dots, \sigma_p\}$ be local frames of E and F respectively over Ω_0. For $1 \le j \le n$, we have

$$\Psi(s_j(z)) = \sum_{i=1}^{p} \psi_{ij}(z)\sigma_i(z).$$

Hence the bundle map Ψ may be represented, with respect to frames s and σ, as a $p \times n$ matrix valued holomorphic (resp. C^∞) function on Ω_0, that is,

$$\psi(z) = \big(\!\big(\Psi_{ij}(z)\big)\!\big)_{1 \le i \le p, 1 \le j \le n}, \quad z \in \Omega_0.$$

Definition 5.5. (1) Let $E \xrightarrow{\pi} \Omega$ be a C^∞ complex vector bundle. A Hermitian metric on E is an assignment of a Hermitian inner product $\langle , \rangle_z$ on each fibre E_z of E such that the function $\langle \xi, \eta \rangle : U \to \mathbb{C}$ given by $\langle \xi, \eta \rangle(z) = \langle \xi(z), \eta(z) \rangle_z$ is smooth for any open subset U of Ω and for any pair of smooth sections ξ, η defined on U.

(2) A C^∞ vector bundle E equipped with a Hermitian metric is called a Hermitian vector bundle.

Definition 5.6. Let $E \xrightarrow{\pi} \Omega$ be a C^∞ complex vector bundle. Then a connection D on E is a $\mathbb{C}$-linear mapping

$$D : \mathcal{E}(\Omega, E) \to \mathcal{E}^1(\Omega, E),$$

which satisfies

$$D(\phi\xi) = d\phi.\xi + \phi D\xi,$$

where $\phi \in \mathcal{A}(\Omega)$ and $\xi \in \mathcal{E}(\Omega, E)$.

Extend the connection $D : \mathcal{E}(\Omega, E) \to \mathcal{E}^1(\Omega, E)$ to a $\mathbb{C}$-linear map

$$D : \mathcal{E}^p(\Omega, E) \to \mathcal{E}^{p+1}(\Omega, E), \quad p \geq 0$$

by setting

$$D(\xi.\omega) = \xi.d\omega + D(\xi) \wedge \omega \ \text{ for } \ \omega \in \mathcal{E}^p(\Omega), \ \xi \in \mathcal{E}(\Omega, E).$$

Using this extended D, define the curvature R of D to be the map

$$R : \mathcal{E}(\Omega, E) \to \mathcal{E}^2(\Omega, E), \quad R = D \circ D.$$

Definition 5.7. Let $E \xrightarrow{\pi} \Omega$ be a C^∞ complex vector bundle with a Hermitian metric. A connection D on E is said to be compatible with the Hermitian metric on E if

$$d\langle \xi, \eta \rangle = \langle D\xi, \eta \rangle + \langle \xi, D\eta \rangle \quad \text{for } \ \xi, \eta \in \mathcal{E}(\Omega, E).$$

5.1.2 The Canonical Connection and Curvature of a Hermitian Holomorphic Vector Bundle

Suppose that $E \xrightarrow{\pi} \Omega$ is a holomorphic vector bundle. If E thought of as a C^∞ vector bundle is equipped with a Hermitian metric, then it is said to be a Hermitian holomorphic vector bundle.

Suppose that we have a connection

$$D : \mathcal{E}(\Omega, E) \to \mathcal{E}^1(\Omega, E) = \mathcal{E}^{1,0}(\Omega, E) \oplus \mathcal{E}^{0,1}(\Omega, E)$$

on a Hermitian holomorphic vector bundle E. Then D splits naturally into $D = D' + D''$, where

$$D' : \mathcal{E}(\Omega, E) \to \mathcal{E}^{1,0}(\Omega, E)$$

$$D'' : \mathcal{E}(\Omega, E) \to \mathcal{E}^{0,1}(\Omega, E).$$

Theorem 5.8. *If $E \xrightarrow{\pi} \Omega$ is a Hermitian holomorphic vector bundle, then the metric on E induces a canonical connection, D_E which satisfies two conditions:*

(a)

$$d\langle \xi, \eta \rangle = \langle D\xi, \eta \rangle + \langle \xi, D\eta \rangle, \quad \xi, \eta \in \mathcal{E}(\Omega_0, E);$$

(b) $D''\xi = 0$ for any local holomorphic local section ξ of E.

Remark 5.9. (1) Let $f = \{s_1, \ldots, s_n\}$ be a local holomorphic frame over an open subset Ω_0 of Ω. Then the metric $h(f)$ of E with respect to the frame f is define as $h(f)(z) = \big(\!\big(\langle s_j(z), s_i(z)\rangle\big)\!\big)_{1\le i,j\le n}$. The proof of Theorem 5.8, which may be found in [21, page 78, Theorem 2.1], gives a simple formula for local representation of the canonical connection and curvature in terms of the metric $h(f)$, namely,

$$\theta(f)(z) = h(f)^{-1}(z)\partial h(f)(z)$$

$$\mathcal{K}_E(f)(z) = \bar\partial(h(f)^{-1}(z)\partial h(f)(z))$$

for a holomorphic frame f.

(2) Let E and $\tilde E$ be Hermitian holomorphic vector bundles on a bounded domain Ω in $\mathbb{C}$ with canonical connections D_E and $D_{\tilde E}$, respectively, and let $\phi : E \to \tilde E$ be a C^∞ isometric bundle map. As pointed out in [7, page 208, Lemma 2.13], ϕ is holomorphic if and only if ϕ is connection-preserving, that is, if and only if

$$D_{\tilde E} \circ \phi = \phi \circ D_E.$$

Definition 5.10. Let $E \xrightarrow{\pi} \Omega$ be a Hermitian holomorphic vector bundle of rank n over a bounded domain Ω in $\mathbb{C}^m$. For $1 \le r \le n$, consider

$$\wedge^r(E) = \bigcup_{x\in\Omega} \wedge^r(\pi^{-1}(x))$$

where $\wedge^r(\pi^{-1}(x))$ is the exterior power of the vector space $\pi^{-1}(x)$. We can give holomorphic and Hermitian structures on $\wedge^r(E)$, so that it becomes a Hermitian holomorphic vector bundle (cf. [21, pp. 19]). For $r = n$, $\wedge^n(E)$ becomes a Hermitian holomorphic line bundle, called the determinant bundle, that is, $\det(E) := \wedge^n(E)$.

Remark 5.11. If $\{s_1, \ldots, s_n\}$ is a frame for the vector bundle $E \xrightarrow{\pi} \Omega$ over some open set U then a frame for the bundle $\det(E)$ over U will be $s_1 \wedge \ldots \wedge s_n$. Hence the metric for the determinant bundle $\det(E)$ takes the form

$$\begin{aligned} h_{\det(E)}(z) &= \langle s_1(z) \wedge \ldots \wedge s_n(z), s_1(z) \wedge \ldots \wedge s_n(z)\rangle \\ &= \det\ \big(\!\big(\langle s_j(z), s_i(z)\rangle\big)\!\big)_{i,j=1}^n \\ &= \det\ h_E(z). \end{aligned}$$

The following Proposition is well known. However we provide a proof for completeness following the informal Lecture Notes of M. J. Cowen.

Proposition 5.12. *Let $E \xrightarrow{\pi} \Omega$ be a Hermitian holomorphic vector bundle of rank n over a bounded domain $\Omega \subset \mathbb{C}^m$. Then the relationship between the curvature of the determinant bundle* $\det(E)$ *and that of the vector bundle E is given by the formula*

$$\mathcal{K}_{\det(E)}(z) = \text{trace}\,(\mathcal{K}_E(z)).$$

Proof. For a given holomorphic frame $\{s_1, \ldots, s_n\}$ of a Hermitian holomorphic vector bundle E defined on some open subset U of Ω, the metric h on U with respect to the frame $\{s_1, \ldots, s_n\}$ is define as $h(z) = \big(\!\big(\langle s_j(z), s_i(z)\rangle\big)\!\big)_{i,j=1}^n$. The curvature of the vector bundle E is given by the formula

$$\mathcal{K}_E(z) = \sum_{i,j=1}^m \tfrac{\partial}{\partial \bar{z}_j}\big(h^{-1}(z)\tfrac{\partial}{\partial z_i}h(z)\big)\, d\bar{z}_j \wedge dz_i.$$

Consider

$$\begin{aligned}\text{trace}(\mathcal{K}_E(z)) &= \sum_{i,j=1}^m \text{trace}\left(\tfrac{\partial}{\partial \bar{z}_j}\big(h^{-1}(z)\tfrac{\partial}{\partial z_i}h(z)\big)\right)\, d\bar{z}_j \wedge dz_i \\ &= \sum_{i,j=1}^m \tfrac{\partial}{\partial \bar{z}_j}\left(\text{trace}\big(h^{-1}(z)\tfrac{\partial}{\partial z_i}h(z)\big)\right)\, d\bar{z}_j \wedge dz_i. \qquad (5.2)\end{aligned}$$

Let z_0 be an arbitrary but fixed point in U. For z in U, Set $\tilde{h}(z) = h^{-1}(z_0)h(z)$. Then for $1 \le i \le m$ and $z \in \Omega$, we have

$$\text{trace}\big(h^{-1}(z)\tfrac{\partial}{\partial z_i}h(z)\big) = \text{trace}\big(\tilde{h}^{-1}(z)\tfrac{\partial}{\partial z_i}\tilde{h}(z)\big)$$

and

$$\tfrac{\partial}{\partial z_i}\big(\log\det\, h(z)\big) = \tfrac{\partial}{\partial z_i}\big(\log\det\, \tilde{h}(z)\big).$$

At z_0, the two equations take the form

$$\begin{aligned}\text{trace}\big(h^{-1}(z)\tfrac{\partial}{\partial z_i}h(z)\big)_{|z=z_0} &= \text{trace}\big(\tilde{h}^{-1}(z)\tfrac{\partial}{\partial z_i}\tilde{h}(z)\big)_{|z=z_0} \\ &= \text{trace}\big(\tfrac{\partial}{\partial z_i}\tilde{h}(z)\big)_{|z=z_0} \qquad (5.3)\end{aligned}$$

and

$$\begin{aligned}\tfrac{\partial}{\partial z_i}\big(\log\det\, h(z)\big)_{|z=z_0} &= \tfrac{\partial}{\partial z_i}\big(\log\det\, \tilde{h}(z)\big)_{|z=z_0} \\ &= \frac{\tfrac{\partial}{\partial z_i}\det\, \tilde{h}(z)_{|z=z_0}}{\det \tilde{h}(z)_{|z=z_0}} \\ &= \tfrac{\partial}{\partial z_i}\det\, \tilde{h}(z)_{|z=z_0} \\ &= \text{trace}\,(\tfrac{\partial}{\partial z_i}\tilde{h}(z))_{|z=z_0}, \qquad (5.4)\end{aligned}$$

the last equality follows from [15, pp. 11]. Hence from equations (5.3) and (5.4), it follows that

$$\operatorname{trace}\big(h^{-1}(z)\tfrac{\partial}{\partial z_i}h(z)\big)_{|z=z_0} = \tfrac{\partial}{\partial z_i}\big(\log\det\, h(z)\big)_{|z=z_0}.$$

Since z_0 is arbitrary point in U, so we have

$$\operatorname{trace}\big(h^{-1}(z)\tfrac{\partial}{\partial z_i}h(z)\big) = \tfrac{\partial}{\partial z_i}\big(\log\det\, h(z)\big) \quad \text{for } z \in U.$$

Hence

$$\operatorname{trace}\big(\mathcal{K}_E(z)\big) = \mathcal{K}_{\det\, E}(z).$$

□

It is possible to pick a holomorphic frame $s = \{s_1, \ldots, s_n\}$ for the Hermitian holomorphic vector bundle E such that the metric h, with respect to the frame s, at an arbitrary but fixed point z_0, has the property that $h(z_0) = 1$, $\frac{\partial}{\partial z_i}h(z_0) = 0$ and $\frac{\partial}{\partial \bar{z}_i}h(z_0) = 0$ for $1 \le i \le m$ (cf. [21, page 80]).

Even a stronger normalization, more in the sprit of this thesis, is possible (see Definition 2.4). It will be useful, in the sequel, to derive the formula given in Proposition 5.12 using this stronger normalization of the metric.

Second Proof of Proposition 5.12. For a given holomorphic frame $\{s_1, \ldots, s_n\}$ of a Hermitian holomorphic vector bundle E defined on some open subset U of Ω, the metric h on U with respect to the frame $\{s_1, \ldots, s_n\}$ is define as $h(z) = \big(\big(\langle s_j(z), s_i(z)\rangle\ \big)\big)_{i,j=1}^n$. The curvature of the vector bundle E is given by the formula

$$\mathcal{K}_E(z) = \sum_{i,j=1}^{m} \tfrac{\partial}{\partial \bar{z}_j}\big(h^{-1}(z)\tfrac{\partial}{\partial z_i}h(z)\big)\, d\bar{z}_j \wedge dz_i.$$

Let $G(z,w)$ be the real analytic function on U, obtained by polarizing the real analytic function h, which is holomorphic in z and antiholomorphic in w. Let z_0 be an arbitrary but fixed point in U. Let U_0 be open neighborhood of z_0 in U such that $G(z,z_0)$ is invertible for all $z \in U_0$. Set

$$\tilde{G}(z,w) = G(z_0,z_0)^{\frac{1}{2}}G(z,z_0)^{-1}G(z,w)G(z_0,w)^{-1}G(z_0,z_0)^{\frac{1}{2}}$$

for $z, w \in U_0$. Clearly $\tilde{G}(z,z_0) = 1$, $z \in U_0$, which implies that $\frac{\partial^{(\alpha_1+\ldots+\alpha_m)}}{\partial z_1^{\alpha_1}\ldots\partial z_m^{\alpha_m}}G(z,z_0) = 0$, $z \in U_0$ and $\alpha_1 + \ldots + \alpha_m \ge 1$. It is easy to see that

$$\sum_{i,j=1}^{m} \tfrac{\partial^2}{\partial z_i \partial \bar{z}_j}\big(\log\det\, \tilde{G}(z,z)\big)d\bar{z}_j \wedge dz_i = \sum_{i,j=1}^{m} \tfrac{\partial^2}{\partial z_i \partial \bar{z}_j}\big(\log\det\, G(z,z)\big)d\bar{z}_j \wedge dz_i \tag{5.5}$$

and

$$\sum_{i,j=1}^{m} \tfrac{\partial}{\partial \bar{z}_j}\big(\tilde{G}^{-1}(z,z)\tfrac{\partial}{\partial z_i}\tilde{G}(z,z)\big)\ d\bar{z}_j \wedge dz_i = \phi(z)^{-1}\big(\sum_{i,j=1}^{m} \tfrac{\partial}{\partial \bar{z}_j}\big(G^{-1}(z,z)\tfrac{\partial}{\partial z_i}G(z,z)\big)\ d\bar{z}_j \wedge dz_i\big)\phi(z),$$

where $\phi(z) = G(z_0,z_0)^{\frac{1}{2}}G(z,z_0)^{-1}$. Hence

$$\begin{aligned} &\operatorname{trace}\big(\sum_{i,j=1}^{m} \tfrac{\partial}{\partial \bar{z}_j}\big(\tilde{G}^{-1}(z,z)\tfrac{\partial}{\partial z_i}\tilde{G}(z,z)\big)\ d\bar{z}_j \wedge dz_i\big) \\ &\qquad = \operatorname{trace}\big(\sum_{i,j=1}^{m} \tfrac{\partial}{\partial \bar{z}_j}\big(G^{-1}(z,z)\tfrac{\partial}{\partial z_i}G(z,z)\big)\ d\bar{z}_j \wedge dz_i\big). \end{aligned} \tag{5.6}$$

For $1 \le i,j \le m$,

$$\operatorname{trace}\big(\tfrac{\partial}{\partial \bar{z}_j}(\tilde{G}^{-1}(z,z)\tfrac{\partial}{\partial z_i}\tilde{G}(z,z))\big)_{|z=z_0} = \operatorname{trace}\big(\tfrac{\partial^2}{\partial \bar{z}_j \partial z_i}\tilde{G}(z,z)\big)_{|z=z_0} \tag{5.7}$$

and

$$\begin{aligned} \tfrac{\partial^2}{\partial \bar{z}_j \partial z_i} \log\ \det\ \tilde{G}(z,z)_{|z=z_0} &= \tfrac{\partial^2}{\partial \bar{z}_j \partial z_i} \det\ \tilde{G}(z,z)_{|z=z_0} \\ &= \operatorname{trace}(\tfrac{\partial^2}{\partial \bar{z}_j \partial z_i}\tilde{G}(z,z))_{|z=z_0}, \end{aligned} \tag{5.8}$$

the last equality follows from [15, pp. 11]. Hence from the equations (5.5), (5.6), (5.7) and (5.8), we have

$$\operatorname{trace}\big(\sum_{i,j=1}^{m} \tfrac{\partial}{\partial \bar{z}_j}\big(G^{-1}(z,z)\tfrac{\partial}{\partial z_i}G(z,z)\big)_{|z=z_0}\ d\bar{z}_j \wedge dz_i\big) = \sum_{i,j=1}^{m} \tfrac{\partial^2}{\partial z_i \partial \bar{z}_j}\big(\log\det\ G(z,z)\big)_{|z=z_0} d\bar{z}_j \wedge dz_i.$$

In other words,

$$\operatorname{trace}\big(\mathcal{K}_E(z_0)\big) = \mathcal{K}_{\det\ E}(z_0).$$

□

5.2 Jet Bundles Over an Open Subset of the Complex Plane

Here we give the definition of a jet bundle closely following [7]. An equivalent description, in a slightly different language, may be found in [6].

Let E be a Hermitian holomorphic bundle of rank n over a bounded domain $\Omega \subset \mathbb{C}$. For each $k = 0,1,\ldots$ we associate to E a $(k+1)n$ -dimensional holomorphic bundle $\mathcal{J}_k(E)$, the holomorphic k-jet bundle of E, defined as follows:

If $\sigma = \{\sigma_1, \ldots, \sigma_n\}$ is a holomorphic frame for E, on an open subset U contained in Ω, then $\mathcal{J}_k(E)$ has an associated frame

$$\mathcal{J}_k(\sigma) = \{\sigma_{10}, \ldots, \sigma_{n0}, \ldots, \sigma_{1k}, \ldots, \sigma_{nk}\}$$

defined on U. If $\widetilde{\sigma}$ is another frame for E defined on $\widetilde{U}$, then on $U \cap \widetilde{U}$, we have $\widetilde{\sigma}_j = \sum a_{ij}\sigma_i$, where $A = (a_{ij})$ is a holomorphic, $n \times n$, nonsingular matrix. Symbolically

$$\widetilde{\sigma} = \sigma A.$$

Let $\mathcal{J}_k(A)$ be the $(k+1)n \times (k+1)n$, non singular, holomorphic matrix

$$\mathcal{J}_k(A) = \begin{pmatrix} A & A' & A'' & \cdots & \binom{k}{k} A^{(k)} \\ \vdots & A & 2A' & \cdots & \binom{k}{k-1} A^{(k-1)} \\ \vdots & & A & \cdots & \binom{k}{k-2} A^{(k-2)} \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & A \end{pmatrix}.$$

Then, by definition, the frames $\mathcal{J}_k(\sigma)$ and $\mathcal{J}_k(\widetilde{\sigma})$ are related on $U \cap \widetilde{U}$ by

$$\mathcal{J}_k(\widetilde{\sigma}) = \mathcal{J}_k(\sigma)\mathcal{J}_k(A).$$

A straightforward computation yields that if A and $\widetilde{A}$ are holomorphic $n \times n$ matrices, then

$$\mathcal{J}_k(A\widetilde{A}) = \mathcal{J}_k(A)\mathcal{J}_k(\widetilde{A})$$

so the bundle $\mathcal{J}_k(E)$ is well-defined.

The Hermitian metric h on E induces a Hermitian form $\mathcal{J}_k(h)$ on $\mathcal{J}_k(E)$ such that if $h(\sigma)$ is the matrix of inner products $\big(\!\big(\langle \sigma_j, \sigma_i \rangle\big)\!\big)_{i,j=1}^{n}$, then

$$\mathcal{J}_k(h)(\mathcal{J}_k(\sigma)) = \begin{pmatrix} h(\sigma) & \cdots & \frac{\partial^k h(\sigma)}{\partial z^k} \\ \vdots & & \vdots \\ \frac{\partial^k h(\sigma)}{\partial \bar{z}^k} & \cdots & \frac{\partial^{2k} h(\sigma)}{\partial z^k \partial \bar{z}^k} \end{pmatrix}$$

is the matrix of $\mathcal{J}_k(h)$ relative to the frame $\mathcal{J}_k(\sigma)$. To see that $\mathcal{J}_k(h)$ is well-defined, we need

$$\mathcal{J}_k(h)(\mathcal{J}_k(\widetilde{\sigma})) = \mathcal{J}_k(A)^*\{\mathcal{J}_k(h)(\mathcal{J}_k(\sigma))\}\mathcal{J}_k(A)$$

which follows from the computation: For $0 \le l_1, l_2 \le k$

$$\frac{\partial^{(l_1+l_2)}}{\partial z^{l_1} \partial \bar{z}^{l_2}} h(\tilde{\sigma}) = \sum_{i=1}^{l_1} \sum_{j=1}^{l_2} \binom{l_1}{i} \binom{l_2}{j} \frac{\partial^j}{\partial \bar{z}^j} A^* \frac{\partial^{l_2+i-j}}{\partial \bar{z}^{l_2-j} \partial z^i} h(\sigma) \frac{\partial^{l_1-i}}{\partial z^{l_1-i}} A. \tag{5.9}$$

Using equation (5.9), we have

$$\mathcal{J}_k(h)(\mathcal{J}_k(\widetilde{\sigma})) = \mathcal{J}_k(A)^*\{\mathcal{J}_k(h)(\mathcal{J}_k(\sigma))\}\mathcal{J}_k(A).$$

In general, the form $\mathcal{J}_k(h)(z)$ on the jet bundle $\mathcal{J}_k(E)$ need not be positive definite for $z \in \Omega$. Thus $\mathcal{J}_k(E)$ has no natural Hermitian metric, just a Hermitian form.

For $\mathcal{H}$ a complex Hilbert space and n a positive integer, let $\mathcal{G}r(n, \mathcal{H})$ denote the Grassmann manifold, the set of all n-dimensional subspaces of $\mathcal{H}$.

Definition 5.13. For Ω an open connected subset of $\mathbb{C}$, we say that a map $f : \Omega \to \mathcal{G}r(n, \mathcal{H})$ is holomorphic at $\lambda_0 \in \Omega$ if there exists a neighborhood U of λ_0 and n holomorphic $\mathcal{H}$-valued functions $\sigma_1, \ldots, \sigma_n$ on U such that $f(\lambda) = \bigvee\{\sigma_1(\lambda), \ldots, \sigma_n(\lambda)\}$ for λ in U. If this holds for each $\lambda_0 \in \Omega$ then we say that f is holomorphic on Ω.

If $f : \Omega \to \mathcal{G}r(n, \mathcal{H})$ is a holomorphic map, then a natural n-dimensional Hermitian holomorphic vector bundle E_f is induced over Ω, namely,

$$E_f = \{(x, \lambda) \in \mathcal{H} \times \Omega : x \in f(\lambda)\}$$

and

$$\pi : E_f \to \Omega \ \text{ where } \ \pi(x, \lambda) = \lambda.$$

Definition 5.14. Let $f : \Omega \to \mathcal{G}r(n, \mathcal{H})$ be a holomorphic map. We say that f is k-nondegenerate if, for each $w_0 \in \Omega$, there exists a neighborhood U of w_0 and n holomorphic $\mathcal{H}$- valued functions $\sigma_1, \ldots, \sigma_n$ on U such that $\sigma_1(w), \ldots, \sigma_n(w), \ldots, \sigma_1^{(k)}(w), \ldots \sigma_n^{(k)}(w)$ are independent for each w in the open set U. If this holds for all $k = 0, 1, \ldots,$ then we say that f is nondegenerate.

If f is k nondegenerate, then f induces a holomorphic map

$$j_k(f) : \Omega \to \mathcal{G}r((k+1)n, \mathcal{H})$$

such that $j_k(f)(w)$ is the span of $\sigma_1(w), \ldots, \sigma_n^{(k)}(w)$. If σ is a frame for E_f on U, let $j_k(\sigma) = \{\sigma_1, \ldots, \sigma_n, \ldots, \sigma_1^{(k)} \ldots, \sigma_n^{(k)}\}$ be the induced frame for $E_{j_k(f)}$. Then $\mathcal{J}_k(E_f)$ and $E_{j_k(f)}$ are naturally equivalent Hermitian holomorphic bundles by identifying σ_{ir} with $\sigma_i^{(r)}$, since $\langle \sigma_{ir}, \sigma_{js}\rangle = \partial^{r+s}\langle \sigma_i, \sigma_j\rangle / \partial z^r \partial \bar{z}^s = \langle \sigma_i^{(r)}, \sigma_j^{(s)}\rangle$. In this case $\mathcal{J}_k(h)$ is a Hermitian metric for $\mathcal{J}_k(E_f)$,that is, $\mathcal{J}_k(h)$ is positive definite.

Definition 5.15. Let $\mathcal{H}$ be a Hilbert space and Ω be a bounded domain in $\mathbb{C}^m$. Let $\mathfrak{G}_n(\Omega, \mathcal{H})$ be the set of all Hermitian holomorphic vector bundles of rank n over Ω which arise as a pull-backs of the tautological bundle by nondegenerate holomorphic maps. That is, for any nondegenerate holomorphic map $f : \Omega \to \mathcal{G}r(n, \mathcal{H})$ the vector bundle $E_f = \{(x, \lambda) \in \mathcal{H} \times \Omega : x \in f(\lambda)\}$ is in $\mathfrak{G}_n(\Omega, \mathcal{H})$.

Remark 5.16. If E_f is in $\mathfrak{G}_n(\Omega, \mathcal{H})$, then the preceding calculation shows that $\mathcal{J}_k(E_f)$ is in $\mathfrak{G}_{n(k+1)}(\Omega, \mathcal{H})$.

5.3 Line Bundles

Let $\mathcal{L}_f$ be a Hermitian holomorphic line bundle over a bounded domain $\Omega \subset \mathbb{C}$. Assume that $\mathcal{L}_f \in \mathfrak{G}_1(\Omega, \mathcal{H})$. Let $\mathcal{J}_k(\mathcal{L}_f)$ be a jet bundle of rank $k+1$ obtained from $\mathcal{L}_f$. Let σ be a frame for $\mathcal{L}_f$ over an open subset Ω_0 of Ω. A frame for $\mathcal{J}_k(\mathcal{L}_f)$ over the open set Ω_0 is easily seen to be the set $\{\sigma, \frac{\partial\sigma}{\partial z}, \frac{\partial^2\sigma}{\partial z^2}, \dots, \frac{\partial^k\sigma}{\partial z^k}\}$. Let h be a metric for $\mathcal{L}_f$, which is of the form

$$h(z) = \langle \sigma(z), \sigma(z) \rangle.$$

The metric for the jet bundle $\mathcal{J}_k(h)$ is then of the form

$$\mathcal{J}_k(h)(z) = \begin{pmatrix} h(z) & \frac{\partial}{\partial z}h(z) & \cdots & \frac{\partial^k}{\partial z^k}h(z) \\ \frac{\partial}{\partial \bar z}h(z) & \frac{\partial^2}{\partial \bar z \partial z}h(z) & \cdots & \frac{\partial^{k+1}}{\partial \bar z \partial z^k}h(z) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^k}{\partial \bar z^k}h(z) & \frac{\partial^{k+1}}{\partial \bar z^k \partial z}h(z) & \cdots & \frac{\partial^{2k}}{\partial \bar z^k \partial z^k}h(z) \end{pmatrix}.$$

Let $\mathcal{K}_{\mathcal{J}_k(\mathcal{L}_f)}$ be the curvature of the jet bundle $\mathcal{J}_k(\mathcal{L}_f)$. An explicit formula for the curvature of a Hermitian holomorphic vector bundle E is given in [21, proposition 2.2, pp. 79]. The curvature $\mathcal{K}_{\mathcal{J}_k(\mathcal{L}_f)}$ of the jet bundle therefore takes the form

$$\mathcal{K}_{\mathcal{J}_k(\mathcal{L}_f)}(z) = \overline{\partial}\{(\mathcal{J}_k(h)(z))^{-1}\partial\mathcal{J}_k(h)(z)\},$$

with respect to the metric $\mathcal{J}_k(h)$ obtained from frame $\{\sigma, \frac{\partial\sigma}{\partial z}, \frac{\partial^2\sigma}{\partial z^2}, \dots, \frac{\partial^k\sigma}{\partial z^k}\}$. Set $\mathbb{J}^k(z) = (\mathcal{J}_k(h)(z))^{-1}\frac{\partial}{\partial z}\mathcal{J}_k(h)(z)$ and note that

$$\begin{aligned} &(\mathcal{J}_k(h)(z))^{-1}\partial\mathcal{J}_k(h)(z) \\ &= \begin{pmatrix} h(z) & \frac{\partial}{\partial z}h(z) & \cdots & \frac{\partial^k}{\partial z^k}h(z) \\ \frac{\partial}{\partial \bar z}h(z) & \frac{\partial^2}{\partial \bar z \partial z}h(z) & \cdots & \frac{\partial^{k+1}}{\partial \bar z \partial z^k}h(z) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^k}{\partial \bar z^k}h(z) & \frac{\partial^{k+1}}{\partial \bar z^k \partial z}h(z) & \cdots & \frac{\partial^{2k}}{\partial \bar z^k \partial z^k}h(z) \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial z}h(z) & \frac{\partial^2}{\partial z^2}h(z) & \cdots & \frac{\partial^{k+1}}{\partial z^{k+1}}h(z) \\ \frac{\partial^2}{\partial \bar z \partial z}h(z) & \frac{\partial^3}{\partial \bar z \partial z^2}h(z) & \cdots & \frac{\partial^{k+2}}{\partial \bar z \partial z^{k+1}}h(z) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^{k+1}}{\partial \bar z^k \partial z}h(z) & \frac{\partial^{k+2}}{\partial \bar z^k \partial z^2}h(z) & \cdots & \frac{\partial^{2k+1}}{\partial \bar z^k \partial z^{k+1}}h(z) \end{pmatrix} dz \\ &= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & (\mathbb{J}^k(z))_{1,k+1} \\ 1 & 0 & 0 & \cdots & 0 & (\mathbb{J}^k(z))_{2,k+1} \\ 0 & 1 & 0 & \cdots & 0 & (\mathbb{J}^k(z))_{3,k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & (\mathbb{J}^k(z))_{k,k+1} \\ 0 & 0 & 0 & \cdots & 1 & (\mathbb{J}^k(z))_{k+1,k+1} \end{pmatrix} dz, \end{aligned}$$

where $(\mathbb{J}^k(z))_{i,k+1}$ is the $(i, k+1)^{\text{th}}$ entry of the matrix $\mathbb{J}^k(z)$. The matrix product in the first equation is of the form $A^{-1}B$, where the first k columns of B are the last k column of A.

Therefore the curvature of the jet bundle $\mathcal{J}_k(\mathcal{L}_f)$ is seen to be of the form

$$\mathcal{K}_{\mathcal{J}_k(\mathcal{L}_f)}(z) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & b_1(z) \\ 0 & 0 & 0 & \cdots & 0 & b_2(z) \\ 0 & 0 & 0 & \cdots & 0 & b_3(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_k(z) \\ 0 & 0 & 0 & \cdots & 0 & \mathcal{K}_{\det(\mathcal{J}_k\mathcal{L})}(z) \end{pmatrix} d\overline{z} \wedge dz,$$

where $b_i(z) = \frac{\partial}{\partial \bar{z}}[(\mathbb{J}^k(z))_{i,k+1}], \;\; 1 \leq i \leq k$.

Theorem 5.17. *As before, let $\mathcal{L}_f$ and $\mathcal{L}_{\tilde{f}}$ be two Hermitian holomorphic line bundles over a bounded domain $\Omega \subset \mathbb{C}$. Let $\mathcal{J}_k(\mathcal{L}_f)$ and $\mathcal{J}_k(\mathcal{L}_{\tilde{f}})$ be the corresponding jet bundles of rank $k+1$. If $\mathcal{J}_k(\mathcal{L}_f)$ is locally equivalent to $\mathcal{J}_k(\mathcal{L}_{\tilde{f}})$, then $\mathcal{J}_{k-1}(\mathcal{L}_f)$ is locally equivalent to $\mathcal{J}_{k-1}(\mathcal{L}_{\tilde{f}})$.*

Proof. Since $\mathcal{J}_k(\mathcal{L}_f)$ and $\mathcal{J}_k(\mathcal{L}_{\tilde{f}})$ are locally equivalent, for each $z_0 \in \Omega$, there exists a neighborhood Ω_0 and a holomorphic bundle map $\phi\colon \mathcal{J}_k(\mathcal{L}_f)_{|\Omega_0} \to \mathcal{J}_k(\mathcal{L}_{\tilde{f}})_{|\Omega_0}$ such that ϕ is an isomorphism. Let $\mathcal{J}_k(\sigma) = \{\sigma, \frac{\partial \sigma}{\partial z}, \frac{\partial^2 \sigma}{\partial z^2}, \ldots, \frac{\partial^k \sigma}{\partial z^k}\}$ and $\mathcal{J}_k(\tilde{\sigma}) = \{\tilde{\sigma}, \frac{\partial \tilde{\sigma}}{\partial z}, \frac{\partial^2 \tilde{\sigma}}{\partial z^2}, \ldots, \frac{\partial^k \tilde{\sigma}}{\partial z^k}\}$ be frames for $\mathcal{J}_k(\mathcal{L}_f)$ and $\mathcal{J}_k(\mathcal{L}_{\tilde{f}})$ over the open subset Ω_0 of Ω respectively.

Now

$$\phi(\tfrac{\partial^j \sigma}{\partial z^j}(z)) = \sum_{i=0}^{k} \phi_{ij}(z) \tfrac{\partial^i \tilde{\sigma}}{\partial z^i}(z). \tag{5.10}$$

So the matrix representing ϕ with respect to the two frames $\mathcal{J}_k(\sigma)$ and $\mathcal{J}_k(\tilde{\sigma})$ is

$$\phi(z) = \begin{pmatrix} \phi_{0,0}(z) & \phi_{0,1}(z) & \phi_{0,2}(z) & \cdots & \phi_{0,k}(z) \\ \phi_{1,0}(z) & \phi_{1,1}(z) & \phi_{1,2}(z) & \cdots & \phi_{1,k}(z) \\ \phi_{2,0}(z) & \phi_{2,1}(z) & \phi_{2,2}(z) & \cdots & \phi_{2,k}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{k,0}(z) & \phi_{k,1}(z) & \phi_{k,2}(z) & \cdots & \phi_{k,k}(z) \end{pmatrix}. \tag{5.11}$$

Therefore we can write

$$\big(\phi(\sigma(z)), \phi(\tfrac{\partial \sigma}{\partial z}(z)), \ldots, \phi(\tfrac{\partial^k \sigma}{\partial z^k}(z))\big) = \big(\tilde{\sigma}(z), \tfrac{\partial \tilde{\sigma}}{\partial z}(z), \ldots, \tfrac{\partial^k \tilde{\sigma}}{\partial z^k}(z)\big)\phi(z). \tag{5.12}$$

But we know that

$$\phi(z)\mathcal{K}_{\mathcal{J}_k(\mathcal{L}_f)}(z) = \mathcal{K}_{\mathcal{J}_k(\mathcal{L}_{\tilde{f}})}(z)\phi(z). \tag{5.13}$$

Now

$$\begin{aligned}\phi(z)\mathcal{K}_{\mathcal{J}_k(\mathcal{L}_f)}(z) &= \begin{pmatrix} \phi_{0,0}(z) & \phi_{0,1}(z) & \cdots & \phi_{0,k}(z) \\ \phi_{1,0}(z) & \phi_{1,1}(z) & \cdots & \phi_{1,k}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k,0}(z) & \phi_{k,1}(z) & \cdots & \phi_{k,k}(z) \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & b_1(z) \\ 0 & 0 & \cdots & b_2(z) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{K}_{\det(\mathcal{J}_k(\mathcal{L}_f))}(z) \end{pmatrix} d\overline{z}\wedge dz \\ &= \begin{pmatrix} 0 & \cdots & 0 & \sum_{i=0}^{k-1} b_{i+1}(z).\phi_{0,i}(z)+\mathcal{K}_{\det(\mathcal{J}_k(\mathcal{L}_f))}(z).\phi_{0,k}(z) \\ 0 & \cdots & 0 & \sum_{i=0}^{k-1} b_{i+1}(z).\phi_{1,i}(z)+\mathcal{K}_{\det(\mathcal{J}_k(\mathcal{L}_f))}(z).\phi_{1,k}(z) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \sum_{i=0}^{k-1} b_{i+1}(z).\phi_{k-1,i}(z)+\mathcal{K}_{\det(\mathcal{J}_k(\mathcal{L}_f))}(z).\phi_{k-1,k}(z) \\ 0 & \cdots & 0 & \sum_{i=0}^{k-1} b_{i+1}(z).\phi_{k,i}(z)+\mathcal{K}_{\det(\mathcal{J}_k(\mathcal{L}_f))}(z).\phi_{k,k}(z) \end{pmatrix} d\overline{z}\wedge dz \quad (5.14)\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}_{\mathcal{J}_k(\mathcal{L}_{\tilde{f}})}(z)\phi(z) &= \begin{pmatrix} 0 & 0 & \cdots & \widetilde{b}_1(z) \\ 0 & 0 & \cdots & \widetilde{b}_2(z) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{K}_{\det(\mathcal{J}_k(\mathcal{L}_{\tilde{f}}))}(z) \end{pmatrix} \begin{pmatrix} \phi_{0,0}(z) & \phi 0,1(z) & \cdots & \phi_{0,k}(z) \\ \phi_{1,0}(z) & \phi_{1,1}(z) & \cdots & \phi_{1,k}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k,0}(z) & \phi_{k,1}(z) & \cdots & \phi_{k,k}(z) \end{pmatrix} d\overline{z}\wedge dz \\ &= \begin{pmatrix} b_1(z).\phi_{k,0}(z) & \cdots & b_1(z).\phi_{k,k}(z) \\ b_2(z).\phi_{k,0}(z) & \cdots & b_2(z).\phi_{k,k}(z) \\ \vdots & \ddots & \vdots \\ b_{k-1}(z).\phi_{k,0}(z) & \cdots & b_{k-1}(z).\phi_{k,k}(z) \\ \mathcal{K}_{\det(\mathcal{J}_k(\mathcal{L}_{\tilde{f}}))}(z).\phi_{k,0}(z) & \cdots & \mathcal{K}_{\det(\mathcal{J}_k(\mathcal{L}_{\tilde{f}}))}(z).\phi_{k,k}(z) \end{pmatrix} d\overline{z}\wedge dz \quad (5.15)\end{aligned}$$

Hence from equations (5.13), (5.14) and (5.15), it follows that

$$\phi_{k,0}(z) = \phi_{k,1}(z) = \cdots = \phi_{k,k-1}(z) = 0.$$

So the bundle map ϕ has the form

$$\phi(z) = \begin{pmatrix} \phi_{0,0}(z) & \phi_{0,1}(z) & \cdots & \phi_{0,k-1}(z) & \phi_{0,k}(z) \\ \phi_{1,0}(z) & \phi_{1,1}(z) & \cdots & \phi_{1,k-1}(z) & \phi_{1,k}(z) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{k-1,0}(z) & \phi_{k-1,1}(z) & \cdots & \phi_{k-1,k-1}(z) & \phi_{k-1,k}(z) \\ 0 & 0 & \cdots & 0 & \phi_{k,k}(z) \end{pmatrix} \quad (5.16)$$

with respect to the frames $\mathcal{J}_k(\sigma)$ and $\mathcal{J}_k(\tilde{\sigma})$. Finally from equations (5.12) and (5.16), we see that

$$\phi_{|\mathcal{J}_{k-1}(\mathcal{L}_f)_{|\Omega_0}} : \mathcal{J}_{k-1}(\mathcal{L}_f)_{|\Omega_0} \to \mathcal{J}_{k-1}(\mathcal{L}_{\tilde{f}})_{|\Omega_0}.$$

Since ϕ is a bundle isomorphism, it follows that

$$\phi_{|\mathcal{J}_{k-1}(\mathcal{L}_f)_{|\Omega_0}} : \mathcal{J}_{k-1}(\mathcal{L}_f)_{|\Omega_0} \to \mathcal{J}_{k-1}(\mathcal{L}_{\tilde{f}})_{|\Omega_0}$$

is also a bundle isomorphism. $\square$

Corollary 5.18. *Let $\mathcal{L}_f$ and $\mathcal{L}_{\tilde{f}}$ be Hermitian holomorphic line bundles. Let $\mathcal{J}_k(\mathcal{L}_f)$ and $\mathcal{J}_k(\mathcal{L}_{\tilde{f}})$ be the corresponding jet bundles of rank $k+1$. The two jet bundles $\mathcal{J}_k(\mathcal{L}_f)$ and $\mathcal{J}_k(\mathcal{L}_{\tilde{f}})$ are locally equivalent as Hermitian holomorphic vector bundles if and only if the two line bundles $\mathcal{L}_f$ and $\mathcal{L}_{\tilde{f}}$ are locally equivalent as Hermitian holomorphic vector bundles.*

Proof. Suppose $\mathcal{J}_k(\mathcal{L}_f)$ and $\mathcal{J}_k(\mathcal{L}_{\tilde{f}})$ are locally equivalent. Then for each $z_0 \in \Omega$ there exists a neighborhood Ω_0 and a holomorphic map $\phi\colon \mathcal{J}_k(\mathcal{L}_f)_{|\Omega_0} \to \mathcal{J}_k(\mathcal{L}_{\tilde{f}})_{|\Omega_0}$ such that ϕ is an isomorphism.

Using Theorem 5.17, $\phi_{|\mathcal{J}_{k-1}(\mathcal{L}_f)_{|\Omega_0}} : \mathcal{J}_{k-1}(\mathcal{L}_f)_{|\Omega_0} \to \mathcal{J}_{k-1}(\mathcal{L}_{\tilde{f}})_{|\Omega_0}$ is an isomorphism. Since $\phi_{|\mathcal{J}_{k-1}(\mathcal{L}_f)_{|\Omega_0}} : \mathcal{J}_{k-1}(\mathcal{L}_f)_{|\Omega_0} \to \mathcal{J}_{k-1}(\mathcal{L}_{\tilde{f}})_{|\Omega_0}$ is an isomorphism, by the same argument which is given in the proof of the Theorem 5.17, it follows that

$$\phi_{|\mathcal{J}_{k-2}(\mathcal{L}_f)_{|\Omega_0}} : \mathcal{J}_{k-2}(\mathcal{L}_f)_{|\Omega_0} \to \mathcal{J}_{k-2}(\mathcal{L}_{\tilde{f}})_{|\Omega_0}$$

is an isomorphism. Repeating this argument, we see that ϕ is an isomorphism from $\mathcal{L}_{f_{|\Omega_0}}$ to $\mathcal{L}_{\tilde{f}_{|\Omega_0}}$. □

Let A be an $n \times n$ matrix and $A_{\hat{i},\hat{j}}$ be the $(n-1)\times(n-1)$ matrix which is obtained from A by removing the i^{th} row and j^{th} column of the matrix A.

Lemma 5.19. *Let A be an $n \times n$ matrix and B be the $(n-2)\times(n-2)$ matrix which is obtained from A by removing the last two rows and last two columns of A. Then*

$$\det(A_{\hat{n},\hat{n}})\det(A_{\widehat{n-1},\widehat{n-1}}) - \det(A_{\hat{n},\widehat{n-1}})\det(A_{\widehat{n-1},\hat{n}}) = \det(B)\det(A).$$

Proof. Case(1): suppose B is invertible. Let

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

and

$$x_1 = (a_{1,n-1}, a_{2,n-1}, \ldots, a_{n-2,n-1})^{\text{tr}}, x_2 = (a_{1,n}, a_{2,n}, \ldots, a_{n-2,n})^{\text{tr}}$$

$$y_1 = (a_{n-1,1}, a_{n-1,2}, \ldots, a_{n-1,n-2}), y_2 = (a_{n,1}, a_{n,2}, \ldots, a_{n,n-2}).$$

Thus the matrix A can be written in the form

$$A = \begin{pmatrix} B & x_1 & x_2 \\ y_1 & a_{n-1,n-1} & a_{n-1,n} \\ y_2 & a_{n,n-1} & a_{n,n} \end{pmatrix}.$$

In this notation, we have the following equalities:

$$\begin{aligned}
\det(A_{\hat{n},\hat{n}}) &= \det\begin{pmatrix} B & x_1 \\ y_1 & a_{n-1,n-1} \end{pmatrix} \\
&= \det(B)(a_{n-1,n-1} - y_1 B^{-1} x_1),
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
\det(A_{\widehat{n-1},\widehat{n-1}}) &= \det\begin{pmatrix} B & x_2 \\ y_2 & a_{n,n} \end{pmatrix} \\
&= \det(B)(a_{n,n} - y_2 B^{-1} x_2),
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
\det(A_{\hat{n},\widehat{n-1}}) &= \det\begin{pmatrix} B & x_2 \\ y_1 & a_{n-1,n} \end{pmatrix} \\
&= \det(B)(a_{n-1,n} - y_1 B^{-1} x_2),
\end{aligned} \tag{5.19}$$

$$\begin{aligned}
\det(A_{\widehat{n-1},\hat{n}}) &= \det\begin{pmatrix} B & x_1 \\ y_2 & a_{n,n-1} \end{pmatrix} \\
&= \det(B)(a_{n,n-1} - y_2 B^{-1} x_1),
\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}
&\det(A) \\
&= \det(B)\det\left\{\begin{pmatrix} a_{n-1,n-1} & a_{n-1,n} \\ a_{n,n-1} & a_{n,n} \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} B^{-1} \begin{pmatrix} x_1 & x_2 \end{pmatrix}\right\} \\
&= \det(B)\det\begin{pmatrix} a_{n-1,n-1} - y_1 B^{-1} x_1 & a_{n-1,n} - y_1 B^{-1} x_2 \\ a_{n,n-1} - y_2 B^{-1} x_1 & a_{n,n} - y_2 B^{-1} x_2 \end{pmatrix} \\
&= \det(B)\left\{(a_{n-1,n-1} - y_1 B^{-1} x_1)(a_{n,n} - y_2 B^{-1} x_2) - (a_{n-1,n} - y_1 B^{-1} x_2)(a_{n,n-1} - y_2 B^{-1} x_1)\right\}.
\end{aligned} \tag{5.21}$$

From equation (5.17),(5.18),(5.19),(5.20) and (5.21), it follows that

$$\det(A) = \det(B)\left\{\frac{\det(A_{\hat{n},\hat{n}})\det(A_{\widehat{n-1},\widehat{n-1}})}{(\det B)^2} - \frac{\det(A_{\widehat{n-1},\hat{n}})\det(A_{\hat{n},\widehat{n-1}})}{(\det B)^2}\right\},$$

that is,

$$\det(A_{\hat{n},\hat{n}})\det(A_{\widehat{n-1},\widehat{n-1}}) - \det(A_{\widehat{n-1},\hat{n}})\det(A_{\hat{n},\widehat{n-1}}) = \det(B)\det(A). \tag{5.22}$$

Case(2): Suppose B is not invertible. Then there exists a sequence of invertible matrices B_m that approximate B, that is, $\|B_m - B\| \to 0$, as $m \to \infty$. Let

$$A_m = \begin{pmatrix} B_m & x_1 & x_2 \\ y_1 & a_{n-1,n-1} & a_{n-1,n} \\ y_2 & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

clearly $\|A_m - A\| \to 0$ as $m \to \infty$. From the proof of the previous case, we have

$$\det\{(A_m)_{\hat{n},\hat{n}}\}\det\{(A_m)_{\widehat{n-1},\widehat{n-1}}\} - \det\{(A_m)_{\hat{n},\widehat{n-1}}\}\det\{(A_m)_{\widehat{n-1},\hat{n}}\} = \det(B_m)\det(A_m).$$

Since determinant is a continuous function, taking $m \to \infty$, it follows that

$$\det(A_{\hat{n},\hat{n}})\det(A_{\widehat{n-1},\widehat{n-1}}) - \det(A_{\widehat{n-1},\hat{n}})\det(A_{\hat{n},\widehat{n-1}}) = \det(B)\det(A).$$

□

Proposition 5.20. *The curvature of the determinant bundle* $\det \mathcal{J}_k(\mathcal{L}_f)$ *is given by the following formula*

$$\mathcal{K}_{\det \mathcal{J}_k(\mathcal{L}_f)}(z) = \frac{(\det \mathcal{J}_{k-1}h)(z)(\det \mathcal{J}_{k+1}h)(z)}{(\det \mathcal{J}_k h)^2(z)}\ d\bar{z} \wedge dz.$$

Proof. The curvature of the determinant bundle $\det(\mathcal{J}_k(\mathcal{L}_f))$ is

$$\mathcal{K}_{\det \mathcal{J}_k(\mathcal{L}_f)}(z) = \frac{(\det \mathcal{J}_k h)(z)(\frac{\partial^2}{\partial z \partial \bar{z}} \det \mathcal{J}_k h)(z) - (\frac{\partial}{\partial \bar{z}} \det \mathcal{J}_k h)(z)(\frac{\partial}{\partial z} \det \mathcal{J}_k h)(z)}{(\det \mathcal{J}_k h)^2(z)}\ d\bar{z} \wedge dz.$$

Here

$$\mathcal{J}_k h = \Big(\big(\tfrac{\partial^{i+j}}{\partial \bar{z}^i \partial z^j} h\big)\Big)_{i,j=0}^{k} \text{ and } \mathcal{J}_{k+1} h = \Big(\big(\tfrac{\partial^{i+j}}{\partial \bar{z}^i \partial z^j} h\big)\Big)_{i,j=0}^{k+1}.$$

Now, we have

$$\begin{aligned} \frac{\partial}{\partial z}(\det \mathcal{J}_k h) &= \det \begin{pmatrix} h & \frac{\partial}{\partial z} h & \cdots & \frac{\partial^{k-1}}{\partial z^{k-1}} h & \frac{\partial^{k+1}}{\partial z^{k+1}} h \\ \frac{\partial}{\partial \bar{z}} h & \frac{\partial^2}{\partial \bar{z} \partial z} h & \cdots & \frac{\partial^k}{\partial \bar{z} \partial z^{k-1}} h & \frac{\partial^{k+2}}{\partial \bar{z} \partial z^{k+1}} h \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^k}{\partial \bar{z}^k} h & \frac{\partial^{k+1}}{\partial \bar{z}^k \partial z} h & \cdots & \frac{\partial^{2k-1}}{\partial \bar{z}^k \partial z^{k-1} h} & \frac{\partial^{2k+1}}{\partial \bar{z} k \partial z^{k+1}} h \end{pmatrix} \\ &= \det((\mathcal{J}_{k+1} h)_{\widehat{k+2},\widehat{k+1}}), \end{aligned} \tag{5.23}$$

$$
\begin{aligned}
\frac{\partial}{\partial \overline{z}}(\det \mathcal{J}_k h) &= \det \begin{pmatrix} h & \frac{\partial}{\partial z}h & \cdots & \frac{\partial^{k-1}}{\partial z^{k-1}}h & \frac{\partial^{k}}{\partial z^{k}}h \\ \frac{\partial}{\partial \overline{z}}h & \frac{\partial^2}{\partial \overline{z}\partial z}h & \cdots & \frac{\partial^{k}}{\partial \overline{z}\partial z^{k-1}}h & \frac{\partial^{k+1}}{\partial \overline{z}\partial z^{k}}h \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^{k-1}}{\overline{z}^{k-1}}h & \frac{\partial^{k}}{\partial \overline{z}^{k-1}\partial z}h & \cdots & \frac{\partial^{2k-2}}{\partial \overline{z}^{k-1}\partial z^{k-1}}h & \frac{\partial^{2k-1}}{\partial \overline{z}^{k-1}\partial z^{k}}h \\ \frac{\partial^{k+1}}{\partial \overline{z}^{k+1}}h & \frac{\partial^{k+2}}{\partial \overline{z}^{k+1}\partial z}h & \cdots & \frac{\partial^{2k}}{\partial \overline{z}^{k+1}\partial z^{k-1}}h & \frac{\partial^{2k+1}}{\partial \overline{z}^{k+1}\partial z^{k}}h \end{pmatrix} \\
&= \det((\mathcal{J}_{k+1}h)_{\widehat{k+1},\widehat{k+2}}), \qquad (5.24)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial \overline{z}\partial z}(\det \mathcal{J}_k h) &= \det \begin{pmatrix} h & \frac{\partial}{\partial z}h & \cdots & \frac{\partial^{k-1}}{\partial z^{k-1}}h & \frac{\partial^{k+1}}{\partial z^{k+1}}h \\ \frac{\partial}{\partial \overline{z}}h & \frac{\partial^2}{\partial \overline{z}\partial z}h & \cdots & \frac{\partial^{k}}{\partial \overline{z}\partial z^{k-1}}h & \frac{\partial^{k+2}}{\partial \overline{z}\partial z^{k+1}}h \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^{k-1}}{\partial \overline{z}^{k-1}}h & \frac{\partial^{k}}{\partial \overline{z}^{k-1}\partial z}h & \cdots & \frac{\partial^{2k-2}}{\partial \overline{z}^{k-1}\partial z^{k-1}}h & \frac{\partial^{2k}}{\partial \overline{z}^{k-1}\partial z^{k+1}}h \\ \frac{\partial^{k+1}}{\partial \overline{z}^{k+1}}h & \frac{\partial^{k+2}}{\partial \overline{z}^{k+1}\partial z}h & \cdots & \frac{\partial^{2k+1}}{\partial \overline{z}^{k+1}\partial z^{k}}h & \frac{\partial^{2k+2}}{\partial \overline{z}^{k+1}\partial z^{k+1}}h \end{pmatrix} \\
&= \det((\mathcal{J}_{k+1}h)_{\widehat{k+1},\widehat{k+1}}), \qquad (5.25)
\end{aligned}
$$

Finally, note that

$$\det \mathcal{J}_k h = \det((\mathcal{J}_{k+1}h)_{\widehat{k+2},\widehat{k+2}}). \qquad (5.26)$$

By Lemma 5.19, we obtain

$$
\begin{aligned}
\det(\mathcal{J}_{k-1}h)\det(\mathcal{J}_{k+1}h) &= \det((\mathcal{J}_{k+1}h)_{\widehat{k+2},\widehat{k+2}})\det((\mathcal{J}_{k+1}h)_{\widehat{k+1},\widehat{k+1}}) \\
&\quad - \det((\mathcal{J}_{k+1}h)_{\widehat{k+2},\widehat{k+1}})\det((\mathcal{J}_{k+1}h)_{\widehat{k+1},\widehat{k+2}}). \qquad (5.27)
\end{aligned}
$$

From equations (5.23), (5.24), (5.25), (5.26) and (5.27), it follows that

$$
\begin{aligned}
&(\det \mathcal{J}_{k-1}h)(z)(\det \mathcal{J}_{k+1}h)(z) \\
&\qquad = (\det \mathcal{J}_k h)(z)(\tfrac{\partial^2}{\partial z\partial \overline{z}}\det \mathcal{J}_k h)(z)(\tfrac{\partial}{\partial \overline{z}}\det \mathcal{J}_k h)(z)(\tfrac{\partial}{\partial z}\det \mathcal{J}_k h)(z).
\end{aligned}
$$

Hence

$$\mathcal{K}_{\det \mathcal{J}_k(\mathcal{L}_f)}(z) = \frac{(\det \mathcal{J}_{k-1}h)(z)(\det \mathcal{J}_{k+1}h)(z)}{(\det \mathcal{J}_k h)^2(z)}\ d\overline{z}\wedge dz.$$

$\square$

Corollary 5.21. *Let $\mathcal{L}_f$ and $\mathcal{L}_{\tilde{f}}$ be Hermitian holomorphic line bundles over a domain $\Omega \subset \mathbb{C}$. The following statements are equivalent:*

(1) $\det \mathcal{J}_k(\mathcal{L}_f)$ *is locally equivalent to* $\det \mathcal{J}_k(\mathcal{L}_{\tilde{f}})$ *and* $\det \mathcal{J}_{k+1}(\mathcal{L}_f)$ *is locally equivalent to* $\det \mathcal{J}_{k+1}(\mathcal{L}_{\tilde{f}})$*, for some* $k \in \mathbb{N}$

(2) $\mathcal{L}_f$ *is locally equivalent to* $\mathcal{L}_{\tilde{f}}$*.*

5.4 Rank n-Vector Bundles

We first recall some well known facts from linear algebra.

Lemma 5.22. *[13, pp. 247] Let V be an inner product space of dimension n. Let $\{x_1, \cdots, x_k\}$ be a set of vectors in V. Then $\{x_1, \cdots, x_k\}$ is independent in V if and only if the gram matrix $\left(\left(\langle x_j, x_i\rangle\right)\right)_{i,j=1}^{n}$ is invertible.*

Lemma 5.23. *[20, pp. 138] Let A, B, C and D be matrices of size $n \times n, n \times m, m \times n$ and $m \times m$ respectively. If A, D and $D^{-1} - CA^{-1}B$ are invertible, then*

$$(A - BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} - CA^{-1}B)^{-1}CA^{-1}.$$

Lemma 5.24. *[20, pp. 138] Let A, B, C and D be matrices of size $n \times n, n \times m, m \times n$ and $m \times m$ respectively. If A, D and $D - CA^{-1}B$ are invertible, then $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is invertible and*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.$$

Lemma 5.25. *[20, pp. 246] Let A, B, C and D be matrices of size $n \times n, n \times m, m \times n$ and $m \times m$ respectively. If A is invertible then*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B)$$

Proof. Let Z be an $n \times m$ matrix. Consider

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & AZ + B \\ C & CZ + D \end{pmatrix}.$$

Choose Z such that $AZ + B = 0$, which implies that $Z = -A^{-1}B$. Thus

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D - CA^{-1}B \end{pmatrix}.$$

Hence

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B).$$

□

Lemma 5.26. *[20, pp. 247] Let A, B, C and D be matrices of size $n \times n, n \times m, m \times n$ and $m \times m$ respectively. If D is invertible then*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\det(A - BD^{-1}C).$$

Lemma 5.27. *[7, pp. 240] If V is a proper, non-zero subspace of an inner product space W then it induces an inner product on the quotient W/V by*

$$([w_1],[w_2]) \;=\; ||v_1 \wedge \ldots \wedge v_n||^{-2}(v_1 \wedge \ldots \wedge v_n \wedge w_1, v_1 \wedge \ldots \wedge v_n \wedge w_2)$$

where $[w_1],[w_2]$ denote the equivalence classes of w_1 and w_2 respectively in W/V and $\{v_1, \ldots, v_n\}$ is a basis for V.

Proof. Apply Gram-Schmidt orthogonalization to the basis $\{v_1, \ldots, v_n\}$ of V. We obtain an orthogonal basis:

$$\begin{aligned}
\widetilde{v}_1 &= v_1 \\
\widetilde{v}_2 &= v_2 - \frac{\langle v_2, \widetilde{v}_1\rangle}{\langle \widetilde{v}_1, \widetilde{v}_1\rangle}\widetilde{v}_1 \\
\widetilde{v}_3 &= v_3 - \frac{\langle v_3, \widetilde{v}_2\rangle}{\langle \widetilde{v}_2, \widetilde{v}_2\rangle}\widetilde{v}_2 - \frac{\langle v_3, \widetilde{v}_1\rangle}{\langle \widetilde{v}_1, \widetilde{v}_1\rangle}\widetilde{v}_1 \\
\widetilde{v}_i &= v_i - \sum_{j=1}^{i-1} \frac{\langle v_i, \widetilde{v}_j\rangle}{\langle \widetilde{v}_j, \widetilde{v}_j\rangle}\widetilde{v}_j, 1 \leq i \leq n
\end{aligned}$$

Let B be the matrix corresponding to the linear transformation taking the basis $\{v_1, \ldots, v_n\}$ to the orthogonal basis $\{\tilde{v}_1, \ldots, \tilde{v}_n\}$. The determinant of B is 1. Let w_1 and w_2 be two vectors in W. Then

$$P_{V^\perp}(w_1) = w_1 - \sum_{i=1}^{n} \frac{\langle w_1, \widetilde{v}_i\rangle}{||\widetilde{v}_i||^2}\widetilde{v}_i$$

and

$$P_{V^\perp}(w_2) = w_2 - \sum_{i=1}^{n} \frac{\langle w_2, \widetilde{v}_i\rangle}{||\widetilde{v}_i||^2}\widetilde{v}_i.$$

We have

$$\langle P_{V^\perp}(w_1), P_{V^\perp}(w_2)\rangle = \frac{\langle w_1, w_2\rangle \prod_{i=1}^{n} ||\widetilde{v}_i||^2 - \sum_{j=1}^{n}(\prod_{i=1, i\neq j}^{n} ||\widetilde{v}_i||^2)(\langle w_1, \widetilde{v}_j\rangle\langle \widetilde{v}_j, w_2\rangle)}{\prod_{i=1}^{n} ||\widetilde{v}_i||^2}.$$

Now

$$\begin{aligned}
\prod_{i=1}^{n} ||\widetilde{v}_i||^2 &= \langle \widetilde{v}_1 \wedge \ldots \wedge \widetilde{v}_n, \widetilde{v}_1 \wedge, \ldots, \widetilde{v}_n\rangle \\
&= \langle v_1 \wedge \ldots \wedge v_n, v_1 \wedge, \ldots, v_n\rangle
\end{aligned}$$

and

$$
\begin{aligned}
\langle w_1, w_2\rangle \prod_{i=1}^{n} \|\widetilde{v}_i\|^2 - \sum_{j=1}^{n}\Big(\prod_{i=1, i\neq j}^{n} \|\widetilde{v}_i\|^2\Big)\big(\langle w_1, \widetilde{v}_j\rangle\langle \widetilde{v}_j, w_2\rangle\big) \\
= \langle \widetilde{v}_1 \wedge \ldots \wedge \widetilde{v}_n \wedge w_1, \widetilde{v}_1 \wedge, \ldots, \widetilde{v}_n \wedge w_2\rangle \\
= \langle v_1 \wedge \ldots \wedge v_n \wedge w_1, v_1 \wedge, \ldots, v_n \wedge w_2\rangle.
\end{aligned}
$$

Hence

$$
\langle P_{V^\perp}(w_1), P_{V^\perp}(w_2)\rangle \;=\; \|v_1 \wedge \ldots \wedge v_n\|^{-2}\langle v_1 \wedge \ldots \wedge v_n \wedge w_1, v_1 \wedge \ldots \wedge v_n \wedge w_2\rangle.
$$

□

Lemma 5.28. *Let W be an inner product space and let V be a subspace of W. Let $\{e_1, \ldots, e_r\}$ be a basis of V and $\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_n\}$ be a basis of W extending the basis of W. Suppose*

$$
\sigma_i = e_1 \wedge \ldots \wedge e_r \wedge e_i, \;\; r+1 \le i \le n
$$

and

$$
\begin{aligned}
A = \big((\langle e_i, e_j\rangle)\big)_{1\le i,j\le r}, \qquad & B = \big((\langle e_i, e_j\rangle)\big)_{r+1\le i\le n, 1\le j\le r}, \\
C = \big((\langle e_i, e_j\rangle)\big)_{1\le i\le r, r+1\le j\le n}, \qquad & D = \big((\langle e_i, e_j\rangle)\big)_{r+1\le i,j\le n}, \\
\boldsymbol{A}_\sigma = \big((\langle \sigma_i, & \sigma_j\rangle)\big)_{r+1\le i,j\le n}.
\end{aligned}
$$

Then

$$
\det\big((\langle e_i, e_j\rangle)\big)_{1\le i,j\le n} = \det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \frac{\det(\boldsymbol{A}_\sigma)}{(\det A)^{n-r-1}}.
$$

Proof. Suppose $x_i = (\langle e_1, e_i\rangle, \ldots, \langle e_r, e_i\rangle)$ and $y_i = \bar{x}_i^{\mathrm{tr}}$, $r+1 \le i \le n$.

$$
\begin{aligned}
\langle \sigma_i, \sigma_j\rangle &= \det\begin{pmatrix} A & y_i \\ x_j & \langle e_i, e_j\rangle\end{pmatrix} \\
&= \det(A)(\langle e_i, e_j\rangle - x_j A^{-1} y_i).
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
\det\big((\langle e_i, e_j\rangle)\big)_{1\le i,j\le n} &= \det\begin{pmatrix} A & B \\ C & D\end{pmatrix} \\
&= \det(A)\det(D - CA^{-1}B) \\
&= \det(A)\det\big((\langle e_i, e_j\rangle - x_j A^{-1} y_i)\big)_{r+1\le i,j\le n} \\
&= \det(A)\det\big((\langle \sigma_i, \sigma_j\rangle/\det(A))\big)_{r+1\le i,j\le n} \\
&= \frac{\det(\mathbf{A}_\sigma)}{(\det A)^{n-r-1}}.
\end{aligned}
$$

$\square$

Proposition 5.29. *Let E be a Hermitian holomorphic vector bundle of rank n over a bounded domain Ω in $\mathbb{C}^m$ and let F be a subbundle of E of rank r. Then*

$$h_{\det(E/F)} = \frac{h_{\det E}}{h_{\det F}}$$

where $h_{\det E}$, $h_{\det(E/F)}$ and $h_{\det F}$ are the metrics of $\det E$, $\det F$ and $\det E/F$ respectively.

Proof. Let $\{s_1, \ldots, s_r\}$ be a frame for F over an open subset U of Ω and let $\{s_1, \ldots, s_r, s_{r+1}, \ldots, s_n\}$ be a frame of E obtained by extending the frame of F. The quotient E/F admits a frame of the form $\{[s_{r+1}], \ldots, [s_n]\}$, where $[s_i], r+1 \leq i \leq n$, denotes the equivalence class of s_i in E/F. Let $h_E = \big(\big(\langle s_j, s_i\rangle\big)\big)_{i,j=1}^n$, $h_F = \big(\big(\langle s_j, s_i\rangle\big)\big)_{i,j=1}^r$ and $h_{E/F} = \big(\big(\langle [s_j], [s_i]\rangle\big)\big)_{i,j=r+1}^n$ be the metrics of E, F and E/F respectively. Then by the definition of the determinant bundle $h_{\det E} = \det h_E$, $h_{\det F} = \det h_F$ and $h_{\det E/F} = \det h_{E/F}$. By Lemma 5.27 and Lemma 5.28, we have

$$\begin{aligned}
h_{\det E/F} &= \det h_{E/F} \\
&= \det \big(\big(\langle [s_j], [s_i]\rangle\big)\big)_{i,j=r+1}^n \\
&= \det \left(\left(\frac{\langle s_1 \wedge \ldots \wedge s_r \wedge s_j, s_1 \wedge \ldots \wedge s_r \wedge s_i\rangle}{||s_1 \wedge, \ldots \wedge s_r||^2}\right)\right)_{i,j=r+1}^n \\
&= \frac{\det \big(\big(\langle s_1 \wedge \ldots \wedge s_r \wedge s_j, s_1 \wedge \ldots \wedge s_r \wedge s_i\rangle\big)\big)_{i,j=r+1}^n}{(\det h_F)^{n-r}} \\
&= \frac{h_{\det E}}{h_{\det F}}.
\end{aligned}$$

$\square$

Corollary 5.30. *Let $0 \to F \to E \to E/F \to 0$ be an exact sequence of Hermitian holomorphic vector bundles. Then*

$$\mathcal{K}_{\det(E/F)} = \mathcal{K}_{\det(E)} - \mathcal{K}_{\det(F)}$$

which is equivalent to

$$\mathrm{trace}(\mathcal{K}_{\mathrm{E/F}}) = \mathrm{trace}(\mathcal{K}_{\mathrm{E}}) - \mathrm{trace}(\mathcal{K}_{\mathrm{F}}).$$

Let E_f be a Hermitian holomorphic vector bundle of rank n over an open subset Ω in $\mathbb{C}$ and let $E_f \in \mathfrak{G}_n(\Omega, \mathcal{H})$. Let $\{\sigma_1, \ldots, \sigma_n\}$ be a frame for E_f over an open subset Ω_0 of Ω. Let h be a metric for E_f which is defined as

$$h(z) = \big(\big(\langle \sigma_j(z), \sigma_i(z)\rangle\big)\big)_{i,j=1}^n$$

We define F_i^k for each $1 \leq k < \infty$ and $1 \leq i \leq n$ by

$$F_i^k = \sigma_1 \wedge \ldots \wedge \sigma_n \wedge \ldots \wedge \frac{\partial^{k-1}\sigma_n}{\partial z^{k-1}} \wedge \frac{\partial^k \sigma_i}{\partial z^k},$$

where wedge products between $\sigma_i's$ and their derivatives are taken in the Hilbert space $\wedge\mathcal{H}$. Let h_k be the matrix

$$h_k(z) = \big(\big(\langle F_j^k(z), F_i^k(z)\rangle\big)\big)_{i,j=1}^n$$

Proposition 5.31. *Let E_f be a Hermitian holomorphic vector bundle of rank n over $\Omega \subset \mathbb{C}$. Then the curvature $\mathcal{K}_{E_f}$ of E_f is given by*

$$\mathcal{K}_{E_f}(z) = (\det h(z))^{-1} h(z)^{-1} h_1(z) \;\; d\bar{z} \wedge dz.$$

Proof. Set $x_i = \big(\frac{\partial}{\partial \bar{z}}\langle \sigma_1, \sigma_i\rangle, \ldots, \frac{\partial}{\partial \bar{z}}\langle \sigma_n, \sigma_i\rangle\big)$ and $y_i = \bar{x}_i^{\mathrm{tr}}$, $1 \leq i \leq n$. For $1 \leq i, j \leq n$

$$\begin{aligned}
\langle F_j^1(z), F_i^1(z)\rangle &= \det\begin{pmatrix} h(z) & y_j \\ x_i & \frac{\partial^2}{\partial z \partial \bar{z}}\langle \sigma_j(z), \sigma_i(z)\rangle \end{pmatrix} \\
&= \det(h(z))\left(\tfrac{\partial^2}{\partial z \partial \bar{z}}\langle \sigma_j(z), \sigma_i(z)\rangle - x_i h(z)^{-1} y_j\right).
\end{aligned}$$

Now we can derive the formula for the curvature of the vector bundle E_f:

$$\begin{aligned}
\mathcal{K}_{E_f}(z) &= h^{-1}(z)\left\{\bar{\partial}\partial h(z) - \bar{\partial} h(z) h^{-1}(z) \partial h(z)\right\} \\
&= h^{-1}(z)\big(\big(\tfrac{\partial^2}{\partial z \partial \bar{z}}\langle \sigma_j(z), \sigma_i(z)\rangle - x_i h(z)^{-1} y_j\big)\big)_{i,j=1}^n d\bar{z} \wedge dz \\
&= h^{-1}(z)\big(\big((\det h(z))^{-1}\langle F_j^1(z), F_i^1(z)\rangle\big)\big)_{i,j=1}^n d\bar{z} \wedge dz \\
&= (\det h(z))^{-1} h^{-1}(z) h_1(z) \;\; d\bar{z} \wedge dz
\end{aligned}$$

□

Corollary 5.32. *Let E_f be a vector bundle of rank n over a bounded domain $\Omega \subset \mathbb{C}$. Then the curvature of the bundle E_f is of rank r if and only if exactly r elements are independent from the set $\{F_1^1, \ldots, F_n^1\}$ of n elements.*

Proof. By Lemma 5.31 the rank of the curvature of the bundle E is same as the rank of h_1. But rank of h_1 is r if and only if r elements are independent from the set $\{F_1^1, \ldots, F_n^1\}$ of n elements. □

A result from [7, page 238, Lemma 4.12], which appeared to be mysterious, now follows from the formula derived for the rank of the curvature. Thus we have the following corollary:

Corollary 5.33. *Let E_f be a vector bundle of rank n over a bounded domain Ω in $\mathbb{C}$. Then the rank of the curvature $\mathcal{K}_{\mathcal{J}_k(E_f)}$ of the jet bundle $\mathcal{J}_k(E_f)$, $1 \leq k < \infty$, is at most n.*

5.4.1 Curvature Formula in General

Let $E_f \xrightarrow{\pi} \Omega$ be a Hermitian holomorphic vector bundle of rank n. Let $\{s_1, \cdots, s_n\}$ be a local frame of E_f over an open subset Ω_0 of Ω. Let h be a metric for E_f which is defined as

$$h(z) = \big(\big(\langle s_i(z), s_j(z)\rangle\big)\big)_{i,j=1}^{n}.$$

For $1 \le p \le n$ and $1 \le j \le m$ set

$$\tau_p^j = s_1 \wedge \cdots \wedge s_n \wedge \frac{\partial s_p}{\partial z_j}.$$

For $1 \le i, j \le m$ set

$$h_{ij}(z) = \big(\big(\langle \tau_p^i(z), \tau_q^j(z)\rangle\big)\big)_{p,q=1}^{n}.$$

Proposition 5.34. *Let $E_f \xrightarrow{\pi} \Omega$ be a Hermitian holomorphic vector bundle of rank n over a domain Ω in $\mathbb{C}^m$. Then curvature $\mathcal{K}_{E_f}$ of the vector bundle E_f is given by*

$$\mathcal{K}_{E_f}(z) = (\det h(z))^{-1} h^{-1}(z) \sum_{i,j=1}^{m} h_{ij}(z)\, d\overline{z}_j \wedge dz_i.$$

Proof. Set $x_p^j = \left(\frac{\partial}{\partial \overline{z}_j}\langle s_1, s_p\rangle, \cdots, \frac{\partial}{\partial \overline{z}_j}\langle s_n, s_p\rangle\right)$ and $y_p^i = \overline{x_p^i}^{\,\mathrm{tr}}$ for $1 \le p \le n$.

For $1 \le i, j \le m$,

$$\begin{aligned}
\tfrac{\partial^2 h}{\partial \bar{z}_j \partial z_i}(z) - \tfrac{\partial h}{\partial \bar{z}_j}(z) h^{-1}(z) \tfrac{\partial h}{\partial z_i}(z) &= \Big(\Big(\tfrac{\partial^2}{\partial \bar{z}_j \partial z_i}\langle s_q(z), s_p(z)\rangle - x_p^j h(z)^{-1} y_q^i\Big)\Big)_{p,q=1}^{n} \\
&= \Big(\Big((\det h(z))^{-1}\langle \tau_q^i(z), \tau_p^j(z)\rangle\Big)\Big)_{p,q=1}^{n} \\
&= (\det h(z))^{-1} h_{ij}(z).
\end{aligned}$$

Hence the curvature of the vector bundle E_f takes the form:

$$\begin{aligned}
\mathcal{K}_{E_f}(z) &= h^{-1}(z) \sum_{i,j=1}^{m} \left(\tfrac{\partial^2 h}{\partial \bar{z}_j \partial z_i}(z) - \tfrac{\partial h}{\partial \bar{z}_j}(z) h^{-1}(z) \tfrac{\partial h}{\partial z_i}(z)\right) d\bar{z}_j \wedge dz_i \\
&= (\det h(z))^{-1} h^{-1}(z) \sum_{i,j=1}^{m} h_{ij}(z)\, d\overline{z}_j \wedge dz_i.
\end{aligned}$$

∎

5.4.2 Curvature of the Jet Bundle

Let $\mathcal{J}_k(E_f)$ be a jet bundle of rank $n(k+1)$ over Ω, where Ω is a bounded domain in $\mathbb{C}$. If $\sigma = \{\sigma_1, \cdots, \sigma_n\}$ is a frame for E_f then a frame for $\mathcal{J}_k(E_f)$ is of the form

$$\mathcal{J}_k(\sigma) = \{\sigma_1, \cdots, \sigma_n, \tfrac{\partial}{\partial z}\sigma_1, \cdots, \tfrac{\partial}{\partial z}\sigma_n, \ldots, \tfrac{\partial^k}{\partial z^k}\sigma_1, \ldots, \tfrac{\partial^k}{\partial z^k}\sigma_n\}.$$

By Lemma 5.31 the curvature $\mathcal{K}_{\mathcal{J}_k(E_f)}$ of the bundle $\mathcal{J}_k(E_f)$ is given by

$$\mathcal{K}_{\mathcal{J}_k(E_f)}(z) = \big(\det \mathcal{J}_k(h)(z)\big)^{-1}(\mathcal{J}_k(h)(z))^{-1}\begin{pmatrix} 0_{nk\times nk} & 0_{nk\times n} \\ 0_{n\times nk} & h_{k+1}(z)\end{pmatrix} d\bar{z}\wedge dz$$

Let $A = \mathcal{J}_{k-1}(h)$,

$$C = \left(\tfrac{\partial^k h}{\partial \bar{z}^k}, \quad \dots, \quad \tfrac{\partial^{2k-1} h}{\partial z^{k-1}\partial \bar{z}^k}\right),$$

$$B = \bar{C}^{\mathrm{tr}}, \;\; D = \tfrac{\partial^{2k}}{\partial z^k \partial \bar{z}^k} h,$$

$$x_i = \big(\tfrac{\partial^k}{\partial \bar{z}^k}\langle \sigma_1, \sigma_i\rangle, \dots, \tfrac{\partial^k}{\partial \bar{z}^k}\langle \sigma_n, \sigma_i\rangle, \dots, \tfrac{\partial^{2k-1}}{\partial z^{k-1}\partial \bar{z}^k}\langle \sigma_n, \sigma_i\rangle\big),\;\; 1\le i\le n,$$

and finally $y_i = \bar{x}_i^{\mathrm{tr}}$, $1 \le i \le n$.

Now

$$\begin{aligned} D - CA^{-1}B &= \tfrac{\partial^{2k}}{\partial z^k \partial \bar{z}^k} h - CA^{-1}B \\ &= \Big(\big(\tfrac{\partial^{2k}}{\partial z^k \partial \bar{z}^k}\langle \sigma_j, \sigma_i\rangle - x_i A^{-1} y_j\big)\Big)_{i,j=1}^n \\ &= \Big(\big((\det \mathcal{J}_{k-1}h)^{-1}\langle F_j^k, F_i^k\rangle\big)\Big)_{i,j=1}^n \\ &= (\det \mathcal{J}_{k-1}h)^{-1} h_k. \end{aligned}$$

Consequently,

$$\begin{aligned} (\mathcal{J}_k h)^{-1} &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \\ &= \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & \det(\mathcal{J}_{k-1}h)h_k^{-1} \end{pmatrix}. \end{aligned}$$

The curvature of the jet bundle $\mathcal{J}_k(E_f)$ is

$$\begin{aligned} &\mathcal{K}_{\mathcal{J}_k(E_f)}(z) \\ &= \begin{pmatrix} 0_{nk\times nk} & -\big(\det \mathcal{J}_k(h)(z)\big)^{-1} A^{-1}(z)B(z)\big(D(z) - C(z)A^{-1}(z)B(z)\big)^{-1} h_{k+1}(z) \\ 0_{n\times nk} & \big(\det \mathcal{J}_k(h)(z)\big)^{-1}\det(\mathcal{J}_{k-1}h(z))h_k^{-1}(z)h_{k+1}(z) \end{pmatrix} \end{aligned}$$

Here

$$\det \mathcal{J}_k h(z) = (\det \mathcal{J}_{k-1} h(z))^{1-n}\det h_k(z)$$

and

$$\begin{aligned} &(\det \mathcal{J}_k h(z))^{-1}\det \mathcal{J}_{k-1}h(z) \\ =\;& (\det h(z))^{n(1-n)^{k-1}}(\det h_1(z))^{n(1-n)^{k-2}}\cdots(\det h_{k-2}(z))^{n(1-n)}(\det h_{k-1}(z))^n(\det h_k(z))^{-1}. \end{aligned}$$

5.4.3 The Trace Formula

Let $\text{trace} \otimes \text{Id}_{n\times n} : \mathcal{M}_{mn}(\mathbb{C}) \cong \mathcal{M}_m(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C}) \to \mathbb{C} \otimes \mathcal{M}_n(\mathbb{C}) \cong \mathcal{M}_n(\mathbb{C})$ be the operator defined as follows

$$\big(\text{trace} \otimes \text{Id}_{n\times n}\big)\Big(\sum_{i,j=1}^{m} E_m(i,j) \otimes A_{i,j}\Big) = \sum_{i=1}^{m} A_{i,i},$$

where $E_m(i,j)$ is the $m \times m$ matrix which is defined as follows

$$\begin{aligned} (E_m(i,j))_{k,l} &= 0 \text{ if } (k,l) \neq (i,j) \\ &= 1 \text{ if } (k,l) = (i,j). \end{aligned}$$

(An arbitrary element A in $\mathcal{M}_m(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$ is of the form $A = \sum_{i,j=1}^{m} E_m(i,j) \otimes A_{i,j}$.)

Theorem 5.35. *Let $0 \to \mathcal{J}_{k-1}(E_f) \to \mathcal{J}_k(E_f) \to \mathcal{J}_k(E_f)/\mathcal{J}_{k-1}(E_f) \to 0$ be an exact sequence of jet bundles. Then we have*

$$\big(\text{trace} \otimes \text{Id}_{n\times n}\big)(\mathcal{K}_{\mathcal{J}_k(E_f)}) - \big(\text{trace} \otimes \text{Id}_{n\times n}\big)(\mathcal{K}_{\mathcal{J}_{k-1}(E_f)}) = \mathcal{K}_{\mathcal{J}_k(E_f)/\mathcal{J}_{k-1}(E_f)}(z).$$

Proof.

$$\begin{aligned} &\big(\text{trace} \otimes \text{Id}_{n\times n}\big)(\mathcal{K}_{\mathcal{J}_k(E_f)}) - \big(\text{trace} \otimes \text{Id}_{n\times n}\big)(\mathcal{K}_{\mathcal{J}_{k-1}(E_f)}) \\ &\quad = \big(\det \mathcal{J}_k(h)(z)\big)^{-1} \det(\mathcal{J}_{k-1}h(z)) h_k^{-1}(z) h_{k+1}(z) \\ &\qquad\quad - \big(\det \mathcal{J}_{k-1}(h)(z)\big)^{-1} \det(\mathcal{J}_{k-2}h(z)) h_{k-1}^{-1}(z) h_k(z) \\ &\quad = \mathcal{K}_{\mathcal{J}_k(E_f)/\mathcal{J}_{k-1}(E_f)}(z). \end{aligned}$$

The last equality follows from [7, page 244, Proposition 4.19]. $\square$

Bibliography

[1] Jim Agler and John E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, vol. 44, American Mathematical Society, Providence, RI, 2002. MR 1882259 (2003b:47001)

[2] Jonathan Arazy, *A survey of invariant Hilbert spaces of analytic functions on bounded symmetric domains*, Multivariable operator theory (Seattle, WA, 1993), Contemp. Math., vol. 185, Amer. Math. Soc., Providence, RI, 1995, pp. 7–65. MR 1332053 (96e:46034)

[3] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404. MR MR0051437 (14,479c)

[4] W. Bertram and J. Hilgert, *Reproducing kernels on vector bundles*, Lie Theory and Its Applications in Physics III (1998), 43–58.

[5] Raoul Bott and S. S. Chern, *Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections*, Acta Math. **114** (1965), 71–112. MR 0185607 (32 #3070)

[6] Karen Chandler and Pit-Mann Wong, *Finsler geometry of holomorphic jet bundles*, A sampler of Riemann-Finsler geometry, Math. Sci. Res. Inst. Publ., vol. 50, Cambridge Univ. Press, Cambridge, 2004, pp. 107–196. MR 2132659 (2006i:32022)

[7] M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Acta Math. **141** (1978), no. 3-4, 187–261. MR MR501368 (80f:47012)

[8] ______, *Operators possessing an open set of eigenvalues*, Functions, series, operators, Vol. I, II (Budapest, 1980), Colloq. Math. Soc. János Bolyai, vol. 35, North-Holland, Amsterdam, 1983, pp. 323–341. MR MR751007 (85k:47033)

[9] R. E. Curto and N. Salinas, *Generalized Bergman kernels and the Cowen-Douglas theory*, Amer. J. Math. **106** (1984), no. 2, 447–488. MR MR737780 (85e:47042)

[10] J. P Demailly, *Complex analytic and differential geometry*, http://www-fourier.ujf-grenoble.fr/ demailly/manuscripts/agbook.pdf.

[11] S. K. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. (3) **50** (1985), no. 1, 1–26. MR 765366 (86h:58038)

[12] R.G. Douglas, G. Misra, and J. Sarkar, *Contractive hilbert modules and their dilations*, Israel Journal of Mathematics (2009), 1–25.

[13] F. R. Gantmacher, *The theory of matrices. Vol. 1*, AMS Chelsea Publishing, Providence, RI, 1998, Translated from the Russian by K. A. Hirsch, Reprint of the 1959 translation. MR 1657129 (99f:15001)

[14] Adam Korányi and Gadadhar Misra, *A classification of homogeneous operators in the Cowen-Douglas class*, Adv. Math. **226** (2011), no. 6, 5338–5360. MR 2775904

[15] S. Kumaresan, *A course in differential geometry and Lie groups*, Texts and Readings in Mathematics, vol. 22, Hindustan Book Agency, New Delhi, 2002. MR 1891361 (2003e:53001)

[16] Hyun-Kyoung Kwon and Sergei Treil, *Similarity of operators and geometry of eigenvector bundles*, Publ. Mat. **53** (2009), no. 2, 417–438. MR 2543858 (2010g:47011)

[17] Rubén A. Martínez-Avendaño and Peter Rosenthal, *An introduction to operators on the Hardy-Hilbert space*, Graduate Texts in Mathematics, vol. 237, Springer, New York, 2007. MR 2270722 (2007k:47051)

[18] G. Misra, *Curvature inequalities and extremal properties of bundle shifts*, J. Operator Theory **11** (1984), no. 2, 305–317. MR 749164 (86h:47057)

[19] K. R. Parthasarathy and K. Schmidt, *Positive definite kernels, continuous tensor products, and central limit theorems of probability theory*, Lecture Notes in Mathematics, Vol. 272, Springer-Verlag, Berlin, 1972. MR 0622034 (58 #29849)

[20] A. Ramachandra Rao and P. Bhimasankaram, *Linear algebra*, second ed., Texts and Readings in Mathematics, vol. 19, Hindustan Book Agency, New Delhi, 2000. MR 1781860

[21] R. O. Wells, Jr., *Differential analysis on complex manifolds*, third ed., Graduate Texts in Mathematics, vol. 65, Springer, New York, 2008, With a new appendix by Oscar Garcia-Prada. MR MR2359489 (2008g:32001)