CURVATURE INEQUALITIES AND EXTREMAL OPERATORS

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ABSTRACT. A curvature inequality is established for contractive commuting tuples of operators $\boldsymbol{T}$ in the Cowen-Douglas class $B_n(\Omega)$ of rank n defined on some bounded domain Ω in $\mathbb{C}^m$. Properties of the extremal operators, that is, the operators which achieve equality, are investigated. Specifically, a substantial part of a well known question due to R. G. Douglas involving these extremal operators, in the case of the unit disc, is answered.

1. INTRODUCTION

For a fixed $n \in \mathbb{N}$, and a bounded domain $\Omega \subseteq \mathbb{C}^m$, the important class of operators $B_n(\Omega^*)$, $\Omega^* = \{\bar{z} : z \in \Omega\}$, defined below, was introduced in the papers [4] and [5] by Cowen and Douglas. An alternative approach to the study of this class of operators is presented in the paper [7] of Curto and Salinas. For $w = (w_1, w_2, \ldots, w_m)$ in Ω^*, let $\mathcal{D}_{\boldsymbol{T}-w\boldsymbol{I}} : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ be the operator: $\mathcal{D}_{\boldsymbol{T}-w\boldsymbol{I}}(h) = \oplus_{k=1}^m (T_k - w_k I)h$, $h \in \mathcal{H}$.

Definition 1.1. A m-tuple of commuting bounded operators $\boldsymbol{T} = (T_1, T_2, \ldots, T_m)$ on a complex separable Hilbert space $\mathcal{H}$ is said to be in $B_n(\Omega^*)$ if

1. $\dim\big(\cap_{k=1}^m \ker(T_k - w_k I)\big) = n$ for each $w \in \Omega^*$;
2. the operator $\mathcal{D}_{\boldsymbol{T}-w\boldsymbol{I}}$, $w \in \Omega^*$, has closed range and
3. $\bigvee_{w\in\Omega^*}\big(\cap_{k=1}^m \ker(T_k - w_k I)\big) = \mathcal{H}$

For any commuting tuple of operators $\boldsymbol{T}$ in $B_n(\Omega^*)$, the existence of a rank n holomorphic Hermitian vector bundle $E_{\boldsymbol{T}}$ over Ω^* was established in [5]. Indeed,

$$E_{\boldsymbol{T}} := \big\{(w, v) \in \Omega^* \times \mathcal{H} : v \in \cap_{k=1}^m \ker(T_k - w_k I)\big\}, \pi(w, v) = w,$$

admits a local holomorphic cross-section. In the paper [4], for $m = 1$, it is shown that two commuting m-tuple of operators $\boldsymbol{T}$ and $\boldsymbol{S}$ in $B_n(\Omega^*)$ are jointly unitarily equivalent if and only if $E_{\boldsymbol{T}}$ and $E_{\boldsymbol{S}}$ are locally equivalent as holomorphic Hermitian vector bundles. This proof works for the case $m > 1$ as well.

Suppose $\mathscr{K} = \mathscr{K}(E_{\boldsymbol{T}}, D)$ is the curvature associated with canonical connection D of the holomorphic Hermitian vector bundle $E_{\boldsymbol{T}}$. Then relative to any C^∞ cross-section σ of $E_{\boldsymbol{T}}$, we have

$$\mathscr{K}(\sigma) = \sum_{i,j=1}^m \mathcal{K}^{i,j}(\sigma)\, dz_i \wedge d\bar{z}_j,$$

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where each $\mathcal{K}^{i,j}$ is a C^∞ cross-section of Hom$(E_{\boldsymbol{T}}, E_{\boldsymbol{T}})$. Let $\boldsymbol{\gamma}(z) = (\gamma_1(z), \ldots, \gamma_n(z))$ be a local holomorphic frame of $E_{\boldsymbol{T}}$ in a neighbourhood $\Omega_0^* \subset \Omega^*$ of some $w \in \Omega^*$. The metric of the bundle $E_{\boldsymbol{T}}$ at $z \in \Omega_0^*$ w.r.t the frame $\boldsymbol{\gamma}$ has the matrix representation

$$h_{\boldsymbol{\gamma}}(z) = \big(\!\!\big(\langle \gamma_j(z), \gamma_i(z)\rangle\big)\!\!\big)_{i,j=1}^n.$$

We write $\partial_i = \frac{\partial}{\partial z_i}$ and $\bar{\partial}_i = \frac{\partial}{\partial \bar{z}_i}$. The co-efficients of the curvature $(1,1)$-form $\mathscr{K}$ w.r.t the frame $\boldsymbol{\gamma}$, are explicitly determined by the formula:

$$\mathcal{K}^{i,j}(\boldsymbol{\gamma})(z) = -\bar{\partial}_j\Big((h_{\boldsymbol{\gamma}}(z))^{-1}\big(\partial_i h_{\boldsymbol{\gamma}}(z)\big)\Big),\ z \in \Omega_0^*.$$

Set $\mathcal{K}_{\boldsymbol{\gamma}}(z) = \big(\!\!\big(\mathcal{K}^{i,j}(\boldsymbol{\gamma})(z)\big)\!\!\big)$.

For a bounded domain Ω in $\mathbb{C}$ and for T in $B_n(\Omega^*)$, recall that $N_w^{(k)}$ is the restriction of the operator $(T - wI)$ to the subspace $\ker(T - wI)^{k+1}$. In general, even if $m = 1$, it is not possible to put the operator $N_w^{(k)}$ into any reasonable canonical form, see [4, sec. 2.19]. Here we show how to do this for any $m \in \mathbb{N}$, assuming that $k = 1$. The canonical form of the operator $N_w^{(1)}$, we find here, is a crucial ingredient in obtaining the curvature inequality for a commuting tuple of operator $\boldsymbol{T}$ in $B_n(\Omega^*)$, which admit $\overline{\Omega}^*$, the closure of Ω^*, as a spectral set.

A commuting m-tuple of operator $\boldsymbol{T}$ in $B_n(\Omega^*)$, may be realized as the m-tuple $\boldsymbol{M}^* = (M_{z_1}^*, \ldots, M_{z_m}^*)$, the adjoint of the multiplication by the m coordinate functions on some Hilbert space of holomorphic functions defined on Ω possessing a reproducing kernel K (cf. [4, 7]). The real analytic function $K(z,z)$ then serves as a Hermitian metric for the vector bundle E_T w.r.t. the holomorphic frame $\gamma_i(\bar{z}) := K(\cdot, z)e_i$, $i = 1, \ldots, n$, $\bar{z}$ in some open subset Ω_0^* of Ω^*. Here the vectors e_i, $i = 1, \ldots, n$, are the standard unit vectors of $\mathbb{C}^n$. For a point $z \in \Omega$, let $\mathscr{K}_T(\bar{z})$ be the curvature of the vector bundle E_T. It is easy to compute the co-efficients of the curvature $\mathscr{K}_T(\bar{z})$ explicitly using the metric $K(z,z)$ for $m = 1, n = 1$, namely,

$$\mathcal{K}_T^{i,j}(\bar{z}) = -\frac{\partial^2}{\partial w_i \partial \bar{w}_j}\log K(w,w)|_{w=z} = -\frac{\|K_z\|^2\,\langle \bar{\partial}_j K_z, \bar{\partial}_i K_z\rangle - \langle K_z, \bar{\partial}_i K_z\rangle\langle \bar{\partial}_j K_z, K_z\rangle}{(K(z,z))^2},\ \ z \in \Omega.$$

First, consider the case of $m = 1$. Assume that $\overline{\Omega}^*$ is a spectral set for an operator T in $B_1(\Omega^*)$, $\Omega \subset \mathbb{C}$. Thus for any rational function r with poles off $\overline{\Omega}^*$, we have $\|r(T)\| \leq \|r\|_{\Omega^*,\infty}$. For such operators T, the curvature inequality

$$\mathcal{K}_T(\bar{w}) \leq -4\pi^2(S_{\Omega^*}(\bar{w}, \bar{w}))^2, \quad \bar{w} \in \Omega^*,$$

where S_{Ω^*} is the Sżego kernel of the domain Ω^*, was established in [11]. Equivalently, since $S_\Omega(z,w) = S_{\Omega^*}(\bar{w}, \bar{z}),\ \ z, w \in \Omega$, the curvature inequality takes the form

$$\frac{\partial^2}{\partial w \partial \bar{w}}\log K(w,w) \geq 4\pi^2(S_\Omega(w,w))^2, \quad w \in \Omega. \tag{1.1}$$

Let us say that a commuting tuple of operators $\boldsymbol{T}$ in $B_n(\Omega^*)$, $\Omega \subset \mathbb{C}^m$, is *contractive* if $\overline{\Omega}^*$ is a spectral set for $\boldsymbol{T}$, that is, $\|f(\boldsymbol{T})\| \leq \|f\|_{\Omega^*,\infty}$ for all functions holomorphic in some neighborhood of $\overline{\Omega}^*$.

In this paper, see Theorem 2.4, we generalize the curvature inequality (1.1) for a contractive tuple of operators $\boldsymbol{T}$ in $B_n(\Omega^*)$, which include the earlier inequalities from [14, 13].

Let U_+ be the forward unilateral shift operator on $\ell^2(\mathbb{N})$. The adjoint U_+^* is the backward shift operator and is in $B_1(\mathbb{D})$. Let ds be the arc length measure on the unit circle of the complex plane and $(H^2(\mathbb{D}), ds)$ denotes the Hardy space. The unilateral shift U_+ is unitarily equivalent to the multiplication operator M on the Hardy space $(H^2(\mathbb{D}), ds)$. The reproducing kernel of the Hardy space is the Sżego kernel $S_{\mathbb{D}}(z,a)$ of the unit disc $\mathbb{D}$. It is given by the formula $S_{\mathbb{D}}(z,a) = \frac{1}{2\pi(1-z\bar{a})}$, $z, a \in \mathbb{D}$. A straightforward computation gives an explicit formula for the curvature $\mathcal{K}_{U_+^*}(w)$:

$$\mathcal{K}_{U_+^*}(w) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log S_{\mathbb{D}}(w,w) = -4\pi^2 (S_{\mathbb{D}}(w,w))^2, \quad w \in \mathbb{D}.$$

Since the closed unit disc is a spectral set for any contraction T (by von Neumann inequality), it follows, from equation (1.1), that the curvature of the operator U_+^* dominates the curvature of every other contraction T in $B_1(\mathbb{D})$,

$$\mathcal{K}_T(w) \leq \mathcal{K}_{U_+^*}(w) = -(1-|w|^2)^{-2}, \ \ w \in \mathbb{D}.$$

Thus the operator U_+^* is the *extremal operator* in the class of contractions in $B_1(\mathbb{D})$. The extremal property of the operator U_+^* prompts the following question due to R. G. Douglas.

Question 1.2 (R. G. Douglas)**.** For a contraction T in $B_1(\mathbb{D})$, if $\mathcal{K}_T(w_0) = -(1-|w_0|^2)^{-2}$ for some fixed w_0 in $\mathbb{D}$, then does it follow that T must be unitarily equivalent to the operator U_+^*?

It is known that the answer is negative, in general, however it has an affirmative answer if, for instance, T is a homogeneous contraction in $B_1(\mathbb{D})$, see [10]. From the simple observation that $\mathcal{K}_T(\bar{\zeta}) = -(1-|\zeta|^2)^{-2}$ for some $\zeta \in \mathbb{D}$ if and only if the two vectors $\tilde{K}_\zeta$ and $\bar{\partial}\tilde{K}_\zeta$ are linearly dependent, where $\tilde{K}_w(z) = (1-z\bar{w})K_w(z)$, it follows that the question of Douglas has an affirmative answer in the class of contractive, co-hyponormal backward weighted shifts. In this paper, we answer Question 1.2 for all those operators T in $B_1(\mathbb{D})$ possessing two additional properties, namely, T^* is 2 hyper-contractive and $(\phi(T))^*$ has the wandering subspace property for any bi-holomorphic automorphism ϕ of $\mathbb{D}$ mapping ζ to 0. This is Theorem 3.6 of this paper.

If the domain Ω is not simply connected, it is not known if there exists an extremal operator T in $B_1(\Omega^*)$, that is, if

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log K(w,w) = 4\pi^2 (S_\Omega(w,w))^2, \ \ w \in \Omega,$$

for some reproducing kernel K defined on $\Omega \times \Omega$, associated with a operator T in $B_1(\Omega^*)$ admitting $\overline{\Omega}^*$ as a spectral set . Indeed, from a result of Suita (cf. [18]), it follows that the adjoint of the multiplication operator on the Hardy space $(H^2(\Omega), ds)$ is not extremal. It was shown in [11] that for any fixed but arbitrary $w_0 \in \Omega$, there exists an operator T in $B_1(\Omega^*)$ for which equality is achieved, at $w = w_0$, in the inequality (1.1). The question of the uniqueness of such an operator was partially answered recently by the second named author in [16]. The precise result is that these "point-wise" extremal operators are determined uniquely within the class of the adjoint of the bundle shifts introduced in [1]. It was also shown in the same paper that each of these bundle shifts can be realized as a multiplication operator on a Hilbert space of weighted Hardy space and conversely. Generalizing these results, in this paper, we prove that the local extremal operators are uniquely determined in a much larger class of operators, namely, the ones that includes all the weighted Bergman spaces along with the weighted Hardy spaces defined on Ω. This is Theorem 5.1. The authors have obtained some preliminary results in the multi-variate case which are not included here.

2. Local operators and Generalized Curvature Inequality

Let Ω be a bounded domain in $\mathbb{C}^m$ and $\boldsymbol{T} = (T_1, T_2, \ldots, T_m)$ be a commuting m-tuple of bounded operators on some separable complex Hilbert space $\mathcal{H}$. Assume that the tuple of operator $\boldsymbol{T}$ is in $B_n(\Omega^*)$.

For an arbitrary but fixed point $w \in \Omega^*$, let

$$\mathcal{M}_w = \bigcap_{i,j=1}^{m} \ker(T_i - w_i)(T_j - w_j). \tag{2.1}$$

Clearly, the joint kernel $\cap_{i=1}^m \ker(T_i - w_i)$ is a subspace of $\mathcal{M}_w$. Fix a holomorphic frame $\boldsymbol{\gamma}$, defined on some neighborhood of w, say $\Omega_0^* \subseteq \Omega^*$, of the vector bundle $E_{\boldsymbol{T}}$. Thus $\boldsymbol{\gamma}(z) = (\gamma_1(z), \ldots, \gamma_n(z))$, for z in Ω_0^*, for some choice $\gamma_i(z)$, $i = 1, 2, \ldots, n$, of joint eigenvectors, that is, $(T_j - z_j)\gamma_i(z) = 0$, $j = 1, 2, \ldots, m$. It follows that

$$(T_j - w_j)(\partial_k \gamma_i(w)) = \gamma_i(w)\delta_{j,k}, \; i = 1, 2, \ldots, n, \text{ and } j, k = 1, \ldots, m. \tag{2.2}$$

The eigenvectors $\boldsymbol{\gamma}(w)$ together with their derivatives, that is $(\partial_1 \boldsymbol{\gamma}(w), \ldots, \partial_m \boldsymbol{\gamma}(w))$ is a basis for the subspace $\mathcal{M}_w$.

The metric of the bundle $E_{\boldsymbol{T}}$ at $z \in \Omega_0^*$ w.r.t the frame $\boldsymbol{\gamma}$ has the matrix representation

$$h_{\boldsymbol{\gamma}}(z) = \big(\!\big(\langle \gamma_j(z), \gamma_i(z) \rangle\big)\!\big)_{i,j=1}^n.$$

Clearly, $\tilde{\boldsymbol{\gamma}}(z) = (\gamma_1(z), \ldots, \gamma_n(z)) h_{\boldsymbol{\gamma}}(w)^{-1/2}$ is also a holomorphic frame for $E_{\boldsymbol{T}}$ with the additional property that $\tilde{\boldsymbol{\gamma}}$ is orthonormal at w, that is, $h_{\tilde{\boldsymbol{\gamma}}}(w) = I_n$. We therefore assume, without loss of generality, that $h_{\boldsymbol{\gamma}}(w) = I_n$.

In what follows, we always assume that we have made a fixed but arbitrary choice of a local holomorphic frame $\boldsymbol{\gamma}(z) = (\gamma_1(z), \ldots, \gamma_n(z))$ defined on a small neighborhood of w, say $\Omega_0^* \subseteq \Omega^*$, such that $h_{\boldsymbol{\gamma}}(w) = I_n$.

Recall that the local operator $\boldsymbol{N}_w = (N_1(w), \ldots, N_m(w))$ is the commuting m-tuple of nilpotent operators on the subspace $\mathcal{M}_w$ defined by $N_i(w) = (T_i - w_i)\,|_{\mathcal{M}_w}$. As a first step in relating the operator $\boldsymbol{T}$ to the vector bundle $E_{\boldsymbol{T}}$, pick a holomorphic frame $\boldsymbol{\gamma}$, satisfying $h_{\boldsymbol{\gamma}}(w) = I_n$, for the holomorphic Hermitian vector bundle $E_{\boldsymbol{T}}$ which also serves as a basis for the joint kernel of $\boldsymbol{T}$. We extend this basis to a basis of $\mathcal{M}_w$. In the following proposition, we determine a natural orthonormal basis in $\mathcal{M}_w$ such that the curvature of the vector bundle $E_{\boldsymbol{T}}$ appears in the matrix representation (obtained with respect to this orthonormal basis) of $\boldsymbol{N}_w$.

Proposition 2.1. *There exists an orthonormal basis in the subspace $\mathcal{M}_w$ such that the matrix representation of $N_l(w)$ with respect to this basis is of the form*

$$N_l(w) = \begin{pmatrix} 0_{n\times n} & \boldsymbol{t}_l(w) \\ 0_{mn\times n} & 0_{mn\times mn} \end{pmatrix}, \; \begin{pmatrix} \mathbf{t}_1(w) \\ \vdots \\ \mathbf{t}_m(w) \end{pmatrix} \left(\overline{\mathbf{t}_1(w)}^t, \ldots, \overline{\mathbf{t}_m(w)}^t\right) = \boldsymbol{t}(w)\overline{\boldsymbol{t}(w)}^{\mathrm{tr}} = -(\mathcal{K}_{\boldsymbol{\gamma}}^t(w))^{-1},$$

where $\boldsymbol{\gamma}$ is a frame of $E_{\boldsymbol{T}}$ defined in a neighborhood of w which is orthonormal at the point w and $\mathcal{K}_{\boldsymbol{\gamma}}^t(z) = \big(\!\big(\mathcal{K}^{j,i}(\boldsymbol{\gamma})(z)\big)\!\big)_{i,j=1}^m$.

Proof. For any $k = (p-1)n + q$, $1 \le p \le m+1$, and $1 \le q \le n$, set $v_k := \partial_{p-1}(\gamma_q(w))$ and $\mathbf{v}_i := \big(v_{(i-1)n+1}, \ldots, v_{(i-1)n+n}\big)$. Thus $\mathbf{v}_i$ is also $\partial_{i-1}\boldsymbol{\gamma}$, where $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$. Hence the set of vectors $\{v_k, \; 1 \le k \le (m+1)n\}$ forms a basis of the subspace $\mathcal{M}_w$. Let P be an invertible matrix of size $(m+1)n \times (m+1)n$ and

$$(\mathbf{u}_1, \ldots, \mathbf{u}_{m+1}) := (\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}) \begin{pmatrix} P_{1,1} & P_{1,2} & \cdots & P_{1,m+1} \\ P_{2,1} & P_{2,2} & \cdots & P_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ P_{m+1,1} & P_{m+1,2} & \cdots & P_{m+1,m+1} \end{pmatrix},$$

where each $P_{i,j}$ is a $n \times n$ matrix. Clearly, $(\mathbf{u}_1, \ldots, \mathbf{u}_{m+1})$ is a basis, not necessarily orthonormal, in the subspace $\mathcal{M}_w$. The set of vectors $\{\mathbf{u} = (\mathbf{u}_1, \ldots \mathbf{u}_{m+1})\}$ is an orthonormal basis in $\mathcal{M}_w$ if and only if $P\bar{P}^t = G^{-1}$, where G is the $(m+1)n \times (m+1)n$, grammian $\left(\!\left(\langle v_j, v_i\rangle\right)\!\right)$, that is,

$$G = \begin{pmatrix} h_{\boldsymbol{\gamma}}(w) & \partial_1 h_{\boldsymbol{\gamma}}(w) & \ldots & \partial_m h_{\boldsymbol{\gamma}}(w) \\ \bar{\partial}_1 h_{\boldsymbol{\gamma}}(w) & \bar{\partial}_1\partial_1 h_{\boldsymbol{\gamma}}(w) & \ldots & \bar{\partial}_1\partial_m h_{\boldsymbol{\gamma}}(w) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\partial}_m h_{\boldsymbol{\gamma}}(w) & \bar{\partial}_m\partial_1 h_{\boldsymbol{\gamma}}(w) & \ldots & \bar{\partial}_m\partial_m h_{\boldsymbol{\gamma}}(w) \end{pmatrix}.$$

In particular, we choose and fix P to be the upper triangular matrix corresponding to the Gram-Schmidt orthogonalization process. Following equation (2.2), the matrix representation of $N_l(w)$ w.r.t. the basis $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_{m+1})$ is $[N_l(w)]_{\mathbf{v}} = \left(\!\left(N_l(w)_{ij}\right)\!\right)$, $l = 1, 2, \ldots, m$, where

$$N_l(w)_{ij} = \begin{cases} 0_{n\times n} & (i,j) \neq (1, l+1) \\ I_n & (i,j) = (1, l+1) \end{cases}, \; 1 \leq i, j \leq m+1.$$

Therefore w.r.t the orthonormal basis $(\mathbf{u}_1, \ldots, \mathbf{u}_{m+1})$, the matrix of N_l is of the form

$$\text{(2.3)} \qquad \left[N_l(w)\right]_{\mathbf{u}} = \begin{pmatrix} 0_{n\times n} & t_l^1(w) & \ldots & t_l^m(w) \\ 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \end{pmatrix} = \begin{pmatrix} 0_{n\times n} & \mathbf{t}_l(w) \\ 0_{mn\times n} & 0_{mn\times mn} \end{pmatrix},$$

where each $t_l^i(w)$ is a square matrix of size n, for $l, i = 1, 2, \ldots, m$ and $\boldsymbol{t}_l(w)$ is a $n \times mn$ rectangular matrix. It is now evident that for $l, r = 1, 2, \ldots, m$, we have

$$\left[N_l(w)N_r(w)^*\right]_{\mathbf{u}} = Q\left[N_l(w)\right]_{\mathbf{v}} G^{-1}\left[N_r(w)\right]_{\mathbf{v}} \overline{Q}^t,$$

where $Q = P^{-1}$. To continue, we write the matrix G^{-1} in the form of a block matrix:

$$\text{(2.4)} \qquad G^{-1} = \begin{pmatrix} *_{n\times n} & *_{n\times n} & *_{n\times n} & \cdots & *_{n\times n} \\ *_{n\times n} & R_{1,1} & R_{1,2} & \ldots & R_{1,m} \\ *_{n\times n} & R_{2,1} & R_{2,2} & \ldots & R_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ *_{n\times n} & R_{m,1} & R_{m,2} & \ldots & R_{m,m} \end{pmatrix} = \begin{pmatrix} *_{n\times n} & *_{n\times mn} \\ *_{mn\times n} & R \end{pmatrix},$$

where each $R_{i,j}$ is a $n \times n$ matrix. Then we have

$$\left[N_l(w)N_r(w)^*\right]_{\mathbf{u}} = \begin{pmatrix} Q_{1,1}R_{l,r}\overline{Q}_{1,1}^t & 0_{n\times mn} \\ 0_{mn\times n} & 0_{mn\times mn} \end{pmatrix}.$$

Since P is upper triangular with $P_{1,1} = I_n$, we have $\mathbf{u}_1 = \mathbf{v}_1 P_{1,1} = \mathbf{v}_1$, that is,

$$(u_1, u_2, \ldots, u_n) \quad = \quad (v_1, v_2, \ldots, v_n).$$

As $P_{1,1} = I_n$, we have $Q_{1,1} = I_n$. Hence w.r.t the orthonormal basis $(\mathbf{u}_1, \ldots, \mathbf{u}_{m+1})$ of the subspace $\mathcal{M}_w$, the linear transformation $N_l(w)N_r(w)^*$ has the matrix representation

$$\text{(2.5)} \qquad [N_l(w)N_r(w)^*]_{\mathbf{u}} = \begin{pmatrix} R_{l,r} & 0_{n\times mn} \\ 0_{mn\times n} & 0_{mn\times mn} \end{pmatrix}.$$

Let $\boldsymbol{t}(w)$ be the $mn \times mn$ matrix given by

$$\boldsymbol{t}(w) = \begin{pmatrix} \mathbf{t}_1(w) \\ \mathbf{t}_2(w) \\ \vdots \\ \mathbf{t}_m(w) \end{pmatrix}.$$

Now combining equation (2.3) and equation (2.5), we then have

$$\boldsymbol{t}(w)\overline{\boldsymbol{t}(w)}^{\mathrm{tr}} = R. \tag{2.6}$$

To complete the proof, we have to relate the block matrix R to the curvature matrix $\mathcal{K}_{\boldsymbol{\gamma}}(w)$ w.r.t the frame $\boldsymbol{\gamma}$. Recalling (2.4), we have that

$$G^{-1} = \begin{pmatrix} *_{n\times n} & *_{n\times mn} \\ *_{mn\times n} & R \end{pmatrix}.$$

The Grammian G admits a natural decomposition as a 2×2 block matrix, namely,

$$G = \begin{pmatrix} h_{\boldsymbol{\gamma}}(w) & \partial_1 h_{\boldsymbol{\gamma}}(w) & \dots & \partial_m h_{\boldsymbol{\gamma}}(w) \\ \bar\partial_1 h_{\boldsymbol{\gamma}}(w) & \bar\partial_1\partial_1 h_{\boldsymbol{\gamma}}(w) & \dots & \bar\partial_1\partial_m h_{\boldsymbol{\gamma}}(w) \\ \vdots & \vdots & \ddots & \vdots \\ \bar\partial_m h_{\boldsymbol{\gamma}}(w) & \bar\partial_m\partial_1 h_{\boldsymbol{\gamma}}(w) & \dots & \bar\partial_m\partial_m h_{\boldsymbol{\gamma}}(w) \end{pmatrix} = \begin{pmatrix} h_{\boldsymbol{\gamma}}(w) & X_{n\times mn} \\ L_{mn\times n} & S_{mn\times mn} \end{pmatrix}.$$

Computing the 2×2 entry of the inverse of this block matrix and equating it to R, we have

$$\begin{aligned} R^{-1} &= S - L h_{\boldsymbol{\gamma}}(w)^{-1} X \\ &= \left(\!\!\left(\bar\partial_i\partial_j h_{\boldsymbol{\gamma}}(w)\right)\!\!\right)_{i,j=1}^m - \left(\!\!\left((\bar\partial_i h_{\boldsymbol{\gamma}}(w)) h_{\boldsymbol{\gamma}}(w)^{-1}(\partial_j h_{\boldsymbol{\gamma}}(w))\right)\!\!\right)_{i,j=1}^m \\ &= \left(\!\!\left(h_{\boldsymbol{\gamma}}(w)\bar\partial_i(h_{\boldsymbol{\gamma}}(w)^{-1}\partial_j h_{\boldsymbol{\gamma}}(w))\right)\!\!\right) \\ &= -\left(\!\!\left(h_{\boldsymbol{\gamma}}(w)\mathcal{K}^{j,i}(\boldsymbol{\gamma})(w)\right)\!\!\right) \end{aligned}$$

where $\left(\!\!\left(\mathcal{K}^{i,j}(\boldsymbol{\gamma})(w)\right)\!\!\right)_{i,j=1}^m$ denote the matrix of the curvature $\mathcal{K}$ at $w \in \Omega_0^*$ w.r.t the frame $\boldsymbol{\gamma}$ of the bundle $E_{\boldsymbol{T}}$ on Ω_0^* and $\mathcal{K}_{\boldsymbol{\gamma}}^t(w) = \left(\!\!\left(\mathcal{K}^{j,i}(\boldsymbol{\gamma})(w)\right)\!\!\right)_{i,j=1}^m$. Also, by our choice of the frame $\boldsymbol{\gamma}$ we have $h_{\boldsymbol{\gamma}}(w) = I_n$. Hence it follows that

$$\boldsymbol{t}(w)\overline{\boldsymbol{t}(w)}^{\mathrm{tr}} = R = -\left(\mathcal{K}_{\boldsymbol{\gamma}}^t(w)\right)^{-1}. \tag{2.7}$$

This completes the proof. □

The matrix representation of the operator $T_{i|\mathcal{M}_w}$ w.r.t. the orthonormal basis $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_{m+1})$ in the subspace $\mathcal{M}_w$ is of the form

$$\left[T_{i|\mathcal{M}_w}\right]_{\mathbf{u}} = \begin{pmatrix} w_i I_n & \boldsymbol{t}_i(w) \\ 0_{mn\times n} & w_i I_{mn} \end{pmatrix}, \ i = 1, \dots, m.$$

It is well known that the curvature $(1,1)$ form determines the local equivalence class of a holomorphic Hermitian vector bundle. Since the class of such vector bundles and those of commuting m - tuples of operators in $B_1(\Omega)$ are in one to one correspondence, one would expect to find a direct proof that the curvature determines the unitary equivalence class of these m - tuple of operators. Such proofs exist (see [4] for the case of $m = n = 1$, [6] for $m = 2, n = 1$ and finally, [12, Theorem 2.1] for arbitrary m but still $n = 1$). It shows that the curvature is indeed obtained from the holomorphic frame and the first order derivatives using the Gram-Schmidt orthonormalization. However, the relationship between the curvature invariant and the operator is not very direct if the rank of the vector bundle is not 1, see [4, section 2.19].

Never the less, using the description of the local operators $N_i(w) := \big[T_{i|\mathcal{M}_w}\big]_{\mathbf{u}}$, $1 \le i \le n$, we obtain the following theorem.

Theorem 2.2. *Suppose that two m-tuples of operators $\boldsymbol{T}$ and $\tilde{\boldsymbol{T}}$ in $\mathrm{B}_n(\Omega)$ are unitarily equivalent. Let $\boldsymbol{\gamma}$ (resp. $\tilde{\gamma}$) be a holomorphic frame for $E_{\boldsymbol{T}}$ (resp. $E_{\tilde{\boldsymbol{T}}}$). Assume, without loss of generality, that the frames $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ are orthonormal at $w \in \Omega$. Then the curvature $\mathcal{K}_{\boldsymbol{\gamma}}(w)$ is unitarily equivalent to $\mathcal{K}_{\tilde{\boldsymbol{\gamma}}}(w)$, $w \in \Omega$.*

Proof. Let $V = \bigcap_{i=1}^{m} \ker(T_i - w_i) \subseteq \mathcal{M}_w$. With respect to the decomposition $\mathcal{M}_w = V \oplus V^{\perp}$, the local operator $(T_i - w_i)_{|\mathcal{M}_w}$ is of the form:

$$[(T_i - w_i I)_{|\mathcal{M}_w}] = \begin{pmatrix} 0_{n\times n} & \mathbf{t}_i(w) \\ 0_{mn\times n} & 0_{mn\times mn} \end{pmatrix},\ i = 1, 2, \ldots, m,$$

where $\boldsymbol{t}_i(w)$ is a $n \times mn$ rectangular matrix, see Equation (2.3).

Suppose that $\boldsymbol{T}$ and $\tilde{\boldsymbol{T}}$ are unitarily equivalent via the unitary U. Since V and $\tilde{V}$ are joint eigenspaces of $\boldsymbol{T}$ and $\tilde{\boldsymbol{T}}$ respectively, U must maps V onto $\tilde{V}$. Thus the matrix representation of $U|_{\mathcal{M}_w}$ is of the form

$$[U|_{\mathcal{M}_w}] = \begin{pmatrix} A_{n\times n} & B_{n\times mn} \\ 0_{mn\times n} & C_{mn\times mn} \end{pmatrix}.$$

But $\mathcal{M}_w$ is a finite dimensional and $U|_{\mathcal{M}_w}$ is a unitary. Hence $B = 0$ and A, C are unitary. Since $UT_i = \tilde{T}_i U$, we have $A\mathbf{t}_i(w) = \tilde{\mathbf{t}}_i(w)C$, $i = 1, 2, \ldots, m$. It follows that

$$A\mathbf{t}_i(w)\overline{\mathbf{t}_j(w)}^{\mathrm{tr}}\bar{A}^{\mathrm{tr}} = \tilde{\mathbf{t}}_i(w)\overline{\tilde{\mathbf{t}}_j(w)}^{\mathrm{tr}}.$$

Let X be the block diagonal unitary matrix given by $X = A \otimes I_m = \mathrm{Diag}(A, \ldots, A)$. Finally, we have

$$X\boldsymbol{t}(w)\overline{\boldsymbol{t}(w)}^{\mathrm{tr}}\overline{X}^{\mathrm{tr}} = \tilde{\boldsymbol{t}}(w)\overline{\tilde{\boldsymbol{t}}(w)}^{\mathrm{tr}}.$$

Thus, using Equation (2.7), we conclude that the curvature $\mathcal{K}_{\boldsymbol{\gamma}}(w)$ is unitarily equivalent to $\mathcal{K}_{\tilde{\boldsymbol{\gamma}}}(w)$. □

Assume that the joint spectrum of the tuple $\boldsymbol{T}$ is contained in $\overline{\Omega}^*$. Then it follows that for any function $f \in \mathcal{O}(\overline{\Omega}^*)$, we have

$$\begin{aligned} f(\boldsymbol{T})_{|\mathcal{M}_w} &= f\big(\boldsymbol{T}_{|\mathcal{M}_w}\big) \\ &= \begin{pmatrix} f(w) & \bigtriangledown f(w)\cdot \boldsymbol{t}(w) \\ 0 & f(w) \end{pmatrix} = f(\boldsymbol{T}_w), \end{aligned}$$

where $\boldsymbol{T}_w$ is the m tuple of operator $\boldsymbol{T}_{|\mathcal{M}_w}$ and

$$\begin{aligned} \bigtriangledown f(w)\cdot \boldsymbol{t}(w) &= \partial_1 f(w)\mathbf{t}_1(w) + \cdots + \partial_m f(w)\mathbf{t}_m(w) \\ &= \big((\partial_1 f(w))I_n, \ldots, (\partial_m f(w))I_n\big)(\boldsymbol{t}(w)) \\ &= (I_n \otimes \bigtriangledown f(w))(\boldsymbol{t}(w)). \end{aligned}$$

From equation (2.7), we also have

$$\boldsymbol{t}(w)\overline{\boldsymbol{t}(w)}^{\mathrm{tr}} = -(\mathcal{K}^t_{\boldsymbol{\gamma}}(w))^{-1}. \tag{2.8}$$

As an application, it is easy to obtain a curvature inequality for those commuting tuples of operators $\boldsymbol{T}$ in the Cowen-Douglas class $B_n(\Omega^*)$ which admit $\overline{\Omega}^*$ as a spectral set. This is easily done via the holomorphic functional calculus.

If $\boldsymbol{T}$ admits $\overline{\Omega}^*$ as a spectral set, then the inequality $I - f(\boldsymbol{T}_w)^* f(\boldsymbol{T}_w) \geq 0$ is evident for all holomorphic functions mapping $\overline{\Omega}^*$ to the unit disc $\mathbb{D}$. As is well-known, we may assume without loss of generality that $f(w) = 0$. Consequently, the inequality $I - f(\boldsymbol{T}_w)^* f(\boldsymbol{T}_w) \geq 0$ with $f(w) = 0$ is equivalent to

$$\left(\overline{I_n \otimes \bigtriangledown f(w)}^{\mathrm{tr}}\right)(I_n \otimes \bigtriangledown f(w)) \leq -(\mathcal{K}^t_{\boldsymbol{\gamma}}(w)). \tag{2.9}$$

Let $V \in \mathbb{C}^{mn}$ be a vector of the form

$$V = \begin{pmatrix} V_1 \\ \vdots \\ V_m \end{pmatrix}, \text{ where } V_i = \begin{pmatrix} V_i(1) \\ \vdots \\ V_i(n) \end{pmatrix} \in \mathbb{C}^n.$$

Definition 2.3 (Carathéodory norm)**.** The Carathéodory norm of the (matricial) tangent vector $V \in \mathbb{C}^{mn}$ at a point z in Ω, is defined by

$$\begin{aligned}(C_{\Omega,z}(V))^2 &= \sup\big\{\langle\left(\overline{I_n \otimes \bigtriangledown f(z)}^{\mathrm{tr}}\right)(I_n \otimes \bigtriangledown f(z))V, V\rangle : f \in \mathcal{O}(\overline{\Omega}), \|f\|_\infty \leq 1, f(z) = 0\big\} \\ &= \sup\big\{\sum_{i,j=1}^m \overline{\partial_i f(z)}\partial_j f(z)\langle V_j, V_i\rangle : f \in \mathcal{O}(\overline{\Omega}), \|f\|_\infty \leq 1, f(z) = 0\big\} \\ &= \sup\big\{\|\sum_{j=1}^m \partial_j f(z) V_j\|^2_{\ell^2} : f \in \mathcal{O}(\overline{\Omega}), \|f\|_\infty \leq 1, f(z) = 0\big\}.\end{aligned}$$

Now we compute the Carathéodory norm of the tangent vector $V \in \mathbb{C}^{mn}$ in the case of Euclidean ball $\mathbb{B}^m$ and of polydisc $\mathbb{D}^m$. For a self map $g = (g_1, g_2, \ldots, g_m) : \Omega \to \Omega$ and

$$V = \begin{pmatrix} V_1 \\ \vdots \\ V_m \end{pmatrix},$$

let $g_*(z)(V)$ be the vector defined by

$$g_*(z)(V) = \begin{pmatrix} \sum_j \partial_j g_1(z) V_j \\ \vdots \\ \sum_j \partial_j g_m(z) V_j \end{pmatrix}.$$

From the definition of the Carathéodory norm, it follows that $C_{\Omega,g(z)}(g_*(z)(V)) \leq C_{\Omega,z}(V)$. In particular we have that $C_{\Omega,\varphi(z)}(\varphi_*(z)(V)) = C_{\Omega,z}(V)$ for any biholomorphic map φ of Ω. The group of biholomorphic automorphisms of both these domains $\mathbb{B}^m$ and $\mathbb{D}^m$ acts transitively. So, it is enough to compute $C_{\Omega,0}(V)$, since there is an explicit formula relating $C_{\Omega,z}(V)$ to $C_{\Omega,0}(V)$, $\Omega = \mathbb{B}^m$ or $\mathbb{D}^m$.

From the Schwarz lemma, it follows that the set $\big\{\bigtriangledown f(0) : f \in \mathcal{O}(\overline{\mathbb{B}^m}), \|f\|_\infty \leq 1, f(z) = 0\big\}$ is equal to the Euclidean unit ball $\mathbb{B}^m$ (cf. [13, Lemma 1.1]. Now for $a = (a_1, a_2, \ldots, a_m) \in \mathbb{B}^m$, note that

$$\|\sum_{j=1}^m a_j V_j\|^2_{\ell^2} = \sum_{i=1}^n |\sum_{j=1}^m a_j V_j(i)|^2 \leq \|a\|^2_{\ell^2} \sum_{i=1}^n \sum_{j=1}^m |V_j(i)|^2.$$

From this it follows that the Carathéodory norm of the tangent vector $V \in \mathbb{C}^{mn}$ at the point 0 in the case of the Euclidean ball $\mathbb{B}^m$ is equal to the Hilbert-Schmidt norm of V, that is, $\|V\|^2_{HS} = \sum_{i=1}^n \sum_{j=1}^m |V_j(i)|^2$. Similarly, in case of polydisc $\mathbb{D}^m$, we have $\big\{\bigtriangledown f(0) : f \in \mathcal{O}(\overline{\mathbb{D}^m}), \|f\|_\infty \leq 1, f(z) = 0\big\}$ is equal to the ℓ^1

unit ball of $\mathbb{C}^m$. For $a = (a_1, a_2, \ldots, a_m) : \|a\|_1 < 1$, we note that

$$\|\sum_{j=1}^{m} a_j V_j\|_{\ell^2} \leq \|a\|_{\ell^1} \max_j \|V_j\|_{\ell^2}.$$

Thus we conclude that the Carathéodory norm of the tangent vector $V \in \mathbb{C}^{mn}$ at the point 0, in the case of the polydisc $\mathbb{D}^m$, is equal to $\max\{\|V_j\|_{\ell^2} : 1 \leq j \leq m\}$. A more detailed discussion on such matricial tangent vectors V and the question ontractivity, complete contractivity of the homomorphism induced by them appears in [13].

From the definition of the Carathéodory norm and Equation (2.9), a proof of the theorem below follows.

Theorem 2.4. *Let $\boldsymbol{T}$ be a commuting tuple of operator in $B_n(\Omega)$ admitting $\bar{\Omega}^*$ as a spectral set. Then for an arbitrary but fixed point $w \in \bar{\Omega}^*$, there exist a frame $\boldsymbol{\gamma}$ of the bundle $E_{\boldsymbol{T}}$, defined in a neighborhood of w, which is orthonormal at w, so that following inequality holds*

$$\langle \mathcal{K}^t_{\boldsymbol{\gamma}}(w)V, V\rangle \leq -(C_{\Omega^*,w}(V))^2 \text{ for every } V \in \mathbb{C}^{mn}.$$

Now we derive a curvature inequality specializing to the case of a bounded planar domains Ω^*. Using techniques from Sz.-Nagy Foias model theory for contractions, Uchiyama [19], was the first one to prove a curvature inequality for operators in $B_n(\mathbb{D})$. To obtain curvature inequalities in the case of finitely connected planar domains Ω, he considered the contractive operator $F_w(T)$, where $F_w : \Omega \to \mathbb{D}$ is the Ahlfors map, $F_w(w) = 0$, for some fixed but arbitrary $w \in \Omega$. The curvature inequality then folllows from the equality $F'_w(w) = S_\Omega(w, w)$. However, the inequality we obtain below follows directly from the functional calculus applied to the local operators. More recently, K. Wang and G. Zhang (cf. [21]) have obtained a series of very interesting (higher order) curvature inequalities for operators in $B_n(\Omega)$.

In the case of bounded finitely connected planar domain with Jordan analytic boundary the carathéodory norm of the tangent vector $V \in \mathbb{C}^n$ at a point z in Ω is given by

$$\begin{aligned}(C_{\Omega,z}(V))^2 &= \sup\left\{|f'(z)|^2\langle V, V\rangle_{\ell^2} : f \in \mathcal{O}(\bar{\Omega}), \|f\|_\infty \leq 1, f(z) = 0\right\}\\ &= 4\pi^2 S_\Omega(z, z)^2\langle V, V\rangle_{\ell^2},\end{aligned}$$

(cf. [3, Theorem 13.1]) where $S_\Omega(z, z)$ denotes the Sz̈ego kernel for the domain Ω which satisfy

$$2\pi S_\Omega(z, z) = \sup\{|r'(z)| : r \in \text{Rat}(\bar{\Omega}), \|r\|_\infty \leq 1, r(z) = 0\}.$$

In consequence, we have the following.

Theorem 2.5. *Let T be a operator in $B_n(\Omega^*)$ admitting $\bar{\Omega}^*$ as a spectral set. Then for an arbitrary but fixed point $w \in \Omega^*$, there exist a frame γ of the bundle E_T, defined on a neighborhood of w, which is orthonormal at w, so that the following inequality holds*

$$\mathcal{K}_\gamma(w) \leq -(S_{\Omega^*}(w, w))^2 I_n, \text{ for every } V \in \mathbb{C}^n.$$

3. Curvature Inequality and The case of unit disc

Let T be an operator in $B_1(\mathbb{D})$ and $\mathcal{H}_K$ be an associated reproducing kernel Hilbert space so that operator T has been realized as M^* on the Hilbert space $\mathcal{H}_K$. Without loss of generality we can assume $K_w \neq 0$ for every $w \in \mathbb{D}$. Let $w_1, \ldots, w_n$ be n arbitrary points in $\mathbb{D}$ and $c_1, \ldots, c_n$ be arbitrary complex numbers. Using the reproducing property of K and the property that $M^*(K_{w_i}) = \bar{w}_i K_{w_i}$ we will have

$$\|M^*(\sum_{i,j=1}^{n} c_i K_{w_i})\|^2 = \sum_{i,j=1}^{n} w_i\bar{w}_j K(w_i, w_j)c_j\bar{c}_i\ ,\ \|\sum_{i,j=1}^{n} c_i K_{w_i})\|^2 = (\sum_{i,j=1}^{n} K(w_i, w_j)c_j\bar{c}_i.$$

Let $\tilde{K}(z,w)$ be the function $(1-z\bar{w})K(z,w)$, $z,w \in \mathbb{D}$. Now it is easy to see that the operator M^* on the Hilbert space $\mathcal{H}_K$ is a contraction if and only if $\tilde{K}$ is non-negative definite.

Lemma 3.1. *Let T be a contraction in $B_1(\mathbb{D})$ and $\mathcal{H}_K$ be an associated reproducing kernel Hilbert space. Then for an arbitrary but fixed $\zeta \in \mathbb{D}$, we have $\mathcal{K}_T(\bar{\zeta}) = -\frac{1}{(1-|\zeta|^2)^2}$ if and only if the vectors $\tilde{K}_\zeta, \bar{\partial}\tilde{K}_\zeta$ are linearly dependent in the Hilbert space $\mathcal{H}_{\tilde{K}}$.*

Proof. Assume $\mathcal{K}_{M^*}(\bar{\zeta}) = -\frac{1}{(1-|\zeta|^2)^2}$ for some $\zeta \in \mathbb{D}$. Contractivity of M^* gives us the function $\tilde{K} : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ defined by

$$\tilde{K}(z,w) = (1-z\bar{w})K(z,w) \quad z,w \in \mathbb{D},$$

is a non negative definite kernel function. Consequently there exist a reproducing kernel Hilbert space $\tilde{\mathcal{H}}$, consisting of complex valued function on $\mathbb{D}$ such that $\tilde{K}$ becomes the reproducing kernel for $\tilde{\mathcal{H}}$. Also note that $\tilde{K}(z,z) = (1-|z|^2)K(z,z) \neq 0$, for $z \in \mathbb{D}$ which gives us $\tilde{K}_z \neq 0$. Let ζ be an arbitrary but fixed point in $\mathbb{D}$. Now, it is straightforward to verify that $\mathcal{K}_T(\bar{\zeta}) = -\frac{1}{(1-|\zeta|^2)^2}$ if and only if $\frac{\partial^2}{\partial z\partial\bar{z}}\log\tilde{K}(z,z)|_{z=\zeta} = 0$. Since we have

$$\frac{\partial^2}{\partial z\partial\bar{z}}\log\tilde{K}(z,z)|_{z=\zeta} = -\frac{\|\tilde{K}_\zeta\|^2\|\bar{\partial}\tilde{K}_\zeta\|^2 - |\langle\tilde{K}_\zeta,\bar{\partial}\tilde{K}_\zeta\rangle|^2}{(\tilde{K}(\zeta,\zeta))^2},$$

Using Cauchy-Schwarz inequality, we see that the proof is complete. □

Remark 3.2. Let $e(w) = \frac{1}{\sqrt{2}}(\tilde{K}_w \otimes \bar{\partial}\tilde{K}_w - \bar{\partial}\tilde{K}_w \otimes \tilde{K}_w)$ for $w \in \mathbb{D}$. A straightforward computation shows that $\|e(w)\|^2_{\tilde{\mathcal{H}}\otimes\tilde{\mathcal{H}}} = \tilde{K}(w,w)^2\frac{\partial^2}{\partial z\partial\bar{z}}\log\tilde{K}(z,z)|_{z=w}$. Now if we define

$$F_K(z,w) := \langle e(z), e(w)\rangle_{\tilde{\mathcal{H}}\otimes\tilde{\mathcal{H}}} \text{ for } z,w \in \mathbb{D},$$

then clearly F_K is a non negative definite kernel function on $\mathbb{D}\times\mathbb{D}$. In view of this, we conclude that $\mathcal{K}_T(\bar{\zeta}) = -(1-|\zeta|^2)^{-2}$ if and only if $F_K(\zeta,\zeta) = 0$.

Proposition 3.3. *Let T be any contractive co-hyponormal unilateral backward weighted shift operator in $B_1(\mathbb{D})$. If $\mathcal{K}_T(w_0) = -(1-|w_0|^2)^{-2}$ for some $w_0 \in \mathbb{D}$, then the operator T is unitarily equivalent to U_+^*, the backward shift operator.*

Proof. Let T be a contraction in $B_1(\mathbb{D})$ and $\mathcal{H}_K$ be the associated reproducing kernel Hilbert space so that T is unitarily equivalent to the operator M^* on $\mathcal{H}_K$. By our hypothesis on T we have that operator M on $\mathcal{H}_K$ is a unilateral forward weighted shift. Without loss of generality, we may assume that the reproducing kernel K is of the form

$$K(z,w) = \sum_{n=0}^{\infty} a_n z^n \bar{w}^n, \quad z,w \in \mathbb{D};\ \text{where } a_n > 0 \ \text{ for all } \ n \geq 0.$$

By our hypothesis on the operator T, we have that the operator M on $\mathcal{H}_K$ is a contraction. So, the function $\tilde{K}$ defined by $\tilde{K}(z,w) = (1-z\bar{w})K(z,w)$ is a non negative definite kernel function. Consequently, following the Remark 3.2, the function $F_K(w,w)$ defined by $F_K(w,w) = \tilde{K}(w,w)^2\frac{\partial^2}{\partial z\partial\bar{z}}\log\tilde{K}(z,z)|_{z=w}$ is also non negative definite. The kernel $K(w,w)$ is a weighted sum of monomials $z^k\bar{w}^k$, $k = 0,1,2,\ldots$. Hence both $\tilde{K}(w,w)$ and $F_K(w,w)$ are also weighted sums of the same form. So, we have

$$F_K(w,w) = \sum_{n=0}^{\infty} c_n |w|^{2n},$$

for some $c_n \geq 0$. Now assume $\mathcal{K}_T(\bar{\zeta}) = -\frac{1}{(1-|\zeta|^2)^2}$ for some ζ in $\mathbb{D}$.

Case 1: If $\zeta \neq 0$, then following Remark 3.2, we have

$$F_K(\zeta,\zeta) = \sum_{n=0}^{\infty} c_n |\zeta|^{2n} = 0.$$

Thus $c_n = 0$ for all $n \geq 0$ since $c_n \geq 0$ and $|\zeta| \neq 0$. It follows that F_K is identically zero on $\mathbb{D} \times \mathbb{D}$, that is, $\frac{\partial^2}{\partial z \partial \bar{z}} \log \tilde{K}(z,z)|_{z=\bar{w}} = 0$ for all $w \in \mathbb{D}$. Hence

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log K(z,z)|_{z=\bar{w}} = \frac{\partial^2}{\partial z \partial \bar{z}} \log S_{\mathbb{D}}(z,z)|_{z=\bar{w}} \text{ for all } w \in \mathbb{D}.$$

Therefore, $\mathcal{K}_T(\bar{w}) = \mathcal{K}_{U_+^*}(\bar{w})$ for all $w \in \mathbb{D}$ making $T \cong U_+^*$.

Now let's discuss the remaining case, that is $\mathcal{K}_T(\bar{\zeta}) = -\frac{1}{(1-|\zeta|^2)^2}$, for $\zeta = 0 \in \mathbb{D}$.

Case 2: If $\zeta = 0$, then by Lemma 3.1, we have $\tilde{K}_0, \bar{\partial}\tilde{K}_0$ are linearly dependent. Now,

$$\tilde{K}(z,w) := (1 - z\bar{w})K(z,w) = \sum_{n=0}^{\infty} b_n z^n \bar{w}^n,$$

where $b_0 = a_0$ and $b_n = a_n - a_{n-1} \geq 0$, for all $n \geq 1$. Consequently, we have $\tilde{K}_0(z) \equiv b_0$ and $\bar{\partial}\tilde{K}_0(z) = b_1 z$. Now $\tilde{K}_0, \bar{\partial}\tilde{K}_0$ are linearly dependent if and only if $b_1 = 0$ that is $a_0 = a_1$.

Since $\{\sqrt{a_n} z^n\}_{n=0}^{\infty}$ is an orthonormal basis for the Hilbert space $\mathcal{H}_K$, the operator M on $\mathcal{H}_K$ is an unilateral forward weighted shift with weight sequence $w_n = \sqrt{\frac{a_n}{a_{n+1}}}$ for $n \geq 0$. So the curvature of M^* at the point zero equal to -1 if and only if $w_0 = \sqrt{\frac{a_0}{a_1}} = 1$. Now if we further assume M is hyponormal, that is, $M^*M \geq MM^*$, then the sequence w_n must be increasing. Also contractivity of M implies that $w_n \leq 1$. Therefore if $\mathcal{K}_{M^*}(0) = -1$ for some contractive hyponormal backward weighted shift M^* in $B_1(\mathbb{D})$, then it follows that $w_n = 1$ for all $n \geq 1$. Thus any such operator is unitarily equivalent to the backward unilateral shift U_+^* completing the proof of our claim. □

The proof of **Case 1** given above, actually proves a little more than what is stated in the proposition, which we record below as a separate Lemma.

Lemma 3.4. *Let T be any contractive unilateral backward weighted shift operator in $B_1(\mathbb{D})$. If $\mathcal{K}_T(w_0) = -(1-|w_0|^2)^{-2}$ for some $w_0 \in \mathbb{D}$, $w_0 \neq 0$, then the operator T is unitarily equivalent to U_+^*, the backward shift operator.*

Let T be a contraction in $B_1(\mathbb{D})$. Let a be a fixed but arbitrary point in $\mathbb{D}$ and ϕ_a be an automorphism of the unit disc taking a to 0. So, we have $\phi_a(z) = \beta(z-a)(1-\bar{a}z)^{-1}$ for some unimodular constant β. For any operator T in $B_1(\mathbb{D})$ and $w \in \mathbb{D}$, the operator $(T-w)$ is Fredholm and the index of $(T-w)$ is 1 by definition. Note that:

$$\begin{aligned}(1-\bar{a}w)(1-\bar{a}T)\big(\phi_a(T) - \phi_a(w)\big) &= \beta\Big((T-a)(1-\bar{a}w) - (w-a)(1-\bar{a}T)\Big) \\ &= \beta(1-|a|^2)(T-w), \;\; w \in \mathbb{D}.\end{aligned}$$

Thus the operator $(\phi_a(T) - \phi_a(w))$ is the product of the Fredholm operator $(T-w)$ of index 1 and the invertible operator $\beta(1-|a|^2)(1-\bar{a}w)^{-1}(1-\bar{a}T)^{-1}$, therefore, it is Fredholm with the same index as that of the operator $(T-w)$.

Also, if $v \in \ker(T-w)$, then for any polynomial p, $p(T)(v) = p(w)v$. Consequently, we have that $v \in \ker(\phi_a(T) - \phi_a(w))$. Hence $\ker(T-w) \subseteq \ker(\phi_a(T) - \phi_a(w))$. Since $\phi_a^{-1} \circ \phi_a(T) = T$, in a similar fashion we will have $\ker(\phi_a(T)-\phi_a(w)) \subseteq \ker(T-w)$. Thus we get that $\ker(\phi_a(T)-\phi_a(w)) = \ker(T-w)$. In consequence,

$$\vee_{w\in\mathbb{D}} \ker(\phi_a(T) - \phi_a(w)) = \vee_{w\in\mathbb{D}} \ker(T - w) = \mathcal{H},$$

which proves that $\phi_a(T)$ is in $B_1(\mathbb{D})$.

Let $\boldsymbol{\gamma}(w)$ be a frame for the associated bundle E_T of T so that $T(\gamma(w)) = w\gamma(w)$ for all $w \in \mathbb{D}$. Now it is easy to see that $\phi_a(T)(\gamma(w)) = \phi_a(w)\gamma(w)$ or equivalently $\phi_a(T)(\gamma\circ\phi_a^{-1}(w)) = w(\gamma\circ\phi_a^{-1}(w))$. So, $\gamma\circ\phi_a^{-1}(w)$ is a frame for the bundle $E_{\phi_a(T)}$ associated with $\phi_a(T)$. Hence the curvature $\mathcal{K}_{\phi_a(T)}(w)$ is equal to

$$\frac{\partial^2}{\partial w\partial\bar{w}}\log\|\gamma\circ\phi_a^{-1}(w)\|^2 = |\phi_a^{-1\prime}(w)|^2\frac{\partial^2}{\partial z\partial\bar{z}}\log\|\gamma(z)\|^2_{|z=\phi_a^{-1}(w)} = |\phi_a^{-1\prime}(w)|^2\mathcal{K}_T(\phi_a^{-1}(w)).$$

This leads us to the following transformation rule for the curvature

$$\mathcal{K}_{\phi_a(T)}(\phi_a(z)) = \mathcal{K}_T(z)|\phi_a'(z)|^{-2},\ \ z\in\mathbb{D}. \tag{3.1}$$

Since $|\phi_a'(a)| = (1-|a|^2)^{-1}$, in particular we have that

$$\mathcal{K}_{\phi_a(T)}(0) = \mathcal{K}_T(a)(1-|a|^2)^2. \tag{3.2}$$

Normalized kernel: Let T be an operator in $B_1(\Omega^*)$ and T has been realized as M^* on a reproducing kernel Hilbert Space $\mathcal{H}_K$ with non degenerate kernel function K. For any fixed but arbitrary $\zeta\in\Omega$, the function $K(z,\zeta)$ is non-zero in some neighborhood, say U, of ζ. The function $\varphi_\zeta(z) := K(z,\zeta)^{-1}K(\zeta,\zeta)^{1/2}$ is then holomorphic. The linear space $(\mathcal{H}, K_{(\zeta)}) := \{\varphi_\zeta f : f\in\mathcal{H}_K\}$ then can be equipped with an inner product making the multiplication operator M_{φ_ζ} unitary from $\mathcal{H}_K$ onto $(\mathcal{H}, K_{(\zeta)})$. It then follows that $(\mathcal{H}, K_{(\zeta)})$ is a space of holomorphic functions defined on $U\subseteq\Omega$, it has a reproducing kernel $K_{(\zeta)}$ defined by

$$K_{(\zeta)}(z,w) = K(\zeta,\zeta)K(z,\zeta)^{-1}K(z,w)\overline{K(w,\zeta)}^{-1},\ \ z,w\in U,$$

with the property $K_{(\zeta)}(z,\zeta) = 1, z\in U$ and finally the multiplication operator M on $\mathcal{H}_K$ is unitarily equivalent to the multiplication operator M on $(\mathcal{H}, K_{(\zeta)})$. The kernel $K_{(\zeta)}$ is said to be normalized at ζ.

The realization of an operator T in $B_1(\Omega^*)$ as the adjoint of the multiplication operator on $\mathcal{H}_K$ is not canonical. However, the kernel function K is determined upto conjugation by a holomorphic function. Consequently, one sees that the curvature $\mathcal{K}_K$ is unambiguously defined. On the other hand, Curto and Salinas (cf. [7, Remarks 4.7 (b)]) prove that the multiplication operators M on two Hilbert spaces $(\mathcal{H}, K_{(\zeta)})$ and $(\hat{\mathcal{H}}, \hat{K}_{(\zeta)})$ are unitarily equivalent if and only if $K_{(\zeta)} = \hat{K}_{(\zeta)}$ in some small neighbourhood of ζ. Thus the normalized kernel at ζ, that is, $K_{(\zeta)}$ is also unambiguously defined. It follows that the curvature and the normalized kernel at ζ serve equally well as a complete unitary invariant for the operator T in $B_1(\Omega^*)$.

To answer Question 1.2, we have to impose two additional conditions on the operator T. These are not too restrictive. However, we don't know if the second of these two conditions follows from the other hypothesis.

First, let us recall the definition of 2 hyper-contraction (cf. [2]). An operator A acting on a Hilbert space $\mathcal{H}$ is said to be 2 hyper-contraction if $I - A^*A \geq 0$ and $A^{*2}A^2 - 2A^*A + I \geq 0$. For example every contractive subnormal operator is a 2 hyper-contraction (cf. [2, Theorem 3.1]). The following lemma will be very useful in establishing our next result.

Lemma 3.5. *Let A be a 2 hyper-contraction and φ be a bi holomorphic automorphism of unit disc $\mathbb{D}$. Then $\varphi(A)$ is also a 2 hyper-contraction.*

Proof. Let A be a 2 hyper-contraction. Let φ be the automorphism of the unit disc $\mathbb{D}$ given by $\varphi(z) = \lambda\frac{z-a}{1-\bar{a}z}$ for some unimodular constant λ and $a \in \mathbb{D}$. So $\varphi(A) = \lambda(A-a)(1-\bar{a}A)^{-1}$. Since A is a contraction, using von-Neuman's inequality we have $\varphi(A)$ is also a contraction. Thus

$$\begin{aligned}
&\varphi(A)^{*2}\varphi(A)^2 - 2\varphi(A)^*\varphi(A) + I \\
&= (1-aA^*)^{-2}\Big\{(A^*-\bar{a})^2(A-a)^2 - 2(1-aA^*)(A^*-\bar{a})(A-a)(1-\bar{a}A) \\
&\qquad\qquad + (1-aA^*)^2(1-\bar{a}A)^2\Big\}(1-\bar{a}A)^{-2} \\
&= (1-aA^*)^{-2}\Big\{(A^*-\bar{a})^2(A-a)^2 - (A^*-\bar{a})(1-aA^*)(1-\bar{a}A)(A-a) \\
&\qquad\qquad -(1-aA^*)(A^*-\bar{a})(A-a)(1-\bar{a}A) + (1-aA^*)^2(1-\bar{a}A)^2\Big\}(1-\bar{a}A)^{-2} \\
&= (1-aA^*)^{-2}\Big\{(A^*-\bar{a})\{(A^*-\bar{a})(A-a) - (1-aA^*)(1-\bar{a}A)\}(A-a) \\
&\qquad\qquad -(1-aA^*)\{(A^*-\bar{a})(A-a) - (1-aA^*)(1-\bar{a}A)\}(1-\bar{a}A)\Big\}(1-\bar{a}A)^{-2} \\
&= (1-aA^*)^{-2}\Big\{(A^*-\bar{a})(A^*A-1)(1-|a|^2)(A-a) \\
&\qquad\qquad -(1-aA^*)(A^*A-1)(1-|a|^2)(1-\bar{a}A)\Big\}(1-\bar{a}A)^{-2} \\
&= (1-aA^*)^{-2}(1-|a|^2)\Big\{(A^*-\bar{a})(A^*A-1)(A-a) \\
&\qquad\qquad -(1-aA^*)(A^*A-1)(1-\bar{a}A)\Big\}(1-\bar{a}A)^{-2} \\
&= (1-aA^*)^{-2}(1-|a|^2)\Big\{(1-|a|^2)(A^{*2}A^2 - 2A^*A + I)\Big\}(1-\bar{a}A)^{-2} \\
&= (1-aA^*)^{-2}(1-|a|^2)(A^{*2}A^2 - 2A^*A + I)(1-|a|^2)(1-\bar{a}A)^{-2}.
\end{aligned}$$

Since A is a 2 hyper-contration, it follows that $\varphi(A)$ is also a 2 hyper-contraction. □

Second, recall that an operator A in $B(\mathcal{H})$ is said to have wandering subspace property if the linear span of $\{A^n(\ker A^*) : n \in \mathbb{Z}_+\}$ is dense in $\mathcal{H}$ (cf. [17].) The following theorem provides a partial answer to Question 1.2.

Theorem 3.6. *Fix an arbitrary point $\zeta \in \mathbb{D}$. Let T be an operator in $B_1(\mathbb{D})$ such that T^* is a 2 hyper-contraction. Suppose that the operator $(\phi_\zeta(T))^*$ has the wandering subspace property for an automorphism ϕ_ζ of the unit disc $\mathbb{D}$ mapping ζ to 0. If $\mathcal{K}_T(\zeta) = -(1-|\zeta|^2)^{-2}$, then T must be unitarily equivalent to U_+^*, the backward shift operator.*

Proof. Let T be an operator in $B_1(\mathbb{D})$ such that the adjoint T^* is a 2 hyper-contraction and $(\phi_\zeta(T))^*$ have wandering subspace property for an automorphism ϕ_ζ of the unit disc $\mathbb{D}$ mapping ζ into 0. Let P be the operator $\phi_\zeta(T)$. We have seen that P is in $B_1(\mathbb{D})$ and from Lemma 3.5, it follows that the adjoint P^* is a 2 hyper-contraction. Now assume $\mathcal{K}_T(\zeta) = -(1-|\zeta|^2)^{-2}$. Following (3.2), we see that $\mathcal{K}_P(0) = -1$.

Without loss of generality, we assume that P is unitarily equivalent to the operator M^* acting on the reproducing kernel Hilbert space $\mathcal{H}_K$, where the kernel function K is normalized at 0. Since $M^* \in B_1(\mathbb{D})$, we have $\ker M^* = \{aK(\cdot, 0) : a \in \mathbb{C}\}$. As K is normalized at 0, that is, $K(z, 0) = 1$ for all z in some neighborhood of 0, we have $\ker M^* = \mathbb{C}$. By our assumption, P^* has the wandering subspace property. As the operator M on $\mathcal{H}_K$ is unitarily equivalent to P^*, the operator M on $\mathcal{H}_K$ also has the wandering subspace property. Thus polynomials are dense in $\mathcal{H}_K$.

Now we claim that $\bar{\partial} K(\cdot, 0) = z$. As $\mathcal{H}_K$ consists of holomorphic function, for any $f \in \mathcal{H}_K$, we have

$$f(z) = \sum_{j=1}^{\infty} a_j z^j, \text{ where, } a_j = \frac{f^{(j)}(0)}{j!} = \langle f, \frac{\bar{\partial}^j K(\cdot, 0)}{j!} \rangle.$$

Let $V_j = \frac{\bar{\partial}^j K(\cdot,0)}{j!}$. To prove $V_1 = \bar{\partial} K(\cdot, 0) = z$, it is sufficient to show that $\langle V_1, V_j \rangle = 0$ for all $j \geq 0$, except $j = 1$. First note that since $K(z, 0) = 1 = K(0, z)$, we have $\bar{\partial} K(0, 0) = 0$. It follows that $\langle V_1, V_0 \rangle = 0$. Since K is normalized at 0, we also have $\mathcal{K}_P(0) = -\partial\bar{\partial} K(0, 0) = -\|V_1\|^2$. Hence we find that $\|V_1\|^2 = 1$. Now to show $\langle V_1, V_j \rangle = 0$ for $j \geq 1$, we need the following lemma.

Lemma 3.7. *Let V and W be two finite dimensional inner product space and $A : V \to W$ be a linear map. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for V and G_v, (resp. G_{Av}) be the grammian $\big(\langle v_j, v_i \rangle_V\big)$ (resp. $\big(\langle Av_j, Av_i \rangle_W\big)$). The linear map A is a contraction if and only if $G_{Av} \leq G_v$.*

Proof. Let $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ be an arbitrary element in V. Then the easy verification that $\|Ax\|_W^2 \leq \|x\|_V^2$ is equivalent $\langle G_{Av} c, c \rangle \leq \langle G_v c, c \rangle$ completes the proof. ☐

As $(M^* - \bar{w})K(\cdot, w) = 0$, it is easily verified that $(M^* - \bar{w})\frac{\bar{\partial} K^j(\cdot,w))}{j!} = \frac{\bar{\partial} K^{j-1}(\cdot,w))}{(j-1)!}$ for all $j \geq 1$. So, we have $M^*(V_j) = V_{j-1}$ for $j \geq 1$ and $M^*(V_0) = 0$. We also have $\|M^*\| \leq 1$. Now applying the lemma 3.7 to the set of vector $\{V_0, V_1, \ldots, V_n\}$ we get that

$$\begin{pmatrix} \langle V_0, V_0 \rangle & \langle V_1, V_0 \rangle & \cdots & \langle V_n, V_0 \rangle \\ \langle V_0, V_1 \rangle & \langle V_1, V_1 \rangle & \cdots & \langle V_n, V_1 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle V_0, V_n \rangle & \langle V_1, V_n \rangle & \cdots & \langle V_n, V_n \rangle \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \langle V_0, V_0 \rangle & \cdots & \langle V_{n-1}, V_0 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \langle V_0, V_{n-1} \rangle & \cdots & \langle V_{n-1}, V_{n-1} \rangle \end{pmatrix} \geq 0.$$

Since $\|V_0\|^2 = K(0, 0) = 1$ and $\|V_1\|^2 = 1$, $(2, 2)$ entry of left hand side is 0. As left hand is a positive semidefinite matrix, this gives that 2nd row and 2nd column must be identically zero (for positive semidefinite matrix B, $\langle Be_2, e_2 \rangle = 0$ gives $\sqrt{B} e_2 = 0$, and hence $Be_2 = 0$.) Consequently we get that $\langle V_j, V_1 \rangle = \langle V_{j-1}, V_0 \rangle$ for all $j = 2, \ldots, n$. But as $K(z, 0) = 1 = K(0, z)$, it follows that $\bar{\partial}^k K(0, 0) = \langle V_k, V_0 \rangle = 0$ for all $k \geq 1$. Hence we get that $\langle V_j, V_1 \rangle = 0$ for all $j \geq 2$. Hence we get that $V_1 = \bar{\partial} K(\cdot, 0) = z$ and $\|z\|^2 = \|V_1\|^2 = 1$. We also have $V_0 = K(\cdot, 0) = 1$ with $\|1\|^2 = \|V_0\|^2 = K(0, 0) = 1$.

By our assumption, the operator M on $\mathcal{H}_K$ is a 2 hyper-contraction. In particular M is also a contraction and $\|1\|_{\mathcal{H}_K} = 1$. Hence we have $\|z^n\|_{\mathcal{H}_K} \leq 1$, for all $n \geq 1$. Since M on $\mathcal{H}_K$ is a 2 hyper-contraction, that is, $I - 2M^*M + M^{*2}M^2 \geq 0$, equivalently, $\|f\|_{\mathcal{H}_K}^2 - 2\|zf\|_{\mathcal{H}_K}^2 + \|z^2 f\|_{\mathcal{H}_K}^2 \geq 0$, for all $f \in \mathcal{H}_K$. Since $\|1\| = \|z\| = 1$, taking $f = 1$, we have $\|z^2\| \geq 1$. But we also have $\|z^2\| \leq 1$, which gives us $\|z^2\| = 1$. Inductively, by choosing $f = z^k$, we obtain $\|z^{k+2}\| = 1$ for every $k \in \mathbb{N}$. Hence we see that $\|z^n\| = 1$ for all $n \geq 0$.

We use Lemma 3.7 to show that $\{z^n \mid n \geq 0\}$ is an orthonormal set in the Hilbert space $\mathcal{H}_K$, Consider the two subspace V and W of $\mathcal{H}_K$, defined by $V = \vee\{1, z, \cdots, z^k\}$ and $W = \vee\{z, z^2, \cdots, z^{k+1}\}$. Since

M is a contraction, applying the lemma we have just proved, it follows that the matrix B defined by

$$B = \left(\langle z^j, z^i\rangle\right)_{i,j=0}^{k} - \left(\langle z^{j+1}, z^{i+1}\rangle\right)_{i,j=0}^{k}$$

is positive semi-definite. But we have $\|z^i\| = 1$, for all $i \geq 0$. Consequently, each diagonal entry of B is zero. Hence $tr(B) = 0$. Since B is positive semi-definite, it follows that $B = 0$. Therefore, $\langle z^j, z^i\rangle = \langle z^{j+1}, z^{i+1}\rangle$ for all $0 \leq i, j \leq k$. We have $K_0(z) \equiv 1$. So, $M^*1 = M^*(K_0) = 0$. From this it follows that for any $k \geq 1$, we have $\langle z^k, 1\rangle = \langle z^{k-1}, M^*1\rangle = 0$. This together with $\langle z^j, z^i\rangle = \langle z^{j+1}, z^{i+1}\rangle$ for all $0 \leq i, j \leq k$, inductively shows that $\langle z^j, z^i\rangle = 0$ for every $i \neq j$. Hence $\{z^n \mid n \geq 0\}$ forms an orthonormal set.

Since polynomials are dense in $\mathcal{H}_K$, the set of vectors $\{z^n \mid n \geq 0\}$ forms an orthonormal basis for $\mathcal{H}_K$. Hence the multiplication operator M on $\mathcal{H}_K$ is unitarily equivalent to U_+, the unilateral forward shift operator. Consequently, P is unitarily equivalent to U_+^*. But U_+^* being a homogeneous operator, we have U_+^* is unitarily equivalent to $\phi_\zeta^{-1}(U_+^*)$ (cf. [10]). Hence, we infer that $T = \phi_\zeta^{-1}(P)$ is unitarily equivalent to U_+^*. □

Corollary 3.8. *Let T be an operator in $B_1(\mathbb{D})$. Assume that T^* is a 2 hyper-contraction and that $(\phi(T))^*$ has the wandering subspace property for every automorphism ϕ of the unit disc $\mathbb{D}$. If $\mathcal{K}_T(\zeta) = -(1-|\zeta|^2)^{-2}$ for an arbitrary but fixed point ζ in $\mathbb{D}$, then T must be unitarily equivalent to U_+^*, the backward shift operator.*

4. Bergman bundle shifts

Let Ω be a finitely connected bounded domain in the complex plane $\mathbb{C}$ whose boundary consist of $n+1$ analytic Jordan curves. Let dv be the Lebesgue area measure in the complex plane $\mathbb{C}$ and ds be the arc length measure on the boundary $\partial\Omega$ of the domain Ω. For a positive continuous function h on Ω which is integrable w.r.t the area measure dv, the weighted Bergman space $(\mathbb{A}^2(\Omega), hdv)$ consists of all holomorphic function f on Ω satisfying $\|f\|_h^2 = \int_\Omega |f(z)|^2 h(z) dv(z) < \infty$. In this section we study the operator M of multiplication by the coordinate function on the weighted Bergman space $(\mathbb{A}^2(\Omega), hdv)$.

Notation 4.1. Let $\mathfrak{h} = \{h : h$ is a positive continuous integrable (w.r.t area measure) function on $\Omega\}$ and similarly let $\hat{\mathfrak{h}} = \{\hat{h} : \hat{h}$ is a positive continuous function on $\partial\Omega\}$. Finally, let $\mathcal{F}_1, \mathcal{F}_2$ be the class of operator defined by $\mathcal{F}_1 = \{M$ on $(\mathbb{A}^2(\Omega), hdv) : h \in \mathfrak{h}\}$ and $\mathcal{F}_2 = \{M$ on $(H^2(\Omega), \hat{h}ds) : \hat{h} \in \hat{\mathfrak{h}}\}$. Set $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$.

It was shown in [16] that the class of operators in $\mathcal{F}_2$ include the bundle shifts introduced in [1]. We conclude this section by showing that the class $\mathcal{F}_1$ includes all the Bergman bundle shifts of rank 1 introduced in [8]. Let $\mathcal{G}$ be the class of operators contained in $\mathcal{F}$ defined by $\mathcal{G} = \{M$ on $(\mathbb{A}^2(\Omega), hdv)$: $\log h$ is harmonic on $\overline{\Omega}\}$. After recalling the definition of of Bergman bundle shift (cf. [8]), we proceed to establish the existence of a surjective map from $\mathcal{G}$ onto the class of Bergman bundle shift of rank 1.

Let $\pi : \mathbb{D} \to \Omega$ be a holomorphic covering map. Bergman bundle shifts is realized as a multiplication operator on a certain subspace of the weighted Bergman space $(\mathbb{A}^2(\mathbb{D}), |\pi'(z)|^2 dv(z))$. Let G denote the group of deck transformation associated to the map π that is $G = \{A \in \mathrm{Aut}(\mathbb{D}) \mid \pi \circ \mathrm{A} = \pi\}$. Let α be a character, that is, $\alpha \in \mathrm{Hom}(G, \mathbb{S}^1)$. A holomorphic function f on unit disc $\mathbb{D}$ satisfying $f \circ A = \alpha(A) f$, for all $A \in G$, is called a modulus automorphic function of index α. Now consider the following subspace of the weighted Bergman space $(\mathbb{A}^2(\mathbb{D}), |\pi'(z)|^2 dv(z))$ which consists of modulus automorphic function of index α, namely

$$\mathbb{A}^2(\mathbb{D}, \alpha) = \{f \in (\mathbb{A}^2(\mathbb{D}), |\pi'(z)|^2 dv(z)) \mid f \circ A = \alpha(A) f, \text{ for all } A \in G\}$$

Let T_α be the multiplication operator by the covering map π on the subspace $\mathbb{A}^2(\mathbb{D}, \alpha)$. The operator T_α is called a Bergman bundle shift of rank 1 associated to the character α.

Like the Hardy bundle shift (cf. [1]) there is another way to realize the Bergman bundle shift as a multiplication operator M on a Hilbert space of multivalued holomorphic function defined on Ω with the property that its absolute value is single valued. A multivalued holomorphic function defined on Ω with the property that its absolute value is single valued is called a multiplicative function. Every modulus automorphic function f on $\mathbb{D}$ induce a multiplicative function on Ω, namely, $f \circ \pi^{-1}$. Converse is also true (cf. [20, Lemma 3.6]). We define the class $\mathbb{A}^2_\alpha(\Omega)$ consisting of multiplicative function in the following way:

$$\mathbb{A}^2_\alpha(\Omega) := \{f \circ \pi^{-1} \mid f \in \mathbb{A}^2(\mathbb{D}, \alpha)\}$$

So the linear space $\mathbb{A}^2_\alpha(\Omega)$ consists of those multiple valued function h on Ω for which $|h|$ is single valued, $|h|^2$ is integrable w.r.t area measure dv on ω and h is locally holomorphic in the sense that each point $w \in \Omega$ has a neighborhood U_w and a single valued holomorphic function g_w on U_w with the property $|g_w| = |h|$ on U_w (cf. [9, p.101]). It follows that the linear space $\mathbb{A}^2_\alpha(\Omega)$ endowed with the norm

$$\|f\|^2 = \int_\Omega |f(z)|^2 dv(z),$$

is a Hilbert space. We denote it by $\big(\mathbb{A}^2_\alpha(\Omega), dv\big)$. In fact the map $f \mapsto f \circ \pi^{-1}$ is a unitary map from $\mathbb{A}^2(\mathbb{D}, \alpha)$ onto $\big(\mathbb{A}^2_\alpha(\Omega), dv\big)$ which intertwine the multiplication by π on $\mathbb{A}^2(\mathbb{D}, \alpha)$ and the multiplication by coordinate function M on $\big(\mathbb{A}^2_\alpha(\Omega), dv\big)$. Thus the multiplication operator M on $\big(\mathbb{A}^2_\alpha(\Omega), dv\big)$ is also called Bergman bundle shift of rank 1.

Let h be a positive function on $\overline{\Omega}$ with $\log h$ harmonic on $\overline{\Omega}$. Now we show that the the multiplication operator M on the weighted Bergman space $(\mathbb{A}^2(\Omega), hdv)$ is unitarily equivalent to a Bergman bundle shift T_α for some character α. In this realization, it is not hard to see that all the Bergman bundle shift of rank 1 are in the same similarity class. First note that as h is both bounded above and below. So, there exist positive constants p, q such that $0 < p \leq h(z) \leq q$ for all $z \in \overline{\Omega}$. Consequently, we have

$$p\|\cdot\|_1 \leq \|\cdot\|_h \leq q\|\cdot\|_1.$$

Thus the norms on weighted Bergman space $(\mathbb{A}^2(\Omega), h\,dv)$ is equivalent to the norm on the Bergman space $(\mathbb{A}^2(\Omega), dv)$. It follows that the identity map is an invertible operator between these two Hilbert spaces and intertwines the associated multiplication operator. This shows that every operator in the class $\mathcal{G}$ is similar to the multiplication operator M on the Bergman space $(\mathbb{A}^2(\Omega), dv)$.

The following lemma is the essential step in proving the existence of a bijective map from $\mathcal{G}$ to the class of Bergman bundle shift of rank 1.

Lemma 4.2. *Let h be a positive function on $\overline{\Omega}$ such that $\log h$ is harmonic on $\overline{\Omega}$, then there exist a function F in $H^\infty_\gamma(\Omega)$ for some character γ such that $|F|^2 = h$ on Ω. In fact F is invertible in the sense that there exist G in $H^\infty_{\gamma^{-1}}(\Omega)$ so that $FG = 1$ on Ω. Furthermore, given any character γ there exists a positive function h on $\overline{\Omega}$ such that $\log h$ is harmonic on $\overline{\Omega}$ and $h = |F|^2$ on Ω for some F in $H^\infty_\gamma(\Omega)$.*

Proof. The proof of the first half of the lemma follows using techniques similar to the ones used in the proof of Lemma 2.4 of [16], therefore, we omit the proof here.

For the proof of the second half of the lemma, recall that there exist functions $\omega_j(z)$ which is harmonic in Ω and has the boundary values 1 on $\partial\Omega_j$ and is 0 on all the other boundary components, for each $j = 1, 2, \ldots, n$. Since the boundary of Ω consists of Jordan analytic curves, we have that the functions $\omega_j(z)$ is also harmonic on $\overline{\Omega}$. Let $p_{i,j}$ be the periods of the harmonic function ω_j around the boundary component $\partial\Omega_i$, that is,

$$p_{i,j} = -\int_{\partial\Omega_i} \frac{\partial}{\partial \eta_z}\big(\omega_j(z)\big) ds_z, \quad \text{for } i,j = 1,2,...,n$$

The negative sign appears in the equation as it is assumed that $\partial\Omega$ is positively oriented, that is, the boundary components $\partial\Omega_j$, $j = 1, 2, \ldots, n$, except the outer one, namely $\partial\Omega_{n+1}$, are oriented in clockwise direction. So the period of the harmonic function $u(z) = a_1\omega_1(z) + a_2\omega_2(z) + \cdots + a_n\omega_n(z)$ around the boundary component $\partial\Omega_i$, is equal to $\sum_j p_{i,j}\alpha_j$. It is well known that the $n \times n$ period matrix $\big(p_{i,j}\big)$ is positive definite and hence invertible (cf. [15, Sec.10,Ch 1]). Thus it follows that for any n- tuple of real number, say $(b_1, b_2, \ldots, b_n)$ we have a harmonic function u on $\overline{\Omega}$ such that its period around boundary component $\partial\Omega_i$, is equal to b_i. Let g be the positive function on $\overline{\Omega}$ defined by $g(z) = \exp(2u(z)),\ z \in \overline{\Omega}$. Now following the first part of the lemma, we have that there exists a F in $H^\infty_\gamma(\Omega)$ such that $|F|^2 = g$ on $\overline{\Omega}$. Furthermore the character γ is determined by

$$\gamma_j = \exp(ib_j),\ \text{for}\, j = 1, 2, \ldots, n.$$

As this is true for arbitrary n- tuple of real number $(b_1, b_2, \ldots, b_n)$, the result follows. □

As a consequence of the previous lemma we have the following theorem.

Theorem 4.3. *There is a bijective correspondence between the multiplication operators on the weighted Bergman spaces $\mathcal{G}$ and the bundle shifts in $\mathcal{B}$.*

Proof. Let h be a positive function on $\overline{\Omega}$ such that $\log h$ is harmonic on $\overline{\Omega}$. We see that there is a F in $H^\infty_\gamma(\Omega)$ with $|F|^2 = h$ on Ω and a G in $H^\infty_{\gamma^{-1}}(\Omega)$ with $|G|^2 = h^{-1}$ on Ω. Now consider the map M_F : $\big(\mathbb{A}^2(\Omega), hdv\big) \to \big(\mathbb{A}^2_\gamma(\Omega), dv\big)$, defined by the equation

$$M_F(g) = Fg, \quad g \in \big(\mathbb{A}^2(\Omega), hdv\big).$$

Clearly, M_F is a unitary operator and its inverse is the operator M_G. The multiplication operator M_F intertwines the corresponding operator of multiplication by the coordinate function on the Hilbert spaces $\big(\mathbb{A}^2(\Omega), hdv\big)$ and $\big(\mathbb{A}^2_\gamma(\Omega), dv\big)$. The character γ is determined by $\gamma_j(h) = \exp(ic_j(h))$, where $c_j(h)$ is given by

$$c_j(h) = -\int_{\partial\Omega_j} \frac{\partial}{\partial \eta_z}\big(\frac{1}{2}\log h(z)\big) ds_z, \quad \text{for } j = 1,2,...,n.$$

Conversely, following second part of the lemma 4.2, for any character γ there exist a positive function h on $\overline{\Omega}$ such that $\log h$ is harmonic on $\overline{\Omega}$ and $h = |F|^2$ on $\overline{\Omega}$ for some function F in $H^\infty_\gamma(\Omega)$. Thus we have established a surjective map from the class $\mathcal{G} = \{M \text{ on} \, (\mathbb{A}^2(\Omega), hdv)\text{: } \log h \text{ is harmonic on } \overline{\Omega}\}$ onto the class $\mathcal{B}$ of Bergman bundle shifts of rank 1, namely, the multiplication operators M on $\big(\mathbb{A}^2_\gamma(\Omega), dv\big)$, where γ is in $\text{Hom}(\pi_1(\Omega), \mathbb{S}^1)$. □

Also, the following corollary is an immediate consequence of [8, Theorem 18].

Corollary 4.4. *Let h_1, h_2 be two positive function on $\overline{\Omega}$. Suppose that $\log h_i$, $i = 1, 2$, are harmonic on $\overline{\Omega}$. Then the operator M on $(\mathbb{A}^2(\Omega), h_1 dv)$ is unitarily equivalent to the operator M on $(\mathbb{A}^2(\Omega), h_2 dv)$ if and only if $\gamma_j(h_1) = \gamma_j(h_2)$ for $j = 1, 2, \ldots, n$.*

5. Curvature Inequality in the case of finitely connected domain

Let h be a positive continuous function on Ω which is integrable w.r.t the Lebesgue area measure dv on Ω. Consider the weighted Bergman space $(\mathbb{A}^2(\Omega), hdv)$. For any compact set $C \subset \Omega$, the function h being bounded below on C. It follows that evaluation at any fixed but arbitrary point in Ω is a locally uniformly bounded linear map on $(\mathbb{A}^2(\Omega), hdv)$. Consequently, $(\mathbb{A}^2(\Omega), hdv)$ is a reproducing kernel Hilbert space. Let $K(z,w)$ be the kernel function for $(\mathbb{A}^2(\Omega), hdv)$. Clearly, the multiplication operator M by co-ordinate function on $(\mathbb{A}^2(\Omega), hdv)$ is a subnormal operator and $\bar{\Omega}$ is a spectral set for M. In this section we will establish the following strict curvature inequality.

$$\partial_z \bar{\partial}_z \log K(z,z)|_{z=w} > 4\pi^2 S(w,w)^2.$$

Let w be an arbitrary but fixed point in Ω. Let $\mathcal{M}_w$ be the closed convex set in $\mathcal{H} = (\mathbb{A}^2(\Omega), hdv)$ defined by $\mathcal{M}_w = \{f \in \mathcal{H} : f(w) = 0, f'(w) = 1\}$. Consider the following extremal problem

$$\inf\{\|f\|^2 : f \in \mathcal{M}_w\}.$$

Let $\mathcal{E}_w$ be the subspace of $\mathcal{H}$ defined by

$$\mathcal{E}_w = \{f \in \mathcal{H} : f(w) = 0, f'(w) = 0\}.$$

Since $f + g \in \mathcal{M}_w$, whenever $f \in \mathcal{M}_w$ and $g \in \mathcal{E}_w$, It is evident that the unique function F which solves the extremal problem must belong to $\mathcal{E}_w^\perp$. From the reproducing property of K, it follows that

$$f(w) = \langle f, K(\cdot,w)\rangle, \;\; f'(w) = \langle f, \bar{\partial} K(\cdot,w)\rangle.$$

Consequently, we have $\mathcal{E}_w^\perp = \vee\{K(\cdot,w), \bar{\partial}K(\cdot,w)\}$. A solution to the extremal problem mentioned above can be found in terms of the kernel function as in [11]:

$$\inf\,\{\|f\|^2 : f \in M_w\} = \left\{K(w,w)\left(\frac{\partial^2}{\partial z \partial \bar{z}} \log K(z,z)|_{z=w}\right)\right\}^{-1}.$$

Now consider the function g in $\mathcal{H}$ defined by

$$g(z) := \frac{K_w(z) F_w(z)}{2\pi S(w,w) K(w,w)}, \quad z \in \Omega,$$

where $F_w(z) = \frac{S_w(z)}{L_w(z)}$ denotes the Ahlfors map for the domain Ω at the point w (cf. [3, Theorem 13.1]). Note that $|F_w(z)| < 1$ on Ω and $|F_w(z)| \equiv 1$ on $\partial\Omega$. As $g \in \mathcal{H}$, we have the inequality

$$\begin{aligned}
\left\{K(w,w)\left(\frac{\partial^2}{\partial z \partial \bar{z}} \log K(z,z)|_{z=w}\right)\right\}^{-1} &\leq \|g\|^2 \\
&= \frac{1}{4\pi^2 S(w,w)^2 K(w,w)^2} \int_\Omega |F_w(z)|^2 |K(z,w)|^2 h(z) dv(z) \\
&< \frac{1}{4\pi^2 S(w,w)^2 K(w,w)^2} \int_\Omega |K(z,w)|^2 h(z) dv(z), \\
&= \frac{1}{4\pi^2 S(w,w)^2 K(w,w)},
\end{aligned}$$

where the last but one strict inequality follows from the inequality $|F_w(z)| < 1$ on Ω. Hence we have $\partial_z \bar{\partial}_z \log K(z,z)|_{z=w} > 4\pi^2 S(w,w)^2$, which is the strict curvature inequality. We obtain the uniqueness of the extremal operator within the class $\mathcal{F}$, defined in Section 4, by combining this with Theorem 2.6 of [16].

Theorem 5.1. *Let ζ be an arbitrary but fixed point in Ω and T be an operator in $B_1(\Omega^*)$. Assume that the adjoint T^* (upto unitary equivalence) is in $\mathcal{F}$. Then $\mathcal{K}_T(\bar{\zeta}) \leq -4\pi^2 S_\Omega(\zeta,\zeta)^2$, equality occurs for a unique operator, upto unitary equivalence.*

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