REPRODUCING KERNEL FOR A CLASS OF WEIGHTED BERGMAN SPACES ON THE SYMMETRIZED POLYDISC

GADADHAR MISRA, SUBRATA SHYAM ROY, AND GENKAI ZHANG

ABSTRACT. A natural class of weighted Bergman spaces on the symmetrized polydisc is isometrically embedded as a subspace in the corresponding weighted Bergman space on the polydisc. We find an orthonormal basis for this subspace. It enables us to compute the kernel function for the weighted Bergman spaces on the symmetrized polydisc using the explicit nature of our embedding. This family of kernel functions include the Szegö and the Bergman kernel on the symmetrized polydisc.

1. INTRODUCTION

Let $\varphi_i, i \geq 0$, be the *elementary symmetric function* of degree i, that is, φ_i is the sum of all products of i distinct variables z_i so that $\varphi_0 = 1$ and

$$\varphi_i(z_1, \ldots, z_n) = \sum_{1 \leq k_1 < k_2 < \ldots < k_i \leq n} z_{k_1} \cdots z_{k_i}.$$

For $n \geq 1$, let $\mathbf{s} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the function of symmetrization given by the formula

$$\mathbf{s}(z_1, \ldots, z_n) = \big(\varphi_1(z_1, \ldots, z_n), \ldots, \varphi_n(z_1, \ldots, z_n)\big).$$

The image $\mathbb{G}_n := \mathbf{s}(\mathbb{D}^n)$ under the map $\mathbf{s}$ of the unit polydisc $\mathbb{D}^n := \{\boldsymbol{z} \in \mathbb{C}^n : \|\boldsymbol{z}\|_\infty < 1\}$ is known as the *symmetrized polydisc*. The restriction map $\mathbf{s}_{|\mathrm{res}\,\mathbb{D}^n} : \mathbb{D}^n \to \mathbb{G}_n$ is a proper holomorphic map [6]. The Bergman kernel for the symmetrized polydisc is computed explicitly in [4]. It is obtained from the transformation rule for the Bergman kernel under proper holomorphic maps [1, Theorem 1].

Here we realize (isometrically) the Bergman space $\mathbb{A}^2(\mathbb{G}_n)$ of the symmetrized polydisc as a subspace of the Bergman space $\mathbb{A}^2(\mathbb{D}^n)$ on the polydisc using the symmetrization map $\mathbf{s}$. Indeed, the map $\Gamma : \mathbb{A}^2(\mathbb{G}_n) \to \mathbb{A}^2(\mathbb{D}^n)$ defined by the formula

$$(\Gamma f)(\boldsymbol{z}) = (f \circ \mathbf{s})(\boldsymbol{z}) J_{\mathbf{s}}(\boldsymbol{z}),\ \boldsymbol{z} \in \mathbb{D}^n,$$

where $J_{\mathbf{s}}$ is the complex Jacobian of the map $\mathbf{s}$, is an isometric embedding. The image $\operatorname{ran}\Gamma \subseteq \mathbb{A}^2(\mathbb{D}^n)$ consists of anti-symmetric functions:

$$\operatorname{ran}\Gamma := \{f : f(\boldsymbol{z}_\sigma) = \operatorname{sgn}(\sigma) f(\boldsymbol{z}),\ \sigma \in \Sigma_n\,, f \in \mathbb{A}^2(\mathbb{D}^n)\},$$

where Σ_n is the symmetric group on n symbols. The range of Γ is a subspace of $\mathbb{A}^2(\mathbb{D}^n)$, we let $\mathbb{A}^2_{\mathrm{anti}}(\mathbb{D}^n)$ be this subspace. An orthonormal basis of $\mathbb{A}^2_{\mathrm{anti}}(\mathbb{D}^n)$ may then be transformed in to an orthonormal basis of the $\mathbb{A}^2(\mathbb{G}_n)$ via the unitary map Γ^*. It is then possible to compute the Bergman kernel for the symmetrized polydisc $\mathbb{G}_n$ by evaluating the sum

$$\sum_{k \geq 0} e_k(\boldsymbol{z})\overline{e_k(\boldsymbol{w})},\ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{G}_n,$$

for some choice of an orthonormal basis in $\mathbb{A}^2(\mathbb{G}_n)$.

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This scheme works equally well for a class of weighted Bergman spaces $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$, $\lambda > 1$, determined by the kernel function

$$\mathbf{B}^{(\lambda)}_{\mathbb{D}^n}(\boldsymbol{z},\boldsymbol{w}) = \prod_{i=1}^{n}(1 - z_i\bar{w}_i)^{-\lambda},\ \boldsymbol{z} = (z_1,\dots,z_n),\ \boldsymbol{w} = (w_1,\dots,w_n) \in \mathbb{D}^n,$$

defined on the polydisc and the corresponding weighted Bergman spaces $\mathbb{A}^{(\lambda)}(\mathbb{G}^n)$ on the symmetrized polydisc.

The limiting case of $\lambda = 1$, as is well-known, is the Hardy space on the polydisc. We show that the reproducing kernel for the Hardy space of the symmetrized polydisc is of the form

$$\mathbb{S}^{(1)}_{\mathbb{G}_n}(\mathbf{s}(\boldsymbol{z}),\mathbf{s}(\boldsymbol{w})) = \prod_{i,j=1}^{n}(1 - z_i\bar{w}_j)^{-1}, \boldsymbol{z},\boldsymbol{w} \in \mathbb{D}^n.$$

This is a consequence of the determinantal identity [5, (4.3), pp. 63]. Indeed, along the way, we obtain a generalization of this well-known identity. We also point out that the Hardy kernel is not a power of the Bergman kernel unlike the case of bounded symmetric domains.

2. Weighted Bergman spaces on the symmetrized polydisc

For $\lambda > 1$, let $dV^{(\lambda)}$ be the probability measure $\left(\frac{\lambda-1}{\pi}\right)^n\left(\prod_{i=1}^n(1-r_i)^{\lambda-2}r_idr_id\theta_i\right)$ on the polydisc $\mathbb{D}^n$. Let $dV^{(\lambda)}_{\mathbf{s}}$ be the measure on the symmetrized polydisc $\mathbb{G}_n$ obtained by the change of variable formula:

$$\int_{\mathbb{G}_n} f\,dV^{(\lambda)}_{\mathbf{s}} = \int_{\mathbb{D}^n}(f\circ\mathbf{s})\,|J_{\mathbf{s}}|^2dV^{(\lambda)},\ \ \lambda > 1$$

where $J_{\mathbf{s}}(\boldsymbol{z}) = \prod_{1\le i<j\le n}(z_i - z_j)$ is the complex Jacobian of the symmetrization map $\mathbf{s}$. Let $\|J_{\mathbf{s}}\|^2_\lambda = \int_{\mathbb{D}^n}|J_{\mathbf{s}}|^2dV^{(\lambda)}$ be the the norm of the jacobian determinant $J_{\mathbf{s}}$ in the Hilbert space $L^2(\mathbb{D}^n, dV^{(\lambda)})$. By a slight abuse of notation, we let $dV^{(\lambda)}_{\mathbf{s}}$ be the measure $\|J_s\|^{-2}_\lambda dV^{(\lambda)}_{\mathbf{s}}$, $\lambda > 1$, on the symmetrized polydisc $\mathbb{G}_n$. The weighted Bergman space $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$, $\lambda > 1$, on the symmetrized polydisc $\mathbb{G}_n$ is the subspace of the Hilbert space $L^2(\mathbb{G}_n, dV^{(\lambda)}_{\mathbf{s}})$ consisting of holomorphic functions. It coincides with the usual Bergman space for $\lambda = 2$. The norm of $f \in \mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ is given by $\|f\|^2 = \int_{\mathbb{G}_n}|f|^2dV^{(\lambda)}_{\mathbf{s}}$. We have normalized the volume measure on $\mathbb{G}_n$ to ensure $\|1\| = 1$.

For $\lambda > 1$, let $\Gamma : \mathbb{A}^{(\lambda)}(\mathbb{G}_n) \longrightarrow \mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ be the operator defined by the rule:

$$(\Gamma f)(\boldsymbol{z}) = \|J_{\mathbf{s}}\|^{-1}_\lambda J_{\mathbf{s}}(\boldsymbol{z})(f\circ\mathbf{s})(\boldsymbol{z}),\quad f \in \mathbb{A}^{(\lambda)}(\mathbb{G}_n),\ \boldsymbol{z} \in \mathbb{D}^n.$$

It is clear from the definition of the norm in $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ that Γ is an isometry. The image of $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ under the isometry Γ in $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ is the subspace $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$ of anti-symmetric functions since $J_{\mathbf{s}}(\boldsymbol{z}_\sigma) = \operatorname{sgn}(\sigma)J_{\mathbf{s}}(\boldsymbol{z})$, $\sigma \in \Sigma_n$. Every function g in $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$ is of the form $J_{\mathbf{s}}h$ for some symmetric function h. For instance, take $h = J_{\mathbf{s}}^{-1}g$ on the open set $\{(z_1,\dots,z_n) \in \mathbb{D}^n : z_i \ne z_j,\ i \ne j\}$. It follows that $g = J_{\mathbf{s}}(f\circ\mathbf{s})$ for some function f defined on $\mathbb{G}_n$. Therefore, the range of the isometry coincides with the subspace $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$. Now, it is easily verified that $\Gamma^*g = \|J_{\mathbf{s}}\|_\lambda f$, where f is chosen satisfying $g(\boldsymbol{z}) = J_{\mathbf{s}}(\boldsymbol{z})(f\circ\mathbf{s})(\boldsymbol{z})$. The operator $\Gamma : \mathbb{A}^{(\lambda)}(\mathbb{G}_n) \longrightarrow \mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$ is evidently unitary. The Hilbert spaces $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$, $\lambda > 1$, are the weigheted Bergman spaces on the symmetrized polydisc $\mathbb{G}_n$.

Since the subspace $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$ is invariant under the multiplication by the elementary symmetric function φ_i, $1 \le i \le n$, we see that it admits a module action via the map

$$(p, f) \mapsto p(\varphi_1,\dots,\varphi_n)f,\ \ f \in \mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n),\ \ p \in \mathbb{C}[\boldsymbol{z}]$$

over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$. The polynomial ring acts naturally via multiplication by the coordinate functions on the Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ making it a module over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$.

The unitary operator Γ intertwines the multiplication by the elementary symmetric functions on the Hilbert space $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$ with the multiplication by the co-ordinate functions on $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$. Thus $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ and $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$ are isomorphic as modules via the unitary map Γ. Moreover, since $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ is a submodule of the $L^2(\mathbb{G}_n, dV^{(\lambda)}_{\mathbf{s}})$, it follows that the map

$$(p, f) \mapsto p \cdot f,\ f \in \mathbb{A}^{(\lambda)}(\mathbb{G}^n),\ p \in \mathbb{C}[\boldsymbol{z}]$$

is contractive. It therefore extends to a continuous map of the function algebra $\mathcal{A}(\mathbb{G}_n)$ obtained by taking the closure of the polynomial ring with respect to the supremum norm on the symmetrized poly-disc.

2.1. **Orthonormal basis and kernel function.** A *partition* $\boldsymbol{p}$ is any finite sequence $\boldsymbol{p} := (p_1, \ldots, p_n)$ of non-negative integers in decreasing order, that is,

$$p_1 \geq \cdots \geq p_n.$$

We let $[n]$ denote the set of all partitions of size n. If a partition $\boldsymbol{p}$ also has the the property $p_1 > p_2 > \cdots > p_n \geq 0$, then we may write $\boldsymbol{p} = \boldsymbol{m} + \boldsymbol{\delta}$, where $\boldsymbol{m}$ is some partition in $[n]$ and $\boldsymbol{\delta} = (n-1, n-2, \ldots, 1, 0)$. Let $[\![n]\!]$ be the set of all partitions of the form $\boldsymbol{m} + \boldsymbol{\delta}$ for $\boldsymbol{m} \in [n]$.

Let $\boldsymbol{z}^{\boldsymbol{m}} := z_1^{m_1} \cdots z_n^{m_n}$, $\boldsymbol{m} \in [n]$, be a monomial. Consider the polynomial $a_{\boldsymbol{m}}$ obtained by anti-symmetrizing the monomial $\boldsymbol{z}^{\boldsymbol{m}}$:

$$a_{\boldsymbol{m}}(\boldsymbol{z}) := \sum_{\sigma \in \sum_n} \operatorname{sgn}(\sigma)\, \boldsymbol{z}^{\boldsymbol{m}_\sigma},$$

where $\boldsymbol{z}^{\boldsymbol{m}_\sigma} = z_1^{m_{\sigma(1)}} \cdots z_n^{m_{\sigma(n)}}$. Thus for any $\boldsymbol{p} \in [\![n]\!]$, we have

$$a_{\boldsymbol{p}}(\boldsymbol{z}) = a_{\boldsymbol{m}+\boldsymbol{\delta}}(\boldsymbol{z}) = \sum_{\sigma \in \sum_n} \operatorname{sgn}(\sigma)\, \boldsymbol{z}^{(\boldsymbol{m}+\boldsymbol{\delta})_\sigma},$$

$\boldsymbol{m} \in [n]$ and it follows that

$$a_{\boldsymbol{p}}(\boldsymbol{z}) = a_{\boldsymbol{m}+\boldsymbol{\delta}}(\boldsymbol{z}) = \det\left((\!(z_i^{p_j})\!)_{i,j=1}^n \right),\ \boldsymbol{p} \in [\![n]\!].$$

The following Lemma clearly shows that the functions $a_{\boldsymbol{p}}$, $\boldsymbol{p} \in [\![n]\!]$, are orthogonal in the Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$.

Lemma 2.1. The set $S := \{m_{\sigma(k)} - m'_{\nu(k)} : \sigma, \nu \in \Sigma_n, m_i > m_j, m'_i > m'_j \text{ for } i < j, m_1 \neq m'_1, 1 \leq k \leq n\} \neq \{0\}$.

Proof. If there exist $\sigma, \nu \in \Sigma_n$ such that $\sigma(k) = \nu(k) = 1$ for some k, $1 \leq k \leq n$, then $m_{\sigma(k)} - m'_{\nu(k)} = m_1 - m'_1 \neq 0$. Therefore, in this case, $S \neq \{0\}$.

Now suppose that there exists no $k, 1 \leq k \leq n$, for which $\sigma(k) = \nu(k) = 1$. In this case, if possible, let $S = \{0\}$. Fix $\sigma, \nu \in \Sigma_n$. Then there exists k such that $\sigma(k) = 1$ and $\nu(k) = j > 1$. Now, $m_{\sigma(k)} - m'_{\nu(k)} = m_1 - m'_j$. Pick $k' \neq k$ such that $\sigma(k') = j, \nu(k') = \ell$, $\ell \neq j$. Thus $m_{\sigma(k')} - m'_{\nu(k')} = m_j - m'_\ell$. Choose $k'' \neq k$ such that $\nu(k'') = 1, \sigma(k'') = r > 1$ and $m_{\sigma(k'')} - m_{\nu(k'')} = m_r - m'_1$. However, we have $m_1 - m'_j = m_j - m'_\ell = m_r - m'_1 = 0$. Clearly, $m_r = m'_1 > m'_j = m_1$. Hence $m_r > m_1$ with $r > 1$, which is a contradiction. $\square$

For $\lambda > 1$, the preceding Lemma says that the vectors $\boldsymbol{z}^{\boldsymbol{p}_\sigma}$ are orthogonal, and hence the set $\{a_{\boldsymbol{p}} : \boldsymbol{p} \in [\![n]\!]\}$ consists of mutually orthogonal vectors in $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$. The linear span these vectors is dense in the Hilbert space $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$. For $\boldsymbol{p} = (p_1, \ldots, p_n) \in [\![n]\!]$. The norm of the vector $a_{\boldsymbol{p}}$ is easily calculated:

$$\begin{aligned}\|a_{\boldsymbol{p}}\|_{\mathbb{A}^{(\lambda)}(\mathbb{D}^n)} &= \Big\| \det\Big((\!(z_i^{p_j})\!)_{i,j=1}^n \Big) \Big\|_{\mathbb{A}^{(\lambda)}(\mathbb{D}^n)} \\ &= \Big\| \sum_{\sigma\in\Sigma_n} \operatorname{sgn}(\sigma) \prod_{k=1}^n z_k^{p_{\sigma(k)}} \Big\|_{\mathbb{A}^{(\lambda)}(\mathbb{D}^n)} = \sqrt{\frac{n!\boldsymbol{p}!}{(\lambda)_{\boldsymbol{p}}}},\end{aligned}$$

where $\boldsymbol{p}! = \prod_{j=1}^n m_j!$ and $(\lambda)_{\boldsymbol{p}} = \prod_{j=1}^n (\lambda)_{m_j}$. Putting $c_{\boldsymbol{p}} = \sqrt{\frac{(\lambda)_{\boldsymbol{p}}}{n!\boldsymbol{p}!}}$, we see that

$$\{e_{\boldsymbol{p}} = c_{\boldsymbol{p}}\, a_{\boldsymbol{p}} : \boldsymbol{p} \in [\![n]\!]\}$$

is an orthonormal basis for $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$. So the reproducing kernel $K^{(\lambda)}_{\text{anti}}$ for $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$ is given by

$$K^{(\lambda)}_{\text{anti}}(\boldsymbol{z},\boldsymbol{w}) = \sum_{\boldsymbol{p}\in[\![n]\!]} e_{\boldsymbol{p}}(\boldsymbol{z})\overline{e_{\boldsymbol{p}}(\boldsymbol{w})}, \text{ for } \boldsymbol{z},\boldsymbol{w}\in\mathbb{D}^n.$$

For all $\sigma \in \Sigma_n$, we have $e_{\sigma(\boldsymbol{p})}(\boldsymbol{z})\overline{e_{\sigma(\boldsymbol{p})}(\boldsymbol{w})} = e_{\boldsymbol{p}}(\boldsymbol{z})\overline{e_{\boldsymbol{p}}(\boldsymbol{w})}$, $\boldsymbol{z},\boldsymbol{w}\in\mathbb{D}^n$. Therefore, it follows that

$$K^{(\lambda)}_{\text{anti}}(\boldsymbol{z},\boldsymbol{w}) = \sum_{\boldsymbol{p}\in[\![n]\!]} e_{\boldsymbol{p}}(\boldsymbol{z})\overline{e_{\boldsymbol{p}}(\boldsymbol{w})} = \frac{1}{n!}\sum_{\boldsymbol{p}\geq 0} e_{\boldsymbol{p}}(\boldsymbol{z})\overline{e_{\boldsymbol{p}}(\boldsymbol{w})}, \tag{2.1}$$

where $\boldsymbol{p}\geq 0$ stands for all multi-indices $\boldsymbol{p} = (p_1,\ldots,p_n)\in\mathbb{Z}^n$ with the property that each $p_i \geq 0$ for $1\leq i\leq n$.

Proposition 2.2. The reproducing kernel $K^{(\lambda)}_{\text{anti}}$ is given explicitly by the formula:

$$K^{(\lambda)}_{\text{anti}}(\boldsymbol{z},\boldsymbol{w}) = \frac{1}{n!}\det\Big((\!((1-z_j\bar{w}_k)^{-\lambda})\!)_{j,k=1}^n \Big), \ \boldsymbol{z},\boldsymbol{w}\in\mathbb{D}^n.$$

Proof. For $\boldsymbol{z},\boldsymbol{w}$ in $\mathbb{D}^n$, we have

$$\begin{aligned}\sum_{\boldsymbol{p}\geq 0} e_{\boldsymbol{p}}(\boldsymbol{z})\overline{e_{\boldsymbol{p}}(\boldsymbol{w})} &= \frac{1}{n!}\sum_{\boldsymbol{p}\geq 0}\frac{(\lambda)_{\boldsymbol{p}}}{\boldsymbol{p}!}\det\Big((\!(z_k^{p_j})\!)_{j,k=1}^n \Big)\det\Big((\!(\bar{w}_k^{p_j})\!)_{j,k=1}^n \Big) \\ &= \frac{1}{n!}\sum_{\boldsymbol{p}\geq 0}\frac{(\lambda)_{\boldsymbol{p}}}{\boldsymbol{p}!}\Big(\sum_{\sigma\in\Sigma_n}\operatorname{sgn}(\sigma)\prod_{i=1}^n z_i^{p_{\sigma(i)}}\Big)\Big(\sum_{\nu\in\Sigma_n}\operatorname{sgn}(\nu)\prod_{i=1}^n \bar{w}_{\nu(i)}^{p_i}\Big) \\ &= \frac{1}{n!}\sum_{\boldsymbol{p}\geq 0}\frac{(\lambda)_{\boldsymbol{p}}}{\boldsymbol{p}!}\sum_{\sigma,\nu\in\Sigma_n}\operatorname{sgn}(\sigma)\operatorname{sgn}(\nu)\prod_{i=1}^n (z_i\bar{w}_{\nu\sigma(i)})^{p_{\sigma(i)}} \\ &= \frac{1}{n!}\sum_{\sigma,\nu\in\Sigma_n}\operatorname{sgn}(\nu\sigma)\sum_{\boldsymbol{p}\geq 0}\frac{(\lambda)_{\boldsymbol{p}}}{\boldsymbol{p}!}\prod_{i=1}^n (z_i\bar{w}_{\nu\sigma(i)})^{p_{\sigma(i)}} \\ &= \frac{1}{n!}\sum_{\sigma,\nu\in\Sigma_n}\operatorname{sgn}(\nu\sigma)\prod_{i=1}^n (1-z_i\bar{w}_{\nu\sigma(i)})^{-\lambda} \\ &= \frac{1}{n!}\sum_{\psi\in\Sigma_n}\operatorname{sgn}(\psi)\sum_{\substack{\nu\sigma=\psi \\ \sigma,\nu\in\Sigma_n}}\prod_{i=1}^n (1-z_i\bar{w}_{\nu\sigma(i)})^{-\lambda} \\ &= \sum_{\psi\in\Sigma_n}\operatorname{sgn}(\psi)\prod_{i=1}^n (1-z_i\bar{w}_{\psi(i)})^{-\lambda} \\ &= \det\Big((\!((1-z_j\bar{w}_k)^{-\lambda})\!)_{j,k=1}^n \Big)\end{aligned}$$

The desired equality follows from (2.1). $\square$

2.2. **Schur function.** The determinant function $a_{\boldsymbol{m}+\boldsymbol{\delta}}$ is divisible by each of the difference $z_i - z_j$, $1 \le i < j \le n$ and hence by the product

$$\prod_{1\le i<j\le n}(z_i - z_j) = \det\left(\!\left(\!\left(z_i^{n-j}\right)\!\right)_{i,j=1}^n\right) = a_{\boldsymbol{\delta}}(\boldsymbol{z}).$$

The quotient $S_{\boldsymbol{p}} := a_{\boldsymbol{m}+\boldsymbol{\delta}}/a_{\boldsymbol{\delta}}$, $\boldsymbol{p} = \boldsymbol{m}+\boldsymbol{\delta}$, is therefore well-defined and is called the Schur function [5, pp. 40]. The Schur function $S_{\boldsymbol{p}}$ is symmetric and defines a function on the symmetrized polydisc $\mathbb{G}_n$. Since the Jacobian of the map $\mathbf{s} : \mathbb{D}^n \to \mathbb{G}_n$ coincides with $a_{\boldsymbol{\delta}}$, it follows from Lemma 2.1 that the Schur functions $\{S_{\boldsymbol{p}} := a_{\boldsymbol{m}+\boldsymbol{\delta}}/a_{\boldsymbol{\delta}} : \boldsymbol{p} \in [\![n]\!]\}$ is a set of mutually orthogonal vectors in $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$. The linear span of these vectors is dense in $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$. Also, the norms of these vectors coincide with those of $a_{\boldsymbol{p}}$ in $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$, modulo the normalizing constant $\|J_{\mathbf{s}}\|_\lambda$, via the unitary map Γ. Hence $\|S_{\boldsymbol{p}}\| = \sqrt{\frac{n!\boldsymbol{p}!}{\|J_{\mathbf{s}}\|_\lambda(\lambda)_{\boldsymbol{p}}}}$, $\boldsymbol{p} \in [\![n]\!]$. The set $\{\hat{e}_{\boldsymbol{p}} = c_{\boldsymbol{p}} S_{\boldsymbol{p}} : \boldsymbol{p} \in [\![n]\!]\}$ is an orthonormal basis for $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$, where $c_{\boldsymbol{p}} = \sqrt{\frac{\|J_{\mathbf{s}}\|_\lambda(\lambda)_{\boldsymbol{p}}}{n!\boldsymbol{p}!}}$. Thus we have proved:

Theorem 2.3. For $\lambda > 0$, the reproducing kernel $\mathbf{B}^{(\lambda)}_{\mathbb{G}_n}$ for the weighted Bergman space $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ on the symmetrized poly-disc is given by the formula:

$$\mathbf{B}^{(\lambda)}_{\mathbb{G}_n}(\mathbf{s}(\boldsymbol{z}), \mathbf{s}(\boldsymbol{w})) = \sum_{\boldsymbol{p}\in[\![n]\!]} c_{\boldsymbol{p}}^2 S_{\boldsymbol{p}}(\boldsymbol{z})\overline{S_{\boldsymbol{p}}(\boldsymbol{w})} \tag{2.2}$$

$$= \frac{\|J_{\mathbf{s}}\|_\lambda^2}{n!}\frac{\det\left(\!\left(\!\left((1 - z_j\bar{w}_k)^{-\lambda}\right)\!\right)_{j,k=1}^n\right)}{a_{\boldsymbol{\delta}}(\boldsymbol{z})\overline{a_{\boldsymbol{\delta}}(\boldsymbol{w})}} \tag{2.3}$$

for $\boldsymbol{z}, \boldsymbol{w}$ in $\mathbb{D}^n$.

The case $\lambda = 2$ corresponds to the Bergman space on the symmetrized polydisc. In this case, $\|J_{\mathbf{s}}\|_2 = 1$ and the formula for the the Bergman kernel, except for the constant factor $\frac{1}{n!}$, was found in [4]. (The factor $\frac{1}{n!}$ appears in our formula because we have chosen the normalization $\|1\| = 1$ for the constant function 1 in the Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$. However, as we will see below, it disappears for the Hardy space on the symmetrized polydisc $\mathbb{G}_n$.) However, the methods of this paper are very different form that of [4], and we hope it sheds some light on the nature of these kernel functions.

Corollary 2.4. The Bergman kernel on the symmetrized polydisc in $\mathbb{C}^2$ is given by the formula

$$\mathbf{B}^{(2)}_{\mathbb{G}_n}(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2}\frac{2(1 + u_2\bar{v}_2) - u_1\bar{v}_1}{((1 - u_2\bar{v}_2)^2 - (u_1 - u_2\bar{v}_1)(\bar{v}_1 - \bar{v}_2u_1))^2},$$

$\boldsymbol{u} = (u_1, u_2), \boldsymbol{v} = (v_1, v_2) \in \mathbb{G}_n$.

This corollary gives an explicit formula for the Bergman kernel function for the symmetrized polydisc which is independent of the symmetriztion map $\mathbf{s}$. It is possible to write down similar formulae for $n > 2$ using the Jacob-Trudy identity [3, pp. 455].

3. The Hardy space and the Szegö kernel for the symmetrized polydisc

Let $d\Theta$ be the normalized Lebesgue measure on the torus $\mathbb{T}^n$, where $\mathbb{T} = \{\alpha : |\alpha| = 1\}$ is the unit circle. Let $d\Theta_{\mathbf{s}}$ be the measure on the symmetrized polydisc $\mathbb{G}_n$ obtained by the change of variable formula:

$$\int_{\partial\mathbb{G}_n} f\, d\Theta_{\mathbf{s}} = \int_{\mathbb{T}^n} (f \circ \mathbf{s})\, |J_{\mathbf{s}}|^2 d\Theta,$$

where, as before, $J_{\mathbf{s}}(\boldsymbol{z})$ is the complex Jacobian of the symmetrization map $\mathbf{s}$. The Hardy space $H^2(\mathbb{G}_n)$ on the symmetrized polydisc $\mathbb{G}_n$ consists of holomorphic functions on $\mathbb{G}_n$ with the property:

$$\sup{}_{0<r<1} \int_{\mathbb{T}^n} |f \circ \mathbf{s}(r\, e^{i\Theta})|^2 |J_{\mathbf{s}}(r\, e^{i\Theta})|^2 d\Theta < \infty,\ e^{i\Theta} \in \mathbb{T}^n.$$

We set the norm of $f \in H^2(\mathbb{G}_n)$ to be

$$\|f\| = \|J_{\boldsymbol{s}}\|^{-1}\Big\{\sup_{0<r<1}\int_{\mathbb{T}^n} |f\circ \mathbf{s}(r\,e^{i\Theta})|^2 |J_{\mathbf{s}}(r\,e^{i\Theta})|^2 d\Theta\Big\}^{1/2},$$

where $\|J_{\boldsymbol{s}}\|^2 = \int_{\mathbb{T}^n}|J_{\mathbf{s}}|^2 d\Theta$. This ensures, as before, $\|1\| = 1$. Let $H^2(\mathbb{D}^n)$ be the Hardy space on the polydisc $\mathbb{D}^n$. The operator $\Gamma : H^2(\mathbb{G}_n) \longrightarrow H^2(\mathbb{D}^n)$ given by $\Gamma(f) = \|J_{\boldsymbol{s}}\|^{-1} J_{\mathbf{s}}\,(f\circ \mathbf{s})$ for $f \in H^2(\mathbb{G}_n)$ is then easily seen to be an isometry. The subspace of anti-symmetric functions $H^2_{\rm anti}(\mathbb{D}^n)$ in the Hardy space $H^2(\mathbb{D}^n)$ coincides with the image of $H^2(\mathbb{G}_n)$ under the isometry Γ. Thus the operator $\Gamma : H^2(\mathbb{G}_n) \longrightarrow H^2_{\rm anti}(\mathbb{D}^n)$ is onto and therefore unitary.

The functions $a_{\boldsymbol{p}}$, $\boldsymbol{p} \in [\![n]\!]$ continue to be an orthogonal spanning set for the subspace $H^2_{\rm anti}(\mathbb{D}^n)$. All of the vectors $a_{\boldsymbol{p}}$ have the same norm, namely, $\sqrt{n}!$. Consequently, the set of vectors $\{e_{\boldsymbol{p}}(\boldsymbol{z}) := \frac{1}{\sqrt{n!}} a_{\boldsymbol{p}}(\boldsymbol{z}) : \boldsymbol{p} \in [\![n]\!]\}$ is an orthonormal basis for the subspace $H^2_{\rm anti}(\mathbb{D}^n)$ of the Hardy space on the polydisc, while the set $\{\hat{e}_{\boldsymbol{p}} := \frac{\|J_{\boldsymbol{s}}\|}{\sqrt{n!}} S_{\boldsymbol{p}} : \boldsymbol{p} \in [\![n]\!]\}$ forms an orthonormal basis for the Hardy space $H^2(\mathbb{G}_n)$ of the symmetrized polydisc $\mathbb{G}_n$ via the unitary map Γ. However, $\|J_{\boldsymbol{s}}\| = \sqrt{n!}$ and consequently, $\hat{e}_{\boldsymbol{p}} = S_{\boldsymbol{p}}$. Thus computations similar to the case $\lambda > 1$ yields an explicit formula for the reproducing kernel $K^{(1)}_{\rm anti}(\boldsymbol{z}, \boldsymbol{w})$ of the subspace $H^2_{\rm anti}(\mathbb{D}^n)$. Indeed,

$$K^{(1)}_{\rm anti}(\boldsymbol{z}, \boldsymbol{w}) = \frac{1}{n!}\det\Big(\big(\!\big((1 - z_j\bar{w}_k)^{-1}\big)\!\big)_{j,k=1}^n\Big).$$

This is the limiting case, as $\lambda \to 1$.

Let $\mathbb{S}_{\mathbb{G}_n}$ be the Sz̈ego kernel for the symmetrized polydisc $\mathbb{G}_n$. Clearly,

$$\mathbb{S}_{\mathbb{G}_n}(\mathbf{s}(\boldsymbol{z}), \mathbf{s}(\boldsymbol{w})) = \frac{\det\Big(\big(\!\big((1 - z_j\bar{w}_k)^{-1}\big)\!\big)_{j,k=1}^n\Big)}{J_{\mathbf{s}}(\boldsymbol{z})\overline{J_{\mathbf{s}}(\boldsymbol{w})}},\ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n.$$

Now, using the well-known identity due to Cauchy [5, (4.3) pp 63], we have

$$\mathbb{S}_{\mathbb{G}_n}(\mathbf{s}(\boldsymbol{z}), \mathbf{s}(\boldsymbol{w})) = \sum_{\boldsymbol{p}\in[\![n]\!]} S_{\boldsymbol{p}}(\boldsymbol{z})\overline{S_{\boldsymbol{p}}(\boldsymbol{w})} = \prod_{j,k=1}^n (1 - z_j\bar{w}_k)^{-1},\ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n.$$

Therefore, we have a formula for the Sz̈ego kernel of the symmetrized polydisc $\mathbb{G}_n$, which we separately record below.

Theorem 3.1. The Sz̈ego kernel $\mathbb{S}_{\mathbb{G}_n}$ of the symmetrized polydisc $\mathbb{G}_n$ is given by the formula

$$\mathbb{S}_{\mathbb{G}_n}(\mathbf{s}(\boldsymbol{z}), \mathbf{s}(\boldsymbol{w})) = \prod_{j,k=1}^n (1 - z_j\bar{w}_k)^{-1}, \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n.$$

4. An alternative approach to the computation of the kernel function

Recall that the weighted Bergman space $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ on the polydisc $\mathbb{D}^n$ is the n-fold tensor product $\otimes_{i=1}^n \mathbb{A}^{(\lambda)}(\mathbb{D})$ of the weighted Bergman spaces $\mathbb{A}^{(\lambda)}(\mathbb{D})$ on the unit disc $\mathbb{D}$. The equivalence class $\widehat{\Sigma}_n$ of finite dimensional irreducible representations of the permutation group Σ_n on n symbols is parametrized by the partitions $\boldsymbol{p} \in [n]$. Let $(V_{\boldsymbol{p}}, \boldsymbol{p})$ be a representation corresponding to the partition $\boldsymbol{p}$. Then we have the decomposition

$$\mathbb{A}^{(\lambda)}(\mathbb{D}^n) = \oplus_{\boldsymbol{p}\in[n]}\mathbb{A}^{(\lambda)}(\mathbb{D}^n, \boldsymbol{p}),$$

where

$$\mathbb{A}^{(\lambda)}(\mathbb{D}^n, \boldsymbol{p}) = \big\{f \in \mathbb{A}^{(\lambda)}(\mathbb{D}^n, V_{\boldsymbol{p}}) : \tau(\boldsymbol{s})f(\boldsymbol{s}^{-1}\cdot \boldsymbol{z}) = f(\boldsymbol{z}), \boldsymbol{s} \in \Sigma_n\big\}$$

and $\mathbb{A}^{(\lambda)}(\mathbb{D}^n, \boldsymbol{p}) \cong \mathbb{A}^{(\lambda)}(\mathbb{D}, V'_{\boldsymbol{p}}) \otimes V'_{\boldsymbol{p}}$. The orthogonal projection $\mathbb{P}_{\boldsymbol{p}} : \mathbb{A}^{(\lambda)}(\mathbb{D}^n) \to \mathbb{A}^{(\lambda)}(\mathbb{D}^n, \boldsymbol{p})$ is given by the formula

$$(\mathbb{P}_{\boldsymbol{p}} f)(\boldsymbol{z}) = \frac{\chi_{\boldsymbol{p}}(1)}{n!} \sum_{\tau} \chi(\tau) f(\tau^{-1} \cdot \boldsymbol{z}),$$

where the sum is over all τ in Σ_n and $\chi_{\boldsymbol{p}}$ is the character corresponding to the representation $V_{\boldsymbol{p}}$. Schur orthogonality relations ensure that $\mathbb{P}_{\boldsymbol{p}}^2 = \mathbb{P}_{\boldsymbol{p}}$ and it follows that $\mathbb{P}_{\boldsymbol{p}}$ is a projection. Let V_{sgn} be the sign representation of the permutation group Σ_n and $\mathbb{P}_{\text{sgn}}$ be the corresponding projection.

Theorem 4.1. The reproducing kernel $K^{(\lambda)}_{\text{sgn}}$ of the Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{D}^n, \text{sgn})$ is given by the formula

$$\begin{aligned} K^{(\lambda)}_{\text{sgn}}(\boldsymbol{z}, \boldsymbol{w}) &= \big(\mathbb{P}_{\text{sgn}} \otimes \mathbb{P}^*_{\text{sgn}}\big)\Big(\prod_{i=1}^n (1 - z_i \bar{w}_i)^{-\lambda}\Big) \\ &= \frac{a_{\boldsymbol{\delta}}(\boldsymbol{z})\overline{a_{\boldsymbol{\delta}}(\boldsymbol{w})}}{n!} \sum_{\boldsymbol{p} \in [\![n]\!]} \frac{(\lambda)_{\boldsymbol{m}+\boldsymbol{\delta}}}{(\boldsymbol{m}+\boldsymbol{\delta})!} S_{\boldsymbol{p}}(z)\overline{S_{\boldsymbol{p}}(w)}, \end{aligned}$$

where $S_{\boldsymbol{p}}$ is the Schur function with $\boldsymbol{p} = \boldsymbol{m} + \boldsymbol{\delta}$.

Proof. Recall that $K^{(\lambda)}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\boldsymbol{m} \geq 0}^{\infty} \frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!} (\boldsymbol{z}\bar{\boldsymbol{w}})^{\boldsymbol{m}}$, $\lambda > 1$, is the reproducing kernel of the weighted Bergman spaces $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$. Therefore, we have

$$\big(\mathbb{P}_{\text{sgn}} \otimes I\big) K^{(\lambda)}_{\boldsymbol{w}}(\boldsymbol{z}) = \sum_{\boldsymbol{m} \geq 0}^{\infty} \frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!} \bar{\boldsymbol{w}}^{\boldsymbol{m}} \mathbb{P}_{\text{sgn}}\big(\boldsymbol{z}^{\boldsymbol{m}}\big).$$

However, $\mathbb{P}_{\text{sgn}}\big(\boldsymbol{z}^{\boldsymbol{m}}\big) = \frac{1}{n!} \det\Big(\big(\!\big(z_i^{m_j}\big)\!\big)\Big)$ which is zero unless $\boldsymbol{m}$ is in the orbit under Σ_n of the weight $\boldsymbol{p}$ in $[\![n]\!]$. So, we conclude that

$$\begin{aligned} \big(\mathbb{P}_{\text{sgn}} \otimes \mathbb{P}^*_{\text{sgn}}\big) K^{(\lambda)}(\boldsymbol{z}, \boldsymbol{w}) &= \sum_{\boldsymbol{m} \geq 0}^{\infty} \frac{(\lambda)_{\boldsymbol{m}}}{\boldsymbol{m}!} \mathbb{P}_{\text{sgn}}\big(\boldsymbol{z}^{\boldsymbol{m}}\big) \mathbb{P}_{\text{sgn}}\big(\bar{\boldsymbol{w}}^{\boldsymbol{m}}\big) \\ &= \sum_{\boldsymbol{p} \in [\![n]\!]} \gamma_{\boldsymbol{p}} \frac{(\lambda)_{\boldsymbol{p}}}{\boldsymbol{p}!}\, a_{\boldsymbol{p}}(\boldsymbol{z})\overline{a_{\boldsymbol{p}}(\boldsymbol{w})} \\ &= a_{\boldsymbol{\delta}}(\boldsymbol{z})\overline{a_{\boldsymbol{\delta}}(\boldsymbol{w})} \sum_{\boldsymbol{p} \in [\![n]\!]} \gamma_{\boldsymbol{p}} \frac{(\lambda)_{\boldsymbol{p}}}{(\boldsymbol{p})!} S_{\boldsymbol{p}}(\boldsymbol{z})\overline{S_{\boldsymbol{p}}(\boldsymbol{w})}. \end{aligned}$$

It is then easy to see that $\gamma_{\boldsymbol{p}} = \frac{1}{n!}$ completing the proof. $\square$

Clearly, the two kernel functions $K^{(\lambda)}_{\text{sgn}}$ and $K^{(\lambda)}_{\text{anti}}$ are equal. As before, the kernel function $K^{(\lambda)}_{\text{sgn}}$, via the unitary map Γ, gives a kernel function for the weighted Bergman spaces $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ on the symmetrized polydisc $\mathbb{G}_n$. Further more, if $\lambda = 1$, then

$$\begin{aligned} \mathbb{S}_{\mathbb{G}_n}\big(\boldsymbol{s}(\boldsymbol{z}), \boldsymbol{s}(\boldsymbol{w})\big) &= \frac{n!}{a_{\boldsymbol{\delta}}(\boldsymbol{z})\overline{a_{\boldsymbol{\delta}}(\boldsymbol{w})}} K^{(1)}_{\text{sgn}}(\boldsymbol{z}, \boldsymbol{w}) \\ &= \sum_{\boldsymbol{p} \in [\![n]\!]} S_{\boldsymbol{p}}(\boldsymbol{z})\overline{S_{\boldsymbol{p}}(\boldsymbol{w})} \\ &= \prod_{i,j=1}^n (1 - z_i \bar{w}_j)^{-1},\ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n, \end{aligned}$$

where the last equality is the formula [5, (4.3), pp. 63].

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(Gadadhar Misra) Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India
E-mail address, Misra: `gm@math.iisc.ernet.in`

(Subrata Shyam Roy) Indian Institute of Science Education and Research, Kolkata, Mohanpur Campus, Mohanpur (West Bengal) - 741 252, India
E-mail address, Subrata Shyam Roy: `ssroy@iiserkol.ac.in`

(Genkai Zhang) Department of Mathematics, Chalmers University of Technology and Gothenburg University, S-412 96 Gothenburg, Sweden
E-mail address, Genkai Zhang: `genkai@chalmers.se`