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CURVATURE INEQUALITIES AND EXTREMAL PROPERTIES OF BUNDLE SHIFTS

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INTRODUCTION

We study a class of Hilbert space operators $B_1(\Omega)$ possessing an open set of eigenvalues Ω. The eigenspaces of an operator T in $B_1(\Omega)$ form a complex holomorphic hermitian line bundle over Ω defined as $E_T = \{(\omega, x) \to \omega : (\omega, x) \in \Omega \times \mathscr{H}$ and $Tx = \omega x\}$. One reason for studying the class $B_1(\Omega)$ is the following theorem due to Cowen and Douglas [8].

THEOREM. *Two operators T and $\tilde{T}$ in $B_1(\Omega)$ are unitarily equivalent if and only if $\mathscr{K}_T(\omega) = \mathscr{K}_{\tilde{T}}(\omega)$, where $\mathscr{K}_T(\omega)\,\mathrm{d}\omega \wedge \mathrm{d}\bar{\omega}$ is the curvature of the bundle E_T .*

We estimate the curvature of an operator T in $B_1(\Omega)$, using the Szegö kernel for Ω, whenever $\mathrm{cl}\,\Omega$ is a spectral set for T. These estimates are then shown to be best possible.

The class $B_1(\Omega)$ was first introduced in Cowen and Douglas [8] as follows,

$$B_1(\Omega) = \{T \in \mathscr{L}(\mathscr{H}) \colon \begin{array}{ll} \text{(i)} & \Omega \subset \sigma(T) \\ \text{(ii)} & \mathrm{ran}(T-\omega) = \mathscr{H}, \quad \omega \in \Omega \\ \text{(iii)} & \bigvee_{\omega \in \Omega} \ker(T-\omega) = \mathscr{H} \\ \text{(iv)} & \dim \ker(T-\omega) = 1\}. \end{array}$$

It is important for us to relate the curvature of the bundle E_T to the operator T. The space $\ker(T-\omega)^2$ is invariant for $T-\omega$ and $(T-\omega) \mid \ker(T-\omega)^2$ is a nonzero nilpotent operator of order two. Relative to an appropriate basis we have the matrix representation

$$N_\omega = T \mid \ker(T-\omega)^2 = \begin{bmatrix} \omega & h_T(\omega) \\ 0 & \omega \end{bmatrix}.$$

Now, $\mathscr{K}_T(\omega)$ bears a simple relationship to $h_T(\omega)$ namely $\mathscr{K}_T(\omega) = -h_T(\omega)^{-2}$.

Our problem is to find reasonable estimates for operators in $B_1(\Omega)$. If $\mathrm{cl}\,\Omega$ is a spectral set for T, then it is also a spectral set for N_ω. We show that $\mathrm{cl}\,\Omega$ is a spectral set for N_ω if and only if $h_T(\omega) \leqslant \hat{K}_\Omega(\omega, \bar{\omega})^{-1}$, where $\hat{K}_\Omega(\omega, \bar{\omega})$ is the Szegö kernel for the region Ω. In turn we obtain the desired curvature inequality. At this point it is natural to ask if $\operatorname{cl}\Omega$ has to be a spectral set for T, whenever T satisfies the curvature inequality. We are not able to resolve this question. (However, see note at the end.)

To show that the estimate $\mathcal{K}_T(\omega) \leqslant -\hat{K}_\Omega(\omega, \bar{\omega})^2$ is best possible we must establish the existence of an operator T in $B_1(\Omega)$ satisfying $\mathcal{K}_T(\omega) = -\hat{K}_\Omega(\omega, \bar{\omega})^2$. First we attempt to compute the curvature in a reasonable manner. For any T in $B_1(\Omega)$, if γ is a nonzero holomorphic cross section of the bundle E_T, then corresponding to γ there is a natural representation Γ of the Hilbert space $\mathcal{H}$ as holomorphic functions on $\bar{\Omega} = \{\omega \mid \bar{\omega} \in \Omega\}$ defined by $(\Gamma x)(\omega) = \langle x, \gamma(\bar{\omega})\rangle$ for $x \in \mathcal{H}$. Moreover since

$$(\Gamma T^* x)(\omega) = \langle x, T\gamma(\bar{\omega})\rangle = \langle x, \bar{\omega}\gamma(\bar{\omega})\rangle = \omega(\Gamma x)(\omega) \quad \text{for } \omega \in \bar{\Omega},$$

it follows that T is the adjoint of multiplication on $\Gamma(\mathcal{H})$. If we set $K(\lambda, \bar{\omega}) = \langle \gamma(\bar{\omega}), \gamma(\bar{\lambda})\rangle$ then K is the reproducing kernel for $\Gamma(\mathcal{H})$.

The map $\omega \to K(\bar{z}, \omega)$ is a holomorphic section of the bundle E_T and $\mathcal{K}_T(\omega)\mathrm{d}\omega\,\mathrm{d}\bar{\omega}$ is the curvature defined with respect to the metric $\langle K, K\rangle_\omega = K(\bar{\omega}, \omega)$. We can express $\mathcal{K}_T(\omega)$ by means of the formula

$$\mathcal{K}_T(\omega) = \frac{\partial^2}{\partial\omega\,\partial\bar{\omega}} \log K(\omega, \bar{\omega})^{-1}.$$

Since all holomorphic bundles over an open subset of $\mathbf{C}$ are trivial (cf. Cowen and Douglas [8]), our problem lies in choosing an appropriate metric for the trivial bundle over Ω so that the curvature with respect to this metric equals $-\hat{K}_\Omega(\omega, \bar{\omega})^2$.

Let $\mathrm{d}s^2 = h^2 \mathrm{d}\omega\,\mathrm{d}\bar{\omega}$ be a metric on Ω. The Gaussian curvature with respect to the metric $\mathrm{d}s^2$ is then given by the formula (Ahlfors [3])

$$C(\mathrm{d}s^2) = -4\frac{\partial^2}{\partial\omega\,\partial\bar{\omega}}(\log h)/h^2.$$

If the region Ω is simply connected and $\hat{K}_\Omega(\omega, \bar{\omega})$ is the Szegö kernel for Ω then $\mathrm{d}s = \hat{K}_\Omega(\omega, \bar{\omega})\mathrm{d}\omega\,\mathrm{d}\bar{\omega}$ is the Poincaré metric. Since the Gaussian curvature for the Poincaré metric equals -4, it follows that

$$\frac{\partial^2}{\partial\omega\,\partial\bar{\omega}} \log \hat{K}_\Omega(\omega, \bar{\omega})^{-1} = -\hat{K}_\Omega(\omega, \bar{\omega})^2.$$

The reproducing kernel for the usual Hardy space $H^2(\Omega)$ is the Szegö kernel function for Ω, which suggests that the adjoint M_z^* of multiplication on $H^2(\Omega)$ is our candidate for an extremal operator. Once we show the operator M_z^* is in $B_1(\bar{\Omega})$ (Corollary 2.1), it follows that

$$\mathcal{K}_{M_z^*}(\omega) = \frac{\partial^2}{\partial\omega\,\partial\bar{\omega}} \log \hat{K}_\Omega(\omega, \bar{\omega})^{-1} = -\hat{K}_\Omega(\omega, \bar{\omega})^2 \quad \text{for all } \omega \in \Omega.$$

To complete the proof that M_z^* is an extremal operator, we have to verify the relation $\hat{K}_\Omega(\omega, \bar{\omega}) = \hat{K}_{\bar{\Omega}}(\bar{\omega}, \omega)$.

When the region Ω is not simply connected, the Szegö kernel does not yield the Poincairé metric and our previous techniques fail. Perhaps even more surprising is the inequality

$$\frac{\partial^2}{\partial\omega\,\partial\bar{\omega}} \log \hat{K}_\Omega(\omega, \bar{\omega})^{-1} < -\hat{K}_\Omega(\omega, \bar{\omega})^2$$

which essentially says that the operator M_z^* on the Hardy space $H^2(\Omega)$ is not an extremal operator. At present we are unable to find a single extremal operator. The next best thing we can do is to ask if the inequality is sharp pointwise, that is given a point ζ in Ω, does there exist an operator T in $B_1(\Omega)$ so that at least

$$\mathscr{K}_T(\zeta) = -\hat{K}_\Omega(\zeta, \bar{\zeta})^2.$$

First, we define a certain Hilbert space $H^2(\partial\Omega, m)$ of analytic functions on Ω determined by a positive measure m on the boundary $\partial\Omega$, depending on the given point ζ. As before the operator M_z^* on the Hilbert space $H^2(\partial\Omega, m)$ satisfies $\mathscr{K}_{M_z^*}(\zeta) = -\hat{K}_\Omega(\zeta, \bar{\zeta})^2$. These operators have been studied in a more general setting (Abrahamse-Douglas [2]), where they are called bundle shifts.

Some of these results (Theorem 1.1, Theorem 2.2 and Corollary 1.1) were reported without proof by Bruce Abrahamse in a private communication (Feb. '79) to Ronald Douglas. This work constitutes part of the author's doctoral dissertation at SUNY, Stony Brook, 1982. The author wishes to thank Ronald G. Douglas for his patient assistance with this research. Also, the author acknowledges many hours of helpful conversation with Douglas N. Clark.

1. CURVATURE INEQUALITIES

We begin with the well known definition of spectral set and reformulate it in various ways suitable for our purpose. Suppose Ω is an open bounded set in $\mathbf{C}$. Let $\mathrm{Hol}(\Omega, \mathbf{D})$ and $\mathrm{Rat}(\mathrm{cl}\,\Omega)$ be the holomorphic functions mapping the region Ω into the disk $\mathbf{D}$ and the rational functions with no poles in $\mathrm{cl}\,\Omega$ respectively.

DEFINITION 1.1. The set $\mathrm{cl}\,\Omega$ is a *spectral set* for T in $\mathscr{L}(\mathscr{H})$ if the spectrum $\sigma(T)$ is contained in $\mathrm{cl}\,\Omega$ and $\|f(T)\| \leqslant \|f\|_\infty$ for all f in $\mathrm{Rat}(\mathrm{cl}\,\Omega)$.

Let us call Ω *reasonable* if for each f in $H^\infty(\Omega)$ there is a sequence of rational functions r_n with poles outside $\mathrm{cl}\,\Omega$ satisfying $\|r_n\|_\infty \leqslant \|f\|_\infty$ and $r_n(\omega) \to f(\omega)$ for ω in Ω. If Ω is a finitely connected Jordan region, that is, the boundary of Ω consists of simple analytic curves, then Ω is reasonable (Gamelin [9]). Let Ω be a Jordan region and $\sigma(T) \subset \Omega$. Since we can define $f(T)$ via the Riesz functional calculus, whenever f is in $H^\infty(\Omega)$, the first of the following two remarks is self evident.

REMARK 1. $\mathrm{cl}\,\Omega$ is a spectral set for T if and only if $\|f(T)\| \leqslant 1$ for all f in $\mathrm{Hol}(\Omega, \mathbf{D})$.

Let $\mathrm{Hol}(\Omega, \omega, \mathbf{D})$ be the set of those functions f in $\mathrm{Hol}(\Omega, \mathbf{D})$ that vanish at ω, that is, $\mathrm{Hol}(\Omega, \omega, \mathbf{D}) = \{f \in \mathrm{Hol}(\Omega, \mathbf{D}) : f(\omega) = 0\}$.

REMARK 2. The following two statements are equivalent.

(a) $\|f(T)\| \leqslant 1$ for f in $\mathrm{Hol}(\Omega, \mathbf{D})$.

(b) $\|f(T)\| \leqslant 1$ for f in $\mathrm{Hol}(\Omega, \omega, \mathbf{D})$.

It is clear that (b) follows from (a). The other way round, let $f(\omega) := \alpha$ and $\alpha \neq 0$. By the maximum principle $|\alpha| < 1$, hence $\varphi_\alpha(z) = (z - \alpha)(1 - \bar{\alpha} z)^{-1}$ is a conformal map of the unit disk. The function $g = \varphi_\alpha \circ f$ lies in $\mathrm{Hol}(\Omega, \omega, \mathbf{D})$ and it follows that $\|g(T)\| \leqslant 1$ or, equivalently $\|f(T)\| = \|(\varphi_\alpha^{-1}(g(T)))\| \leqslant 1$.

Let $\gamma = \mathrm{Sup}\{|\varphi(\zeta)|^2 : \varphi \in \mathrm{Hol}(\Omega, \omega, \mathbf{D})\}$ and

$$\alpha(\omega, \zeta) = (\gamma^{-1} - 1)^{1/2}|\omega - \zeta|, \quad \zeta \text{ in } \Omega.$$

THEOREM 1.1. *For ω, ζ in Ω, $\mathrm{cl}\,\Omega$ is a spectral set for the matrix* $\begin{bmatrix} \omega & \mu \\ 0 & \zeta \end{bmatrix}$ *if and only if* $|\mu| \leqslant \alpha(\omega, \zeta)$.

Proof. Let $p(x) = a_0 + a_1(x) + \ldots + a_n x^n$ be a polynomial.

$$p\begin{bmatrix} \omega & \mu \\ 0 & \zeta \end{bmatrix} = \begin{bmatrix} p(\omega) & \dfrac{\mu}{\omega - \zeta}(p(\omega) - p(\zeta)) \\ 0 & p(\zeta) \end{bmatrix}.$$

For any $\varphi = p/q$ in $\mathrm{Rat}(\mathrm{cl}\,\Omega)$, the rational functional calculus yields the following:

$$\varphi\begin{bmatrix} \omega & \mu \\ 0 & \zeta \end{bmatrix} = \left(p\begin{bmatrix} \omega & \mu \\ 0 & \zeta \end{bmatrix}\right) q\left(\begin{bmatrix} \omega & \zeta \\ 0 & \mu \end{bmatrix}\right)^{-1} =$$

$$= \begin{bmatrix} \varphi(\omega) & \dfrac{\mu}{\omega - \zeta}(\varphi(\omega) - \varphi(\zeta)) \\ 0 & \varphi(\zeta) \end{bmatrix}.$$

We can now apply the Riesz functional calculus to see that the same representation holds when φ is in $\mathrm{Hol}(\Omega, \mathbf{D})$. In view of Remarks 1 and 2, $\mathrm{cl}\,\Omega$ will be a spectral set for $\begin{bmatrix} \omega & \mu \\ 0 & \zeta \end{bmatrix}$ if and only if

$$\left\|\varphi\begin{bmatrix} \omega & \mu \\ 0 & \zeta \end{bmatrix}\right\| \leqslant 1 \quad \text{for all } \varphi \text{ in } \mathrm{Hol}(\Omega, \omega, \mathbf{D})$$

which is equivalent to

$$\left\| \begin{bmatrix} 0 & \dfrac{\mu}{\omega - \zeta}\varphi(\zeta) \\ 0 & \varphi(\zeta) \end{bmatrix} \right\| \leqslant 1 \quad \text{for all } \varphi \text{ in } \mathrm{Hol}(\Omega, \omega, \mathbf{D}).$$

The desired inequality follows immediately.

COROLLARY 1.1. *For* ω *in* Ω, $\mathrm{cl}\,\Omega$ *is a spectral set for* $\begin{bmatrix} \omega & h(\omega) \\ 0 & \omega \end{bmatrix}$ *if and only if*

$$|h(\omega)| \leqslant [\sup\{|f'(\omega)| : f \in \mathrm{Hol}(\Omega, \omega, \mathbf{D})\}]^{-1}.$$

Proof. We take limit as ζ approaches ω in the theorem and the corollary follows.

COROLLARY 1.2. *For any* T *in* $B_1(\Omega)$, *if* $\mathrm{cl}\,\Omega$ *is a spectral set for* T, *then*

$$\mathscr{K}_T(\omega) \leqslant -\mathrm{Sup}\{|f'(\omega)|^2 : f \in \mathrm{Hol}(\Omega, \omega, \mathbf{D})\}.$$

Proof. Since $\ker(T-\omega)^2$ is a rationally invariant subspace of T, $N_\omega = T|\ker(T-\omega)^2$ admits $\mathrm{cl}\,\Omega$ as a spectral set. It is shown in Cowen and Douglas [8] that N_ω has the matrix representation $\begin{bmatrix} \omega & h_T(\omega) \\ 0 & \omega \end{bmatrix}$ with respect to an appropriate basis. We apply Corollary 1.1 and use $h_T(\omega) = (-\mathscr{K}_T(\omega))^{-1/2}$ to obtain the curvature inequality.

We wish to reformulate the curvature inequality in terms of the Szegö kernel function for the domain Ω. We proceed with the relevant definitions and theorems, most of which appear in Bergman [6].

Let m be a positive measure defined on the boundary $\partial\Omega$, that is mutually absolutely continuous with respect to arc length measure on $\partial\Omega$. The space $L^2(\partial\Omega, m)$ consists of complex functions on $\partial\Omega$ that are square integrable with respect to the measure m. We also introduce the Hardy class $H^2(\Omega)$ consisting of analytic functions f on the region Ω such that $|f|^2$ admits a harmonic majorant. The properties of $H^2(\Omega)$ are well known (Rudin [10]), in particular each f in $H^2(\Omega)$ possesses a well behaved nontangential boundary value. It is natural to define $H^2(\partial\Omega, m)$ to be the class of functions f^* in $L^2(\partial\Omega, m)$ that are boundary values of some f in $H^2(\Omega)$. Let $\{e_n(\omega)\}$ be an orthonormal basis for $H^2(\Omega, |dz|)$, define the Szegö kernel to be

$$\hat{K}_\Omega(\omega, \bar{\zeta}) = \sum e_n(\omega)\overline{e_n(\zeta)}\,.$$

That the kernel is well defined and has the reproducing property is established in Bergman [6, p. 108]. On our domain Ω there exists an adjoint kernel $\hat{L}(\omega, \zeta)$ [6, p. 111] determined by the following properties:

(i) $\hat{L}(\omega, \zeta)$ is regular in $\mathrm{cl}\,\Omega$ with the exception of a single pole at $\alpha = \zeta$ with residue 1;

(ii) For ω on the boundary $\partial\Omega$, the two kernels $\hat{K}_\Omega(\omega, \bar{\zeta})$ and $\hat{L}(\omega, \zeta)$ satisfy the relation

$$(*) \qquad \hat{K}_\Omega(\omega, \bar{\zeta})|d\omega| = -i\,\hat{L}(\omega, \zeta)\,d\omega.$$

We record here one more relationship which will be useful to us later.

$$(**) \qquad -i\,\hat{L}(\omega, \zeta)\hat{K}_\Omega(\omega, \bar{\zeta})\,d\omega > 0 \quad \text{on } \partial\Omega.$$

The function $F_\zeta(\omega) = \hat{K}_\Omega(\omega, \bar{\zeta})/\hat{L}(\omega, \zeta)$ maps the region Ω onto the n-times covered disk, where n is the connectivity of the region Ω and $F_\zeta(\zeta) = 0$. The problem of finding, among all functions f in $\mathrm{Hol}(\Omega, \omega, \mathbf{D})$, the one that maximizes $|f'(\omega)|$ is solved by means of a generalization of Schwarz Lemma.

Schwarz Lemma (for multiply connected domains). *If the function f is in* $\mathrm{Hol}(\Omega, \omega, \mathbf{D})$ *then*

$$|f'(\omega)| \leqslant \frac{d}{dz} F_\omega(z)\Big|_{z=\omega} = \hat{K}_\Omega(\omega, \bar{\omega}).$$

Whenever Ω_0 is a subset of Ω, it follows that $\mathrm{Hol}(\Omega, \omega, \mathbf{D})$ is a subset of $\mathrm{Hol}(\Omega_0, \omega, \mathbf{D})$ and we obtain the monotonicity of the kernel function, that is,

$$\hat{K}_\Omega(\omega, \bar{\omega}) = \mathrm{Sup}\{|f'(\omega)| : f \in \mathrm{Hol}(\Omega, \omega, \mathbf{D})\} \leqslant$$

$$\leqslant \mathrm{Sup}\{|f'(\omega)| : f \in \mathrm{Hol}(\Omega_0, \omega, \mathbf{D})\} = \hat{K}_{\Omega_0}(\omega, \bar{\omega}),$$

where equality can occur only if Ω and Ω_0 differ by a null set (Ahlfors and Beurling [4]).

We now restate Corollaries 1.1 and 1.2 using the Szegö kernel function $\hat{K}_\Omega(\omega, \bar{\zeta})$ for Ω.

Corollary 1.1′. *For ω in Ω, $\mathrm{cl}\,\Omega$ is a spectral set for $\begin{bmatrix} \omega & h(\omega) \\ 0 & \omega \end{bmatrix}$ if and only if*

$$\left\| F_\omega \begin{bmatrix} \omega & h(\omega) \\ 0 & \omega \end{bmatrix} \right\| \leqslant 1, \quad F_\omega(z) = \hat{K}_\Omega(z, \bar{\omega})/\hat{L}(z, \omega) \quad \textit{and} \quad F(\omega, \omega) = 0,$$

or equivalently

$$|h(\omega)| \leqslant \hat{K}_\Omega(\omega, \bar{\omega})^{-1}.$$

Corollary 1.2′. *If $\mathrm{cl}\,\Omega$ is a spectral set for T in $B_1(\Omega)$, then*

$$\mathscr{K}_T(\omega) \leqslant -\hat{K}_\Omega(\omega, \bar{\omega})^2.$$

We apply Corollary 1.1′ to show that $\|T\| \leqslant 1$ and $\|T^{-1}\| \leqslant 1/r$ are not sufficient for T to admit the annulus $A = \{\omega \mid r \leqslant |\omega| \leqslant 1\}$ as a spectral set. The following lemma appears in Williams [13].

LEMMA 1.1. *If U_2 is the two dimensional shift with matrix representation $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ relative to an orthonormal basis then*

$$\|\alpha + \beta U_2\| = \frac{1}{2}\{|\beta| + \sqrt{4|\alpha|^2 + |\beta|^2}\}.$$

To produce the desired examples, let us consider

$$T = \begin{bmatrix} \omega & 1 - |\omega|^2 \\ 0 & \omega \end{bmatrix},$$

where $\omega \in A_0 = \{\omega \mid \sqrt{r} < |\omega| < 1\}$. We can apply Lemma 1.1 to verify that $\|T\| = 1$ and $\|T^{-1}\| \leqslant 1/r$.

Since $\hat{K}_{\mathbf{D}}(\omega, \bar{\omega})^{-1} > \hat{K}_A(\omega, \bar{\omega})^{-1}$, in view of Corollary 1.1′ it follows that the annulus A can not be a spectral set for T.

We point out that the following theorem due to Williams [13] yields the same result.

THEOREM 1.2. *If an operator T on a finite dimensional vector space is completely non normal and $\|T\| = 1$, then the unit disk is a minimal spectral set.*

2. EXTREMAL PROPERTIES OF BUNDLE SHIFTS

We discuss some extremal problems arising in classical Hilbert space theory (Bergman [6]). Let Ω be a Jordan region and m be a positive measure on the boundary $\partial\Omega$, that is mutually absolutely continuous with respect to the arc length measure. We have already defined the Hardy class $H^2(\partial\Omega, m)$ with respect to the measure m. For fixed point ζ in Ω, define

$$\mathcal{M}_0 = \{f \in H^2(\partial\Omega, m) : f(\zeta) = 1\}$$

and

$$\mathcal{M}_1 = \{f \in H^2(\partial\Omega, m) : f(\zeta) = 0 \text{ and } f'(\zeta) = 1\}.$$

Consider the two problems

0. To find the minimum $\|f\|^2$ over $\mathcal{M}_0$.
1. To find the minimum $\|f\|^2$ over $\mathcal{M}_1$.

The existence and uniqueness of the solution to both these problems are well known. We quote here a slight generalization of a lemma due to Suita [12], that establishes the extremality of certain functions.

LEMMA 2.1. *The function F in $H^2(\partial\Omega, m)$ is a solution to*
(i) *Problem* 0, *if and only if $F \in \mathcal{M}_0$ and F is orthogonal to*

$$\{f \in H^2(\partial\Omega,\ m) : f(\zeta) = 0\};$$

(ii) *Problem* 1, *if and only if $F \in \mathcal{M}_1$ and F is orthogonal to*

$$\{f \in H^2(\partial\Omega,\ m) : f(\zeta) = f'(\zeta) = 0\}.$$

Since we have chosen the measure m to be mutually absolutely continuous with respect to $|dz|$, it follows that the point evaluation functional on $H^2(\partial\Omega, m)$ is continuous for each ω in Ω. Consequently $H^2(\partial\Omega, m)$ has a well defined reproducing kernel $K_m(\cdot, \bar\zeta)$. We define two functions f_0 and f_1, which turn out to be the extremal functions for Problems 0 and 1 respectively.

Let K_m^{jk} denote the partial derivatives $\dfrac{\partial^{j+k}}{\partial\omega^j\,\partial\bar\omega^k} K_m(\omega, \bar\omega)\big|_{\omega=\zeta}$. Set

$$f_0(\omega) = K_m(\omega, \bar\zeta)$$

and

$$f_1(\omega) = \begin{vmatrix} K_m(\omega, \bar\zeta) & \dfrac{\partial}{\partial\bar z} K_m(\omega,\ \bar z)\Big|_{z=\zeta} \\ K_m^{00} & K_m^{10} \end{vmatrix} \begin{vmatrix} K_m^{00} & K_m^{01} \\ K_m^{10} & K_m^{11} \end{vmatrix}^{-1}.$$

If g is any function in $H^2(\partial\Omega, m)$ satisfying $g(\zeta) = 0$ then $\langle g, f_0\rangle = 0$. Similarly if h is any function in $H^2(\partial\Omega, m)$ satisfying $h(\zeta) = h'(\zeta) = 0$ then $\langle h, f_1\rangle = 0$. We apply Lemma 2.1 to conclude that f_0 and f_1 are the extremal functions. If $\lambda_0(\zeta)$ and $\lambda_1(\zeta)$ are the solutions to Problems 0 and 1, it is not difficult to compute the values

$$\lambda_0(\zeta) = \|f_0\|^2 = K_m(\zeta, \bar\zeta)^{-1},$$

$$\lambda_1(\zeta) = \|f_1\|^2 = K_m^{00}(K_m^{00}K_m^{11} - |K_m^{01}|^2)^{-1} = \left(K_m^{00}\,\frac{\partial^2}{\partial\omega\,\partial\bar\omega}\log K_m(\omega,\ \bar\omega)\Big|_{\omega=\zeta}\right)^{-1}.$$

Assuming that the adjoint M_z^* of multiplication on $H^2(\partial\Omega, m)$ is in $B_1(\bar\Omega)$, we obtain the curvature inequality

$$\mathcal{K}_{M_z^*}(\zeta) = -\frac{\partial^2}{\partial\omega\,\partial\bar\omega}\log K_m(\omega, \bar\omega)\big|_{\omega=\zeta} \leqslant -\hat{K}_\Omega(\zeta, \bar\zeta)^2.$$

It is interesting to note that here we have a different proof of Theorem 2 of Suita [12] and parts of Theorem 2 of Burbea [7], when $m = |dz|$. In this case, they have shown further that the inequality is strict whenever Ω is not simply connected.

To find when equality holds, at least pointwise, we have actually to use some of the ideas developed by Suita [12] and Burbea [7]. For each fixed point ζ in the region Ω, let m be the measure $|\hat{K}_\Omega(\omega, \bar{\zeta})|^2|d\omega|$. For this particular choice of the measure m we have

THEOREM 2.1. $\mathscr{K}_{M_z^*}(\zeta) = -\hat{K}_\Omega(\zeta, \bar{\zeta})^2$.

Proof. Let φ be the function

$$\varphi(\omega) = F(\omega, \zeta)K_m(\omega, \bar{\zeta})/\hat{K}_\Omega^{00}K_m^{00}.$$

The function F has a zero at ζ and therefore $\varphi(\zeta) = 0$. Also

$$\varphi'(\zeta) = [F'(\zeta, \zeta)K_m(\zeta, \bar{\zeta}) + F(\zeta, \zeta)K_m'(\zeta, \bar{\zeta})]/\hat{K}_\Omega^{00}K_m^{00} = 1.$$

The function φ is therefore in the subspace $\mathscr{M}_1$. Since $\|f_1\|^2$ is a solution to the problem of minimizing $\{\|f\|^2 : f \in \mathscr{M}_1\}$, we obtain

$$\|\varphi\|^2 \geqslant \|f_1\|^2 = \lambda_1(\zeta) = \left[K_m^{00}\frac{\partial^2}{\partial\omega\,\partial\bar{\omega}}\log K(\omega, \bar{\omega})|_{\omega=\zeta}\right]^{-1} = -[K_m(\zeta, \bar{\zeta})\mathscr{K}_{M_z^*}](\zeta)^{-1}.$$

On the other hand,

$$\|\varphi\|^2 = [\hat{K}_\Omega(\zeta, \bar{\zeta})K_m(\zeta, \bar{\zeta})]^{-2}\int_{\partial\Omega}|F(\omega, \zeta)|^2|K_m(\omega, \bar{\zeta})|^2\,dm = [\hat{K}_\Omega(\zeta, \bar{\zeta})^2K_m(\zeta, \bar{\zeta})]^{-1}.$$

Putting these together,

$$\|\varphi\|^2 = [\hat{K}_\Omega(\zeta, \bar{\zeta})^2K_m(\zeta, \bar{\zeta})]^{-1} \geqslant -[K_m(\zeta, \bar{\zeta})\mathscr{K}_{M_z^*}(\zeta)]^{-1}.$$

Equality will hold if φ can be shown to be the extremal function. An application of Lemma 2.1 reduces the problem to showing that φ is orthogonal to the subspace $\{f \in H^2(\partial\Omega, m) : f(\zeta) = f'(\zeta) = 0\}$. For any function f in this subspace, we compute

$$\langle f, \varphi\rangle = (2\pi\hat{K}_\Omega^{00}K_m^{00})^{-1}\int_{\partial\Omega} f(\omega)\overline{F(\omega, \bar{\zeta})K_m(\omega, \bar{\zeta})}|\hat{K}_\Omega(\omega, \bar{\zeta})|^2|d\omega| =$$

$$= (2\pi i\hat{K}_\Omega^{00}K_m^{00})^{-1}\int_{\partial\Omega} f(\omega)\hat{L}(\omega, \zeta)^2\overline{K_m(\omega, \bar{\zeta})}\,d\omega.$$

For every function f in $H^2(\partial\Omega, m)$, we also have

$$\langle f, (\hat{K}_\Omega^{00})^{-1}\rangle = (2\pi\hat{K}_\Omega^{00})^{-1}\int_{\partial\Omega} f(\omega)|\hat{K}_\Omega(\omega, \bar{\zeta})|^2|d\omega| =$$

$$= (2\pi i\hat{K}_\Omega^{00})^{-1}\int_{\partial\Omega} f(\omega)\hat{K}_\Omega(\omega, \bar{\zeta})\hat{L}(\omega, \zeta)\,d\omega.$$

Since the functions f and $\hat{K}_\Omega(\cdot, \bar{\zeta})$ are holomorphic, while $\hat{L}(\cdot, \zeta)$ is meromorphic in the region Ω with a simple pole at ζ and residue 1, it follows that

$$\langle f, (\hat{K}_\Omega^{00})^{-1}\rangle = f(\zeta).$$

Uniqueness of the kernel function implies

$$K_m(\omega, \zeta) = (\hat{K}_\Omega^{00})^{-1} \quad \text{for all } \omega \in \Omega.$$

Continuing, we obtain

$$\langle f, \varphi\rangle = (2\pi i \hat{K}_\Omega^{00} K_m^{00})^{-1} \int_{\partial\Omega} f(\omega)\hat{L}(\omega, \zeta)^2 \overline{K_m(\omega, \bar{\zeta})}\, d\omega =$$

$$= (2\pi i K_m^{00})^{-1}(\hat{K}_\Omega^{00})^{-2} \int f(\omega)\hat{L}(\omega, \zeta)^2\, d\omega.$$

The function $\hat{L}(\omega, \zeta)^2$ has a double pole at $\omega = \zeta$, where f has zero of order at least two. Therefore, the product $f(\omega)\hat{L}(\omega, \zeta)^2$ is holomorphic on all of Ω. It follows that $\langle f, \varphi\rangle = 0$.

This theorem suggests that the operator M_z^* on the Hardy space $H^2(\partial\Omega, m)$ is a natural candidate for the extremal operator.

THEOREM 2.2. *For each fixed point ζ in the region Ω, let m be the measure $|\hat{K}_\Omega(\omega, \bar{\zeta})|^2|d\omega|$, then the operator M_z^* on the Hardy space $H^2(\partial\Omega, m)$ satisfies*

$$\mathscr{K}_{M_z^*}(\zeta) = -\hat{K}_{\bar{\Omega}}(\bar{\zeta}, \zeta)^2.$$

We reiterate that it remains for us only to show that the operator M_z^* is in the class $B_1(\bar{\Omega})$ and $\hat{K}_\Omega(\zeta, \bar{\zeta}) = \hat{K}_{\bar{\Omega}}(\bar{\zeta}, \zeta)$. Let us recall that every operator T in $B_1(\Omega)$ is unitarily equivalent to the adjoint of the multiplication operator on a certain Hilbert space of analytic functions on $\bar{\Omega}$. With a little effort we can prove a converse.

PROPOSITION 2.1. *Let $\mathscr{H}$ be a Hilbert space of analytic functions on the region Ω, equipped with a reproducing kernel K. If the operator M_z maps $\mathscr{H}$ into itself, then it is bounded and each $\bar{\omega}$ in $\bar{\Omega}$ is an eigenvalue for the operator M_z^*. If in addition, ω is a simple eigenvalue, then the operator M_z^* lies in $B_1(\bar{\Omega})$.*

Proof. The first part of the proposition is a result due to Shields and Wallen [11]. Since the point evaluations are bounded on $\mathscr{H}$ and $\mathrm{ran}(M_z - \omega) = \{f \in \mathscr{H} | f(\omega) = 0\}$, it follows that $\mathrm{ran}(M_z - \omega)$ is closed. The operator $M_z - \omega$ is one to one, therefore $\mathrm{ran}(M_z^* - \bar{\omega}) = \mathscr{H}$. The eigenvector corresponding to the eigenvalue $\bar{\omega}$ is $K(\cdot, \bar{\omega})$ and we have $\mathrm{span}\ker(M_z^* - \bar{\omega}) = \mathrm{span}\, K(\cdot, \bar{\omega}) = \mathscr{H}$. Assuming $\bar{\omega}$ is a simple eigenvalue, M_z^* lies in $B_1(\bar{\Omega})$.

COROLLARY 2.1. *The operator $\mathring{M}_z^*$ on the Hardy space $H^2(\partial\Omega, m)$ is in $B_1(\bar{\Omega})$.*

Proof. Each function f in $H^2(\partial\Omega, m)$ can be written as $f(z) = f(\omega) + (z-\omega)\dfrac{f(z)-f(\omega)}{z-\omega}$, where $\dfrac{f(z)-f(\omega)}{z-\omega} \in H^2(\partial\Omega, m)$. Thus, $\operatorname{ran}(M_z - \omega)$ has codimension 1 or in other words $\bar{\omega}$ is a simple eigenvalue for M_z^*.

PROPOSITION 2.2. *For all ω in $\bar{\Omega}$, we have $\hat{K}_\Omega(\bar{\omega}, \omega) = \hat{K}_{\bar{\Omega}}(\omega, \bar{\omega})$.*

Proof. Let F be the function that maps the region Ω onto the n-times covered unit disk. The corresponding function F^* mapping the region $\bar{\Omega}$ is given by the formula $F^*(\omega, \zeta) = \overline{F(\bar{\omega}, \bar{\zeta})}$, where ω and ζ are now in $\bar{\Omega}$. Since $\hat{K}_\Omega(\bar{\omega}, \omega) = \dfrac{d}{d\bar{z}} F(\bar{z}, \bar{\omega})\big|_{z=\omega}$, it follows that

$$\hat{K}_{\bar{\Omega}}(\omega, \bar{\omega}) = \frac{d}{dz} F^*(z, \omega)\Big|_{z=\omega} = \frac{d}{dz}\overline{F(\bar{z}, \bar{\omega})}\Big|_{z=\omega} = \frac{d}{d\bar{z}} F(\bar{z}, \bar{\omega})\Big|_{z=\omega} = \hat{K}_\Omega(\bar{\omega}, \omega).$$

We apply the existence of extremal operators to show that the two notions of spectral sets and complete spectral sets are actually the same for some 2×2 matrices. Or, equivalently if $\operatorname{cl}\Omega$ is a spectral set for such a matrix M then M possesses a normal $R(\operatorname{cl}\Omega)$-dilation (Arveson [5]).

DEFINITION 2.1. The region $\operatorname{cl}\Omega$ is a *complete spectral set* for the operator T if the map $\sigma \otimes I_n\colon \operatorname{Rat}(\operatorname{cl}\Omega) \otimes M_n \to \mathscr{L}(\mathscr{H}) \otimes M_n$ is contractive for each n.

Here, σ is the map $\varphi \to \varphi(T)$ and M_n is the C^*-algebra of $n\times n$ complex matrices.

THEOREM 2.3. *If $\operatorname{cl}\Omega$ is a complete spectral set for the matrix $\begin{bmatrix}\omega & \mu\\ 0 & \zeta\end{bmatrix}$, then it is also a complete spectral set for $\begin{bmatrix}\omega & \lambda\\ 0 & \zeta\end{bmatrix}$ whenever $|\lambda| \leqslant |\mu|$.*

Proof. First, for $|\lambda| \leqslant |\mu|$ and any three $k\times k$ matrices A, B and C, we claim

$$\left\|\begin{bmatrix}A & \lambda B\\ 0 & C\end{bmatrix}\right\| \leqslant \left\|\begin{bmatrix}A & \mu B\\ 0 & C\end{bmatrix}\right\|.$$

Let $\begin{bmatrix}x\\ y\end{bmatrix}$ be any unit vector in $\mathbf{C}^{2k}$, then

$$\left\|\begin{bmatrix}A & \lambda B\\ 0 & C\end{bmatrix}\right\| = \|Ax\|^2 + \|Cx\|^2 + |\lambda|^2\|By\|^2 + 2\operatorname{Re}\lambda\langle Ax, By\rangle.$$

If we set $\lambda = |\lambda|\bar{\alpha}$, $\mu = |\mu|\bar{\beta}$ and $\langle Ax, \alpha By\rangle = |\langle Ax, \alpha By\rangle|\bar{\delta}$, then

$$\|Ax\|^2 + \|Cy\|^2 + |\lambda|^2\|By\|^2 + 2\mathrm{Re}\,\lambda\langle Ax, By\rangle =$$
$$= \|Ax\|^2 + \|Cy\|^2 + |\lambda|^2\|By\|^2 + 2|\lambda|\mathrm{Re}\langle Ax, \alpha By\rangle \leqslant$$
$$\leqslant \|Ax\|^2 + \|Cy\|^2 + |\lambda|^2\|By\|^2 + 2|\lambda|\; |\langle Ax, \alpha By\rangle| =$$
$$= \|Ax\|^2 + \|Cy\|^2 + |\lambda|^2\|By\|^2 + 2|\lambda|\langle Ax, (\alpha/\delta)By\rangle \leqslant$$
$$\leqslant \|Ax\|^2 + \|C(\alpha/\beta\delta)y\|^2 + |\mu|^2\|B(\alpha/\beta\delta)y\|^2 + 2|\mu|\bar{\beta}\langle Ax, (\alpha/\delta\beta)By\rangle =$$
$$= \left\|\begin{bmatrix} A & \mu B \\ 0 & C\end{bmatrix}\begin{bmatrix} x \\ (\alpha/\delta\beta)y\end{bmatrix}\right\|^2.$$

Since $\begin{bmatrix} x \\ (\alpha/\delta\beta)y\end{bmatrix}$ is again a unit vector in $\mathbf{C}^{2k}$, this proves the inequality for the two matrices in the operator norm.

To prove the statement about complete spectral sets we have to show

$$(*) \qquad \left\|\left(\varphi_{ij}\begin{bmatrix} \omega & \lambda \\ 0 & \zeta\end{bmatrix}\right)\right\| \leqslant \left\|\left(\varphi_{ij}\begin{bmatrix} \omega & \mu \\ 0 & \zeta\end{bmatrix}\right)\right\|$$

for all $\varphi_{ij} \in \mathrm{Rat}(\mathrm{cl}\,\Omega) \otimes M_n$.

Set $A = (\varphi_{ij}(\omega))_{k\times k}$, $B = (\varphi_{ij}(\omega) - \varphi_{ij}(\zeta))_{k\times k}$ and $C = (\varphi_{ij}(\zeta))_{k\times k}$. As in Theorem 1.1,

$$\varphi_{ij}\begin{bmatrix} \omega & \lambda \\ 0 & \zeta\end{bmatrix} = \begin{bmatrix} \varphi_{ij}(\omega) & \dfrac{\lambda}{\omega - \zeta}(\varphi_{ij}(\omega) - \varphi_{ij}(\zeta)) \\ 0 & \varphi_{ij}(\zeta)\end{bmatrix}.$$

Applying elementary row operations we obtain

$$\left(\varphi_{ij}\begin{bmatrix} \omega & \lambda \\ 0 & \zeta\end{bmatrix}\right) \sim \begin{bmatrix} A & \dfrac{\lambda}{\omega - \zeta}B \\ 0 & C\end{bmatrix}.$$

Similarly,

$$\left(\varphi_{ij}\begin{bmatrix} \omega & \mu \\ 0 & \zeta\end{bmatrix}\right) \sim \begin{bmatrix} A & \dfrac{\mu}{\omega - \zeta}B \\ 0 & C\end{bmatrix}.$$

Since $\dfrac{|\lambda|}{|\omega - \zeta|} \leqslant \dfrac{|\mu|}{|\omega - \zeta|}$, the inequality $(*)$ is an immediate consequence of the claim.

We wish to determine whether the region $\mathrm{cl}\,\Omega$ is a complete spectral set for $\begin{bmatrix} \omega & \lambda \\ 0 & \zeta\end{bmatrix}$ whenever $\mathrm{cl}\,\Omega$ is a spectral set. Theorem 1.1 and 2.3 reduce the problem to finding whether the region $\mathrm{cl}\,\Omega$ is a spectral set for the matrix $\begin{bmatrix} \omega & \alpha(\omega, \zeta) \\ 0 & \zeta \end{bmatrix}$. We are able to resolve the question only if $\omega = \zeta$ and $\alpha(\omega, \omega) = \hat{K}_\Omega(\omega, \bar{\omega})^{-1}$.

COROLLARY 2.1. *If the region* $\mathrm{cl}\,\Omega$ *is a spectral set for* $\begin{bmatrix} \omega & \lambda \\ 0 & \omega \end{bmatrix}$, *then it is also a complete spectral set.*

Proof. We have to prove only that the region $\mathrm{cl}\,\Omega$ is a complete spectral set for $\begin{bmatrix} \omega & \hat{K}_\Omega(\omega, \bar{\omega})^{-1} \\ 0 & \omega \end{bmatrix}$. Recall that, for any point ω in Ω there is a subnormal operator T in $B_1(\Omega)$ with curvature $-\hat{K}_\Omega(\omega, \bar{\omega})^2$. The normal extension N of the operator T provides a dilation for $\begin{bmatrix} \omega & \hat{K}_\Omega(\omega, \bar{\omega})^{-1} \\ 0 & \omega \end{bmatrix}$, Arveson's result [5] now implies that $\mathrm{cl}\,\Omega$ is a complete spectral set.

Added in poofs. J. Alger has found an example of a weighted shift operator T in $B_1(\mathbf{D})$ with $\|T\| < 1$ which satisfies the curvature inequality.

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