Flag structure for operators in the Cowen-Douglas class

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Abstract

The explicit description of homogeneous operators and localization of a Hilbert module naturally leads to the definition of a class of Cowen-Douglas operators possessing a flag structure. These operators are irreducible. We show that the flag structure is rigid in the sense that the unitary equivalence class of the operator and the flag structure determine each other. We obtain a complete set of unitary invariants which are somewhat more tractable than those of an arbitrary operator in the Cowen-Douglas class.

La description explicite des oprateurs homognes et la localisation d'un module de Hilbert conduit naturellement la dfinition d'une classe des oprateurs Cowen-Douglas possdant une structure flag. Ces oprateurs sont irrductibles. Nous montrons que la structure flag est rigide en ce sens que la classe d'quivalence unitaire de l'oprateur et de la structure du pavillon de dterminer une de l'autre. Nous obtenons un ensemble complet d'invariants unitaires qui sont un peu plus docile que ceux d'un oprateur arbitraire dans la classe Cowen-Douglas.

Keywords: The Cowen-Douglas class, strongly irreducible, homogeneous operator, curvature, second fundamental form

The Cowen-Douglas class $B_n(\Omega)$ consists of those bounded linear operators T on a complex separable Hilbert space $\mathcal{H}$ which possess an open set $\Omega \subset \mathbb{C}$ of eigenvalues of constant multiplicity n and admit a holomorphic choice of eigenvectors : $s_1(w), \ldots, s_n(w)$, $w \in \Omega$, in other words, there exists holomorphic functions $s_1, \ldots, s_n : \Omega \to \mathcal{H}$ which span the eigenspace of T at $w \in \Omega$.

The holomorphic choice of eigenvectors $s_1, \ldots, s_n$ defines a holomorphic Hermitian vector bundle E_T via the map

$$s : \Omega \to \mathrm{Gr}(n, \mathcal{H}),\ \ s(w) = \ker(T - w) \subseteq \mathcal{H}.$$

In the paper [3], Cowen and Douglas show that there is a one to one correspondence between the unitary equivalence classes of the operators T in $B_n(\Omega)$ and the equivalence classes of the holomorphic Hermitian vector bundles E_T determined by them. They also find a set of complete invariants for this equivalence

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consisting of the curvature $\mathcal{K}$ of E_T and a certain number of its covariant derivatives. Unfortunately, these invariants are not easy to compute unless n is 1. Also, it is difficult to determine, in general, when an operator T in $B_n(\Omega)$ is irreducible, again except in the case $n = 1$. In the latter case, the rank of the vector bundle is 1 and therefore it is irreducible and so is the operator T.

Finding similarity invariants for operators in the class $B_n(\Omega)$ has been somewhat difficult from the beginning. Counter examples to the similarity conjecture in [3] were given in [1, 2]. More recently, significant progress on the question of similarity has been made (cf. [6, 8, 9]).

We isolate a subset of irreducible operators in the Cowen-Douglas class $B_n(\Omega)$ for which a complete set of tractable unitary invariants is relatively easy to identify. We also determine when two operators in this class are similar.

We introduce below this smaller class $\mathcal{F}B_2(\Omega)$ of operators in $B_2(\Omega)$ leaving out the more general definition of the class $\mathcal{F}B_n(\Omega)$, $n > 2$, for now.

Definition 1. *We let $\mathcal{F}B_2(\Omega)$ denote the set of all bounded operators T on some Hilbert space $\mathcal{H}$, which satisfy one of the following equivalent conditions.*

1. *The operator T admits a decomposition of the form $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ for some choice of operators T_0, T_1 in $B_1(\Omega)$ with $T_0 S = S T_1$.*
2. *The operator T is in $B_2(\Omega)$. There exists a frame $\{\gamma_0, \gamma_1\}$ of the vector bundle E_T such that $\gamma_0(w)$ and $t_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - \gamma_1(w)$ are orthogonal for all w in Ω.*
3. *The operator T is in $B_2(\Omega)$. There exists a frame $\{\gamma_0, \gamma_1\}$ of the vector bundle E_T such that $\frac{\partial}{\partial w}\|\gamma_0(w)\|^2 = \langle \gamma_1(w), \gamma_0(w)\rangle$, $w \in \Omega$.*

It follows, from the definition, that $\mathcal{F}B_2(\Omega) \subseteq B_2(\Omega)$. Any operator T in $B_2(\Omega)$ admits a decomposition of the form $\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ for some pair of operators T_0 and T_1 in $B_1(\Omega)$ (cf. [9, Theorem 1.49, pp. 48]). In defining the new class $\mathcal{F}B_2(\Omega)$, we are merely imposing one additional condition, namely that $T_0 S = S T_1$. Our first main theorem on unitary classification is given below, where we have set $\mathcal{K}_{T_0}(z) = -\frac{\partial^2}{\partial z \partial \bar{z}} \log \|\gamma_0(z)\|^2$.

Theorem 1. *Let $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ and $\tilde{T} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}$ be two operators in $\mathcal{F}B_2(\Omega)$. Also let t_1 and $\tilde{t}_1$ be non-vanishing sections of the holomorphic Hermitian vector bundles E_{T_1} and $E_{\tilde{T}_1}$ respectively. The operators T and $\tilde{T}$ are equivalent if and only if $\mathcal{K}_{T_0} = \mathcal{K}_{\tilde{T}_0}$ (or $\mathcal{K}_{T_1} = \mathcal{K}_{\tilde{T}_1}$) and $\frac{\|S(t_1)\|^2}{\|t_1\|^2} = \frac{\|\tilde{S}(\tilde{t}_1)\|^2}{\|\tilde{t}_1\|^2}$.*

In any decomposition $\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$, of an operator $T \in \mathcal{F}B_2(\Omega)$, let t_1 be a non-vanishing section of the holomorphic Hermitian vector bundle E_{T_1}. We assume, without loss of generality, that $S(t_1)$ is a non-vanishing section of E_{T_0} on some open subset of Ω. Following the methods of [7, pp. 2244], the second fundamental form of E_{T_0} in E_T is easy to compute. It is the $(1,0)$-form $\frac{\mathcal{K}_{T_0}(z)}{\left(-\mathcal{K}_{T_0}(z) + \frac{\|t_1(z)\|^2}{\|S(t_1(z))\|^2}\right)^{1/2}} d\bar{z}$. Thus the second fundamental form of E_{T_0} in E_T together with the curvature of E_{T_0} is a complete set of invariants for the operator T. The inclusion of the line bundle E_{T_0} in the vector bundle E_T of rank 2 is the flag structure of E_T.

Proposition 1. *The operators in the class $\mathcal{F}B_2(\Omega)$ are irreducible. Furthermore, if S is invertible, then T is strongly irreducible, that is, there is no non-trivial idempotent commuting with T.*

Recall that an operator T in the Cowen-Douglas class $B_n(\Omega)$, up to unitary equivalence, is the adjoint of the multiplication operator M on a Hilbert space $\mathcal{H}$ consisting of holomorphic functions on $\Omega^* := \{\bar{w} : w \in \Omega\}$ possessing a reproducing kernel K. A model for operators in $\mathcal{F}B_2(\Omega)$ is given in the Proposition following the discussion below.

Let $\gamma = (\gamma_0, \gamma_1)$ be a holomorphic frame for the vector bundle E_T, $T \in \mathcal{F}B_2(\Omega)$. Then the operator T is unitarily equivalent to the adjoint of the multiplication operator M on a reproducing kernel Hilbert space $\mathcal{H}_\Gamma \subseteq \mathrm{Hol}(\Omega^*, \mathbb{C}^2)$ possessing a reproducing kernel $K_\Gamma : \Omega^* \times \Omega^* \to \mathbb{C}^{2\times 2}$ of the special form that we describe explicitly now. For $z, w \in \Omega^*$,

$$
\begin{aligned}
K_\Gamma(z,w) &= \begin{pmatrix} \langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle & \langle \gamma_1(\bar{w}), \gamma_0(\bar{z})\rangle \\ \langle \gamma_0(\bar{w}), \gamma_1(\bar{z})\rangle & \langle \gamma_1(\bar{w}), \gamma_1(\bar{z})\rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle & \frac{\partial}{\partial \bar{w}} \langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle \\ \frac{\partial}{\partial z}\langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle & \frac{\partial^2}{\partial z \partial \bar{w}}\langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle + \langle t_1(\bar{w}), t_1(\bar{z})\rangle \end{pmatrix},
\end{aligned}
$$

where t_1 is a non-vanishing section of the line bundles E_{T_1} and $S(t_1)$ is a non-vanishing section of E_{T_0}. It follows that $t_1(w)$ is orthogonal to $\gamma_0(w)$, $w \in \Omega$ and that $\{\frac{\partial}{\partial w}\gamma_0(w) - t_1(w), \gamma_0(w)\}$ is a holomorphic frame for the bundle E_T.

Proposition 2. *An operator in the class $\mathcal{F}B_2(\Omega)$, upto unitary equivalence, is the adjoint of the multiplication operator M on a Hilbert space of holomorphic functions on Ω taking values in $\mathbb{C}^2$ and possessing a reproducing kernel K_Γ of the form :*

$$
K_\Gamma(z,w) = \begin{pmatrix} K_0(z,w) & \frac{\partial}{\partial \bar{w}} K_0(z,w) \\ \frac{\partial}{\partial z} K_0(z,w) & \frac{\partial^2}{\partial z \partial \bar{w}} K_0(z,w) + K_1(z,w) \end{pmatrix},
$$

where $K_0(z,w) = \langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle$ and $K_1(z,w) = \langle t_1(\bar{w}), t_1(\bar{z})\rangle$.

This special form of the kernel K_Γ for an operator in the class $\mathcal{F}B_2(\Omega)$ entails that a change of frame between any two frames $\{\gamma_0, \gamma_1\}$ and $\{\sigma_0, \sigma_1\}$ of the vector bundle E_T, which have the property $\gamma_0 \perp (\partial\gamma_0 - \gamma_1)$ and $\sigma_0 \perp (\partial\sigma_0 - \sigma_1)$, must be induced by a holomorphic $\Phi : \Omega \to \mathbb{C}^{2\times 2}$ of the form $\Phi = \left(\begin{smallmatrix} \phi & \phi' \\ 0 & \phi \end{smallmatrix}\right)$ for some holomorphic function $\phi : \Omega \to \mathbb{C}$. As an immediate corollary, we see that an unitary operator intertwining two of these operators, represented in the form $T := \left(\begin{smallmatrix} T_0 & S \\ 0 & T_1 \end{smallmatrix}\right)$ and $\tilde{T} := \left(\begin{smallmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{smallmatrix}\right)$, must be diagonal with respect to the implicit decomposition of the two Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$. As a second corollary, we see that if $T_0 = \tilde{T}_0$ and $T_1 = \tilde{T}_1$, then the operators T and $\tilde{T}$ are unitarily equivalent if and only if $\tilde{S} = e^{i\theta} S$ for some real θ.

We now give examples of natural classes of operators that belong to $\mathcal{F}B_2(\Omega)$. Indeed, we were led to the definition of this new class $\mathcal{F}B_2(\Omega)$ of operators by trying to understand these examples better.

Definition 2. *An operator T is called* homogeneous *if $\phi(T)$ is unitarily equivalent to T for all ϕ in Möb which are analytic on the spectrum of T.*

If an operator T is in $B_1(\mathbb{D})$, then T is homogeneous if and only if $\mathcal{K}_T(w) = -\lambda(1-|w|^2)^{-2}$ for some $\lambda > 0$. The similarity and unitary classifications of homogeneous operators in $B_n(\mathbb{D})$ were obtained in [10] using non-trivial results from representation theory of semi-simple Lie groups. A model for homogeneous operators in $B_n(\mathbb{D})$ is also given in that paper. Homogeneous operators in $B_2(\mathbb{D})$, upto unitary equivalence, are listed in the following proposition (cf. [10]).

Proposition 3. *(i) Every irreducible homogeneous operator T in $B_2(\mathbb{D})$ belongs to $\mathcal{F}B_2(\mathbb{D})$.*

(ii) Such an operator T, up to unitary equivalence, may be realized as the adjoint of the multiplication operator on a Hilbert space possessing the reproducing kernel K_Γ, where $K_0(z,w) = (1-z\bar{w})^{-\lambda}$ and $K_1(z,w) = \mu(1-z\bar{w})^{-\lambda-2}$ for some $\lambda > 1$ and $\mu > 0$.

(iii) The pair $\{\lambda, \mu\}$ is a set of complete unitary invariants for these operators

Theorem 1 provides a direct verification that the operators listed in Proposition 3 is a complete (upto unitary equivalence) list of homogeneous operators in $B_2(\mathbb{D})$.

Definition 3. *Let $\mathcal{F}B_n(\Omega)$ be the set of all operators T in the Cowen-Douglas class $B_n(\Omega)$ for which there exists operators $T_0, T_1, \ldots, T_{n-1}$ in $B_1(\Omega)$ and a decomposition of the form*

$$T = \begin{pmatrix} T_0 & S_{0\,1} & S_{0\,2} & \ldots & S_{0\,n-1} \\ 0 & T_1 & S_{1\,2} & \ldots & S_{1\,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & T_{n-2} & S_{n-2\,n-1} \\ 0 & \ldots & \ldots & 0 & T_{n-1} \end{pmatrix}$$

such that none of the operators $S_{i\,i+1}$ are zero and $T_i S_{i\,i+1} = S_{i\,i+1} T_{i+1}$.

In the following theorem, we describe the nature of intertwining invertible (resp. unitary) operators between any two operators in the class $\mathcal{F}B_n(\Omega)$.

Theorem 2. *Suppose $T, \tilde{T}$ are two operators in $\mathcal{F}B_n(\Omega)$ and that there exists an invertible bounded linear operator X such that $XT = \tilde{T}X$. Then X must be upper triangular with respect to the decomposition mandated in the definition of the class $\mathcal{F}B_n(\Omega)$. Moreover, if X is unitary, then it must be diagonal with respect to this decomposition.*

Thus we see that the two operators T and $\tilde{T}$ are unitarily equivalent if and only if there exists unitary operators $U_i : \mathcal{H}_i \to \tilde{\mathcal{H}}_i$, $i = 0, 1, \cdots n-1$, such that $U_i^* \widetilde{T}_i U_i = T_i$ and $U_i S_{i,j} = \widetilde{S}_{i,j} U_j$, $i < j$. This provides a list of unitary invariants, not necessarily complete, for operators in the class $\mathcal{F}_n(\Omega)$.

For an operator T in $\mathcal{F}B_n(\Omega)$, pick a holomorphic section t_{n-1} for the line bundle $E_{T_{n-1}}$ corresponding to the operator T_{n-1} (in $B_1(\Omega)$) appearing in the decomposition of T. Set $t_{i-1} = S_{i-1\,i}(t_i)$, $i = n-1, \ldots, 1$.

Theorem 3. *Let T and $\tilde{T}$ be two operators in $\mathcal{F}B_n(\Omega)$. If T is unitarily equivalent to $\tilde{T}$ then*

$$\mathcal{K}_{T_0} = \mathcal{K}_{\tilde{T}_0}, \frac{\|t_{i-1}\|}{\|t_i\|} = \frac{\|\tilde{t}_{i-1}\|}{\|\tilde{t}_i\|},\ i = 1, \ldots, n-1,$$

and

$$\|S_{k\,l}(t_l)\| = \frac{\|t_0\|}{\|\tilde{t}_0\|} \|\tilde{S}_{k\,l} \tilde{t}_l\|,\ 0 \le k \le n-3,\ 2 \le l \le n-1.$$

The first set of conditions in the theorem imply that $\mathcal{K}_{T_i} = \mathcal{K}_{\tilde{T}_i}$, $i = 0, \ldots, n-1$. They are therefore unitary invariants for the operator T. The second set of conditions are somewhat more mysterious and is related to a finite number of second fundamental forms inherent in our description of the operator T. In what follows, we make this a little more explicit after making some additional assumptions. With these somewhat more restrictive assumptions, we obtain a complete set of unitary invariants.

Let T be an operator acting on a Hilbert space $\mathcal{H}$. Assume that there exists a representation of the form

$$T = \begin{pmatrix} T_0 & S_{0\,1} & 0 & \ldots & 0 \\ 0 & T_1 & S_{1\,2} & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & T_{n-2} & S_{n-2\,n-1} \\ 0 & \ldots & 0 & 0 & T_{n-1} \end{pmatrix} \tag{1}$$

for the operator T with respect to some orthogonal decomposition $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{n-1}$. Suppose also that the operator T_i is in $B_1(\Omega)$, $0 \le i \le n-1$, the operator $S_{i-1,i}$ is non zero and $T_{i-1} S_{i-1,i} = S_{i-1,i} T_i$, $1 \le i \le n-1$. Then we show that the operator T must be in the Cowen-Douglas class $B_n(\Omega)$. We can also relate the frame of the vector bundle E_T to those of the line bundles E_{T_i}, $i = 0, 1, \ldots, n-1$. Indeed, we show that there is a frame $\{\gamma_0, \gamma_1, \cdots, \gamma_{n-1}\}$ of E_T such that

$$t_i(w) := \gamma_i(w) + \cdots + \frac{1}{j!} \gamma_{i-j}^{(j)}(w) + \cdots + \frac{1}{i!} \gamma_0^{(i)}(w)$$

is a non-vanishing section of the line bundle E_{T_i} and it is orthogonal to $\gamma_i(w)$, $i = 0, 1, 2, \ldots, k-1$. We also have $t_{i-1} := S_{i-1\,i}(t_i)$, $1 \le i \le n-1$. In this special case, we can extract a complete set of invariants explicitly.

Theorem 4. *Pick two operators T and $\tilde{T}$ which admit a decomposition of the form given in* (1)*. Find an orthogonal frame $\{\gamma_0, t_1, \cdots, t_{n-1}\}$ (resp. $\{\tilde{\gamma}_0, \tilde{t}_1, \cdots, \tilde{t}_{n-1}\}$) for the vector bundle $\bigoplus_{i=0}^{n} E_{T_i}$ (resp. $\bigoplus_{i=0}^{n} E_{\tilde{T}_i}$) as above. Then the operators T and $\tilde{T}$ are unitarily equivalent if and only if*

$$\mathcal{K}_{T_0} = \mathcal{K}_{\tilde{T}_0} \text{ and } \frac{\|S_{i-1\,i}(t_i)\|^2}{\|t_i\|^2} = \frac{\|\tilde{S}_{i-1\,i}(\tilde{t}_i)\|^2}{\|\tilde{t}_i\|^2},\ 1 \leq i \leq n-1.$$

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