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Part I

Automorphic Analysis

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Chapter 0

Introduction

0.1 Motivation: Discrete groups in complex analysis and mathematical physics

0.1.1 Universal covering of Kähler manifolds

A complex Kähler manifold M (not necessarily compact) has a universal covering manifold $D = \tilde{M}$ such that

$$M = D/\Gamma$$

where $\Gamma = \pi_1(M)$ is a discrete group of holomorphic **deck transformations** acting on D. Thus complex analysis on M is related to 'automorphic' analysis on D. We call Γ **co-compact** if the quotient space $M = D/\Gamma$ is compact. More generally, we call Γ of **finite covolume** if D/Γ has finite volume under the volume form induced by the Kähler metric. For 'hyperbolic' Kähler manifolds, D can often be realized as a bounded domain in $\mathbf{C}^d$, and in important cases as a **bounded symmetric domain**

$$D = K\backslash G,$$

where G is a semi-simple real Lie group and K is a maximal compact subgroup. In this case we have a discrete subgroup

$$\Gamma \subset G$$

which is called a **lattice** if it is co-compact. In the **1-dimensional case** a compact Riemann surface M is hyperbolic iff it has genus > 1. By the uniformization theorem, $D = \tilde{M}$ is the unit disk (or upper half-plane). Therefore

$$\Gamma \subset G = \mathrm{SL}_2(\mathbf{R})$$

becomes a discrete group of Möbius transformations. One (i.e., Poincaré) calls Γ of Klein type if it is co-compact and of Fuchs type if it has finite co-volume. Thus the all-important **modular group**

$$\mathrm{SL}_2(\mathbf{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{Z}^{2\times 2} : \ ad - bc = 1\}$$

is Fuchsian, but not Kleinian. In **higher dimensions**, there exist differential geometric criteria for M to ensure that $D = \tilde{M}$ is the unit ball in $\mathbf{C}^d$, or more generally a bounded symmetric domain. These criteria involve important geometric invariants of the underlying Kähler manifold.

0.1.2 Teichmüller space

For a compact Riemann surface X the **Teichmüller space**

$$\mathcal{T}(X) = \mathrm{Conf(X)}/\mathrm{Diff}^0(X)$$

consists of all conformal structures on X modulo equivalence by diffeomorphisms which are isotopic to the identity. Via Beltrami differentials, $\mathcal{T}(X)$ can be realized as a convex bounded domain. However, the 'true' **moduli space**

$$\mathcal{M}(X) = \mathrm{Conf(X)}/\mathrm{Diff}(X)$$

consists of all conformal structures modulo equivalence by the full diffeomorphism group. Thus

$$\mathcal{M}(X) = \mathcal{T}(X)/\Gamma$$

where

$$\Gamma = \mathrm{Diff}(X)/\mathrm{Diff}^0(X)$$

is the discrete group of components of Diff(X), also called the **mapping class group**. The Teichmüller space becomes the universal covering

$$\mathcal{T}(X) = \widetilde{\mathcal{M}(X)}.$$

As an important step, this moduli space has to be **compactified** to a projective algebraic variety (or stack) by adding points at infinity, so $\mathcal{M}(X)$ becomes a Zariski-open (dense) subset of the compactification. Actually, the important case is where X is not compact, but arises from a compact Riemann surface by removing finitely many punctures. Then the compactification $\overline{\mathcal{M}}_{g,n}$ is the Mumford-Deligne moduli space.

0.1.3 String theory duality groups

In the preceeding two examples, Γ is a discrete group of holomorphic transformations. These give rise to automorphic forms, which are better regarded as sections of a holomorphic line bundle (therefore 'forms' instead of 'functions'). On the other hand, in number theory (Langlands program) and mathematical physics one encounters discrete subgroups of 'real' Lie groups which give rise to real automorphic (or better, invariant) functions. For example, for a metric g on a pseudo-Riemannian manifold of Minkowski signature, the Einstein field equation

$$\mathrm{Ric}(g) = 0$$

arises from a variational principle under the Einstein-Hilbert action

$$\mathcal{L}(g) = \int\limits_X dVol_g \ \mathrm{Scal}(g).$$

Extending this concept to **super-gravity** in 10 dimensions, the corresponding solutions, when compactified on tori $\mathbf{T}^n$ of dimension $0 \leqslant n \leqslant 10$ have scalar moduli which transform under the **super-gravity duality groups**

$$A_n(\mathbf{R}),\ D_n(\mathbf{R}),\ E_n(\mathbf{R}),$$

the real forms of algebraic groups of ADE-type. Now super-gravity is regarded as the low-energy limit of string theory. Passing to string theory, which is a quantum field theory, one expects again that the corresponding solutions have scalar moduli which transform under the **super-string duality groups**

$$A_n(\mathbf{Z}),\ D_n(\mathbf{Z}),\ E_n(\mathbf{Z}),$$

which form a **lattice** within the real duality groups. More precisely, for string-theory backgrounds of the form

$$\mathbf{R}^{1,9-d} \times \mathbf{T}^d$$

we obtain the following duality groups

$$\mathrm{SU}_2^{\mathbf{R}} \backslash \mathrm{SL}_2^{\mathbf{R}} / \mathrm{SL}_2^{\mathbf{Z}}, \quad d = 0$$

$$\mathrm{SU}_2^{\mathbf{R}} \backslash \mathrm{GL}_2^{\mathbf{R}} / \mathrm{GL}_2^{\mathbf{Z}}, \quad d = 1$$

$$(\mathrm{SU}_2^{\mathbf{R}} \times \mathrm{SU}_3^{\mathbf{R}}) \backslash (\mathrm{SL}_2^{\mathbf{R}} \times \mathrm{SL}_3^{\mathbf{R}}) / (\mathrm{SL}_2^{\mathbf{Z}} \times \mathrm{SL}_3^{\mathbf{Z}}), \quad d = 0$$

$$\mathrm{SU}_5^{\mathbf{R}} \backslash \mathrm{SL}_5^{\mathbf{R}} / \mathrm{SL}_5^{\mathbf{Z}}, \quad d = 3$$

$$(\mathrm{SU}_5^{\mathbf{R}} \times_{\mathbf{Z}_2} \mathrm{SU}_5^{\mathbf{R}}) \backslash \mathrm{SU}_{5,5}^{\mathbf{R}} / \mathrm{SU}_{5,5}^{\mathbf{Z}}, \quad d = 4$$

$$(\mathrm{U}_4^{\mathbf{H}} / \mathbf{Z}_2) \backslash \mathrm{E}_6^{\mathbf{R}} / \mathrm{E}_6^{\mathbf{Z}}, \quad d = 5$$

$$(\mathrm{SU}_8^{\mathbf{C}} / \mathbf{Z}_2) \backslash \mathrm{E}_7^{\mathbf{R}} / \mathrm{E}_7^{\mathbf{Z}}, \quad d = 6$$

$$(\mathrm{SU}_{16}^{\mathbf{R}} / \mathbf{Z}_2) \backslash \mathrm{E}_8^{\mathbf{R}} / \mathrm{E}_8^{\mathbf{Z}}, \quad d = 7$$

In general these can be written as the series E_{d+1} with Dynkin diagram

0.1.4 Free group von Neumann algebras

For any free group Γ in ℓ-generators (more generally, every group with only infinite conjugacy classes) the group von Neumann algebra $W^*(\Gamma) = \Gamma''$ (bicommutant) is a von Neumann factor of type II_1. We will show that this arises in the Berezin quantization on weighted Bergman spaces $H^2_\nu(\mathbf{D})$ over the unit disk (or upper half-plane), where ν becomes the number of generators.

0.2 Basic concepts

0.2.1 Holomorphic automorphism groups

For a complex manifold D (or even more general, a complex analytic space) we let $\mathrm{Aut}(D)$ denote the 'automorphism' group of all biholomorphic transformations of D, acting from the right: $(z, g) \mapsto z \cdot g$. It is known that for a bijective holomorphic map $g : D \to D$ the inverse map $g^{-1} : D \to D$ is also holomorphic.

If D is a locally compact and locally connected topological space, then Arens has shown that the homeomorphism group $\mathrm{Top}(X)$, endowed with the so-called **compact-open topology**, is a topological group and the evaluation map $D \times G \to D$ is jointly continuous. In particular, for a domain $D \subset \mathbf{C}^d$ we consider the identity component

$$G = \mathrm{Aut}(D)^0$$

of the holomorphic automorphism group $\mathrm{Aut}(D) \subset \mathrm{Top}(D)$. By Arens' result this is a connected topological group. In general, it is not a Lie group. For example, $\mathrm{Aut}(\mathbf{C}^2)$ has infinite dimension, since for every entire function $f : \mathbf{C} \to \mathbf{C}$ the mapping

$$\Phi_f(z, w) := (z, w + f(z))$$

is an automorphism of $\mathbf{C}^2$, with inverse $\Phi_f^{-1} = \Phi_{-f}$. On the other hand, if $D \subset \mathbf{C}^d$ is a **bounded domain**, then G is a (finite-dimensional) Lie group by a deep theorem of H. Cartan. The first step in the proof is the following:

Lemma 1. *Let $A, B \subset D$ be compact subsets of D. Then the set*

$$G_{A,B} := \{g \in G : \ A.g \cap B \neq \emptyset\}$$

is compact.

Proof. By separability, it is enough to show that $G_{A,B}$ is sequentially compact. Consider a sequence $g_n \in G_{A,B}$. Then there exist sequences $a_n \in A, \ b_n \in B$ such that $a_n \cdot g_n = b_n$. Since $g_n^\pm$ are bounded holomorphic maps on D we may choose by Montel's theorem convergent subsequences satisfying $g_n^\pm \to g_\pm : D \to \overline{D}$. Since A, B are compact, taking further subsequences we may assume in addition that $a_n \to a \in A,\ b_n \to b \in B$. Then $a \cdot g_+ = b,\ b \cdot g_- = a$ since the evaluation map $D \times G \to D$ is jointly continuous. Choose open sets $a \in U \subset W \subset\subset D,\ b \in V \subset\subset D$ satisfying $U \cdot g_+ \subset V$ and $V \cdot g_- \subset W$.Then joint continuity implies $g_+ \circ g_-|_V = \mathrm{id},\ g_- \circ g_+|_U = \mathrm{id}$. A similar argument shows that $D \cdot g_\pm \subset D$. □

Corollary 2. *For every* $a \in D$ *the* **isotropy subgroup**

$$G_a := G_{a,a} = \{g \in G : \ a \cdot g = a\}$$

is compact.

Let D be a complex manifold, for example a bounded domain $D \subset \mathbf{C}^d$. A group $\Gamma \subset \mathrm{Aut}(D)$ of holomorphic transformations of D is called **properly discontinuous** if for all compact subsets $A, B \subset D$ the set

$$\Gamma_{A,B} := \{\gamma \in \Gamma : \ A\gamma \cap B \neq \emptyset\}$$

is finite. Note that in general this is only a subset of Γ. For $A = B$ we obtain a (finite) subgroup

$$\Gamma_A := \Gamma_{A,A} = \{\gamma \in \Gamma : \ A\gamma \cap A \neq \emptyset\}.$$

In particular, for each point $a \in D$ the **isotropy subgroup**

$$\Gamma_a := \Gamma_{a,a} = \{\gamma \in \Gamma : \ a\gamma = a\}$$

is finite. The same concepts apply to more general 'analytic spaces' which may have singularities.

Proposition 3. *For a bounded domain* D, *every discrete subgroup* $\Gamma \subset \mathrm{Aut}(D)$ *acts properly discontinuous on* D

Proof. For all compact subsets $A, B \subset D$ the set

$$\Gamma_{A,B} = \Gamma \cap G_{A,B}$$

is compact and discrete, hence finite. □

0.2.2 Holomorphic automorphic forms

Consider first a connected complex manifold D and a properly discontinuous group $\Gamma \subset \mathrm{Aut}(D)$. An **automorphic cocycle** $J : \Gamma \times D \to \mathbf{C}$ consists of holomorphic functions $J_\gamma : D \to \mathbf{C}$ which satisfy the cocycle property

$$J_{\gamma\gamma'}(z) = J_\gamma(\gamma' z) \ J_{\gamma'}(z).$$

The standard example, for a domain D, is given by the Jacobian

$$J_g(z) := \det g'(z)$$

where $g'(z)$ is the holomorphic derivative of $g \in \mathrm{Aut}(D)$ at $z \in D$. Relative to the cocycle J, a holomorphic function $f : D \to \mathbf{C}$ is called an **m-automorphic form** if

$$J_\gamma(z)^m \ f(\gamma z) = f(z)$$

for all $\gamma \in \Gamma$ and all $z \in D$. This means that f is a holomorphic section of the m-th power of a line bundle determined by J. For $m = 0$ one would say invariant function, but typically automorphic forms exist for large m. Let $\mathcal{O}_\Gamma^m(D, \mathbf{C})$ denote the vector space of all m-automorphic forms. Then $\mathcal{O}_\Gamma^m(D, \mathbf{C}) \cdot \mathcal{O}_\Gamma^n(D, \mathbf{C}) \subset \mathcal{O}_\Gamma^{m+n}(D, \mathbf{C})$ and hence

$$\mathcal{O}_\Gamma(D, \mathbf{C}) := \sum_{m \geqslant 0} \mathcal{O}_\Gamma^m(D, \mathbf{C}) \subset \mathcal{O}(D, \mathbf{C})$$

is a graded subalgebra of holomorphic functions.

0.2.3 Holomorphic Eisenstein series on bounded domains

Let $D \subset \mathbf{C}^d$ be a bounded domain and $\Gamma \subset \mathrm{Aut}(D)$ a discrete, hence properly discontinuous, subgroup. Let $f \in H^\infty(D)$ be a holomorphic function. For $m \geqslant 2$ define the **Poincaré-Eisenstein series**

$$f_\Gamma^m(z) := \sum_{\gamma \in \Gamma} J_\gamma(z)^m \ f(z\gamma).$$

Note that Γ is acting from the right.

Proposition 4. *For $m \geqslant 2$ the series*

$$1_\Gamma^m(z) := \sum_{\gamma \in \Gamma} \ J_\gamma(z)^m$$

is compactly $|\cdot|$-convergent on D.

Proof. Let $A \subset\subset B \subset\subset D$ be compact subsets. Then for each $z \in A$ there exists an open polydisk (product of disks) $P_z \subset B$. If $P_z\sigma \cap P_z \cdot \tau \neq \emptyset$ then $B \cdot (\sigma\tau^{-1}) \cap B \supset P_z \cdot (\sigma\tau^{-1}) \cap P_z \neq \emptyset$ and hence $\sigma\tau^{-1} \in \Gamma_B$. Therefore the collection $(P_z\gamma)_{\gamma \in \Gamma}$ covers D at most $|\Gamma_B|$ times. This implies for the volume $|\cdot|$

$$\sum_{\gamma \in \Gamma} |P_z\gamma| \leqslant |\Gamma_B| \ |D|.$$

The mean value theorem and integral transformation formula imply

$$|J_\gamma(z)|^2 \leqslant \frac{1}{|P_z|} \int\limits_{P_z} dw |J_\gamma(w)|^2 = \frac{|P_z\gamma|}{|P_z|}$$

since $|J_\gamma(w)|^2$ is the real Jacobian determinant. It follows that

$$\sum_{\gamma\in\Gamma} |J_\gamma(z)|^2 \leqslant \sum_{\gamma\in\Gamma} \frac{|P_z\gamma|}{|P_z|} \leqslant \frac{1}{|P_z|}\ |\Gamma_B|\ |D|.$$

Since A is covered by finitely many polydisks P_z, this proves uniform convergence on A for $m = 2$. This in turn implies $\sup_{z\in A}|J_\gamma(z)| < 1$ for almost all $\gamma \in \Gamma$ and therefore $|J_\gamma(z)|^m \leqslant |J_\gamma(z)|^2$ of $m \geqslant 2$. ☐

Corollary 5. *If $f \in H^\infty(D)$ is a bounded holomorphic function, then for $m \geqslant 2$ the series*

$$f_\Gamma^m := \sum_{\gamma\in\Gamma} J_\gamma^m\ \gamma\cdot f, \quad f_m^\Gamma(z) := \sum_{\gamma\in\Gamma} J_\gamma(z)^m\ f(z\gamma)$$

is compactly $|\cdot|$-convergent on D.

Proof.

$$\sum_{\gamma\in\Gamma} |J_\gamma(z)|^m\ |f(z\gamma)| \leqslant \sup_D |f|\ \sum_{\gamma\in\Gamma} |J_\gamma(z)|^m$$

☐

In case $D \subset Z$ is a bounded domain, all polynomials $f \in \mathcal{P}(Z)$ restricted to D are bounded.

0.2.4 Poincaré series on Lie groups

Proposition 6. *Let G be a unimodular group and $f \in L^1(G)$ be integrable (could be vector-valued). Then the Poincaré series*

$$f_\Gamma := \sum_{\gamma\in\Gamma}\ \gamma\cdot f, \quad f_\Gamma(g) := \sum_{\gamma\in\Gamma} f(g\gamma)$$

$\|\cdot\|$-converges compactly on G and is bounded.

Proof. Since Γ is discrete there exists a symmetric compact e-neighborhood $P \subset G$ such that $\Gamma \cap P^2 = \{e\}$. By a deep result of Harish-Chandra [1, Theorem 19, p. 154] there exists a 'Dirac' like function $\delta \in \mathcal{C}^\infty(G)$ with compact support $\mathrm{supp}(\delta) \subset P$ (which is K-invariant $\delta(k^{-1}gk) = \delta(k)\ \forall\ k \in K$) and satisfies the convolution equation

$$f * \delta = f.$$

Putting $h' = h\gamma$, it follows that

$$f(g\gamma) = (f*\delta)(g\gamma) = \int_G dh'\ f(g\gamma h'^{-1})\ \delta(h')$$

$$= \int_G dh\ f(gh^{-1})\ \delta(h\gamma) = \int_{P\gamma^{-1}} dh\ f(gh^{-1})\ \delta(h\gamma)$$

Therefore

$$\|f(g\gamma)\| \leqslant \int_{P\gamma^{-1}} dh\ \|f(gh^{-1})\|\ |\delta(h\gamma)| \leqslant \sup_G |\delta| \int_{P\gamma^{-1}} dh\ \|f(gh^{-1})\|$$

If $\gamma_1, \gamma_2 \in \Gamma$ are distinct, then $P\gamma_1^{-1} \cap P\gamma_2^{-1} = \emptyset$. Putting $h'' = gh^{-1}$, it follows that

$$\|f\|_\Gamma(g) := \sum_{\gamma\in\Gamma} \|f(g\gamma)\| \leqslant \sup_G |\delta| \sum_{\gamma\in\Gamma} \int_{P\gamma^{-1}} dh\ \|f(gh^{-1})\|$$

$$\leqslant \sup_G |\delta| \int_G dh\ \|f(gh^{-1})\| = \sup_G |\delta| \int_G dh''\ \|f(h'')\| = \sup_G |\delta|\ \|f\|_1$$

using that G is unimodular. This shows that the series converges normally on G. Since f is integrable, for any $\epsilon > 0$ there exists a compact set $Q \subset G$ such that

$$\int_{G\sim Q} dh\ \|f(h)\| \leqslant \epsilon.$$

For any compact subset $C \subset G$ the set

$$A := \{\gamma \in \Gamma :\ C\gamma P \cap Q \neq \emptyset\}$$

is finite. For $\gamma \in \Gamma \sim A$ the sets $g\gamma P$ are pairwise disjoint and contained in $G \sim Q$. Therefore, for any $g \in C$

$$\sum_{\gamma\in\Gamma\sim A} \|f(g\gamma)\| \leqslant \sup_G |\delta| \sum_{\gamma\in\Gamma\sim A} \int_{g\gamma P} dh\ \|f(h)\| \leqslant \sup_G |\delta| \int_{G\sim Q} dh\ \|f(h)\| \leqslant \epsilon\ \sup_G |\delta|.$$

Hence the series converges uniformly on C □

In general, it is difficult to decide whether these Poincaré series do not vanish identically. This can be studied, e.g., by Fourier expansions to be considered later.

Chapter 1

Quotients of Complex Analytic Spaces

1.1 Overview

The quotient space D/Γ of a complex manifold D (e.g., a domain $D \subset Z = \mathbf{C}^d$) by a properly discontinuous group Γ is in general not a complex manifold, because of singularities arising at fixed points $a \in D$ where the (finite) isotropy group Γ_a is not trivial. Nevertheless, it will be shown that D/Γ is always a so-called analytic space. More precisely,

- The quotient Z/Γ by a **finite linear group** $\Gamma \subset \mathrm{GL}(Z) = \mathrm{GL}_n(\mathbf{C})$ (not necessarily a reflection group) is a complex analytic space.
- As a consequence, the quotient D/Γ of any complex analytic space D by a **properly discontinuous group** $\Gamma \subset \mathrm{Aut}(D)$ (not necessarily finite or linear) is again a complex analytic space.
- If D is a **bounded domain** and $\Gamma \subset \mathrm{Aut}(D)$ is a **co-compact discrete** subgroup, then D/Γ is a projective algebraic variety. This deep result of H. Cartan was a primary motivation for Kodaira's embedding theorem.
- If $D = K\backslash G$ is a **bounded symmetric domain** and $\Gamma \subset G$ is an 'arithmetic' discrete subgroup (of finite co-volume) then D/Γ is a Zariski-dense open subset of a projective algebraic variety.

In this chapter we prove the first three assertions. The fourth assertion (Satake compactification) lies deeper and will be proved later.

1.2 Commutative algebra

1.2.1 Integral closure and Krull topology

We consider unital commutative rings A. For an integral domain A let

$$\underline{A} := \{\frac{a}{b} : \ a, b \in A, \ b \neq 0\}$$

denote its field of fractions. For a commutative ring extension $A \subset B$ let

$$\overline{A}^B := \{b \in B : \ A[b] = A\langle fin\rangle\}$$

denote the **integral closure** of A in B. This shorthand notation means that the algebra $A[b]$ generated by A and $b \in A$ (in short, the **A-algebra** generated by b) is a finitely generated **A-module**. One can show that

$$A \subset \overline{A}^B \subset B$$

is a subring of B. We define the notion of **integrally closed** and **integrally dense** by looking at the extreme cases

$$A \underset{\text{closed}}{\overset{\text{int}}{\subset}} B \Leftrightarrow A = \overline{A}^B,$$

$$A \underset{\text{dense}}{\overset{\text{int}}{\subset}} B \Leftrightarrow \overline{A}^B = B.$$

An integral domain A is called **normal** if

$$A = \overline{A}^{\underline{A}} \underset{\text{closed}}{\overset{\text{int}}{\subset}} A^{\underline{A}}$$

is integrally closed in its field of fractions. Consider a group $\Gamma \subset \mathrm{Aut}(A)$ of ring automorphisms of A. Then

$$A^\Gamma := \{a \in A : \ \gamma \cdot a = a \ \forall \ \gamma \in \Gamma\}$$

is a subring of A.

Lemma 7. *Let A be a normal ring. Then the subring*

$$A^\Gamma := \{a \in A : \ \gamma \cdot a = a \ \forall \ \gamma \in \Gamma\}$$

is also normal.

Proof. Since A is an integral domain, its subring A^Γ is also an integral domain. Now let $f = \frac{p}{q} \in \overline{A^\Gamma}^{\underline{A}^\Gamma}$, where $p, q \in A^\Gamma$ and $q \neq 0$. Then $f \in \overline{A}^{\underline{A}} = A$ and for all $\gamma \in \Gamma$ we have

$$\gamma \cdot f = \frac{\gamma \cdot p}{\gamma \cdot q} = \frac{p}{q} = f$$

Therefore $f \in A^\Gamma$ and hence $A^\Gamma = \overline{A^\Gamma}^{\underline{A}^\Gamma}$ □

Lemma 8. *Let A be a noetherian ring, and $M = A\langle fin\rangle$ a finitely generated A-module. Then every A-submodule $N \subset M$ is also finitely generated, $N = A\langle fin\rangle$.*

The following **integrality criterion** will often be used.

Proposition 9. *Let $A \subset B$ be a commutative ring extension. Then $B = A\langle fin\rangle$ is a finitely generated A-module if and only if $B = A[fin]$ is a finitely generated A-algebra and $B = \overline{A}^B$. In short,*

$$B = A\langle fin\rangle \Leftrightarrow B = A[fin] = \overline{A}^B$$

Now we study **ring completions** under the so-called Krull topology. For any ring A and ideal $\mathfrak{m} \lhd A$ the $\mathfrak{m}$-closure of an ideal $\mathfrak{a} \lhd A$ is given by

$$\overline{\mathfrak{a}} = \bigcap_{n \geqslant 0} (\mathfrak{a} + \mathfrak{m}^n A)$$

More generally, a submodule $N \subset M$ of an A-module M has the $\mathfrak{m}$-closure

$$\overline{N} = \bigcap_{n \geqslant 0} (N + \mathfrak{m}^n M)$$

The following **closure criterion** is proved in [Zariski-Samuel, p. 262, Theorem 9].

Proposition 10. *Consider a noetherian ring A and an ideal $\mathfrak{m} \lhd A$ contained in every maximal ideal. An equivalent condition is that*

$$1 + \mathfrak{m} \subset \mathring{A}$$

is invertible. Then every ideal $\mathfrak{a} \lhd A$ is $\mathfrak{m}$-closed

$$\mathfrak{a} = \bigcap_{n \geqslant 0} (\mathfrak{a} + \mathfrak{m}^n A)$$

More generally, every submodule $N \subset M = A\langle fin\rangle$ of a finitely generated A-module M is $\mathfrak{m}$-closed:

$$N = \bigcap_{n \geqslant 0} (N + \mathfrak{m}^n M)$$

An important special case is a (noetherian) **local ring** A with a unique maximal ideal

$$\mathfrak{m} = A \sim \mathring{A}.$$

Here $\mathring{A}$ denotes the group of units in A.

1.2.2 Power series and germs of analytic functions

For a field K and indeterminates $x = (z_1, \ldots, z_n)$ we denote by

$$K[z] = K[z_1, \ldots, z_n]$$

$$K|z| = K|z_1, \ldots, z_n|$$

$$\mathbf{C}\{z\} = \mathbf{C}\{z_1, \ldots, z_n\}$$

the ring of polynomials/formal power series/convergent power series

$$f(z) = \sum_{\nu \in \mathbf{N}^n} f_\nu \; x^\nu$$

with coefficients $f_\nu \in K$. These rings are integral domains (no zero divisors). Putting

$$x = (z_1, \; z_n) = (z', z_n),$$

with $z' = (z_1, \ldots, z_{n-1})$, we have

$$K[z] = K[z'][z_n]$$

$$K|z| = K|z'||z_n|$$

$$\mathbf{C}\{z\} = \mathbf{C}\{z'\}\{z_n\}$$

The **Weierstrass division theorem** states

Theorem 11. *Let $f, g \in K|z|$ such that $f(0, z_n) \neq 0$, i.e., $\mathfrak{o}(f(0', z_n)) = k < \infty$. Then there exist unique $q \in K|z|$ and $r \in K|z'|[z_n]$ such that the order*

$$\mathfrak{o}(g(0', z_n) - r(0', z_n)) \geqslant k$$

and

$$f = qg + r.$$

Similarly for convergent power series.

Thus the Taylor coefficients in the z_n-variable satisfy $g_i(0') = r_i(0')$ for $0 \leqslant i < k$.

Corollary 12. *We have*

$$f = q \; z_n^k + r$$

with $q(0) \neq 0$, i.e., q is a unit.

Proposition 13. *The rings $K|z|$ and $\mathbf{C}\{z\}$ are noetherian.*

Proof. Use induction on n and, in the convergent setting, the Weierstrass theorem. □

Proposition 14. *The rings $K[z]$, $K|z|$, $\mathbf{C}\{z\}$ are normal.*

For any $a \in \mathbf{C}^n$ let

$$\mathcal{O}_a = \mathcal{O}_a^{\mathbf{C}^n} \approx \mathbf{C}\{z-a\}$$

denote the local ring of **germs of analytic functions** at a. Given an open subset $U \subset \mathbf{C}^n$ a closed subset $X \subset U$ is called **analytic** if for every $a \in X$ there exist $a \in U_a \subset_{\text{open}} U$ and holomorphic functions $h_i \in \mathcal{O}(U_a),\ i \in I$ such that

$$X \cap U_a = \{z \in U_a :\ h_i(z) = 0 \ \forall\ i \in I\}.$$

By the noetherian property, one may always choose I to be a finite set. For an analytic set X we denote by $\mathcal{O}_a^X$ the ring of germs of analytic functions on X. We write

$$X \underset{\text{ana}}{\subset} U \underset{\text{open}}{\subset} \mathbf{C}^n$$

There are **two basic ways** to construct local rings of analytic functions. Suppose first that $\pi : D \to D/\Gamma$ is a quotient map. Endow D/Γ with the quotient topology, and for $a \in D$, let $\mathcal{C}_{\pi a}^{D/\Gamma}$ denote the ring of germs of continuous functions. Then we define

$$\mathcal{O}_{\pi a}^{D/\Gamma} := \{f \in \mathcal{C}_{\pi a}^{D/\Gamma} :\ f \circ \pi \in \mathcal{O}_a^D\} =: \pi_*(\mathcal{O}_a^D).$$

On the other hand, for an analytic subset $X \subset U$, with inclusion map $\iota : X \to U$, and $b \in X$ we define

$$\mathcal{O}_b^X := \{f|_X = f \circ \iota :\ f \in \mathcal{O}_b^U\} =: \iota^* \mathcal{O}_b^U.$$

The maximal ideal $\mathfrak{m}$ in the local power series ring $K|z|/\mathbf{C}\{z\}$ are the power series f without constant term, i.e. $f(0) = 0$. Given power series $f_i \in \mathbf{C}\{z\}$ without constant term we put

$$f_* = (f_1, \ldots, f_m).$$

Since $f_* 0 = 0$ we have the **substitution homomorphism**

$$\mathbf{C}\{z\} \xleftarrow{\circ f_*} \mathbf{C}\{w\}, \quad g(w) \mapsto g(f_* z)$$

for z near 0, inducing a commuting diagram

$$\begin{tikzcd}
\mathbf{C}\{z\} & \mathbf{C}\{w\} \arrow[l, "\circ f_*"'] \arrow[d] \\
\mathbf{C}\{f_*\} \arrow[u] & \mathbf{C}\{w\}/\underline{\circ f_*} \arrow[l, "\approx"]
\end{tikzcd}$$

where the range

$$\mathbf{C}\{f_*\} = \mathbf{C}\{f_1, \ldots, f_m\} := \mathbf{C}\{w\} \circ f_*$$

is a subring of $\mathbf{C}\{z\}$ and the kernel

$$\underline{\circ f_*} := \ker(\circ f_*) = \{g \in \mathbf{C}\{w\} :\ g \circ f_* = 0\} \lhd \mathbf{C}\{w\}$$

is called the **ideal of analytic relations** between $f_1, \ldots, f_m$.

For a polynomial ideal $\mathcal{I} \lhd K[z]$ we denote by

$$\mathcal{I}^{\perp\!\!\perp} := \{z \in \; Z: \; f(z) = 0 \; \forall \; f \in \mathcal{I}\}$$

the **algebraic variety** in $Z = K^d$. If $f_* = (f_1, \ldots, f_m)$ we also write

$$f_*^{\perp\!\!\perp} = K[z]\langle f_* \rangle^{\perp\!\!\perp} = \{z \in \; T: \; f_i(z) = 0 \; \forall \; i\}$$

by considering the ideal $K[z]\langle f_* \rangle$ generated by the f_i. For convergent power series $f_i \in \mathbf{C}\{z\}$ we have instead the **analytic variety** (germ)

$$f_*^{\perp\!\!\perp} := \{z: \; f_1(z) = \ldots = f_m(z) = 0\}$$

near 0. Then $0 \in f_*^{\perp\!\!\perp}$ and f_* defines an analytic mapping into $\underline{\circ f_*}^{\perp\!\!\perp}$.

The following **Zariski criterion** is proved in [1, Corollary, p. 19].

Proposition 15. *0 is isolated in $f_*^{\perp\!\!\perp}$ if and only if the ring $\mathbf{C}\{z\}$ is integral over its subring $\mathbf{C}\{f_*\}$, i.e.,*

$$\mathbf{C}\{f_*\} \underset{\text{dense}}{\overset{\text{int}}{\subset}} \mathbf{C}\{z\} = \overline{\mathbf{C}\{f_*\}}^{\mathbf{C}\{z\}}$$

1.3 Quotient by a finite linear group

Let $\Gamma \subset \mathrm{GL}_d(\mathbf{C})$ be a finite group of linear transformations. More generally, let K be a field, not necessarily of characteristic 0 or algebraically closed. We often write $Z = K^d$ (resp., $Z = \mathbf{C}^d$) since the coordinates play no distinguished role. Thus $\Gamma \subset \mathrm{GL}(Z)$.

Via substitution

$$(\gamma \cdot p)(z) := p(z\gamma)$$

the group Γ acts by ring automorphisms on the polynomials $K[z]$. Consider the invariant subalgebra

$$K[z]^\Gamma := \{p \in K[z]: \; \gamma \cdot p = p \; \forall \; \gamma \in \Gamma\}.$$

Since the Γ-action preserves degrees, the homogeneous terms of a Γ-invariant polynomial are also Γ-invariant. It follows that $K[z]^\Gamma$ is a graded K-algebra.

Lemma 16. *The ring extension $K[z]^\Gamma \subset K[z]$ is integral, i.e.*

$$K[z]^\Gamma \underset{\text{dense}}{\overset{\text{int}}{\subset}} K[z] = \overline{K[z]^\Gamma}^{K[z]}$$

Proof. For $p \in K[z]$ the monic polynomial

$$\hat{p}(t) = \prod_{\gamma \in \Gamma} (t - \gamma \cdot p) = (t - p) \prod_{1 \neq \gamma \in \Gamma} (t - \gamma \cdot p) \in K[z]^\Gamma[t]$$

satisfies $\hat{p}(p) = 0$. Therefore $p \in \overline{K[z]^\Gamma}^{K[z]}$. $\square$

Lemma 17.

$$K[z] = K[z]^\Gamma \langle fin \rangle$$

is a finitely generated $K[z]^\Gamma$-module.

Proof. Since

$$K[z] = K[fin] = K[z]^\Gamma[fin] = \overline{K[z]^\Gamma}^{K[z]},$$

the assertion follows from the 'integrality criterion'. □

The **'polynomial' finite generation theorem** is

Theorem 18. *There exist finitely many homogeneous polynomials $p_* = (p_1, \ldots, p_m)$ such that*

$$K[z]^\Gamma = K[p_1, \ldots, p_m] = K[p_*]$$

is a finitely generated K-algebra. Thus the substitution homomorphism

$$K[z]^\Gamma \xleftarrow{\circ p_*} K[w], \;\; f(w) \mapsto f \circ p_*(z)$$

is surjective and induces a commuting diagram

$$\begin{tikzcd}
 & \underline{\circ p_*} \arrow[dl, "0"'] \arrow[d] \\
K[z]^\Gamma & K[w] \arrow[l, "\circ p_*"'] \arrow[d] \\
 & K[w]/\underline{\circ p_*} \arrow[ul, "\approx"]
\end{tikzcd}$$

where

$$\underline{\circ p_*} = \{f \in K[w] : \; f \circ p_* = 0\} \lhd K[w]$$

denotes the kernel of the substitution homomorphism.

Proof. Applying (**??**) to the coordinate functions z_i we obtain

$$\hat{z}_i(t) = \sum_{n \geqslant 0} t^n \; z_{i,n} \in K[z]^\Gamma[t]$$

where $z_{i,n} \in K[z]^\Gamma$. Define the unital K-algebra

$$A := K[z_{i,n}] \subset K[z]^\Gamma.$$

For each i we have $z_i \in \overline{A}^{K[z]}$ since $\hat{z}_i(t) \in A[t]$ by definition of A. Since the integral closure $\overline{A}^{K[z]}$ is a subring and even a K-subalgebra, it follows that

$$K[z] = \overline{A}^{K[z]}. \tag{1.3.1}$$

Therefore

$$K[z] = K[fin] = A[fin] = \overline{A}^{K[z]}$$

and the 'integrality criterion' implies that

$$K[z] = A\langle fin\rangle \tag{1.3.2}$$

is a finitely generated A-module. Now A is a homomorphic image of a polynomial ring, hence noetherian. By (2.1.2) it follows that $K[z]$ is a noetherian A-module. Hence the A-submodule $K[z]^\Gamma \subset K[z]$ is also noetherian. Now the Lemma implies

$$K[z]^\Gamma = A\langle fin\rangle = K[fin]\langle fin\rangle = K[fin]$$

This yields finitely many algebra-generators $p_1, \ldots, p_m$, which may be assumed homogeneous, since $K[z]^\Gamma$ is a graded algebra. □

Corollary 19. *The quotient ring $K[w]/\underline{\circ p_*}$ is normal.*

Proof. This follows from $K[w]/\underline{\circ p_*} \approx K[z]^\Gamma$. □

For $1 \leqslant 1 \leqslant m$ define

$$d_j = \deg p_j > 0.$$

For any multi-index $\mu = (\mu_1, \ldots, \mu_m)$ of length m put

$$d \cdot \mu := \sum_j d_j \mu_j.$$

A polynomial of the form

$$\phi(w) = \sum_{d\cdot\mu = k} \phi_\mu w^\mu$$

for some integer k is called **k-isobaric**. Let

$$(j) := (0, \ldots, 0, 1_j, 0, \ldots, 0).$$

Lemma 20. *Let $\phi \in K[w]$ be k-isobaric. If there exists $1 \leqslant j \leqslant m$ with coefficient $\phi_{(j)} \neq 0$, then*

$$\phi - \phi_{(j)} w_j \in K[w_1, \ldots, \hat{w}_j, \ldots, w_m]$$

Proof. If $\phi_{(j)} \neq 0$, then $d_j = k$. Now

$$\phi - \phi'(0) = \sum_{|\mu|>1} \phi_\mu w^\mu.$$

Let $|\mu| > 1$ satisfy $\mu_j > 0$. If $\mu_j > 1$ then $d \cdot \mu \geqslant d_j \mu_j > k$. Therefore $\phi_\mu = 0$. If $\mu_j = 1$ there is another index $i \neq j$ such that $\mu_i > 0$. Then $d \cdot \mu \geqslant d_j + d_i \mu_i > d_j = k$. Therefore $\phi_\mu = 0$ □

We say that a set of generators $p_* = (p_1, \ldots, p_m)$ of $K[z]^\Gamma$ is **reduced** if every isobaric polynomial $\phi \in K[w]$ satisfying $\phi \circ p_* = 0$ has a vanishing linear term $\phi'(0) = 0$.

Lemma 21. *Every homogeneous set* $p_1, \ldots, p_m$ *of generators of* $K[z]^\Gamma$ *contains a reduced set of generators.*

Proof. If $p_1, \ldots, p_m$ is not reduced, there exists an isobaric polynomial $\phi \in K[w]$, satisfying $\phi \circ p_* = 0$, with non-vanishing linear term $\phi'(0) \neq 0$. Hence $\phi_{(j)} \neq 0$ for some j. By the Lemma we have

$$0 = \phi(p_*) = (\phi - \phi_{(j)} w_j)(p_1, \ldots, \hat{p}_j, \ldots, p_m) + \phi_{(j)} p_j.$$

Since $\phi_{(j)} \neq 0$ it follows that $p_j \in K[p_1, \ldots \hat{p}_j, \ldots, p_m]$. Therefore $p_1, \ldots \hat{p}_j, \ldots, p_m$ is a smaller set of generators. Repeating this argument, we obtain a reduced set of generators. □

From now on we assume that the generators p_* are homogeneous and reduced.

Lemma 22. *Let* $A \subset Z$ *be a finite set of* Γ*-inequivalent elements. For each* $a \in A$ *let* $\phi_a \in K[z]^{\Gamma_a}$ *satisfy* $\mathfrak{o}_a(\phi_a) > r$*. Then there exists* $\psi \in K[z]^\Gamma$ *such that* $\mathfrak{o}_a(\psi - \phi_a) > r$ *for all* $a \in A$*.*

Proof. For each $b \in A$, the finite set $A\Gamma \sim b$ is Γ_b-invariant. There exists a polynomial p_b such that

$$\mathfrak{o}_b(p_b - 1) > r, \ \ \mathfrak{o}_{A\Gamma \sim b}(p_b) > r.$$

The polynomial

$$q_b := \prod_{\gamma \in \Gamma_b} \gamma \cdot p_b \in K[z]^{\Gamma_b}$$

has the same vanishing properties, since

$$p_1 \cdots p_n - 1 = (p_1 - 1)p_2 \cdots p_n + (p_2 - 1)p_3 \cdots p_n + \ldots + p_n - 1$$

Define

$$\psi_b := \sum_{\Gamma_b \backslash \Gamma} \gamma \cdot (\phi_b q_b) = \phi_b q_b + \sum_{b\gamma \neq b} \gamma \cdot (\phi_b q_b) \in K[z]^\Gamma.$$

Then

$$\sum_b \psi_b - \phi_a = \psi_a - \phi_a + \sum_{b \neq a} \psi_b = \phi_a(q_a - 1) + \sum_{a\gamma \neq a} \gamma \cdot (\phi_a q_a) + \sum_{b \neq a} \sum_{\Gamma_b \backslash \Gamma} \gamma \cdot (\phi_b q_b).$$

For the first term we have

$$\mathfrak{o}_a(\phi_a(q_a - 1)) \geqslant \mathfrak{o}_a(q_a - 1) > r.$$

For the second term, if $a\gamma \neq a$ then $a\gamma \in A\Gamma \sim a$ and therefore

$$\mathfrak{o}_a(\gamma \cdot (\phi_a q_a)) = \mathfrak{o}_{a\gamma}(\phi_a q_a) \geqslant \mathfrak{o}_{a\gamma}(q_a) \geqslant \min \mathfrak{o}_{A\Gamma \sim a}(q_a) > r.$$

For the third term, if $b \neq a$ we have $a\gamma \in A\Gamma \sim b$ since a, b are Γ-inequivalent. Therefore

$$\mathfrak{o}_a(\gamma \cdot (\phi_b q_b)) = \mathfrak{o}_{a\gamma}(\phi_b q_b) \geqslant \mathfrak{o}_{a\gamma}(q_b) \geqslant \min \mathfrak{o}_{A\Gamma \sim b}(q_b) > r.$$

In summary, $\mathfrak{o}_a(\sum\limits_{b \in A} \psi_b - \phi_a) > r$. ☐

Lemma 23. *For* $0 \neq a \in Z$ *there exists* $q \in K[x]^{\Gamma_a}$ *such that*

$$\mathfrak{o}_0(q) > 0, \ \mathfrak{o}_a(q-1) > 0.$$

Proof. Take any polynomial $p \in K[x]$ and let

$$q = \prod_{\gamma \in \Gamma_a} \gamma \cdot p = p \prod_{e \neq \gamma \in \Gamma_a} \gamma \cdot p \in K[x]^{\Gamma_a}$$

Then $\mathfrak{o}_0(q) \geqslant \mathfrak{o}_0(p) > 0$. With $\Gamma_a = \{\gamma_1, \ldots, \gamma_N\}$ we have $q = \prod\limits_{i=1}^{N} (\gamma_i \cdot p)$ and hence

$$q - 1 = \prod_{i=1}^{N} (\gamma_i \cdot p - 1)(\gamma_{i+1} \cdot p) \cdots (\gamma_N \cdot p).$$

For each i we have

$$\mathfrak{o}_a\Big((\gamma_i \cdot p - 1)(\gamma_{i+1} \cdot p) \cdots (\gamma_N \cdot p)\Big) \geqslant \mathfrak{o}_a(\gamma_i \cdot p - 1) = \mathfrak{o}_a(\gamma_i \cdot (p-1)) = \mathfrak{o}_{a\gamma_i}(p-1) > 0$$

since $a\gamma_i = a$. This implies $\mathfrak{o}_a(q-1) > 0$. ☐

For every $a \neq 0$ there exists $p \in K[z]^{\Gamma}$ such that $\mathfrak{o}_0(p) > 0$, i.e., $p \in \mathcal{I}$, and $\mathfrak{o}_a(p-1) > 0$, i.e., $p(a) = 1$. Hence

$$\mathcal{V}(p_1, \ldots, p_\ell) = \mathcal{V}(\mathcal{I}) = \{0\}$$

showing that $p_1, \ldots, p_\ell$ have no common zero $\neq 0$. By Zariski's theorem, the ring extension $K[p_1, \ldots, p_\ell] \subset K[z]$ is integral. Therefore, the ring extension $K[z]^{\Gamma} \subset K[z]$ is also integral, proving the first assertion. Similarly, the ring extension $K[p_1, \ldots, p_\ell] \subset K[z]^{\Gamma}$ is integral and $K[p_1, \ldots, p_\ell]$ is noetherian. Therefore

$$K[z]^{\Gamma} = K[p_1, \ldots, p_\ell]\langle p_{\ell+1}, \ldots, p_m \rangle = K[p_1, \ldots, p_m]$$

is a finitely generated K-algebra.

Next we obtain the **'power series' finite generation theorem**.

Theorem 24. *For formal/convergent power series we have*

$$K|z|^{\Gamma} = K|p_*| \ / \ \mathbf{C}\{z\}^{\Gamma} = \mathbf{C}\{p_*\}.$$

Thus for every $f(z) \in K|z|^\Gamma \ / \ \mathbf{C}\{z\}^\Gamma$ *there exists* $\hat{f} \in K|w| \ / \ \mathbf{C}\{w\}$ *such that*

$$f(z) = \hat{f}(p_* z).$$

Equivalently, the substitution homomorphism

$$K|z|^\Gamma \xleftarrow{\circ p_*} K|w| \ / \ \mathbf{C}\{z\}^\Gamma \xleftarrow{\circ p_*} \mathbf{C}\{w\}$$

is surjective and induces a commuting diagram

$$\begin{tikzcd}
 & \underline{\circ p_*} \arrow[dl, "0"'] \arrow[d] & & \underline{\circ p_*} \arrow[dl, "0"'] \arrow[d] & . \\
K|z|^\Gamma & K|w| \arrow[l, "\circ p_*"'] \arrow[d] & \mathbf{C}\{z\}^\Gamma & \mathbf{C}\{w\} \arrow[l, "\circ p_*"'] \arrow[d] & \\
 & K|w|/\underline{\circ p_*} \arrow[ul, "\approx"] & & \mathbf{C}\{w\}/\underline{\circ p_*} \arrow[ul, "\approx"] &
\end{tikzcd}$$

Note that $p_* 0 = 0$ *is needed to define these rings.*

Proof. The assertion for formal power series follows from the expansion into homogeneous terms. In the convergent setting $\mathbf{C}\{p_*\} \subset \mathbf{C}\{z\}^\Gamma$, Taylor expansion into homogeneous terms shows that

$$\mathbf{C}[p_*] = \mathbf{C}[z]^\Gamma \overset{\mathfrak{m}}{\underset{\text{dense}}{\subset}} \mathbf{C}\{z\}^\Gamma$$

in the topology induced by the powers of the maximal ideal

$$\mathfrak{m} = \{f \in \mathbf{C}\{z\} : \ f(0) = 0\} \lhd \mathbf{C}\{z\}.$$

A fortiori, we obtain

$$\mathbf{C}\{p_*\} \overset{\mathfrak{m}}{\underset{\text{dense}}{\subset}} \mathbf{C}\{z\}^\Gamma.$$

We will now show that $\mathbf{C}\{p_*\}$ is also $\mathfrak{m}$-closed in $\mathbf{C}\{z\}^\Gamma$. For any $f \in \mathbf{C}\{z\}^\Gamma$ consider the algebra $\mathbf{C}\{p_*\}[f]$. Then (**??**) implies

$$\mathbf{C}\{p_*\} \overset{\mathfrak{m}}{\underset{\text{dense}}{\subset}} \mathbf{C}\{p_*\}[f].$$

Since $\mathbf{C}\{z\}$ is a noetherian ring, its homomorphic image $\mathbf{C}\{p_*\}$ is also noetherian. Since 0 is isolated in $V(p_*)$, the 'Zariski criterion' implies

$$\mathbf{C}\{p_*\} \overset{\text{int}}{\underset{\text{dense}}{\subset}} \mathbf{C}\{z\} = \overline{\mathbf{C}\{p_*\}}^{\mathbf{C}\{z\}}$$

is integrally dense in $\mathbf{C}\{z\}$. A fortiori,

$$\mathbf{C}\{p_*\} \overset{\text{int}}{\underset{\text{dense}}{\subset}} \mathbf{C}\{p_*\}[f] = \overline{\mathbf{C}\{p_*\}}^{\mathbf{C}\{p_*\}[f]}$$

is also integrally dense in the subring $\mathbf{C}\{p_*\}[f] \subset \mathbf{C}\{z\}$. Thus

$$\mathbf{C}\{p_*\}[f] = \mathbf{C}\{p_*\}[fin] = \overline{\mathbf{C}\{p_*\}}^{\mathbf{C}\{p_*\}[f]}$$

and the 'integrality criterion' implies that

$$\mathbf{C}\{p_*\}[f] = \mathbf{C}\{p_*\}\langle fin\rangle$$

is a finitely generated $\mathbf{C}\{p_*\}$-module. Applying the 'closure criterion' to the noetherian ring $\mathbf{C}\{p_*\}$ and its maximal ideal $\mathfrak{m} \cap \mathbf{C}\{z\}$ it follows that the $\mathbf{C}\{p_*\}$-submodule

$$\mathbf{C}\{p_*\} \underset{\text{closed}}{\overset{\mathfrak{m}}{\subset}} \mathbf{C}\{p_*\}[f]$$

is $\mathfrak{m}$-closed. Since it is also $\mathfrak{m}$-dense, we obtain $\mathbf{C}\{p_*\} = \mathbf{C}\{p_*\}[f]$. Therefore $f \in \mathbf{C}\{p_*\}$. Since $f \in \mathbf{C}\{z\}^\Gamma$ is arbitrary, it follows that $\mathbf{C}\{p_*\} = \mathbf{C}\{z\}^\Gamma$. □

Proposition 25. *Consider power series* $q_1, \ldots, q_m \in K|z|^\Gamma \ / \ \mathbf{C}\{z\}^\Gamma$ *which satisfy*

$$\mathfrak{o}(q_j - p_j) > d_j.$$

Then there exist power series $\Lambda_j(w) \in K|w| \ / \ \mathbf{C}\{w\}$ *such that*

$$\Lambda_* \circ p_* = q_*, \quad \Lambda_j(p_* z) = q_j(z)$$

and the linear term $\Lambda'_*(0)$ *is invertible. Here we put*

$$q_*(z) := (q_1(z), \ldots, q_m(z)), \ \Lambda_*(w) := (\Lambda_1(w), \ldots, \Lambda_m(w)).$$

Proof. We may assume that $d_1 \leqslant \ldots \leqslant d_m$. Write

$$q_i(z) = \sum_{n \geqslant 0} q_i^{(n)}(z)$$

where $q_i^{(n)} \in K[z]^\Gamma$ is n-homogeneous. Then $q_i^{(n)} \in K[z]^\Gamma$ can be (non-uniquely) written as

$$q_i^{(n)} = \sum_{d \cdot \mu = n} a_{i,\mu}^n \ p_*^\mu$$

with coefficients $a_{i,\mu}^n \in K$. Define formal power series

$$\Lambda_i(w) := \sum_\mu a_{i,\mu}^{d\cdot\mu} \ w^\mu \in K|w|.$$

Then

$$\Lambda_i(p_* z) = \sum_\mu a_{i,\mu}^{d\cdot\mu} \ (p_* z)^\mu = \sum_n q_i^{(n)}(z) = q_i(z).$$

For each fixed i the assumption

$$\mathfrak{o}(\Lambda_i \circ p_* - p_i) = \mathfrak{o}(q_i - p_i) > d_i$$

implies that the isobaric polynomial

$$\phi(w) := \begin{cases} \sum\limits_{d\cdot\mu=n} a^n_{i,\mu}\ w^\mu & n < d_i \\ \sum\limits_{d\cdot\mu=d_i} a^{d_i}_{i,\mu}\ w^\mu - w_i & \end{cases}$$

satisfy $\phi \circ p_* = 0$. By reducedness, the linear term vanishes:

$$0 = \phi'(0)y = \begin{cases} \sum\limits_{d_j=n} a^n_{i,(j)} w_j & n < d_i \\ \sum\limits_{d_j=d_i} a^{d_i}_{i,(j)} w_j - w_i & \end{cases}.$$

Therefore

$$\begin{cases} a^n_{i,(j)} = 0 \ \forall\ d_j = n & n < d_i \\ a^{d_i}_{i,(i)} = 1 & a^{d_i}_{i,(j)} = 0 \ \forall\ j \neq i, d_j = d_i \end{cases}. \tag{1.3.3}$$

We claim that the linear terms

$$\Lambda_i'(0)y = \sum_j a^{d_j}_{i,(j)} w_j$$

form a **unipotent upper triangular** matrix. On the diagonal we have $a^{d_i}_{i,(i)} = 1$ by (2.1.2). Now let $j < i$. Then $d_j \leqslant d_i$. If $d_j < d_i$, then $a^{d_j}_{i,(j)} = 0$ by (2.1.2). If $d_j = d_i$, then $a^{d_j}_{i,(j)} = a^i_{d_i,(j)} = 0$ by (2.1.2). □

Note that this argument needs coordinates $w_1, \ldots, w_m$ (instead of just a complex vector space W of dimension m) in order to define upper triangular matrices. The deeper reason is that the degrees d_j of the generators p_j will in general be distinct.

Corollary 26. *For formal/convergent power series we have*

$$K|z|^\Gamma = K|q_*| \ /\ \mathbf{C}\{z\}^\Gamma = \mathbf{C}\{q_*\}.$$

Thus for every $f(z) \in K|z|^\Gamma \ /\ \mathbf{C}\{z\}^\Gamma$ *there exists* $\tilde{f} \in K|w| \ /\ \mathbf{C}\{w\}$ *such that*

$$f(z) = \tilde{f}(q_* z).$$

Equivalently, the substitution homomorphism

$$K|z|^\Gamma \xleftarrow{\circ q_*} K|w| \ /\ \mathbf{C}\{z\}^\Gamma \xleftarrow{\circ q_*} \mathbf{C}\{w\}$$

is surjective and induces a commuting diagram

$$\begin{tikzcd} & \underline{\circ q_*} \arrow[ld, "0"'] \arrow[d] \\ K|z|^\Gamma & K|w| \arrow[l, "\circ q_*"'] \arrow[d] \\ & K|w|/\underline{\circ q_*} \arrow[lu, "\approx"] \end{tikzcd} , \qquad \begin{tikzcd} & \underline{\circ q_*} \arrow[ld, "0"'] \arrow[d] \\ \mathbf{C}\{z\}^\Gamma & \mathbf{C}\{w\} \arrow[l, "\circ q_*"'] \arrow[d] \\ & \mathbf{C}\{w\}/\underline{\circ q_*} \arrow[lu, "\approx"] \end{tikzcd} .$$

Note that $q_* 0 = 0$ *is needed to define these rings.*

Proof. Since the power series map

$$w \mapsto \Lambda_*(w) = (\Lambda_1(w), \ldots, \Lambda_m(w))$$

satisfies $\Lambda_*(0) = 0$ and $\Lambda_*'(0)$ is invertible, the inverse mapping theorem for formal/convergent power series implies that the (composition) inverse $\Lambda_*^{-1}(w)$ exists as a formal/convergent power series near 0. Now the 'power series' finite generation theorem implies

$$f = \hat{f} \circ p_* = f \circ (\Lambda_*^{-1} \circ q_*) = (f \circ \Lambda_*^{-1}) \circ q_* = \tilde{f} \circ q_*$$

with $\tilde{f} = \hat{f} \circ \Lambda_*^{-1} \in K|z| \ / \ \mathbf{C}\{z\}$. □

1.3.1 Z/Γ as an algebraic variety

Let $\pi : Z = K^d \to Z/\Gamma$ be the canonical projection. The map $p_* := (p_1, \ldots p_m) : Z \to K^m$ is Γ-invariant and therefore has a factorization

$$\begin{tikzcd}
Z \arrow[r, "p_*"] \arrow[d, "\pi"'] & K^m \\
Z/\Gamma \arrow[ur, "\overline{p}_*"'] &
\end{tikzcd}$$

Consider the associated affine algebraic variety

$$\underline{\circ p_*}^{\perp\!\!\perp} := \{w \in K^m : \ \underline{\circ p_*}(w) = 0\}$$

Lemma 27. *The map $\overline{p}_* : Z/\Gamma \to K^m$ is injective.*

Proof. Let $\pi(a) \neq \pi(b)$. Then $a\Gamma \cap b\Gamma = \emptyset$. Choose a polynomial $\phi \in K[z]$ such that $\phi(a\Gamma) = 0$ and $\phi(b\Gamma) = 1$. Then

$$f(z) := \prod_{\gamma \in \Gamma} \phi(z\gamma) \in K[z]^\Gamma$$

satisfies $f(a\Gamma) = 0$ and $f(b\Gamma) = 1$. The 'polynomial' finite generation theorem implies $f(z) = \tilde{f}(p_* z)$ for some $\tilde{f} \in K[w]$. Therefore $p_* a \neq p_* b$. □

Theorem 28. *Suppose that K is algebraically closed. Then the range*

$$\overline{p}_*(Z/\Gamma) = p_* Z = \underline{\circ p_*}^{\perp\!\!\perp}$$

is the algebraic variety determined by the ideal $\underline{\circ p_*}$.

Proof. Let $a \in Z$ and $f \in \underline{\circ p_*}$. Then $f \circ p_* = 0$ and therefore

$$f(\overline{p}_*(\pi a) = f(p_* a) = (f \circ p_*)(a) = 0.$$

It follows that $p_*Z \subset \underline{\circ p_*}^{\perp\!\!\!\perp}$. Conversely, let $b \in \underline{\circ p_*}^{\perp\!\!\!\perp}$. Then $\underline{\circ p_*}(b) = 0$. Hence there is a commuting diagram

$$\begin{tikzcd}
 & \underline{\circ p_*} \arrow[dl, "0"'] \arrow[d] \\
K & K|w| \arrow[l, "\epsilon_b"'] \arrow[d] \\
K[z]^\Gamma \arrow[u] & K[w]/\underline{\circ p_*} \arrow[l, "\approx", "\overline{\circ p_*}"'] \arrow[ul, "\overline{\epsilon}_b"']
\end{tikzcd}$$

for the evaluation map ϵ_b. Hence

$$\ker \overline{\epsilon}_b \underset{\max}{\lhd} K[w]/\underline{\circ p_*}$$

which implies

$$\mathfrak{m} := \ker(\epsilon_b \circ p_*) = (\ker \overline{\epsilon}_b) \circ \overline{p}_* \underset{\max}{\lhd} K[z]^\Gamma$$

We claim that $K[z]\mathfrak{m} \lhd K[z]$ is a proper ideal. In fact, if $1 = \sum\limits_i u_i a_i$ with $a_i \in K[z]$ and $u_i \in \mathfrak{m}$ then for each $\gamma \in \Gamma$ we have $1 = \gamma \cdot 1 = \sum\limits_i u_i(\gamma \cdot a_i)$ and therefore

$$1 = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \sum_i u_i(\gamma \cdot a_i) = \frac{1}{|\Gamma|} \sum_i u_i \sum_{\gamma\in\Gamma} \gamma \cdot a_i \in \mathfrak{m}$$

since $\sum\limits_{\gamma\in\Gamma} \gamma \cdot a_i \in K[z]^\Gamma$. This contradiction shows $1 \notin K[z]\mathfrak{m}$. By Zorn's Lemma there exists a maximal ideal

$$K[z]\mathfrak{m} \lhd \mathfrak{n} \underset{\max}{\lhd} K[z].$$

Then $\mathfrak{m} \subset \mathfrak{n} \cap K[z]^\Gamma \lhd_{\neq} K[z]^\Gamma$, since $1 \notin \mathfrak{n} \cap K[z]^\Gamma$. It follows that

$$(\ker \epsilon_b) \circ p_* = \mathfrak{m} = \mathfrak{n} \cap K[z]^\Gamma.$$

Since K is algebraically closed, Hilbert's Nullstellensatz implies $\mathfrak{n} = \ker \epsilon_a$ for some $a \in Z$. For each j the affine polynomial $\lambda(w) := w_j - b_j$ belongs to $\ker \epsilon_b$, showing that $\lambda \circ p_* \in \mathfrak{n} = \ker \epsilon_a$. Therefore $0 = (\lambda \circ p_*)(a) = p_j(a) - b_j$ for all j showing that $p_*a = b$. $\square$

Together with Lemma **??** it follows that the continuous map

$$Z/\Gamma \xrightarrow{\overline{p}_*} \underline{\circ p_*}^{\perp\!\!\!\perp}$$

is bijective.

Proposition 29. *For $K = \mathbf{C}$ the map $\overline{p}_*$ in the diagram*

$$\begin{tikzcd}
Z \arrow[d, "\pi"'] \arrow[dr, "p_*"] & \mathbf{C}^m \\
Z/\Gamma \arrow[r, "\overline{p}_*", "\approx"'] & \underline{\circ p_*}^{\perp\!\!\!\perp} \arrow[u, "\iota"']
\end{tikzcd}$$

is a homeomorphism for the quotient topology on Z/Γ and the relative topology on $\underline{\circ p_*}^{\perp\!\!\!\perp}$.

Proof. For $t > 0$ define **homotheties**

$$\rho_t(z) := tx, \quad \sigma_t(w_j)_{j=1}^m := (t^{d_j} w_j)_{j=1}^m.$$

Then the diagram

$$\begin{tikzcd}
Z/\Gamma \arrow[r, "\overline{p}_*"] \arrow[d, "\rho_t"'] & \underline{\circ p_*}^{\perp\!\!\perp} \arrow[d, "\sigma_t"] \\
Z/\Gamma \arrow[r, "\overline{p}_*"'] & \underline{\circ p_*}^{\perp\!\!\perp}
\end{tikzcd}$$

commutes. Let $C \subset Z$ be a compact 0-neighborhood. Then there exists $r > 0$ such that

$$\{z \in Z : \ \|z\| \leqslant r\} \subset C.$$

Suppose there exists a sequence $w_\ell \in p_*Z \sim p_*C$ such that $w_\ell \to 0$. Then $w_\ell = p_* z_\ell$ for some $z_\ell \in Z \sim C$. Hence $\|z_\ell\| > r$ and $r\frac{z_\ell}{\|z_\ell\|} \in C$ has a convergent subsequence

$$r\frac{z_\ell}{\|z_\ell\|} \to a \in C$$

with $\|a\| = r > 0$. Then $\overline{p}_*(\pi a) \neq 0$ since 0 is the only common zero of p_*. On the other hand,

$$\overline{p}_*(r\frac{z_\ell}{\|z_\ell\|}) = \overline{p}_*(\rho_{r/\|z_\ell\|} z_\ell \Gamma) = \sigma_{r/\|z_\ell\|} \overline{p}_*(z_\ell \Gamma) = \sigma_{r/\|z_\ell\|} w_\ell \to 0$$

since $\frac{r}{\|z_\ell\|} < 1$. This contradiction shows that p_*C is a neighborhood of $0 \in \underline{\circ p_*}^{\perp\!\!\perp}$. Since $\overline{p}_*$ is bijective and continuous, it follows that $\pi C \xrightarrow{\overline{p}_*} p_*C$ is a homeomorphism. Using the homotheties again, we can reach any point in Z, and the assertion follows. □

1.3.2 Z/Γ as a ringed space

The preceding theorem shows that Z/Γ is isomorphic to a normal affine algebraic variety $\underline{\circ p_*}^{\perp\!\!\perp} \subset K^m$ as a set. We will now show that this isomorphism holds on the level of **ringed analytic spaces**, if K is an algebraically closed, non-discrete, complete valuation field, e.g. $K = \mathbf{C}$. This more difficult part of Cartan's theorem proceeds by investigating the isotropy subgroups

$$\Gamma_a := \{\gamma \in \Gamma : \ a\gamma = 0\}$$

at all points $a \in Z$. Since Γ_a is also a finite linear group, the preceding results apply to Γ_a as well. Note that $\Gamma_0 = \Gamma$.

We make Z/Γ into a ringed topological space. For an open subset $V \subset Z/\Gamma$ define

$$\mathcal{O}_V^{Z/\Gamma} := \{f : V \to \mathbf{C} : \ f \circ \pi : \pi^{-1}(V) \to \mathbf{C} \text{ holomorphic}\}$$

This yields a (pre)sheaf on Z/Γ. The local ring $\mathcal{O}_{\pi a}^{Z/\Gamma}$ can be described as follows. The translation action

$$\mathfrak{t}_a \phi(x) := \phi(x - a)$$

yields an isomorphism

$$\mathcal{O}_a^{\Gamma_a} \xleftarrow[\approx]{\mathfrak{t}_a} \mathbf{C}\{z\}^{\Gamma_a} = \mathcal{O}_0^{\Gamma_a}$$

On the other hand, the averaging

$$f^{\Gamma_a\backslash\Gamma} := \sum_{\gamma\in\Gamma_a\backslash\Gamma} \gamma \cdot f, \quad f^{\Gamma_a\backslash\Gamma}(x) := \sum_{\gamma\in\Gamma_a\backslash\Gamma} f(x\gamma)$$

is a surjective map

$$\mathcal{O}_{\pi a}^{Z/\Gamma} \xleftarrow{(\cdot)^{\Gamma_a\backslash\Gamma}} \mathcal{O}_a^{\Gamma_a}.$$

Thus we have

$$\begin{tikzcd}
 & \mathcal{O}_a^{\Gamma_a} \arrow[dl, "(\cdot)^{\Gamma_a\backslash\Gamma}"'] \\
\mathcal{O}_{\pi a}^{Z/\Gamma} & \mathbf{C}\{z\}^{\Gamma_a} \arrow[l] \arrow[u, "\approx", "\mathfrak{t}_a"']
\end{tikzcd}$$

and

$$\mathcal{O}_{\pi a}^{Z/\Gamma} = \{\tilde{f} : \; f \in \mathbf{C}\{z\}^{\Gamma_a}\}.$$

consists of all germs

$$\tilde{f}(\pi z) := \sum_{\gamma\in\Gamma_a\backslash\Gamma} f(z\gamma - a),$$

where $f \in \mathbf{C}\{z\}^{\Gamma_a}$, since $(\tilde{f} \circ \pi)(z) = \tilde{f}(\pi z)$ is holomorphic near a.

On the other hand, the (affine) algebraic variety $\underline{\circ p_*}^{\perp\!\!\!\perp}$ has the regular functions

$$K[\underline{\circ p_*}^{\perp\!\!\!\perp}] := K[w]/\underline{\circ p_*}.$$

At any point $b \in \underline{\circ p_*}^{\perp\!\!\!\perp}$ we may form the **localization**

$$K[\underline{\circ p_*}^{\perp\!\!\!\perp}]_b := \{\frac{\phi}{\psi} : \; \phi, \psi \in K[\underline{\circ p_*}^{\perp\!\!\!\perp}], \; \psi(b) \neq 0\}$$

These local rings form a coherent sheaf over $\underline{\circ p_*}^{\perp\!\!\!\perp}$. Passing to convergent power series, the algebraic variety $Y := \underline{\circ p_*}^{\perp\!\!\!\perp} \subset \mathbf{C}^m$ is a ringed space with local rings

$$\mathcal{O}_b^{\underline{\circ p_*}^{\perp\!\!\!\perp}} := \{(\mathfrak{t}_b\psi)|_Y : \; \psi \in \mathbf{C}\{w\}\},$$

where we define

$$\mathfrak{t}_b\psi(y) := \psi(y - b)$$

For each $a \in Z$ define a ring homomorphism Λ_a by the commuting diagram

$$\begin{tikzcd}
\mathbf{C}\{z\}^{\Gamma_a} \arrow[d, "\mathfrak{t}_a"', "\approx"] & \mathbf{C}\{w\} \arrow[l, "\Lambda_a"'] \arrow[d, "\approx"', "\mathfrak{t}_{p_*a}"] \\
\mathcal{O}_a^{\Gamma_a} & \mathcal{O}_{p_*a}^{\mathbf{C}^m} \arrow[l, "\circ p_*"]
\end{tikzcd}$$

where $\mathfrak{t}$ denotes the translation actions. Similarly, for formal power series.

Theorem 30. *For any $a \in Z$ the homomorphism Λ_a is surjective: If $f \in K|z|^{\Gamma_a}$ / $\mathbf{C}\{z\}^{\Gamma_a}$, there exists $\tilde{f} \in K|w|$ / $\mathbf{C}\{w\}$ such that*

$$f(x-a) = \tilde{f}(p_* x - p_* a)$$

In other words, we have

$$\mathfrak{t}_a f = (\mathfrak{t}_{p_* a} \tilde{f}) \circ p_*$$

Proof. Applying the 'polynomial' finite generation theorem to Γ_a it follows that

$$K[z]^{\Gamma_a} = K[r_*]$$

for a finite reduced set of homogeneous polynomials $r_*(z) = (r_1(z), \ldots, r_{m_a}(z))$. By Lemma (**??**), there exist invariant polynomials $\mathfrak{t}_a s_k \in K[z]^\Gamma$ with

$$\mathfrak{o}(s_k - r_k) = \mathfrak{o}_a(\mathfrak{t}_a s_k - \mathfrak{t}_a r_k) > \deg r_k.$$

Then $s_k \in K[z]^{\Gamma_a} \subset K|z|^{\Gamma_a}$. It follows from Proposition (**??**) applied to Γ_a that

$$r_k = h_k \circ s_*$$

for some power series $h_k \in K|z|$ / $\mathbf{C}\{z\}$. Since $\mathfrak{t}_a s_k \in K[z]^\Gamma$ we can write

$$s_k(x-a) = (\mathfrak{t}_a s_k)(x) = g_k(p_* x - p_* a)$$

for polynomials $g_k \in K[w]$. Note that $p_* x - p_* a$ (unlike $p_*(x-a)$) is still a set of (inhomogeneous) Γ-invariant generators. Then

$$g_k(0) = g_k(r_* a - r_* a) = s_k(0) = 0.$$

Thus we may form the formal power series $\tilde{r}_k = h_k \circ g_*$ and obtain

$$r_k(x-a) = h_k(s_*(x-a)) = h_k(g_*(p_* x - p_* a)) = (h_k \circ g_*)(p_* x - p_* a) = \tilde{r}_k(p_* x - p_* a).$$

This proves the assertion for the generators r_k. Since $K|z|^{\Gamma_a} = K|r_*|$ / $\mathbf{C}\{z\}^{\Gamma_a} = \mathbf{C}\{r_*\}$ by the 'power series' finite generation theorem applied to Γ_a, this suffices for the assertion in general. □

Corollary 31. *For each $a \in Z$ there is a ring isomorphism*

$$\mathcal{O}^{Z/\Gamma}_{\pi a} = \pi_* \mathcal{O}^Z_a \underset{\approx}{\xleftarrow{\circ p_*}} \mathcal{O}^{\underline{\circ p_*}^{\perp\!\!\perp}}_{p_* a} = \iota^* \mathcal{O}^{\mathbf{C}^m}_{p_* a}.$$

Here $Z \xrightarrow{\pi} Z/\Gamma$ and $\underline{\circ p_*}^{\perp\!\!\perp} \xrightarrow{\iota} \mathbf{C}^m$ denote the canonical projection/injection, respectively.

Proof. Every germ $\psi \in \mathcal{O}^{\circ p_*^{\perp\!\!\perp}}_{\overline{p_* a}}$ is of the form

$$\psi(y) = (\mathsf{t}_{p_* a} g)(y) := g(y - p_* a)$$

where $g \in \mathbf{C}\{w\}$. Thus we have

$$\mathcal{O}^{\circ p_*^{\perp\!\!\perp}}_{\overline{p_* a}} \xleftarrow[\text{onto}]{\iota^*} \mathcal{O}^{\mathbf{C}^m}_{p_* a} \xleftarrow[\approx]{\mathsf{t}_{p_* a}} \mathbf{C}\{w\}$$

Then the convergent power series

$$f_a(z) := g(p_*(z + a) - p_* a) \in \mathbf{C}\{z\}^{\Gamma_a}$$

satisfies

$$(\psi \circ p_*)(x) = g(p_* x - p_* a) = f_a(x - a)$$

□

Proposition 32. *At any point $b = p_* a \in Y$ the power series completion*

$$K|Y|_b := \widehat{K[Y]_b} \approx K|z|^{\Gamma_a}$$

is normal.

Proof. The formal power series ring $K|z|$ is normal. By Lemma, its subring $K|z|^{\Gamma_a}$ is also normal. □

1.4 Quotients of Analytic Spaces

A topological ringed space D is called an **analytic space** if around every $a \in D$ there exists an isomorphism

$$D \underset{\text{open}}{\supset} U \underset{\approx}{\xrightarrow{\sigma}} \sigma U \underset{\text{ana}}{\subset} V \underset{\text{open}}{\subset} Z$$

for some vector space Z, such that $\sigma(a) = 0$. In short, D is locally isomorphic (as a topological ringed space) to an analytic subset of an open subset in some $\mathbf{C}^d$. We sometimes write

$$D \underset{\text{open}}{\supset} U \underset{\approx}{\xrightarrow{\sigma}} \sigma U \overset{\text{loc}}{\underset{\text{ana}}{\subset}} Z.$$

A group Γ acting by holomorphic transformations on D is called **properly discontinuous** if the following two conditions hold:

$$a\Gamma \cap b\Gamma = \emptyset \Rightarrow \exists_{\text{neighborhoods}}\ a \in U, b \in V : U\Gamma \cap V\Gamma = \emptyset$$

Moreover every isotropy group Γ_a is finite, and there exists a neighborhood $U = U\Gamma_a$ such that

$$\Gamma_U = \Gamma_a.$$

The first condition means that D/Γ is a Hausdorff space. The second condition means that the canonical projection $\pi : D \to D/\Gamma$ satisfies

$$\pi(U) = U/\Gamma_a$$

It follows that the local structure of D/Γ is determined by quotients of the form U/Γ_a, where U is again a complex analytic space invariant under the finite group Γ_a, which leaves the 'base point' $a \in U$ fixed. The main idea is now to realize Γ_a as a **linear** group. Consider first the easy case that $a \in D$ is a regular point, i.e., not a singularity. The set of regular points is an open dense subset of D. Around a regular point a the above simplifies to

$$D \underset{\text{open}}{\supset} U \underset{\approx}{\xrightarrow{\sigma}} \sigma U \underset{\text{open}}{\subset} Z.$$

Lemma 33. *Let Γ be a finite group acting on an open set $0 \in U \subset Z = \mathbf{C}^d$ and fixing 0. Then there exists an isomorphism $U \xrightarrow{\sigma} U'$ onto an open set $0 \in U' \subset Z$ such that the diagram*

$$\begin{tikzcd} U \arrow[r, "\sigma"] \arrow[d, "\gamma"'] & U' \arrow[d, "\gamma'_0"] \\ U \arrow[r, "\sigma"'] & U' \end{tikzcd}$$

commutes, i.e., we have

$$\sigma(z\gamma) = \sigma(z)\gamma'_0$$

for all $\gamma \in \Gamma$.

Proof. Define $\sigma : U \to Z$ by

$$\sigma(z) := \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}(z\gamma)\ \gamma_0'^{-1}$$

Then

$$\sigma'_0 = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma'_0\ \gamma_0'^{-1} = \mathrm{id}$$

Hence σ is a local isomorphism on an open set $0 \in U \subset Z$. Put $U' := \sigma(U)$. Now let $\gamma \in \Gamma$. Putting $\tau = \gamma\beta \in \Gamma$, with $\tau'_0 = \gamma'_0\beta'_0$ we obtain

$$\sigma(z\gamma) = \frac{1}{|\Gamma|}\sum_{\beta\in\Gamma}((z\gamma)\beta)\ \beta_0'^{-1} = \frac{1}{|\Gamma|}\sum_{\tau\in\Gamma}(z\tau)\ \tau_0'^{-1}\ \gamma'_0 = \sigma(z)\ \gamma'_0.$$

□

For arbitrary, not necessarily regular points $a \in D$, one uses the following 'linearization trick' due to Serre. For $Z := \mathbf{C}^d$ let Z^{Γ_a} denote the finite-dimensional vector space of all maps $\psi : \Gamma_a \to Z,\ \gamma \mapsto \psi_\gamma$, endowed with the linear right action

$$Z^{\Gamma_a} \times \Gamma_a \to Z^{\Gamma_a},\ (\psi\tilde{\gamma})_\beta := \psi_{\gamma\beta}.$$

induced by permutation of the 'coordinates' $\beta \in \Gamma_a$. For any subset $V \subset Z$ we define the invariant subset

$$V^{\Gamma_a} = \{\psi : \Gamma_a \to V\} \subset Z^{\Gamma_a}$$

If D is a complex analytic space, then for each $a \in D$ there is an open Γ_a-invariant set $a \in U \subset D$ and an injective holomorphic map $\sigma : U \to Z$ into a complex vector space Z wich $\sigma(a) = 0$, such that

$$U \underset{\approx}{\xrightarrow{\sigma}} \sigma U \underset{\text{ana}}{\subset} V \underset{\text{open}}{\subset} Z.$$

Lemma 34. *Define an analytic map* $\tilde{\sigma} : U \to Z^{\Gamma_a}$ *by*

$$(\tilde{\sigma} z)_\gamma := \sigma(z\gamma).$$

Then the diagram

$$\begin{tikzcd} U \arrow[r, "\tilde{\sigma}"] \arrow[d, "\gamma"'] & V^{\Gamma_a} \arrow[r, "\subset"'] \arrow[d, "\tilde{\gamma}"'] & Z^{\Gamma_a} \arrow[d, "\tilde{\gamma}"] \\ U \arrow[r, "\sigma"'] & V^{\Gamma_a} \arrow[r, "\subset"'] & Z^{\Gamma_a} \end{tikzcd}$$

commutes, i.e., for all $\gamma \in \Gamma_a$ *we have* $\tilde{\sigma}(z\gamma) = (\tilde{\sigma} z)\tilde{\gamma}$, *and*

$$\tilde{\sigma} U \underset{\text{ana}}{\subset} V^{\Gamma_a} \underset{\text{open}}{\subset} Z^{\Gamma_a}.$$

Proof. Let $\beta \in \Gamma_a$. Then

$$(\tilde{\sigma}(z\gamma))_\beta = \sigma((z\gamma)\beta) = \sigma(z(\gamma\beta)) = (\tilde{\sigma} z)_{\gamma\beta} = ((\tilde{\sigma} z)\tilde{\gamma})_\beta.$$

Since β is arbitrary, the equivariance property (**??**) follows. We claim that

$$\tilde{\sigma} U = \{\psi \in (\sigma U)^{\Gamma_a} : \ (\sigma^{-1}(\psi_\gamma))\beta = \sigma^{-1}(\psi_{\gamma\beta})\}.$$

In fact, let $\psi = \tilde{\sigma}(z)$ for some $z \in U$. Then $\psi_\gamma = (\tilde{\sigma} z)_\gamma = \sigma(z\gamma)$ and hence

$$(\sigma^{-1}(\psi_\gamma))\beta = (z\gamma)\beta = z(\gamma\beta).$$

On the other hand, we have $\psi_{\gamma\beta} = (\tilde{\sigma} z)_{\gamma\beta} = \sigma(z(\gamma\beta))$ and hence $\sigma^{-1}(\psi_{\gamma\beta}) = z(\gamma\beta)$. This proves the claim. It follows that

$$\tilde{\sigma} U \underset{\text{ana}}{\subset} (\sigma U)^{\Gamma_a} \underset{\text{ana}}{\subset} V^{\Gamma_a} \Rightarrow \tilde{\sigma} U \underset{\text{ana}}{\subset} V^{\Gamma_a}$$

□

Note that by the linearization trick the embedding dimension of the underlying analytic set increases considerably, so will not be optimal anymore. Also, the notion of analytic subset is well-adapted to this process, since all one has to check is the analytic equations of the image.

Theorem 35. *The quotient D/Γ of any complex analytic space D by a properly discontinuous group $\Gamma \subset \mathrm{Aut}(D)$ (not necessarily finite or linear) is again a complex analytic space.*

Proof. Keeping the above notation, for each fixed $a \in D$, consider the linear action of the finite group Γ_a on Z^{Γ_a}. Since $\tilde{\sigma}$ is Γ_a-equivariant, it induces an isomorphism

$$U/\Gamma_a \underset{\approx}{\xrightarrow{\tilde{\sigma}}} (\tilde{\sigma}U)/\tilde{\Gamma}_a$$

for the ringed structure induced by the projection. By Theorem **??** we have a polynomial map p_*^a

$$\pi(U) = U/\Gamma_a \underset{\approx}{\xrightarrow{\tilde{\sigma}}} (\tilde{\sigma}U)/\tilde{\Gamma}_a \underset{\text{ana}}{\subset} V^{\Gamma_a}/\tilde{\Gamma}_a \underset{\text{open}}{\subset} Z^{\Gamma_a}/\tilde{\Gamma}_a \underset{\approx}{\xrightarrow{p_*^a}} \underline{\circ p_*^{a\perp\!\!\!\perp}} \underset{\text{alg}}{\subset} \mathbf{C}^m$$

□

1.5 Compact Quotients of Bounded Domains

We now specialize to a bounded domain $D \subset Z = \mathbf{C}^d$ and a properly discontinuous group $\Gamma \subset \mathrm{Aut}(D)$. Consider an automorphic cocycle $J_\gamma(z)$. Then for each $a \in D$ the map $\gamma \mapsto J_\gamma(a)$ is a character of Γ_a. Since this is a finite group by assumption, there exists an integer $\dot{a} \in \mathbf{N}$ (for example, the order $|\Gamma_a|$) such that

$$J_\gamma(a)^{\dot{a}} = 1$$

for all $\gamma \in \Gamma_a$. By the linearization Lemma, the isotropy group Γ_a acts by linear transformations in a local chart near a. In this chart the Jacobian $J_\gamma(z)$ for $\gamma \in \Gamma_a$ is independent of z. Therefore we have

$$J_\gamma(z)^{\dot{a}} = 1$$

for all $\gamma \in \Gamma_a$ and z in a neighborhood of $a \in D$. Consider a graded subalgebra

$$\mathcal{A} = \sum_{m \geqslant 0} \mathcal{A}^m \subset \mathcal{O}(D, \mathbf{C}),$$

where $\mathcal{A}^m \subset \mathcal{O}_\Gamma^m(D, \mathbf{C})$ consists of m-automorphic forms relative to the cocycle J. To ease notation, we sometimes write

$$\mathcal{A}_q^\ell := \mathcal{A}^{\ell q}.$$

We assume that the following two conditions hold:

$$(*) \quad \forall\ a, b \in D,\ \pi(a) \neq \pi(b),\ \forall_{\text{integer}} \ell \geqslant \ell_{a,b}\ \forall\ \alpha, \beta \in \mathbf{C}$$

$$\exists\ f \in \mathcal{A}_{\dot{a} \vee \dot{b}}^\ell :\ f(a) = \alpha,\ f(b) = \beta.$$

Here $m \vee n = l.c.m.(m,n)$ denotes the least common multiple of integers m, n. The second condition is

$$(**) \quad \forall\, a \in D\ \forall\, d \in \mathbf{N}\ \forall_{\text{integer}} \ell \geqslant \ell_a^d\ \forall h \in \mathcal{O}_a^{\Gamma_a}$$

$$\exists\, f \in \mathcal{A}^\ell_{\dot{a}} : \mathfrak{o}_a(f-h) > d.$$

Example 36. *For a bounded domain* $D \subset Z := \mathbf{C}^d$ *consider Poincaré-Eisenstein series*

$$\phi^m_\Gamma := \sum_{\gamma\in\Gamma} J^m_\gamma\ \gamma \cdot \phi, \quad \phi^m_\Gamma(z) := \sum_{\gamma\in\Gamma} J^m_\gamma(z)\ \phi(z\gamma)$$

where $m \geqslant 2$ *and* $\phi \in \mathcal{P}(Z)$ *is a polynomial. Define*

$$\mathcal{A}^m := \sum_{m_1+\ldots+m_k=m} \mathcal{P}(Z)^{m_1}_\Gamma \cdots \mathcal{P}(Z)^{m_k}_\Gamma$$

$$= \langle (\phi_1)^{m_1}_\Gamma \cdots (\phi_k)^{m_k}_\Gamma :\ \phi_i \in \mathcal{P}(Z),\ m_i \geqslant 2,\ m_1 + \ldots + m_k = m\rangle.$$

Then the conditions (*) and (**) are satisfied.

Proposition 37. *For all* $a, b \in D$ *there exist* $D \supset_{\text{open}} U \ni a, b$ *and* $\ell_0 \in \mathbf{N}$ *such that for all* $\ell \geqslant \ell_0$ *there exists* $f \in \mathcal{A}^\ell_{\dot{a}\vee\dot{b}}$ *with* $0 \notin f(U)$.

Proof. By (*) there exists ℓ' and $f_1 \in \mathcal{A}^{\ell'}_{\dot{a}\vee\dot{b}}$, $f_2 \in \mathcal{A}^{\ell'+1}_{\dot{a}\vee\dot{b}}$ such that $f_{1/2}(a) = f_{1/2}(b) = 1$. Hence there exists $D \supset_{\text{open}} U \ni a, b$ such that $0 \notin f_{1/2}(U)$. Every integer $\ell \geqslant \ell_0 := \ell'(1+\ell')$ can be written as $\ell = m_1\ell' + m_2(\ell'+1)$ for positive integers $m_{1/2}$. Then

$$f := f_1^{m_1}\ f_2^{m_2} \in \mathcal{A}^{m_1\ell'}_{\dot{a}\vee\dot{b}} \mathcal{A}^{m_2(\ell'+1)}_{\dot{a}\vee\dot{b}} \subset \mathcal{A}^\ell_{\dot{a}\vee\dot{b}}$$

satisfies $0 \notin f(U)$. □

Proposition 38. *Suppose* $a_1, a_2 \in D$ *are not* Γ*-equivalent. Then there exist* $D \supset_{\text{open}} U_i \ni a_i$ *and* $\ell' \in \mathbf{N}$ *such that for all* $\ell \geqslant \ell'$ *there exist* $f_i \in \mathcal{A}^\ell_{\dot{a}_1\vee\dot{a}_2}$ *with* $f_i(a_i) \neq 0$ *and for* $i \neq j$

$$|f_j|_{U_i} < |f_i|_{U_i}$$

Proof. By (*) there exist ℓ' and $h_i \in \mathcal{A}^{\ell'}_{\dot{a}_1\vee\dot{a}_2}$ such that $h_i(a_i) = 1,\ h_j(a_i) = 0$ if $j \neq i$. By Proposition **??** there exists $D \supset_{\text{open}} U \ni a_1, a_2$ and $\ell_0 > \ell'$ such that for all $\ell \geqslant \ell_0$ there exists $f \in \mathcal{A}^{\ell-\ell'}_{\dot{a}_1\vee\dot{a}_2}$ with $0 \notin f(U)$. Choose smaller neighborhoods $U \supset_{\text{open}} U_i \ni a_i$ with $|h_i(U_i)| > \frac{1}{2} > |h_j(U_i)|$. Then

$$f_i := f\ h_i \in \mathcal{A}^\ell_{\dot{a}_1\vee\dot{a}_2}$$

satisfies the requirements. □

Proposition 39. *For each* $a \in D \subset Z$ *there exists* $D \supset_{\text{open}} U_a \ni a$ *and* $\ell_a \in \mathbf{N}$ *such that for all* $\ell \geqslant \ell_a$ *there exist finitely many* $h_i^a \in \mathcal{A}_{\dot{a}}^{\ell}$, $0 \leqslant i \leqslant n_a$ *with* $0 \notin h_0^a(U_a)$ *and the homogeneous coordinates yield an isomorphic embedding*

$$[h_0^a : \ldots : h_{n_a}^a] = [h_*^a] : \pi(U_a) \to \mathbf{P}^{n_a}$$

onto a locally analytic subset of projective space.

Proof. By Proposition **??** there exist

$$D \underset{\text{open}}{\supset} U = U\Gamma_a \underset{\approx}{\xrightarrow{\sigma}} \sigma(U) \underset{\text{open}}{\subset} Z$$

around a such that $\sigma(a) = 0$ and $\Gamma'_a := \sigma \circ \Gamma_a \circ \sigma^{-1} \subset \mathrm{GL}_d(\mathbf{C})$ is a (finite) linear group leaving $\sigma(U)$ invariant. By the 'polynomial finite generation theorem' there exist finitely many homogeneous polynomials $p_1, \ldots, p_m \in \mathcal{P}(Z)$ such that

$$\mathcal{P}(Z)^{\Gamma'_a} = \mathbf{C}[p_i] = \mathbf{C}[p_*].$$

Put $d_i = \deg p_i$ and choose $\ell_a^{d_i}$ as in condition (**). Since $p_i \circ \sigma \in \mathcal{O}_a^{\Gamma_a}$ it follows that for $\ell_0 \geqslant \max \ell_a^{d_i}$ there exist $f_i \in \mathcal{A}_{\dot{a}}^{\ell_0}$ such that

$$\mathfrak{o}_0(f_i \circ \sigma^{-1} - p_i) = \mathfrak{o}_a(f_i - p_i \circ \sigma) > d_i.$$

In particular, $f_i(a) = 0$ for all i. For any $D \supset_{\text{open}} U \ni a$ the intersection $\bigcap_{\Gamma \in \Gamma_a} U\gamma$ is Γ_a-invariant and still open, since Γ_a is finite. It follows that Γ_a-invariant neighborhoods form a neighborhood basis, so we may by Proposition **??** assume that $f_* = (f_1, \ldots, f_m)$ defines an isomorphic embedding

$$\begin{tikzcd}
\pi(U) = U/\Gamma_a \arrow[r, "\sigma", "\approx"'] \arrow[rr, bend right, "f_*"'] & \sigma(U)/\Gamma'_a \arrow[r, "(f\circ\sigma^{-1})_*"] & \mathbf{C}^m
\end{tikzcd}$$

onto a locally analytic subset of $\mathbf{C}^m$. The f_* may still have a common zero in U. For each $\ell \geqslant \ell_0$ there exist $h_0 \in \mathcal{A}_{\dot{a}}^{\ell}$ and $h \in \mathcal{A}_{\dot{a}}^{\ell-\ell_0}$ with $0 \notin h_0(U\Gamma) \cup h(U\Gamma)$. Then $h_i := hf_i \in \mathcal{A}_{\dot{a}}^{\ell}$ and (h_0, hf_*) has no common zero on U, so that the projectivation

$$[h_0, hf_*] : \pi(U) \to \mathbf{P}^m$$

defines an isomorphism onto a locally analytic set in $\mathbf{P}^m$. More precisely, if the range $f_*(U)$ is defined by the equations $h_j(w_1, \ldots, w_m) = 0$, for $(w_1, \ldots, w_m) \in \mathbf{C}^m$ then the range

$$[h_0(z) : h(z)f_1(z) : \ldots : h(z)f_m(z)] = [\frac{h_0(z)}{h(z)} : f_1(z) : \ldots : f_m(z)]$$

is described by the additional equation

$$w_0 = \frac{h_0(f_*^{-1}(w_1, \ldots, w_n))}{h(f_*^{-1}(w_1, \ldots, w_n)}$$

in $m+1$-variables $w_0, w_1, \ldots w_m$. Here f_*^{-1} is a local analytic inverse for f_*. $\square$

Remark 40. *The last, somewhat cumbersome, argument can be avoided in case D/Γ is compact. In this case we produce an injective holomorphic map $[f_*] : D/\Gamma \to \mathbf{P}^N$ whose range, by the proper mapping theorem, is automatically an analytic (in fact, algebraic) subset of $\mathbf{P}^N$.*

Lemma 41. *For $1 \leqslant j \leqslant k$ let $f_*^j = (f_0^j, \ldots, f_{n_j}^j)$ be Γ-automorphic of weight m_j. Take all monomials*

$$f' = \prod_{j=1^k} \prod_{i=0}^{n_j} (f_i^j)^{\alpha_i^j}$$

of total weight

$$\sum_{j=1}^{k} m_j \sum_{i=0}^{n_j} \alpha_i^j = m'$$

Then if one of the family f_*^j has no common zero on a subset $U \subset D$, then f'_* also has no common zero on U, and moreover the projectivation $[f'_*] : D/\Gamma \to \mathcal{P}^{n'}$ is injective wherever $[f_*^j] : D/\Gamma \to \mathbf{P}^{n_j}$ is injective.

Proof. For any $0 \leqslant i \leqslant n_j$ let $\alpha_{i'}^{j'} := m' \ \delta_{i'}^{i} \delta_{j}^{j'}$. Then $(f_i^j)^{m'}$ is an allowed monomial and the monomials $(f_i^j)^{m'}$ have no common zero on D, proving the first assertion. Now assume that the full monomial family f'_* satisfies $[f'_*(a)] = [f'_*(b)]$ for some $a, b \in D$. Then there exists a non-zero $\lambda \in \mathbf{C}$ such that $f'(a) = \lambda f'(b)$ for all admissible mononomials f'. In particular,

$$(f_i^j(a))^{m'} = \lambda \ (f_i^j(b))^{m'}$$

and more generally, for the same j

$$\prod_{i=0}^{n_j} (f_i^j(a))^{\alpha_i} = \prod_{i=0}^{n_j} (f_i^j(b))^{\alpha_i}$$

whenever $\sum\limits_{i=0}^{n_j} \alpha_i = m'$. This implies $f_i^j(a) = \lambda \ f_i^j(b)$, so that $[f_*^j(a)] = [f_*^j(b)]$. This proves the second assertion. □

Let us call the preceding procedure the **monomial construction**.

Proposition 42. *Assume that D/Γ is compact. Then for large enough $m \geqslant m_0$ there exists finitely many $f_0, \ldots, f_N \in \mathcal{A}_\Gamma^m(D)$ without common zero on D. Hence the projectivation*

$$[f_0, \ldots, f_N] : D/\Gamma \to \mathbf{P}^N$$

is a well-defined holomorphic map.

Proof. Since D/Γ is compact there exists a compact set $K \subset D$ such that $\pi(K) = D/\Gamma$. Equivalently, $K\Gamma = D$. There exists a finite covering $K \subset \bigcup\limits_{\mathcal{U}} U$ of open sets $U \subset D$ such that for each $U \in \mathcal{U}$ there exist automorphic forms $f_*^U = (f_0^U, \dots, f_{n_U}^U) \in \mathcal{A}^{m_U}$ without common zero on U so that the projectivation

$$[f_*^U] = [f_0^U : \dots : f_{n_U}^U] : \pi(U) \to \mathbf{P}^{n_U}$$

is a holomorphic isomorphism onto a locally analytic subset of $\mathbf{P}^{n_U}$. By increasing the weights if necessary, or by applying the 'monomial construction' to the finitely many families f_*^U we obtain family f'_* without common zeros on K. Since $K\Gamma = D$ it follows that the projectivation

$$[f'_*] : D/\Gamma \to \mathbf{P}^N$$

is a well-defined holomorphic map. □

Theorem 43. *Assume that D/Γ is compact. Then for large enough $m \geqslant m_0$ there exists finitely many $f_0, \dots, f_N \in \mathcal{A}_\Gamma^m(D)$ without common zero on D such that the projectivation*

$$[f_0, \dots, f_N] : D/\Gamma \to \mathbf{P}^N$$

is an injective holomorphic map, hence an isomorphism onto an analytic (in fact, algebraic) subset of $\mathbf{P}^N$.

Proof. Let $\mathcal{F}$ denote the collection of all maps $[f_*] = [f_0, \dots, f_n] : D/\Gamma \to \mathbf{P}^n$, where $f_i \in \mathcal{A}_\Gamma^m(D, \mathbf{C})$ are automorphic of the same weight m (depending on f_*) and have no common zeros on D. Define

$$(K \times K)_{f_*} := \{(z, w) \in K \times K : \ [f_*(z)] = [f_*(w)]\}.$$

By Proposition **??** $\mathcal{F}$ is non-empty. Now assume $[f_*] \in \mathcal{F}$ is not injective. Then there exist $a_1, a_2 \in K$ not Γ-related, such that $[f_*(a_1)] = [f_*(a_2)]$. In other words, $(a_1, a_2) \in (K \times K)_{f_*}$. By Proposition **??** there exist $D \supset_{\text{open}} U_i \ni a_i$ and automorphic forms h_i of the same weight m such that for $\{i, j\} = \{1, 2\}$ we have

$$|h_j|_{U_i} < |h_i|_{U_i}.$$

Applying the 'monomial construction' to the two families (h_1, h_2) and f_*, we obtain a new family f'_* with a common weight which has no common zeros on D, and the (well-defined) projectivation

$$[f'_*] : D/\Gamma \to \mathbf{P}^{n'}$$

is injective where $[f_*]$ is injective, i.e., $(K \times K)_{f'_*} \subset (K \times K)_{f_*}$. Since $[h_1, h_2]$ separates a_1 and a_2, Lemma **??** asserts that

$$[f'_*(a_1)] \neq [f'_*(a_2)].$$

It follows that $(a_1, a_2) \in (K \times K)_{f_*} \sim (K \times K)_{f_*}$., so that $(K \times K)_{f'_*} \subset (K \times K)_{f_*}$ is a proper subset. Now assume by contradiction, that there is no injective holomorphic map $[f_0, \ldots, f_N] : D/\Gamma \to \mathbf{P}^N$. Then the above construction produces a sequence $[f_*^k]$ for $k \in \mathbf{N}$ such that

$$(K \times K)_{f_*^0} \underset{\neq}{\supset} (K \times K)_{f_*^1} \underset{\neq}{\supset} \ldots \underset{\neq}{\supset} (K \times K)_{f_*^n} \underset{\neq}{\supset} (K \times K)_{f_*^{n+1}} \underset{\neq}{\supset} \ldots$$

On the other hand this is a decreasing sequence of analytic subsets meeting a compact set $K \times K$, which therefore must become stationary (noetherian property). Contradiction! □

Theorem 44. *Assume in addition that D/Γ is compact. Then there exist $h_i \in \mathcal{A}^m$ without common zero on D such that the projective map associated with $h_* = (h_0, \ldots, h_N)$ gives an isomorphism*

$$D/\Gamma \underset{\approx}{\xrightarrow{[h_*]}} [h_*](D/\Gamma) \underset{\text{alg}}{\subset} \mathbf{P}^N$$

onto a locally analytic (in fact, algebraic) subset of $\mathbf{P}^N$.

Proof. It remains to construct a global **injective** embedding. For any finite subset $A := \{a_1, \ldots, a_k\} \subset D$ of pairwise inequivalent points there exist neighborhoods $a_i \in U_i \subset_{\text{open}} D$ and, for $\ell \geqslant \ell_A$, there exist $f_i \in \mathcal{A}^\ell_{\vee \dot{a}_i}$ with $|f_i||_{U_i} > |f_j||_{U_j}$ for all $j \neq i$. By Proposition **??** we may assume that

$$\pi(U_i) \underset{\approx}{\xrightarrow{[h_0^i : \ldots : h_{n_i}^i]}} [h_*^i](\pi U_i) \underset{\text{ana}}{\subset} V_i \underset{\text{open}}{\subset} \mathbf{P}^{n_i},$$

where all $h_j^i \in \mathcal{A}^\ell_{\vee \dot{a}_i}$ have the same weight as f_i. Since D/Γ is compact, we may assume that $U_i\Gamma,\ 1 \leqslant i \leqslant k$ cover all of D. This implies that the functions $f_1, \ldots, f_k$ have no common zero on D and thus we may form the projective map

$$[f_1 : h_*^1 : f_2 : h_*^2 : \ldots : f_k : h_*^k] : D/\Gamma \to \mathbf{P}^N.$$

We claim that this map is injective and hence an isomorphism. Let $z, w \in D$ satisfy

$$(f_1(z), h_*^1(z), \ldots, f_k(z), h_*^k(z)) = \lambda(f_1(w), h_*^1(w), \ldots, f_k(w), h_*^k(w))$$

for some non-zero $\lambda \in \mathbf{C}$. If $z, w \in U_i$ for some i then $[h_*^i(z)] = [h_*^i(w)]$ implies $\pi(z) = \pi(w)$ since $[h_*^i]$ is an embedding when restricted to $\pi(U_i)$. Now suppose $z \in U_i,\ w \in U_j$ with $i \neq j$. Since $f_i(z) = \lambda f_j(z)$ we obtain $|\lambda|\ |f_j(z)| = |f_i(z)| > |f_j(z)|$ and $|\lambda|\ |f_j(w)| = |f_i(w)| < |f_j(w)|$. This contradiction shows that (**??**) is an embedding. □

Lemma 45. *There exist an open neighborhood $U_a \subset D$ of a which is invariant under the isotropy group Γ_a and a local chart $\phi_a : \tilde{U}_a \to U_a$ from a 0-neighborhood $\tilde{U}_a$ such that $J(0, \phi_a) = \mathrm{id}$ and for each $\gamma \in \Gamma_a$ the transformation $\tilde{\gamma} := \phi_a^{-1} \circ \gamma \circ \phi_a$ is linear.*

For each $a \in D$ there exists a biholomorphic map $\lambda_a : U_a \to U'_a \subset_{\text{open}} Z$ with $a\lambda_a = 0$ such that U_a is Γ_a-stable and for each $\gamma \in \Gamma_a$ the diagram

$$\begin{tikzcd}
U_a \arrow[r, "\lambda_a"] \arrow[d, "\gamma"'] & U'_a \arrow[d, "\gamma'_a"] \\
U_a \arrow[r, "\lambda_a"'] & U'_a
\end{tikzcd}$$

commutes, where $\gamma'_a \in \mathrm{GL}(Z)$ is linear. Putting $z = \zeta\lambda$ this implies

$$\lambda'_a(\zeta)\ \gamma'(\zeta\lambda) = (\lambda_a\gamma)'(\zeta) = (\gamma'_a\lambda_a)'(\zeta) = \gamma'_a(\lambda_a)'(\zeta\gamma'_a)$$

and hence for the Jacobians

$$J_{\lambda_a}(\zeta)\ J_\gamma(\zeta\lambda) = \det(\gamma'_a)\ J_{\lambda_a}(\zeta\gamma'_a).$$

Taking m-th powers we obtain

$$J^m_{\lambda_a}(\zeta)\ J^m_\gamma(\zeta\lambda_a) = \det(\gamma'_a)^m\ J^m_{\lambda_a}(\zeta\gamma'_a) = J^m_{\lambda_a}(\zeta\gamma'_a)$$

since m is a multiple of $|\Gamma_a|$. Put $\Gamma'_a := \{\gamma'_a :\ \gamma \in \Gamma_a\} = \lambda\Gamma_a\lambda^{-1}$. Then λ_a defines an isomorphism

$$\mathcal{O}_0^{\Gamma'_a} \xleftarrow{\Lambda_a} \mathcal{O}_a^{\Gamma_a,m}$$

by putting

$$(\Lambda_a f)(\zeta) := J^m_{\lambda_a}(\zeta)\ f(\zeta\lambda_a)$$

This isomorphism preserves the respective maximal ideals and their higher powers, and is therefore bicontinuous in the Krull topology. The finite-dimensional quotient space

$$\mathcal{O}_a^{\Gamma_a,m}/(\mathfrak{m}_a^{\Gamma_a,m})^{\ell+1}$$

consists of all jets at a up to order ℓ.

Proposition 46. *Let $A \subset D$ be a finite set of Γ-inequivalent points. Let $a \in A$ and $\phi_a \in \mathcal{O}_0^{\Gamma'_a}$. Choose $\ell \in \mathbf{N}$. Then there exists $p \in \mathbf{C}[z]$ such that for all $a \in A$ we have $\mathfrak{o}_a(p^\Gamma_m - f_a) > \ell$. Here we write $\phi_a \in \mathcal{O}_0^{\Gamma'_a}$ as $\phi_a = \Lambda_a f_a$ for a unique germ $f_a \in \mathcal{O}_a^{\Gamma_a,m}$ and m is any large multiple of all $\dot{a},\ a \in A$.*

Proof. Let $a \in U_a \subset_{\text{open}} D$ be a linearizing neighborhood around a and let $V_a \subset\subset U_a$. Then

$$\eta_a := \min_{z \in V_a} |J_{\lambda_a}(z)| \leqslant 1$$

since $|J_{\lambda_a}(0)| = 1$. The set (not a group)

$$\Gamma^a := \{\gamma \in \Gamma :\ \exists\ z \in V_a,\ |J_\gamma(z)| > \frac{\eta_a}{2}\}$$

is finite since $\sup_{z \in V_a} |J_\gamma(z)| < \frac{\eta_a}{2}$ for almost all $\gamma \in \Gamma$. Moreover, $\Gamma_a \subset \Gamma^a$, since for $\gamma \in \Gamma_a$ we have $J_\gamma(a)^{|\Gamma_a|} = 1$ and therefore $|J_\gamma(a)| = 1 > \frac{1}{2} \geqslant \frac{\eta_a}{2}$. For each $a \in A$ the set $A\Gamma^a \sim a$ is finite and Γ_a-invariant. There exists a polynomial $p \in \mathbf{C}[z]$ such that for all $a \in A$

$$\mathfrak{o}_a(p - f_a) > \ell,\ \mathfrak{o}_{A\Gamma^a \sim a}(p) > \ell.$$

Consider the m-weighted average

$$p^m_{\Gamma^a} - |\Gamma_a|\ f_a = (p^m_{\Gamma_a} - |\Gamma_a|\ f_a) + p^m_{\Gamma^a \sim \Gamma_a}.$$

For the first term we have

$$\mathfrak{o}_a(p^m_{\Gamma_a} - |\Gamma_a|\ f_a) > \ell$$

since the m-automorphy of f_a under Γ_a implies

$$p^m_{\Gamma_a} - |\Gamma_a|\ f_a = \sum_{\gamma\in\Gamma_a} J^m_\gamma(\gamma\cdot p - \gamma\cdot f_a)$$

and for each γ we have

$$\mathfrak{o}_a(J^m_\gamma(\gamma\cdot p - \gamma\cdot f_a) \geqslant \mathfrak{o}_a(\gamma\cdot p - \gamma\cdot f_a) = \mathfrak{o}_{a\gamma}(p - f_a) = \mathfrak{o}_a(p - f_a) > \ell.$$

For the second term we have $a\gamma \neq a$ and therefore

$$\mathfrak{o}_a(J^m_\gamma\ \gamma\cdot p) \geqslant \mathfrak{o}_a(\gamma\cdot p) = \mathfrak{o}_{a\gamma}(p) \geqslant \min \mathfrak{o}_{(a\Gamma^a)\sim a}(p) > \ell.$$

It follows that $p^m_{\Gamma^a} - |\Gamma_a|\ f_a$ vanishes of order $> \ell$ at a. Therefore the ℓ-jet (Taylor polynomial up to order ℓ) at a satisfies

$$j^\ell_a\Big(p^m_\Gamma - |\Gamma_a|f_a\Big) = j^\ell_a\Big(p^m_{\Gamma\sim\Gamma^a} + p^m_{\Gamma^a} - |\Gamma_a|f_a\Big) = j^\ell_a\Big(p^m_{\Gamma\sim\Gamma^a}\Big) + j^\ell_a\Big(p^m_{\Gamma^a} - |\Gamma_a|f_a\Big) = j^\ell_a\ p^m_{\Gamma\sim\Gamma^a}$$

since the second term has vanishing Taylor polynomial. In order to estimate the first term, for all $z \in V_a$ and $\gamma \in \Gamma \sim \Gamma^a$ we have $|J_\gamma(z)| \leqslant \frac{\eta}{2}$ by definition, and therefore

$$|1^m_{\Gamma\sim\Gamma^a}(z)| = \sum_{\gamma\in\Gamma\sim\Gamma^a} |J^m_\gamma(z)| \leqslant \Big(\frac{\eta}{2}\Big)^{m-2} \sum_{\gamma\in\Gamma\sim\Gamma^a} |J^2_\gamma(z)| \leqslant M\ 2^{-m}$$

uniformly for $z \in V_a$. Since p is bounded on D we also have

$$|p^m_{\Gamma\sim\Gamma^a}(z)| \leqslant M'\ 2^{-m}$$

for all $z \in V_a$. Thus for any $\epsilon > 0$ there exists a multiple m of all $\dot a,\ a \in A$, such that the ℓ-th Taylor polynomial of $p^m_{\Gamma\sim\Gamma^a}$ at a has norm $\leqslant \epsilon$. The same holds therefore for $p^m_\Gamma - |\Gamma_a| f_a$. Now consider the finite-dimensional vector space $\prod\limits_{a\in A} j^\ell_a\ \mathcal{O}^{\Gamma_a,m}_a$ and the linear mapping

$$\prod_{a\in A} j^\ell_a\ \mathcal{O}^{\Gamma_a,m}_a \xleftarrow{\Lambda} \mathbf{C}[z]$$

given by

$$\Lambda p := (j^\ell_a\ p^m_\Gamma)_{a\in A}$$

We have proved that this map has a dense linear range. By finite dimension, Λ is surjective. Hence for any given $f_a,\ a \in A$ there exists $p \in \mathbf{C}[z]$ such that for all $a \in A$ we have $j^\ell_a f_a = j^\ell_a p^m_\Gamma$, i.e., $\mathfrak{o}_a(p^m_\Gamma - f_a) > \ell$. Now (*) and (**) are easy consequences. $\square$

Proof. Let $A \subset\subset D$ be compact. For each $a \in A$ there exists a polynomial p and a weight d such that $p_d^{\Gamma}(a) \neq 0$. More generally, there exists polynomials p_i and weights d_i such that

$$(p_0)_{d_0}^{\Gamma}, \ldots, (p_k)_{d_k}^{\Gamma}$$

have no common zeros on A. Put $d := \text{l.c.m.}(d_i)$. Then

$$(p_1^{\Gamma,d_0})^{d/d_0}, \ldots, (p_k^{\Gamma,d_k})^{d/d_k}$$

have no common zeros on A and are automorphic of the same weight d. Thus we obtain a holomorphic map

$$D' \to \mathbf{P}^k, \quad z \mapsto [(p_1^{\Gamma,d_0}(z))^{d/d_0}, \ldots, (p_k^{\Gamma,d_k}(z))^{d/d_k}]$$

defined on an open neighborhood $A \subset D' \subset D$. Now suppose

$$F : D' \to \mathbf{P}^k, \quad z \mapsto [(f_1(z), \ldots, f_k(z)]$$

is given by automorphic forms $f_0, \ldots f_k$ of the same weight d. Suppose that this map is not injective on A and let $a, b \in A$ not Γ-related satisfy $f_i(a) = f_i(b)$ for $0 \leqslant i \leqslant k$. Choose polynomials p_1, p_2 and weights d_1, d_2 such that

$$p_1^{\Gamma,d_1}(a) = 0 = p_2^{\Gamma,d_2}(b), \quad p_1^{\Gamma,d_1}(b) \neq 0 \neq p_2^{\Gamma,d_2}(a).$$

Let $\tilde{d} = \text{l.c.m.}(d, d_1, d_2)$ and consider all monomials in $f_0, \ldots, f_k, p_1^{\Gamma,d_1}, p_2^{\Gamma,d_2}$ of total weight $\tilde{d}$. These finitely many monomials $g_0, \ldots g_\ell$ define a holomorphic map

$$G : D'' \to \mathbf{P}^\ell, \quad z \mapsto [g_0(z), \ldots, g_\ell(z)]$$

in the algebra generated by Poincaré series which satisfies

$$F(z) \neq F(w) \Rightarrow G(z) \neq G(w)$$

and, in addition, $G(a) \neq G(b)$. The first fact follows since the powers $f_i^{\tilde{d}/d_i}$ occurs as a monomial. The second fact follows since $p_1^{\Gamma,d_1}(a_j) \neq p_2^{\Gamma,d_2}(a_j)$ and the power $(p_j^{\Gamma,d_j})^{\tilde{d}/d_j}$ occurs as a monomial. Repeating this process, we obtain an injective map

$$G : D'' \to \mathbf{P}^m, \quad z \mapsto [h_0(z), \ldots, h_m(z)]$$

since A is met by only finitely many Γ-orbits. □

Chapter 2

Construction of Automorphic forms

2.1 Automorphic forms on semi-simple Lie groups

Let G be a semi-simple real Lie group of non-compact type. Its Lie algebra $\mathfrak{g}$ is identified with the right-invariant vector fields on G, by associating with $X \in \mathfrak{g}$ the first order differential operator

$$(X^\partial f)(g) := \partial_t^0 f(g \exp(tX)).$$

It follows that the universal enveloping algebra $\hat{\mathfrak{g}}$ is identified with the right-invariant differential operators (of any order) on G. Its **center** $\hat{\mathfrak{g}}^\circ$ is the commutative subalgebra of all bi-invariant differential operators on G. By Chevalley, this is a free polynomial algebra with rank(G) generators. For any function $f \in \mathcal{C}^\infty(G, V)$ into some vector space V, the set

$$(\hat{\mathfrak{g}}^\circ)^\perp_f := \{Y \in \hat{\mathfrak{g}}^\circ : \ Y^\partial f = 0\}$$

is an ideal (since $(XY)^\partial = X^\partial Y^\partial$ and the center is commutative) called the **annihilator ideal** of f. The linear evaluation map

$$\hat{\mathfrak{g}}^\circ \xrightarrow{\epsilon_f} (\hat{\mathfrak{g}}^\circ)^\partial f, \quad Y \mapsto Y^\partial f$$

induces a commuting diagram

$$\begin{tikzcd}
\hat{\mathfrak{g}}^\circ \arrow[d] \arrow[dr, "\epsilon_f"] & (\hat{\mathfrak{g}}^\circ)^\perp_f \arrow[l] \arrow[d, "0"] \; . \\
\hat{\mathfrak{g}}^\circ/(\hat{\mathfrak{g}}^\circ)^\perp_f \arrow[r, "\approx"'] & (\hat{\mathfrak{g}}^\circ)^\partial f
\end{tikzcd}$$

We say that f is **$\hat{\mathfrak{g}}^\circ$-finite** if

$$\text{codim } (\hat{\mathfrak{g}}^\circ)^\perp_f = \dim \ \hat{\mathfrak{g}}^\circ/(\hat{\mathfrak{g}}^\circ)^\perp_f = \dim \ (\hat{\mathfrak{g}}^\circ)^\partial f < \infty.$$

In the important case when the annihilator ideal has **codimension** 1 there exists a **character** (unital algebra homomorphism)

$$\begin{tikzcd}
\hat{\mathfrak{g}}^\circ \arrow[d] \arrow[rd, "\chi_f"] & (\hat{\mathfrak{g}}^\circ)^\perp_f \arrow[l] \arrow[d, "0"] \\
\hat{\mathfrak{g}}^\circ/(\hat{\mathfrak{g}}^\circ)^\perp_f \arrow[r, "\approx"'] & \mathbf{C}
\end{tikzcd}$$

such that $(\hat{\mathfrak{g}}^\circ)^\perp_f = \ker \chi_f$ and

$$X^\partial f = \epsilon_f(X) = \chi_f(X) f$$

for all $X \in \hat{\mathfrak{g}}^\circ$. Thus f is an **eigenfunction** under $\hat{\mathfrak{g}}^\circ$.

Let $K \subset G$ be a maximal compact subgroup, so that $K\backslash G$ is a Riemannian symmetric space. Let $\varkappa : K \to U(\underline{K}_\varkappa)$ be a unitary representation of K on $V := \underline{K}_\varkappa$. Let $\Gamma \subset G$ be a discrete subgroup of finite co-volume. A smooth function $f : G/\Gamma \to V$ is called **automorphic** if it satisfies the following three conditions: The first is an **invariance condition**

$$f(kg\gamma) = k^\varkappa\ f(g) \quad \forall\ k \in K,\ \gamma \in \Gamma. \tag{2.1.1}$$

The second is a **finiteness condition**

$$\text{codim}\ (\hat{\mathfrak{g}}^\circ)^\perp_f = \dim\ (\hat{g}^\circ/(\hat{\mathfrak{g}}^\circ)^\perp_f < \infty. \tag{2.1.2}$$

The third is a **growth condition** at ∞

$$\|f(g)\| \leqslant c \cdot \text{tr}_{\mathfrak{g}}(\text{Ad}_g^* \text{Ad}_g)^{m/2} \tag{2.1.3}$$

for some $c > 0$ and $m \in \mathbf{N}$. Here $\text{Ad} : G \to \text{GL}(\mathfrak{g})$ is the adjoint representation and the adjoint

$$g^* = \theta(g^{-1})$$

where θ is the Cartan involution. These conditions make sense for distributions, but one can show that automorphic forms are automatically real-analytic. Here the finiteness condition (2.1.2) is essential. We note that there exist the important special **scalar case** where $V = \mathbf{C}$ is 1-dimensional and codim $(\hat{\mathfrak{g}}^\circ)^\perp_f = 1$.

In order to understand the condition (2.1.1) in geometric terms, consider the associated vector bundle

$$G \times_K V = \{[g, v] = [kg, k^\varkappa v] :\ k \in K\}$$

over $K\backslash G$. Define

$$\mathcal{C}^\infty_K(G, V) = \{\Psi \in \mathcal{C}^\infty(G, V) :\ \Psi(kg) = k^\varkappa\ \Psi(g)\ \forall\ k \in K\}.$$

Every smooth section $\psi \in \mathcal{C}^\infty(G \times_K V)$ has a 'homogeneous lift' $\tilde{\psi} : G \to V$ defined by

$$\psi_{Kg} = [g, \tilde{\psi}(g)].$$

Then $[kg, \tilde{\psi}(kg)] = \psi_{Kg} = [g, \tilde{\psi}(g)] = [kg, k\tilde{\psi}(g)]$. Hence $\tilde{\psi} \in \mathcal{C}^\infty_K(G, V)$ and we obtain a linear isomorphism

$$\mathcal{C}^\infty_K(G, V) \xleftarrow{\approx} \mathcal{C}^\infty(G \times_K V).$$

The first part of (2.1.1) says that f is the homogeneous lift of a (unique) section $\psi \in \mathcal{C}^\infty(G \times_K V)$. Now consider the left action

$$(\gamma \cdot \Psi)(g) := \Psi(g\gamma)$$

of $\gamma \in G$ on $\mathcal{C}^\infty(G, V)$, which leaves the subspace $\mathcal{C}^\infty_K(G, V)$ invariant, since $(\gamma.\Psi)(kg) = \Psi(kg\gamma) = k^{\varkappa}\ \Psi(g\gamma) = k^{\varkappa}\ (\gamma \cdot \Psi)(g)$. Via the isomorphism (**??**) we obtain a left action $(\gamma, \psi) \mapsto \gamma \cdot \psi$ of G on $\mathcal{C}^\infty(G \times_K V)$ which is indirectly determined by

$$\psi_K h = [h, (\gamma \cdot \tilde{\psi})(h\gamma^{-1})]$$

for all $h \in G$. The second part of (2.1.1) says that the section corresponding to f is Γ-invariant under this action. In summary, automorphic forms are Γ-invariant sections of a homogeneous vector bundle, which are generalized eigensections and satisfy a growth condition.

To make contact with the standard notion using cocycles, consider a right action $X \times G \to X$ of G on a (smooth/real-analytic/complex analytic) space X, and a group H with a linear representation $\rho : H \to \mathrm{GL}(V)$ on some complex vector space V. Consider a map

$$J : G \times X \to H, \quad (g, z) \mapsto J_g(z)$$

satisfying the cocycle condition

$$J_{gg'}(z) = J_g(z)\ J_{g'}(zg)$$

for all $z \in X$ and $g.g' \in G$. Note that H is non-commutative, so that the order is important. Then a (smooth/real-analytic/holomorphic) function $f : X \to V$ is called J-automorphic if it satisfies

$$f(z) = J_\gamma(z)^\rho\ f(z\gamma)$$

for all $z \in X$ and $\gamma \in G$. Let $\mathcal{A}^J_\Gamma(X, V)$ denote the vector space of all J-automorphic functions. The assignment

$$(z, v) \cdot \gamma := (z\gamma, J_\gamma(z)^{-1}v)$$

defines a right action of G on $X \times G$ since

$$((z, v) \cdot \gamma) \cdot \gamma' = (z\gamma; J_\gamma(z)^{-1}v) \cdot \gamma' = ((z\gamma)\gamma', J_{\gamma'}(z\gamma)^{-1}J_\gamma(z)^{-1}v)$$

$$= ((z\gamma)\gamma', (J_\gamma(z)\ J_{\gamma'}(z\gamma))^{-1}v) = (z(\gamma\gamma'), J_{\gamma\gamma'}(z)^{-1}v) = (z, v) \cdot (\gamma\gamma')$$

Let

$$X \times_\Gamma V := (X \times V)/\Gamma = \{[z, v] = [z\gamma, J_\gamma(z)^{-1}v]\}$$

denote the quotient, regarded as a bundle over X/Γ via the map $X\times_\Gamma V \to X/\Gamma, \quad [z,v] \mapsto z\Gamma$. Its sections $\mathcal{C}^\infty(X \times_\Gamma V)$ can be identified with

$$\mathcal{C}^\infty_\Gamma(X,V) := \{\Phi : X \to V : \ \Phi(z\gamma) = J_\gamma(z)^{-1}\Phi(z)\}$$

by putting

$$\phi_z = [z, \tilde{\phi}(z)$$

for all $\phi \in \mathcal{C}^\infty(X \times_\Gamma V)$ and $z \in X$. Comparison with (**??**) shows that J-automorphic functions are just the (homogeneous lifts of) (smooth/real-analytic/holomorphic) sections of $X \times^J_\Gamma V$.

Now consider the special case $X = K\backslash G$ endowed with its natural right G-action.

Lemma 47. *The H-valued automorphy factors on $K\backslash G$ are in 1-1 correspondence with homomorphisms $j : K \to H$ together with a cross-section (= trivialization) θ of the associated (principal) H-bundle*

$$G \times_K H = \{[g,h] = [kg, j_k\ h] : \ g \in G, \ h \in H, \ k \in K\}$$

such that $\tilde{\theta}(k) = j_k$.

Proof. Let J be a factor of automorphy. Then for $k, k' \in K$ and the fixed point $o := K \in K\backslash G$ we obtain

$$J_{kk'}(o) = J_k(o)\ J_{k'}(ok) = J_{gg'}(z) = J_k(o)\ J_{k'}(o).$$

It follows that $k \mapsto J_k(o)$ is a homomorphism $K \to H$. Consider the associated (principal) H-bundle

$$P = G \times_K H = \{[g,h] = [kg, J_k(o)h] : \ g \in G, \ h \in H, \ k \in K\}.$$

The cross sections $\mathcal{C}^\infty(G \times_K H)$ are identified with

$$\mathcal{C}^\infty_K(G,H) = \{\Theta : G \to H : \ \Theta(kg) = J_k(o)\Theta(g)\}$$

by putting

$$\vartheta_g = [g, \tilde{\vartheta}(g)]$$

for all $\vartheta \in \mathcal{C}^\infty(G \times_K H)$. The map $G \ni g \mapsto J_g(o) \in H$ satisfies

$$J_{kg}(o) = J_k(o)\ J_g(ok) = J_k(o)\ J_g(o)$$

and hence belongs to $\mathcal{C}^\infty_K(G,H)$. It follows that

$$\theta_{Kg} = [g, J_g(o)]$$

defines a cross-section of $G \times_K H$ such that $\tilde{\theta}(g) = J_g(o)$. In particular, $\tilde{\theta}(k) = J_k(o)$.

Conversely, let $j : K \to H$ be a homomorphism. Then the cross sections $\mathcal{C}^\infty(G\times_K H)$ are identified with

$$\mathcal{C}^\infty_K(G, H) = \{\Theta : G \to H : \ \Theta(kg) = j_k\ \Theta(g)\}$$

by putting

$$\vartheta_g = [g, \tilde{\vartheta}(g)]$$

for all $\vartheta \in \mathcal{C}^\infty(G \times_K H)$. Assume there is a cross-section θ such that its homogeneous lift $\tilde{\theta} \in \mathcal{C}^\infty_K(G, H)$ satisfies $\tilde{\theta}(k) = j_k$. Define

$$J_\gamma(og) := \tilde{\theta}(g)^{-1}\tilde{\theta}(g\gamma) \in H$$

Then

$$J_\gamma(okg) := \tilde{\theta}(kg)^{-1}\tilde{\theta}(kg\gamma) = (j(k)\tilde{\theta}(g))^{-1}\ j(k)\tilde{\theta}(g\gamma) = \tilde{\theta}(g)^{-1}\ \tilde{\theta}(g\gamma) = J_\gamma(og)$$

so that $J : G\times K\backslash G \to H$ is well-defined, and satisfies $J_k(o) = \tilde{\theta}(e)^{-1}\tilde{\theta}(k) = j(e)^{-1}j(k) = j(k)$. Moreover, the automorphy property becomes

$$J_\gamma(z)\ J_{\gamma'}(z\gamma) = J_\gamma(og)\ J_{\gamma'}(og\gamma) = \tilde{\theta}(g)^{-1}\tilde{\theta}(g\gamma)\tilde{\theta}(g\gamma)^{-1}\tilde{\theta}(g\gamma\gamma') = \tilde{\theta}(g)^{-1}\tilde{\theta}(g\gamma\gamma') = J_{\gamma\gamma'}(og) = J_{\gamma\gamma'}(z).$$

□

In view of the above Lemma, we write the homomorphism $j : K \to H$ as $j(k) = J_k(o)$ for a (unique) H-valued cocycle J.

Lemma 48. *The map*

$$(Kg, h) \mapsto [g, J_g(o)h]$$

induces a trivialization

$$K\backslash G \times H \approx G \times_K H$$

as a principal H-bundle.

Proof. This map is well-defined, since $[kg, J_{kg}(o)h] = [kg, J_k(o)J_g(o)h] = [g, J_g(o)H]$ for all $k \in K$. By construction, the map is also H-equivariant. □

Lemma 49. *The map*

$$(Kg, v) \mapsto [g, J_g(o)^\rho v]$$

induces a vector bundle trivialization

$$K\backslash G \times V \approx G \times_K V \approx (G \times_K H) \times_H V.$$

The induced isomorphism on the sections

$$\mathcal{C}^\infty(G, V)^{id}_K \leftarrow \mathcal{C}^\infty(K\backslash G, V)$$

has the form $\tilde{f}(g) = J_g(o)^\rho \; f(Kg)$ *for all* $f \in \mathcal{C}^\infty(K\backslash G, V)$*. Moreover, for the right translation action*

$$(g \cdot \tilde{f})(g') := \tilde{f}(g'g)$$

on $\mathcal{C}^\infty(G,V)_K^{id}$*, and the* G*-action on* $CL^\infty(K\backslash G, V)$ *induced by*

$$(K\backslash G \times V) \times G \to K\backslash G \times V, \quad (z,v)\cdot g := (zg, J_g(z)^{-\rho} v),$$

the isomorphism (**??**) *is* G*-equivariant.*

Proof. This map is well-defined, since $[kg, J_{kg}(o)^\rho \; v] = [kg, (J_k(o) J_g(o))^\rho \; v] = [kg, J_k(o)^\rho \; J_g(o)^\rho \; v] = [g, J_g(o)^\rho v]$ for all $k \in K$. □

Thus every section of $(G \times_K H) \times_H V$ over $(G \times_K H)/H \approx K\backslash G$ is of the form $\tilde{\theta}[g,h] = h^{-\rho} \; f(J_g(o))$ for some function $f : K\backslash G \to V$. There is a left action of G on these sections by

$$g' \cdot \tilde{\theta}[g,h] := \tilde{\theta}[g'g, h]$$

Put

$$f \circ \pi(g) := f(Kg)$$

where $\pi : G \to K\backslash G$ is the canonical projection. Then the automorphy condition becomes

$$(f\circ\pi)(g) = f(Kg) = J_\gamma(Kg)^\rho \; f(Kg\gamma) = J_\gamma(Kg)^\rho \; (f\circ\pi)(g\gamma).$$

Equivalently,

$$(f\circ\pi)(g\gamma) = J_\gamma(Kg)^{-\rho} \; (f\circ\pi)(g)$$

which shows that $f \circ \pi \in \mathcal{C}_\Gamma^\infty(G,V)$ or, equivalently, its homogeneous lift is Γ-invariant under the action specified in Lemma **??**.

*** Now assume in addition that H has a representation $\rho : H \to \mathrm{GL}(V)$ and consider the associated vector bundle

$$(G \times_K H) \times HV = \{[p,v] = [ph, h^{-\rho} v] : \; p \in P, \; v \in V, \; h \in H\}$$

over P/H, whose cross-sections $\mathcal{C}^\infty(P \times HV))$ are identified with

$$\mathcal{C}_H^\infty(P,V) = \{\Theta \in \mathcal{C}^\infty(P,V) : \; \Theta(ph) = h^{-\rho}\Theta(p)\}$$

via the assignment

$$\vartheta_{pH} = [h, \tilde{\vartheta}(p)].$$

On the other hand, the representation $k \mapsto J_k(o)^\rho$ allows to form the associated vector bundle

$$G \times_K V = \{[g,v] = [kg, J_k(o)^\rho v] : \; g \in G, \; v \in V, \; k \in K\}$$

over $K\backslash G$, whose cross-sections $\mathcal{C}^\infty(G \times_K V))$ are identified with

$$\mathcal{C}^\infty_K(G,V) = \{\Sigma \in \mathcal{C}^\infty(G,V) :\ \Sigma(kg) = J_k(o)^\rho \Sigma(g)\}$$

via the assignment

$$\sigma_{Kg} = [g, \tilde{\sigma}(g)].$$

Let $[g,h]h' := [g,hh]$ be the canonical right H-action on $G \times_K H$. Then the map

$$Kg \mapsto [g, J_g(o)]H$$

is an isomorphism of the quotient spaces $K\backslash G \to (G \times_K H)/H$.

There is a natural identification

$$G \times_K V = (G \times_K H) \times_H V.$$

Then $P/H = K\backslash G$. Now let $f : K\backslash G \to V$ and write $\tilde{f}(g) := f(og)$ where $o = K \in K\backslash G$ is the midpoint. Then the automorphy condition is

$$\tilde{f}(g) = f(og) = J_\gamma(og)^\rho f(og\gamma) = J_\gamma(og)^\rho \tilde{f}(g\gamma)$$

with

$$J_{g\gamma}(o) = J_g(o)\ J_\gamma(og).$$

Thus we obtain $J_g(o)^\rho \tilde{f}(g) = J_{g\gamma}(o)^\rho \tilde{f}(g\gamma)$ so that ***

The space of all automorphic functions (with values in $\underline{K}_\varkappa$) of type $\mathcal{I}$, resp. χ, is denoted by

$$\mathcal{C}^\infty_\chi(G/\Gamma, \underline{K}_\varkappa), \quad \mathcal{C}^\infty_\mathcal{I}(G/\Gamma, \underline{K}_\varkappa),$$

One has to show that this space has finite dimension (and to compute its dimension). This was done (in the codim 1 case) by Selberg for $\mathrm{SL}_2^\mathbf{Z} \subset \mathrm{SL}_2^\mathbf{R}$, by Gelfand-Pjatetski-Shapiro (GPS) for $\mathrm{SL}_n^\mathbf{Z} \subset \mathrm{SL}_n^\mathbf{R}$, and in the holomorphic case by Siegel for $\mathrm{Sp}_{2n}^\mathbf{Z} \subset \mathrm{Sp}_{2n}^\mathbf{R}$. The general case is due to Langlands.

2.1.1 The holomorphic case

Let $K\backslash G$ be a hermitian bounded symmetric domain. The Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

into the $\pm$-eigenspaces of the symmetry s_o at the origin $o = K \in K\backslash G$ induces a splitting

$$\mathfrak{g}^\mathbf{C} = \mathfrak{k}^\mathbf{C} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

where $\mathfrak{k}^\mathbf{C}$ consists of linear vector fields on $D = k\backslash G \subset J$, in its Harish-Chandra realization

$$D \mapsto \mathfrak{p}^+, \quad \tanh(v) \mapsto \partial_v \quad (v \in J),$$

$\mathfrak{p}^+$ consists of all constant vector fields, and $\mathfrak{p}^-$ contains all quadratic vector fields induced by the Jordan triple product. Considering the associated subgroups of the conformal group $G^{\mathbf{C}}$ we have

$$P^- K^{\mathbf{C}} P^+ \underset{\text{open}}{\subset} G^{\mathbf{C}}.$$

Every $g \in G$ has a unique decomposition

$$g = h\ g'(o)\ \mathfrak{t}_{g(o)}$$

with $h \in P^-$. This follows from the properties $h(0) = 0,\ h'(0) = 0$ of $h \in P^-$. Therefore

$$G \subset P^- K^{\mathbf{C}} P^+.$$

Proposition 50. *The assignment*

$$J_g(z) := g'(z) \in K^{\mathbf{C}}$$

defines a holomorphic factor of automorphy with values in the complex Lie group $K^{\mathbf{C}}$*.*

Proof. The automorphy condition

$$J_{gg'}(z) = J_g(z)\ J_{g'(zg)}$$

follows since we use the right action $(z, g) \mapsto zg$. $\square$

The above decomposition shows that the anti-holomorphic tangent space

$$\overline{T}_o(D) = \mathfrak{p}^-.$$

Regarding $\mathfrak{p}^-$ as complexified vector fields on D, let $Y^{\overline{\partial}} f$ for each $Y \in \mathfrak{p}^-$ denote the anti-holomorphic Wirtinger derivative of functions $f : D \to V$. Then

$$\mathcal{O}(D, V) = \{f \in \mathcal{C}^\infty(D, V) :\ Y^{\overline{\partial}} f = 0\ \forall\ Y \in \mathfrak{p}^-\} = \mathcal{C}^\infty(D, V)^\perp_{\mathfrak{p}^-}.$$

Proposition 51. *Let* f *be a holomorphic automorphic function with respect to* J*. Then its homogeneous lift*

$$\tilde{f}(g) := J_g(o)\ f(og)$$

is automatically $\hat{\mathfrak{g}}^\circ$*-finite (a generalized eigenfunction)*

Proof. One first shows that satisfies

$$(Y^{\dot{\rho}} \tilde{f})(g) = J_g(o)\ (Y^{\overline{\partial}} f)(Kg)$$

Thus for holomorpic f we have

$$Y^{\dot{\rho}} \tilde{f} = 0$$

for all $Y \in \mathfrak{p}^-$. The decomposition (**??**) induces a vector space decomposition

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}^{\mathbf{C}} = \hat{\mathfrak{p}}^- \otimes \hat{\mathfrak{k}} \otimes \hat{\mathfrak{p}}^+$$

with $\hat{\mathfrak{p}}^{\pm}$ actually symmetric algebras, since $\mathfrak{p}^{\pm}$ is abelian. Therefore every $Y \in \hat{\mathfrak{g}}$ has a finite representation

$$Y = \sum_i p_- \ k_i \ p_i^+.$$

One shows that for $Y \in \hat{\mathfrak{g}}^\circ$ the terms $p_i^{\pm}$ occur always both ore not, so there is a linear map

$$\lambda : \hat{\mathfrak{g}}^\circ \hat{\mathfrak{k}}$$

such that

$$Y - \lambda Y \in \hat{\mathfrak{g}} \otimes \hat{\mathfrak{p}}^-$$

By holomorphy, this implies

$$(Y - \lambda Y)^{\dot{\rho}} \tilde{f} = 0.$$

Since $\tilde{f}$ is supposed to be K-finite, it follows that the annihilator ideal $\hat{\mathfrak{k}}_{\tilde{f}}^{\perp}$ has finite codimension. By (**??**) the same is true for $(\hat{\mathfrak{g}}^\circ)_{\tilde{f}}^{\perp}$. □

2.1.2 Poincaré and Eisenstein series

In the **holomorphic case**, let $J : K\backslash G \times G \to \mathrm{GL}(V)$ be a (holomorphic) automorphy factor, and let $\phi : K\backslash g \to V$ be holomorpihc, but not necessarily Γ-invariant. We know that $\tilde{f}$ is automatically $\hat{\mathfrak{g}}^\circ$-finite. Then the series

$$\phi_\Gamma^J(z) := \sum_{\gamma \in \Gamma} J_\gamma(z) \ \phi(z\gamma)$$

if convergent, defines a holomorphic J-automorphic function on $K\backslash G$. If $V = \mathbf{C}$, we can also take

$$\phi_\Gamma^m(z) := \sum_{\gamma \in \Gamma} J_\gamma^m(z) \ \phi(z\gamma)$$

since in this case J^m is again a (holomorphic) automorphy factor. If ϕ is already invariant under a subgroup $\Gamma_\infty \subset \Gamma$ and $J_\gamma = 1$ for $\gamma \in \Gamma_\infty$, then we take instead

$$\phi_{\Gamma/\Gamma_\infty}(z) := \sum_{\gamma \in \Gamma/\Gamma_\infty} J_\gamma(z) \ \phi(z\gamma)$$

In the **homogeneous case** let $f : G \to V$ be K-equivariant and $\hat{\mathfrak{g}}^\circ$-finite, but not necessarily Γ-invariant. Then the series

$$f_\Gamma(g) := \sum_{\gamma \in \Gamma} f(g\gamma)$$

if convergent, defines an automorphic function on G. If f is already invariant under a subgroup $\Gamma_\infty \subset \Gamma$, then we take instead

$$f_{\Gamma/\Gamma_\infty}(g) := \sum_{\gamma \in \Gamma/\Gamma_\infty} f(g\gamma)$$

Theorem 52. *Let $f \in L^1(G, V)$ be left K-finite and $\hat{\mathfrak{g}}^\circ$-finite. Then*

$$f_\Gamma(g) := \sum_{\gamma \in \Gamma} f(g\gamma)$$

converges absolutely and uniformly on compact subsets.

Proof. Since

$$\int\limits_G dg|f(g)| = \int\limits_{G/\Gamma} dg \sum_{\gamma \in \Gamma} |f(g\gamma)| < \infty$$

it follows that $\sum\limits_{\gamma\in\Gamma} |f(g\gamma)|$ converges in $L^1(G/\Gamma)$, in particular almost everywhere. Now the two finiteness conditions imply that f is annihilated by an elliptic operator L. By general (closed graph) principles this implies that the series $\sum\limits_{\gamma\in\Gamma} |f(g\gamma)|$ converges in the $\mathcal{C}^\infty$-topology, hence also uniformly on compact subsets. For the second assertion, assume that f is right K-finite. By Harish-Chandra's Lemma, for any e-neighborhood U there exists $\delta \in \mathcal{C}_c^\infty(U)$ invariant under $\mathrm{Int}(K)$ such that

$$f(g) = (f * \delta)(g) = \int\limits_G ds\ f(gs^{-1})\ \delta(s)$$

For $\gamma \in \Gamma$ it follows that

$$f(g\gamma) = \int\limits_G ds\ f(g\gamma s^{-1})\ \delta(s) = \int\limits_G dt\ f(gt^{-1})\ \delta(t\gamma)$$

If U □

2.1.3 Root decomposition and parabolic subgroups

For any torus $S \subset G$, with character group $S^\sharp$, we have the root decomposition

$$\mathfrak{g} = \mathfrak{s} \oplus \sum_{\alpha \in S^\sharp_{\mathfrak{g}}} \mathfrak{g}_S^\alpha$$

where, for $\alpha \in S^\sharp$, we put

$$\mathfrak{g}_S^\alpha := \{X \in \mathfrak{g} :\ \mathrm{Ad}_s X = s^\alpha\ X\ \forall\ s \in S\}$$

and

$$S^\sharp_\mathfrak{g} := \{\alpha \in S^\sharp : \ \mathfrak{g}^\alpha_S \neq 0\}.$$

For an algebraic group G a subgroup $P \subset G$ is called **parabolic** if G/P is projectively algebraic. Then we have a Levi decomposition

$$P = SP_>$$

where $S \subset G$ is a torus and $P_>$ is the unipotent radical. For the Lie algebra this means

$$\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{p}_>.$$

The **minimal parabolic** (Borel) subgroups contain a **maximal torus** $T \subset G$. Consider the associated root decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in T^\sharp_\mathfrak{g}} \mathfrak{g}^\alpha_T.$$

The Weyl group $G^\bullet_T/G^\circ_T$ acts simply transitively on the set of minimal parabolic subgroups by selecting a Weyl chamber. Thus

$$\mathfrak{p} = \mathfrak{t} \oplus \sum_{\alpha \in T^\sharp_\mathfrak{n}} \mathfrak{g}^\alpha_T$$

where $T^\sharp_\mathfrak{n} \subset T^\sharp_\mathfrak{g}$ denotes the set of positive roots. Choose a subset $\underset{\sim}{T}^\sharp_\mathfrak{n} \subset T^\sharp_\mathfrak{n}$ of simple (positive) roots. For any subset $\Theta \subset \underset{\sim}{T}^\sharp_\mathfrak{n}$ we define a subtorus

$$T^\Theta := \bigcap_{\alpha \in \Theta} \ker \alpha \subset T$$

of dimension $\mathrm{rk}(G) - |\Theta|$ and obtain the **standard parabolic**

$$P^\Theta = \langle G^\circ_{T^\Theta}, U \rangle = G^\circ_{T^\Theta} P^\Theta_>$$

with

$$\mathfrak{n}^\Theta = \sum_{\alpha \in T^\sharp_\mathfrak{n} \sim \langle \Theta \rangle} \mathfrak{g}^\alpha_T.$$

In the real case $G = G^\mathbf{R}$ we have $A \subset T \subset G$ for some maximal $\mathbf{R}$-split torus T and obtain

$$G^\bullet_A = G^\bullet_T = K^\bullet_A A$$
$$G^\circ_A = K^\circ_A A$$

It follows that

$$G^\bullet_A / G^\circ_A = K^\bullet_A / K^\circ_A$$

and

$$G = KAN \supset K^\circ_A AN$$

is a minimal parabolic.

2.2 Eisenstein series

For a matrix $x \in \mathbf{C}^{n \times n}$ we put

$$\|x\|_2 := \operatorname{tr}(x^* x)^{1/2}.$$

If $g \in \mathrm{GL}_n^{\mathbf{C}}$ we have $\|g\| \geqslant 1$ and $\|g^{-1}\|_2 \leqslant c\ \|g\|^N$ for some c and N. Let G be a connected semi-simple Lie group with finite center and Lie algebra $\mathfrak{g}$. We regard G as the group of real points of some algebraic subgroup of $\mathrm{GL}_n^{\mathbf{C}}$.

To construct such automorphic functions, consider the **Iwasawa decomposition**

$$G = KAN$$

of G. For example, if $G = \mathrm{SL}_n^{\mathbf{R}}$, then $K = \mathrm{SU}_n^{\mathbf{R}}$, $A \approx \mathbf{R}_+^n$ is realized as diagonal matrices and $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ consists of all unipotent upper triangular matrices. Writing $g = kan$ we put $a =: g_A,\ k =: g_K$. By [6, p.4] we have

$$dg = a^{2\rho} dk\ da\ dn$$

where $\rho : \mathfrak{a} \to \mathbf{R}$ is the half-sum of positive restricted roots. Now fix $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ (linear dual) and define the 'conical' function $N_\lambda : G/N \to \mathrm{End}(K_\varkappa)$ by

$$N_\lambda(g) := g_K^\varkappa\ g_A^{\lambda-\rho}.$$

Here we use $(gn)_K = g_K,\ (gn)_A = g_A$. Then there exists a character

$$\chi_\lambda : \mathcal{U}_{\mathfrak{g}}^\circ \to \mathbf{C},$$

satisfying $\chi_{s\lambda} = \chi_\lambda$ for all $s \in W = K_A^\bullet / K_A^\circ$ (Weyl group), such that

$$X^\partial N_\lambda = \chi_\lambda(X)\ N_\lambda$$

for all $X \in \mathcal{U}_{\mathfrak{g}}^\circ$. The Γ-invariant **Eisenstein series** is now (formally) defined by

$$N_\lambda^{(\Gamma\cap N)\backslash\Gamma}(g) = \sum_{\gamma \in (\Gamma\cap N)\backslash\Gamma} N_\lambda(g\gamma) = \sum_{\gamma \in (\Gamma\cap N)\backslash\Gamma} (g\gamma)_K^\varkappa\ (g\gamma)_A^{\lambda-\rho}$$

Then we still have

$$X^\partial N_\lambda^{(\Gamma\cap N)\backslash\Gamma} = \chi_\lambda(X)\ N_\lambda^{(\Gamma\cap N)\backslash\Gamma}$$

since X is acting from the left. One first proves convergence for λ in a non-empty open subset of $\mathfrak{a}_{\mathbf{C}}^*$. However, for these λ the associated character χ_λ is not the infinitesimal character of a unitary representation of G. Thus one needs analytic continuation as a meromorphic function in $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ and prove unitarity on a suitable 'imaginary' subspace.

2.3 Siegel domains

Let $P \subset G$ be a cuspidal parabolic subgroup. By [6, p. 5] we have decompositions

$$G = KP,$$

$$P = MAU \text{ Langlands decomposition}$$

where the A-component of $g = kmau$ is uniquely determined. Put

$$A_t := \{a \in A : \ \Sigma^O | \log a \leqslant t\}$$

For a bounded domain $\Omega \subset P^O$ the associated **Siegel domain** is defined by

$$\mathcal{S} := KA_t\Omega \subset G.$$

A subgroup $\Gamma \subset G^{\mathbf{Q}}$ which is commensurable with $G^{\mathbf{Z}}$ is called **arithmetic**. By a theorem of Borel [6, p. 5] the double quotient

$$P_{\mathbf{Q}} \backslash G_{\mathbf{Q}} / \Gamma$$

is finite and for a (finite) subset $\Lambda \in G_{\mathbf{Q}}$ we have

$$G = \mathcal{S}\Lambda\Gamma$$

for a Siegel domain $\mathcal{S} \subset G$ if and only if

$$G_{\mathbf{Q}} = P_{\mathbf{Q}}\Lambda\Gamma.$$

2.4 Theta Functions

In this section we generalize the classical theta function and its transformation properties to a multi-variable setting. Let X be a euclidean Jordan algebra of rank r, with positive definite cone $\acute{X}$, and tube domain

$$\acute{U} = X + i\acute{X} = \{z + iy : \ y > 0\}$$

in the complexification $U = X \otimes \mathbf{C}$. Let V be a hermitian vector space, with inner product $(v|b)$, endowed with a conjugation $v \mapsto \overline{v}$ and real form

$$V_{\mathbf{R}} := \{v \in V : \ \overline{v} = v\}.$$

Define the **Fourier transform** $L^2(V_{\mathbf{R}}) \to L^2(V_{\mathbf{R}}^\sharp)$ by

$$\hat{f}(\beta) := \int\limits_{V_{\mathbf{R}}} db \ e^{-2\pi i (b|\beta)} \ f(b).$$

The inverse Fourier transform $L^2(V_{\mathbf{R}}^\sharp) \to L^2(V_{\mathbf{R}})$ is given by

$$\check{\phi}(b) := \int\limits_{V_{\mathbf{R}}^\sharp} d\zeta\; e^{2\pi i(b|\beta)}\; \phi(\beta).$$

Consider the dual lattice

$$L^\sharp := \{\lambda \in V_{\mathbf{R}}^\sharp :\; (L|\lambda) \subset \mathbf{Z}\}.$$

Then we have the **Poisson summation formula**

$$|L|^{1/2} \sum_{\ell \in L} f(\ell) = |L^\sharp|^{1/2} \sum_{\lambda \in L^\sharp} \hat{f}(\lambda)$$

Here $|L| = \mathrm{Vol}(V_{\mathbf{R}}/L)$ is the volume of a fundamental domain for L in $V_{\mathbf{R}}$.

The set $\mathcal{H}(V_{\mathbf{R}})$ of all self-adjoint endomorphisms of $V_{\mathbf{R}}$ is a euclidean Jordan algebra under the anti-commutator product. Consider an injective unital **representation**

$$\rho : X \to \mathcal{H}(V_{\mathbf{R}}), \quad z \mapsto \rho_x = \tilde{x}$$

of X on $V_{\mathbf{R}}$, satisfying $\widetilde{x^2} = \tilde{x}^2$ for all $z \in X$. Let $u \mapsto \tilde{u} \in \mathrm{End}_{\mathrm{sym}}(V)$ be the **C**-linear extension. Every simple euclidean Jordan algebra $X \neq \mathcal{H}_3(\mathbf{O})$ has such a faithful representation.

A positive definite $w \in \mathcal{H}^+(V_{\mathbf{R}})$ is called an **intertwiner** if

$$w\; \tilde{u} = \tilde{u}\; w$$

for all $u \in U$.

Let $w : V_{\mathbf{R}} \to V_{\mathbf{R}}^\sharp$ be a linear isomorphism, such that for all $u \in U$ the bilinear form

$$(\tilde{u}v)(wb) = (\tilde{u}b)(wv)$$

on V is symmetric, and is positive definite when $u \in \acute{X}$.. For the unit element $u = c$ we obtain in particular

$$v(wb) = b(wv).$$

Let $L \subset V_{\mathbf{R}}$ be a lattice and define the multi-variable **theta function**

$$\Theta_w^L(u, v) := |L|^{1/2} \sum_{\ell \in L} e^{\pi i(\tilde{u}\ell + 2v)\cdot\tilde{w}\ell}$$

for all $u \in \acute{U}$ and $v \in V$. This series $|.|$-converges compactly on $\acute{U} \times V$ since for $u = z + iy \in \acute{U}$ we have $y \in \acute{X}$ and therefore

$$i\tilde{u}\ell \cdot \tilde{w}\ell = i\tilde{z}\ell \cdot \tilde{w}\ell - \tilde{y}\ell \cdot \tilde{w}\ell$$

with $\tilde{y}\ell \cdot \tilde{w}\ell$ positive definite and $e^{\pi i \tilde{z}\ell\cdot\tilde{w}\ell}$ of modulus 1. For the special case $v = 0$ we obtain the **theta nullwerte**

$$\Theta_w^L(u,0) := |L|^{1/2} \sum_{\ell\in L} e^{\pi i \tilde{u}\ell\cdot\tilde{w}\ell}$$

as a holomorphic function on $\acute{U}$, which is also called the **theta function of the lattice** L.

Consider the inverse isomorphism $w^{-1} : V_{\mathbf{R}}^{\sharp} \to V_{\mathbf{R}}$ and the dual action $\tilde{u}^{\sharp}$ on $V_{\mathbf{R}}^{\sharp}$ defined by

$$v(\tilde{u}^{\sharp}\nu) = (\tilde{u}v)\nu$$

for all $v \in V_{\mathbf{R}},\ \nu \in V_{\mathbf{R}}^{\sharp}$.

Lemma 53.

$$(w^{-1}\beta)(wb) = b\beta.$$

$$(w^{-1}\beta)(\tilde{u}^{\sharp}\nu) = (w^{-1}\nu)(\tilde{u}^{\sharp}\beta)$$

Proof. In fact, put $v := w^{-1}\beta$. Then $(w^{-1}\beta)(wb) = v(wb) = b(wv) = b\beta$. For the second assertion, put $w^{-1}\nu = v$ and $w^{-1}\beta = b$. Then

$$(w^{-1}\beta)(\tilde{u}^{\sharp}\nu) = (\tilde{u}w^{-1}\beta)\nu = (\rho^u b)(\tilde{w}v)$$

is symmetric in (b,v) by (**??**) and hence symmetric in β,ν. □

By Lemma (**??**), we can also define the dual theta function

$$\Theta_{w^{-1}}^{L^{\sharp}}(u,\nu) = |L^{\sharp}|^{1/2} \sum_{\lambda\in L^{\sharp}} e^{\pi i (w^{-1}\lambda|\tilde{u}\lambda+2\nu)}$$

for all $u \in \acute{U}$ and $\nu \in V^{\sharp}$.

Lemma 54.

$$\Theta_w^L(u,v) = \det w^{-1/4}\ \Theta_{\mathrm{id}}^{w^{1/2}L}(u, w^{1/2}v)$$

Proof. Use the formulas

$$(\tilde{u}\ell + 2v|w\ell) = (w^{1/2}\tilde{u}\ell + 2w^{1/2}v|w^{1/2}\ell) = (\tilde{u}w^{1/2}\ell + 2w^{1/2}v|w^{1/2}\ell)$$

and

$$|V_{\mathbf{R}}/w^{1/2}L| = \det w^{1/2}\ |V_{\mathbf{R}}/L|$$

□

For example,

$$(M\mathbf{Z}^{n\times 1})^\sharp = M^{-T}\mathbf{Z}^{n\times 1}.$$

An automorphism of ρ is a pair $\sigma \in \mathrm{GL}(X),\ \tau \in \mathrm{GL}(V)$ satisfying

$$\rho\sigma(z) = \tau\rho(z)\tau^T$$

for all $z \in X$.

Proposition 55. *For an automorphism (σ, τ) of ρ we have*

$$\Theta_w^L(u, v) = \det \tau^{1/2}\ \Theta_{\tau^T w\tau}^{\tau^T L}(u, v)$$

For $v, b \in V_{\mathbf{R}}$ we have

$$(\tilde{u}v|w^* b) = (\tilde{u}b|w^* v),$$

since both sides are **C**-linear in $u \in U$, and for $z \in X$ we have $(v|w^*\tilde{x}b) \in \mathbf{R}$ and

$$(\tilde{x}v|w^* b) = (w\tilde{x}v|b) = (\tilde{x}wv|b) = (wv|\tilde{x}b) = (\tilde{x}b|wv).$$

Proposition 56. *For $\ell \in L$ and $\lambda \in L^\sharp$ the* **translation formulas**

$$\Theta_w^L(u, v) = \Theta_w^L(u, v + w^{-1}\lambda) = e^{\pi i(\tilde{u}\ell + 2v|w^*\ell)}\ \Theta_w^L(u, v + \tilde{u}\ell)$$

hold.

Proof. Let $\ell \in L$ and $\lambda \in L^\sharp$. Then (**??**) implies $(w^{-1}\lambda|w\ell) = (\ell|\lambda) \in \mathbf{Z}$. Hence the first assertion follows from

$$(\tilde{u}\ell + 2(v + w^{-1}\lambda|w^*\ell) = (\tilde{u}\ell + 2v|w^*\ell) + 2(w^{-1}\lambda|w^*\ell) = (\tilde{u}\ell + 2v|w^*\ell) + 2(\ell|\lambda).$$

For $\ell, \ell' \in L$ we have $(\tilde{u}\ell|w^*\ell') = (\tilde{u}\ell'|w^*\ell)$ by (**??**). Hence the second assertion follows from

$$\begin{gathered}(\tilde{u}\ell + 2v|w^*\ell) + (\tilde{u}\ell' + 2(v + \tilde{u}\ell)|w^*\ell') \\ = (\tilde{u}\ell|w^*\ell) + (\tilde{u}\ell'|w^*\ell') + 2(\tilde{u}\ell|w^*\ell') + 2(v|w^*\ell + w^*\ell') \\ = (\tilde{u}(\ell + \ell')|w^*(\ell + \ell')) + 2(v|w^*(\ell + \ell')) = (\tilde{u}(\ell + \ell') + 2v|w^*(\ell + \ell'))\end{gathered}$$

using $\ell + L = L$. □

Proposition 57. *We have the* **inversion formula**

$$\Theta_{w^{-1}}^{L^\sharp}(-u^{-1}, \tilde{u}^{-1}wv) = |w|^{1/2}\ |\rho(-iu)|^{1/2}\ e^{\pi i(\tilde{u}^{-1}v|w^*v)}\ \Theta_w^L(u, v)$$

Proof. Consider the function

$$f(b) = e^{\pi i(i\tilde{w}^{-1}b - 2i\tilde{w}^{-1}\tilde{w}v)\cdot\tilde{w}^{-1}b} = e^{-\pi\tilde{w}^{-1}\cdot\tilde{w}^{-1}b}\ e^{2\pi\tilde{w}^{-1}\tilde{w}v\cdot\tilde{w}^{-1}b} = e^{-\pi(\tilde{w}\tilde{y})^{-1}b\cdot b}\ e^{2\pi\tilde{w}^{-1}v\cdot b}$$

for $b \in V_{\mathbf{R}}$. Applying [**?**, Theorem 1, p. 256] to the matrix $(\tilde{w}\tilde{y})^{-1}$ and the vector $\beta + i\tilde{w}^{-1}v$, we obtain the Fourier transform

$$\hat{f}(\beta) = \int\limits_{V_{\mathbf{R}}} db\ e^{-2\pi i\beta\cdot b}\ f(b)$$

$$= \int\limits_{V_{\mathbf{R}}} db\ e^{-2\pi i\beta\cdot b}\ e^{-\pi(\tilde{w}\tilde{y})^{-1}b\cdot b}\ e^{2\pi\tilde{w}^{-1}v\cdot b} = \int\limits_{V_{\mathbf{R}}} db\ e^{-2\pi i(\beta + i\tilde{w}^{-1}v)\cdot b}\ e^{-\pi(\tilde{w}\tilde{y})^{-1}b\cdot b}$$

$$= |\tilde{w}\tilde{y}|^{1/2}\ e^{-\pi(\beta + i\tilde{w}^{-1}v)\cdot\tilde{w}\tilde{y}(\beta + i\tilde{w}^{-1}v)} = |\tilde{w}\tilde{y}|^{1/2}\ e^{-\pi(\beta + i\tilde{w}^{-1}v)\cdot\tilde{w}(\tilde{y}\beta + iv)}$$

$$= |\tilde{w}\tilde{y}|^{1/2}\ e^{\pi\tilde{w}^{-1}v\cdot\tilde{w}v}\ e^{-\pi\beta\cdot\tilde{w}\tilde{y}\beta}\ e^{-2\pi i\beta\cdot\tilde{w}v} = |\tilde{w}\tilde{y}|^{1/2}\ e^{\pi\tilde{w}^{-1}v\cdot\tilde{w}v}\ e^{\pi i(i\tilde{y}\beta + 2v)\cdot\tilde{w}\beta}$$

The Poisson summation formula yields

$$\Theta^L_{w^{-1}}(-iw^{-1}, \tilde{w}^{-1}\tilde{w}v) = |L|^{1/2}\sum_{\ell\in L} f(\ell) = |L^\sharp|^{1/2}\sum_{\lambda\in L^\sharp}\hat{f}(\lambda)$$

$$= |\tilde{w}\tilde{y}|^{1/2}\ |L^\sharp|^{1/2}\ e^{\pi\tilde{w}^{-1}v\cdot\tilde{w}v}\sum_{\lambda\in L^\sharp} e^{\pi i(i\tilde{y}\lambda + 2v)\cdot\tilde{w}\lambda} = |\tilde{w}\tilde{y}|^{1/2}\ e^{\pi\tilde{w}^{-1}v\cdot\tilde{w}v}\ \Theta^{L^\sharp}_w(iy, v).$$

□

Let Z be a J^*-triple, with triple product $\{u; v; w\} =: u_v w$. Any idempotent $c \in Z$ of rank $k \leqslant r$ induces a Peirce deomposition

$$Z = Z_2^c \oplus Z_1^c \oplus Z_0^c.$$

One can show that the mapping

$$\tilde{u}v := u_c v = \{u; c; v\}$$

defines a homomorphism $Z_2^c \to \mathrm{End}(Z_1^c)$. For each $w \in Z_0^c$ and $v, b \in Z_1^c$ put

$$v(\tilde{w}b) := (v_w b|c) = (v_c b|w).$$

Lemma 58. *For each invertible* $w \in \overset{\circ}{Z}{}_0^c$ *the transformation* $\tilde{w} : Z_1^c \to Z_1^{c\sharp}$ *is an isomorphism, with inverse* $\tilde{w}^{-1}$*. Moreover*

$$(u_c v)(\tilde{w}b) = (v_{u^*} b|w)$$

Proof. Consider matrices $u = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix},\ v = \begin{pmatrix} 0 & v_1 \\ v_2 & 0 \end{pmatrix},\ b = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix},\ w = \begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix}$. Then

$$u_c v = uv + vu = \begin{pmatrix} 0 & uv_1 \\ v_2 u & 0 \end{pmatrix},$$

$$v_w b = vw^* b + bw^* v = \begin{pmatrix} v_1 w^* b_2 + b_1 w^* v_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$v(\tilde{w}b) = (\begin{pmatrix} v_1 w^* b_2 + b_1 w^* v_2 & 0 \\ 0 & 0 \end{pmatrix} | \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = (v_1 w^* b_2 + b_1 w^* v_2 | 1) = \mathrm{tr}(v_1 w^* b_2 + b_1 w^* v_2)$$

This implies

$$(u_c v)(\tilde{w}b) = \begin{pmatrix} 0 & uv_1 \\ v_2 u & 0 \end{pmatrix} (\tilde{w}b) = \mathrm{tr}(uv_1 w^* b_2 + b_1 w^* v_2 u) = \mathrm{tr}(v_1 w^* b_2 u + u b_1 w^* v_2) = (u_c b)(\tilde{w}v).$$

The identity

$$v_u b = vu^* b + bu^* v = \begin{pmatrix} 0 & 0 \\ 0 & v_2 u^* b_1 + b_2 u^* v_1 \end{pmatrix}$$

shows that

$$(v_u b | w) = \left(\begin{pmatrix} 0 & 0 \\ 0 & v_2 u^* b_1 + b_2 u^* v_1 \end{pmatrix} 000| \begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix} \right) = (v_2 u^* b_1 + b_2 u^* v_1 | w)$$

$$= \mathrm{tr}(v_2 u^* b_1 + b_2 u^* v_1) w^* = \mathrm{tr}(b_1 w^* v_2 u^*) + \mathrm{tr}(b_2 u^* v_1 w^* b_2)$$

It follows that

$$(v_{u^*} b | w) = (u_c v)(\tilde{w}b)$$

for all $u \in Z_2^c,\ v, b \in Z_1^c$ and $w \in Z_0^c$. $\square$

hence obtain the theta function

$$\Theta_w^L(u, v) = |L|^{1/2} \sum_{\ell \in L} e^{\pi i (\ell_{u^*} \ell + 2 v_c \ell | w)}$$

Note that $\ell_{u^*} \ell + 2 v_c \ell \in Z_0^c$.

Let $L \subset X$ be a lattice, with dual lattice

$$L' := \{\lambda \in X : \ (L|\lambda) \subset 2\mathbf{Z}\}.$$

Assume that $\ell \in L \Rightarrow \ell^2 \in L$. Put $L_i := L \cap X_i^c$ and suppose the lattice L_1^c is self-dual. Now fix $\omega \in \acute{X}_0^c$.

Proposition 59. *Consider the* $\mathcal{O}(\acute{Z}_2^c)$*-module* $\mathcal{M}$ *consisting of all holomorphic functions* $\vartheta : \acute{Z}_2^c \times Z_1^c \to \mathbf{C}$ *which satisfy the two invariance properties*

$$\vartheta(u,v) = \vartheta(u, v+\ell) = e^{\pi i(P_\ell u + \{\ell;e;v\})}\ \vartheta(u, v + \{u;e;\ell\})$$

for all $\ell \in L_1$*. Then*

$$\dim_{\mathcal{O}(\acute{Z}_2^c)} \mathcal{M} = \frac{|\{L_1;e;\omega\}|}{|L_1|}.$$

Define the **theta function**

$$\Theta_\mu(v,w) := \sum_{\nu \in L_1^\sharp} e^{\pi i(\frac{\{\lambda;w;\lambda\}}{2} + \{v;e-c;\lambda\}|\mu)}$$

Note that $\{\lambda;w;\lambda\} \in \{Z_1;Z_0;Z_1\} \subset Z_2$ and $\{v;e-c;\lambda\} \in \{Z_1;Z_0;Z_1\} \subset Z_2$.

Proposition 60. *The functions*

$$(u,v) \mapsto e^{\pi i(v|\ell)}\ \Theta_\omega(u, v + \{\omega^{-1};e;\{v;e;\ell\}\}),$$

for $\ell \in L_1/\{L_1;e;\omega\}$*, form a basis of* $\mathcal{M}$ *over* $\mathcal{O}(\acute{Z}_2^c)$*. For* $\ell = 0$ *we obtain the standard* Θ*-function.*

The theta function has the following invariance properties

$$\Theta_\omega(u,v) = \Theta_\omega(u+\ell_2, v) = \Theta_\omega(u, v+\ell_1)$$

$$= e^{\pi i(P_{\ell_1} u + \{\ell_1;e;v\}|\omega)}\ \Theta_\omega(u, v+\{u;e;\ell_1\}) = \Theta_{g\omega}(gu, gv)$$

whenever $\ell_2 \in L_2'$, $\ell_1 \in L_1 = L_1'$ and $g \in \mathrm{GL}(\acute{X})$ satisfies $P_{gz} = g\ P_z\ g^+$, $g(X_i^c) = X_i^c$, $g^+ L_1 = L_1$ and $g|_{X_0^c}$ is a Jordan algebra automorphism, equivalently, $g(e-c) = e-c$. The important inversion formula is

$$\Theta(\omega^{-1})(-u^{-1}, \{u^{-1};e;\{v;e;\omega\}\}) = N(e-c-iu)^{d/r-d_2/k}\ N(c+\omega)^{d/r-d_0/(r-k)}\ e^{\pi i(P_v u^{-1}|\omega)}\ \Theta_\omega(u,v)$$

2.5 Algebraic groups

Let V be a finite-dimensional vector space defined by linear equations over $\mathbf{Q}$. Thus $V_\mathbf{Q}$ is a $\mathbf{Q}$-vector space and we put

$$V_K := V_\mathbf{Q} \otimes K$$

for any field $K \supset \mathbf{Q}$. A subgroup $G \subset \mathrm{GL}(V)$ defined by algebraic equations with rational coefficients is called a **(linear) algebraic group** defined over $\mathbf{Q}$. We define

$$G_K := G_\mathbf{Q} \otimes K \subset \mathrm{GL}(V_K)$$

defined by the same equations over $K \supset \mathbf{Q}$. Examples are the full linear group and the orthogonal/symplectic subgroups, but not the unitary group. A discrete subgroup $lL \subset V_{\mathbf{Q}}$ is called a **lattice** if $V_{\mathbf{R}}/lL$ is compact. This means that Λ is a free abelian group of full rank. Put

$$G_\Lambda := \{g \in G : \ g\Lambda = \Lambda\}$$

A subgroup $\Gamma \subset G$ is called an **arithmetic subgroup** if Γ is commensurable with G_Λ for some lattice $\Lambda \subset V^{\mathbf{Q}}$.

Now assume that G is of hermitian type. Equivalently, G is the conformal group of a hermitian Jordan triple. Then we have

$$G^0_{\mathbf{R}} = \mathrm{Aut}^0(D)$$

for some bounded symmetric domain D. The image $\Gamma \subset G^0_{\mathbf{R}}$ of an arithmetic subgroup is a discrete subgroup and hence acts properly discontinuous on D. It follows that the quotient space D/Γ is a Zariski-open subset of an algebraic projective variety.

2.6 Satake compactification

Let $\Gamma \subset G = \mathrm{Aut}(D)^0$ be an arithmetic discrete subgroup. Let $c \in Z$ be a rational tripotent and $F = c + D^c_0 \subset \partial_{\mathbf{Q}} D$ be the associated rational boundary component. The Cayley transformation γ_c maps D onto the Siegel domain

$$D_c := \gamma_c(D) = \{(u,v,w) \in Z^c_2 \oplus Z^c_1 \oplus Z^c_0 : \ \mathrm{Im}(u) - L_w(u,u) \in \acute{X}_2\}.$$

For an open neighborhood $\zeta \in U \subset F$ and $a \in \acute{X}_2$ we consider the cylindrical set

$$D^{a,U}_c := \{(u,v,w) \in Z^c_2 \oplus Z^c_1 \oplus Z^c_0 : \ \mathrm{Im}(u) - L_w(u,u) \in a + \acute{X}^c_2\}$$

in D_c. There exist finitely many rational boundary components $Z_i \subset \partial_{\mathbf{Q}} D$ such that $F \subset \partial_{\mathbf{Q}} E_i$, and for every rational boundary component $Z \subset \partial_{\mathbf{Q}} D$ such that $F \subset \partial_{\mathbf{Q}} Z$ there exists $\gamma \in \Gamma$ such that $Z = E_i \gamma$ for some i. Write

$$Z_i = c_i) + D^{c_i}_0$$

where c_i is a rational tripotent covered by c. The Peirce 0-space $Z^{c_i}_0$ has itself a Peirce decomposition

$$Z^{c_i}_0 = Z^{c-c_i}_2 \oplus Z^{c-c_i}_1 \oplus Z^c_0$$

with respect to the rational tripotent $c - c_i \in Z^{c_i}_0$. Consider the Siegel domain realization

$$(D^{c_i}_0)_{c-c_i} := \gamma_{c-c_i}(D^{c_i}_0) = \{(u,v,w) \in Z^{c-c_i}_2 \oplus Z^{c-c_i}_1 \oplus Z^c_0 : \ \mathrm{Im}(u) - L_w(u,u) \in \acute{X}^{c-c_i}_2\}.$$

Choosing $a_i \in \acute{X}^{c-c_i}_2$ for each i, we may consider cylindrical sets

$$(D^{c_i}_0)^{a_i,U}_{c-c_i} := \{(u,v,w) \in Z^{c-c_i}_2 \oplus Z^{c-c_i}_1 \oplus Z^c_0 : \ \mathrm{Im}(u) - L^i_w(u,u) \in a_i + \acute{X}^{c-c_i}_2\}$$

Note that U is unchanged and independent of i.

$$\begin{pmatrix} c_i & 0 & 0 \\ 0 & c - c_i & 0 \\ 0 & 0 & D_0^c \end{pmatrix}$$

Then a basis of open neighborhoods of $\zeta\Gamma \in (D \cup \partial_{\mathbf{Q}} D)/\Gamma$ is given by

$$\Big(U \cup D_c^{a,U} \cup \bigcup_i c_i + (D_0^{c_i})_{c-c_i}^{a_i,U}\Big)\Gamma$$

where $\zeta \in U \subset D_0^c,\ a \in \acute{X}_2^c$ and $a_i \in \acute{X}_2^{c-c_i}$.

2.7 Siegel domains

Let $u \in \mathring{X}_2,\ x, y \in X_1$. By Lemma 4.1 we have

$$u_e(u_e^{-1}x) = x$$

Therefore also $u_e^{-1}(u_e x) = x$. Moreover, the Jordan triple identity yields

$$z_e(w_e u) - w_e(z_e u) = (z_e y)_e u - w_{e_x e} u$$

Since $e_x e = 0$ and $z_e y \in X_2 \oplus X_0$ it follows that

$$z_e(w_e u) - w_e(z_e u) \in X_2.$$

Replacing $z \mapsto u_e^{-1}x$ yields

$$(u_e^{-1}x)_e(u_e y) - w_e x = (u_e^{-1}x)_e(u_e y) - w_e(u_((u_e^{-1}x)) \in X_2.$$

Replacing $z \mapsto u_e x, u \mapsto u^{-1}$ yields

$$(u_e x)_e(u_e^{-1}y) - w_e x = (u_e x)_e(u_e^{-1}y) - w_e(u_e^{-1}(u_e x)) \in X_2.$$

Now the assertion follows by subtraction.

[*]

2.8 Automorphic Forms on Bounded Symmetric Domains

2.8.1 Harish-Chandra realization

Every $g \in G$ can be represented as $g = \tilde{\mathfrak{t}}_b \cdot h \cdot \mathfrak{t}_a$ with $a = 0 \cdot g$ and $h = {}^0\underline{g}$. Now let $z \in D$ and $g \in G$. Choose

$$\gamma = \tilde{\mathfrak{t}}_z \cdot B_{z,z}^{1/2} \cdot \mathfrak{t}_z$$

and

$$\gamma g = \tilde{\mathfrak{t}}_b \cdot h \cdot \mathfrak{t}_{z\gamma}$$

Then we have

$$h = {}^0\underline{\gamma g} = {}^0\underline{\gamma} \cdot {}^z\underline{g} = B_{z,z}^{1/2} \cdot {}^z\underline{g}$$

Therefore

$$\gamma g = \tilde{\mathfrak{t}}_b \cdot (B_{z,z}^{1/2} \cdot {}^z\underline{g}) \cdot \mathfrak{t}_{z\gamma}$$

and

$${}^z\underline{g} = B_{z,z}^{-1/2} \cdot h = Ad_{\mathfrak{p}^+}(B_{z,z}^{-1/2} \cdot h)$$

2.9 Boundary Components and Fourier-Jacobi Series

2.9.1 Boundary components

The boundary components of D have the form

$$F = c + \check{Z}_0^c$$

where $c \in S_k$ is a non-zero tripotent and $\check{Z}_0^c$ is the open unit ball of the Peirce 0-space Z_0^c of rank $r - k$. The associated **Cayley transform** is defined by

$$\gamma_c = \exp(\frac{\pi}{4}(c + c^*)) = \mathfrak{t}_c \circ B_{c,-c}^{1/2} \circ \tilde{\mathfrak{t}}_c$$

Under the Peirce decomposition

$$Z = Z_2^c \oplus Z_1^c \oplus Z_0^c \ni (u, v, w)$$

the Cayley transform has the explicit rational realization

$$\gamma_c(u,v,w) = (c+u)\circ(c-u)^{-1}, \sqrt{2}D((c-u)^{-1}, c)v, w+P_v(c-u)^{-1}) = (\{c+u; c; ; (c-u)^{-1}, \sqrt{2}\{(e-u)^{-1}; e; v\}, u$$

The Siegel domain of type III is

$$D_c := \{(u,v,w) : \ w \in \check{Z}_0^c, \ u - \frac{1}{2}\{v; (\mathrm{id} + Q_{c,w})^{-1}; v\} \in \underline{\acute{X}_c}\}$$

where the conjugate-linear endomorphism $Q_{c,w}v := \{c; v; w\}$ acting on V has norm < 1 since $\|w\| < 1$. The Cayley transform satisfies

$$\gamma_c(0) = c$$

The Peirce 0-projection

$$\gamma_c(\check{Z}) \xrightarrow{P_0^c} \check{Z}_0^c, \quad (u, v, w) \mapsto w$$

yields a holomorphic projection

$$\check{Z} \xrightarrow{(c+P_0^c)\circ\gamma_c} c + \check{Z}_0^c$$

which is equivariant under $N(c + \check{Z}_0^c)$. For any boundary component F of D the normalizer $N(F)$ is a parabolic subgroup of $G^0_{\mathbf{R}}$, realized as a semi-direct product

$$N(F) = U(F) \times Z(S_F)$$

of its unipotent radical $U(F)$ and the centralizer of a 1-dimensional $\mathbf{R}$-split torus S_F. Let α_F be the positive simple $\mathbf{R}$-root on G such that $\alpha_F|S_F$ is non-trivial. Let $A_F = (S_F)^0_{\mathbf{R}}$ be the identity component of the group of real points of S_F. Then A_F acts on $D_F := c_F(D)$ by

$$(u, v, w) \cdot a = (\alpha^i_F(a)u, \alpha^J_F(a)v, w)$$

where $i, j > 0$. This implies

$$J(u + v + w, a) = \alpha^m_F(a)$$

for some $m < 0$. Now let $\Gamma \subset G^0_{\mathbf{R}}$ be an arithmetic subgroup. Then $N(F) \cap \Gamma$ is arithmetic in $N(F)_{\mathbf{R}}$. For each $\gamma \in X \cap \Gamma$ the action on D_F is

$$(u + v + w)\gamma := (u + \ell_\gamma, v, w)$$

for some translation vector $\ell_\gamma \in X$. These vectors span a lattice

$$X_{\mathbf{Z}} := \{\ell_\gamma : \ \gamma \in X \cap \Gamma\} \subset X$$

such that $X/X_{\mathbf{Z}}$ is compact. Consider the dual lattice

$$\Lambda := \{\lambda \in X : \ (\lambda|X_{\mathbf{Z}}) \in \mathbf{Z}\} \subset X = X^\sharp.$$

2.9.2 Fourier-Jacobi series

Consider a Γ-automorphic form f on D_F. Then we have

$$f(u, v, w) = f((u, v, w)\gamma) = f(u + \ell, v, w)$$

for all $\ell \in L_2$. Therefore we obtain a **Fourier-Jacobi expansion**

$$f(u, v, w) = \sum_{\mu \in L_2^\sharp} f^\bullet_\mu(v, w)\ e^{2\pi i(u|\mu)}$$

over the dual lattice $L_2^\sharp$. If $\dim Z > 1$, i.e., $Z \neq \mathbf{C}$, then 'Koecher's principle' asserts that

$$f(u, v, w) = \sum_{\mu \in \Lambda_+} f^\bullet_\mu(v, w)\ e^{2\pi i(u|\mu)},$$

where

$$\Lambda_+ := L^\sharp \cap \acute{X}_2^-$$

is the intersection of $L^\sharp$ with the closed convex cone $\acute{X}_2^- := \overline{\acute{X}_2}$ associated with the Jordan algebra X_2. Moreover, the 0-th Fourier coefficient satisfies

$$f_0^\bullet(v+\alpha, w) = f_0^\bullet(v, w)$$

for a lattice $U \cap \Gamma$ in $V = Z_1^c$. By Liouville it follows that $f_0^\bullet(v,w) = f_0^\bullet(w)$ is independent of v. Now consider the 'cylindrical' set

$$S := \{(u,v,w) \in D_F : \ \mathrm{Im}(u) - \frac{1}{2} L_w(v,v) \in \omega + \acute{X}, \ \|v\| \leqslant K, \ w \in Q\}$$

where $\omega\acute{X}, \ K < \infty$ and $Q \subset \check{Z}_0^c$ is compact. One can show that the constants

$$M_\lambda := \sup_{(u,v,w)\in S} |f_\lambda^\bullet(v,w) \ e^{2\pi i(u|\lambda)}|$$

satisfy

$$\sum_{\lambda \in \Lambda_+} M_\lambda < \infty.$$

2.9.3 Jacobi Forms

The functions $f_\mu^\bullet$ occurring in (**??**) are called 'Jacobi forms'. In general, let $V_{\mathbf{R}}$ be a real vector space endowed with a symmetric bilinear form $(v|v')$ and let $V_{\mathbf{Z}} \subset V_{\mathbf{R}}$ be an even lattice, satisfying $(\ell|\ell) \in 2\mathbf{Z}$ for all $\ell \in \mathbf{Z}$. Let

$$V_{\mathbf{Z}}^\sharp := \{\lambda \in V_{\mathbf{R}} : \ (V_{\mathbf{Z}}|\lambda) \subset \mathbf{Z}\}$$

be the dual lattice in $V_{\mathbf{R}}^\sharp = V_{\mathbf{R}}$ (via the inner product). A holomorphic function

$$f : V_{\mathbf{C}} \times \acute{\mathbf{C}} \to \mathbf{C}$$

is called a **Jacobi k-form** if for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma := \mathrm{PSL}_2(\mathbf{Z})$ and for all $\ell_1, \ell_2 \in V_{\mathbf{Z}}$ we have

$$f(v,w) = (cw+d)^{-k} \ f(\frac{aw+b}{cw+d}) \ \exp(-\pi i \frac{c(v|\overline{v})}{cw+d}) = f(v + w\ell_1 + \ell_2, w),$$

and there is a Fourier expansion

$$f(v,w) = \sum_{\nu \in V_{\mathbf{Z}}^\sharp} f_\nu^\bullet \ e^{2\pi i(\frac{(\nu|\nu)}{2}w + (v|\nu))}$$

Example 61. *Let $Z = \mathbf{C}^{r\times r}_{\mathrm{sym}}$ be the J^*-triple of symmetric matrices. Write the elements of Z as*

$$z = \begin{pmatrix} u & v \\ v^+ & w \end{pmatrix}$$

where $u \in \mathbf{C},\ v \in \mathbf{C}^{r-1}$ and $w \in \mathbf{C}^{(r-1)\times(r-1)}_{\mathrm{sym}}$. A k-automorphic function $f(z)$, under $\Gamma := \mathrm{Sp_{2r}}(\mathbf{Z})$, has a Fourier-Jacobi expansion

$$f \begin{pmatrix} u & v \\ v^+ & w \end{pmatrix} = \sum_{\mu \geqslant 0} f_l m^\bullet(v,w)\ e^{2\pi i \mathrm{tr}(u\mu)}.$$

If $\mu > 0$ is positive definite, then $f^\bullet_\mu(v,w)$ is a Jacobi k-form on $\mathbf{C}^{r-1} \times \acute{\mathbf{C}}$, for the inner product

$$(v_1|v_2) := 2v_1^+\mu v_2$$

Example 62. *Let $Z = \mathbf{C}^{r\times r}$ be the J^*-triple of square matrices. Write the elements of Z as*

$$z = \begin{pmatrix} u & v_1 + iv_2 \\ v_1^* + iv_2^* & w \end{pmatrix}$$

where $u \in \mathbf{C},\ v_1, v_2 \in \mathbf{C}^{r-1}$ and $w \in \mathbf{C}^{(r-1)\times(r-1)}$. A k-automorphic function $f : \acute{Z} \to \mathbf{C}$, under the imaginary quadratic field K, has a Fourier-Jacobi expansion

$$f \begin{pmatrix} u & v_1 + iv_2 \\ v_1^* + iv_2^* & w \end{pmatrix} = \sum_{\omega \geqslant 0} f^\bullet_\omega(u, v_1 + iv_2)\ e^{2\pi i \mathrm{tr}(w\omega)}.$$

If $\omega > 0$ is positive definite, then $f^\bullet_\omega(u, v_1 + iv_2)$ is a Jacobi k-form on $\acute{\mathbf{C}} \times (\mathbf{C}^{r-1} \times \overline{\mathbf{C}}^{r-1}$, for the inner product

$$(v_1 + iv_2 | v_1' + iv_2') := (v_1^* - iv_2^*)\omega(v_1' + iv_2') + (v_1'^* - iv_2'^*)\omega(v_1 + iv_2)$$

Example 63. *Similar for $Z = \mathbf{C}^{2r\times 2r}_{asym}$*

Example 64. *Let $Z = \mathcal{H}_3(\mathbf{O}) \otimes \mathbf{C}$ be the exceptional J^*-triple of tube type. Write the elements of Z as*

$$z = \begin{pmatrix} u & v_1 + iv_2 \\ v_1^* + iv_2^* & w \end{pmatrix}$$

where $u \in \mathbf{C},\ v_1, v_2 \in \mathbf{O}^2_{\mathbf{R}}$ and $w \in \mathcal{H}_2(\mathbf{O}) \otimes \mathbf{C}$. A k-automorphic function $f : \acute{Z} \to \mathbf{C}$, under the integer Cayley numbers $\mathbf{O}_{\mathbf{Z}}$, has a Fourier-Jacobi expansion

$$f \begin{pmatrix} u & v_1 + iv_2 \\ v_1^* + iv_2^* & w \end{pmatrix} = \sum_{\omega \geqslant 0} f^\bullet_\omega(u, v_1 + iv_2)\ e^{2\pi i (w|\omega)}.$$

If $\omega > 0$ is positive definite, then $f^\bullet_\omega(u, v_1 + iv_2)$ is a Jacobi $18k$-form on $\acute{\mathbf{C}} \times \mathbf{O}^2_{\mathbf{C}}$, for the inner product

$$(a|b) := (ab^* + ba^*|\omega).$$

Example 65. *Let R be an even unimodular positive definite $2k \times 2k$-matrix, and let $G \in \mathbf{Z}^{2k\times n}$ have rank n. Then the theta series*

$$\Theta_{R,G}(u,v) := \sum_{\lambda\in\mathbf{Z}^{2k\times 1}} e^{\pi i\lambda^+ R(\lambda u+2Gv)}$$

is a Jacobi k-form for $V_{\mathbf{C}} = \mathbf{C}^{n\times 1}$ and $(a|b) := a^+G^+RGb$.

Now define the **level**

$$q := \min\{0 < n \in \mathbf{N} : \ \frac{n}{2}(\ell|\ell) \in \mathbf{Z} \ \forall\ \ell \in V_{\mathbf{Z}}\}$$

and put

$$\Gamma_q := \{\gamma \in \dot{\mathbf{Z}}_2^{\mathrm{G}} : \ \gamma - 1 \in (q\mathbf{Z})^{2\times 2}\}$$

Then every Jacobi k-form $f(u,v)$ has a theta expansion

$$f(u,v) = \sum_{\mu\in\Lambda^\sharp/\Lambda} f_\mu^\bullet(u)\ \vartheta_\mu(u,v)$$

where

$$f_\mu(u) = \sum_{0\leqslant m\in q(\mathbf{Z}-(\mu|\mu)/2)} f^\bullet_{m/q+(\mu|\mu)/2,\mu}\ e^{2\pi ium/q}$$

$$\vartheta_\mu(u,v) = \sum_{\lambda\in\mu+\Lambda} e^{2pi(u(\lambda|\lambda)/2)+(v|\lambda)}$$

Choose representatives $\mu_1,\ldots,\mu_d$ of $\Lambda^\sharp/\Lambda$, where $d = [\Lambda^\sharp : \Lambda]$. Then there exists a linear isomorphism

$$\mathcal{J}_k^\Lambda(\acute{\mathbf{C}} \times V_{\mathbf{C}}) \xleftarrow{\begin{pmatrix} \vartheta_{\mu_1} \\ \hline \vdots \\ \hline \vartheta_{\mu_d} \end{pmatrix}} \{F \in {}_d\mathcal{M}_{k-n/2}^{\Gamma_q}(\acute{\mathbf{C}}) : \ \gamma^{k-n/2}F = \chi(\gamma)F \ \forall\ \gamma \in \Gamma\}$$

Here $\chi : \Gamma \to U(d)$ and $\mathcal{M}_k^{\Gamma_q}(\acute{\mathbf{C}})$ is the finite-dimensional vector space of all elliptic modular k-forms on $\acute{\mathbf{C}}$.

Chapter 3

Automorphic forms in Toeplitz-Berezin quantization

3.1 II_1-factors and discrete groups

A von Neumann algebra $M \subset \mathcal{B}(H)$ carries the **ultraweak topology** generated by the seminorms

$$a \mapsto \sum_{i=1}^{\infty} |(\zeta_i|a\eta_i)|$$

where $\zeta_i, \eta_i \in H$ satisfy $\sum\limits_{i=1}^{\infty} \|\zeta_i\|^2 < \infty$, $\sum\limits_{i=1}^{\infty} \|\zeta_i\|^2 < \infty$. Any equivalent representation of M induces the same ultraweak topology, and every $*$-representation of M on a separable Hilbert space is ultraweakly continuous.

A II_1-factor is a von Neumann algebra M with trivial center $\mathbf{C} \cdot 1$ and a normal faithful finite trace $\tau : M \to \mathbf{C}$, normalized by $\tau(1) = 1$. Let

$$M^\tau = L^2(M, \tau)$$

be the GNS-Hilbert space with inner product

$$(a|b)_\tau := \tau(a^*b).$$

Then M acts on M^τ by left multiplication, and the commutant

$$M' := \{T \in \mathcal{L}(M^\tau) : \ [M, T] = 0\}$$

is again a II_1-factor. If M is a II_1-factor, with a $*$-representation $M \to \mathcal{B}(H)$ there exists a **formal dimension** $\dim_M H$ of H as a left Hilbert module over M. For $H = M^\tau$ we obtain

$$\dim_M M^\tau = 1.$$

Example 66. *A group* Γ *is called an* **icc-group** *(infinite conjugacy classes) if for each* $e \neq \gamma \in \Gamma$ *the conjugacy class* $\{g\gamma g^{-1} : \ g \in \Gamma\}$ *is infinite. For each icc group* Γ *the left group von Neumann algebra*

$$M := W^*_\lambda(\Gamma) \subset \mathcal{B}(\ell^2(\Gamma))$$

is a II_1*-factor, with trace*

$$\tau \sum_{\gamma \in \Gamma} a_\gamma \gamma := a_e.$$

In this case we have

$$M^\tau = \ell^2(\Gamma)$$

and therefore

$$\dim_{W^*_\lambda(\Gamma)} \ell^2(\Gamma) = 1.$$

The commutant

$$W^*_\lambda(\Gamma)' = \{T \in \mathcal{B}(\ell^2(\Gamma)) : \ [T, W^*_\lambda(\Gamma)] = 0\} = W^*_\rho(\Gamma)$$

is the right convolution W^**-algebra.*

For a semisimple, non-compact Lie group G a subgroup $\Gamma \subset G$ is called a **lattice** if G/Γ has finite volume (with respect to Haar measure)

Proposition 67. *A lattice* Γ *in a semi-simple Lie group* G *with trivial center is an icc group.*

Proof. G is an algebraic group (more precisely, its real points) and for each $h \in \Gamma$ the map

$$\alpha_h : G \to G, \ \alpha_h(g) := ghg^{-1}$$

is Zariski-continuous. Let $C_h := \alpha_h(\Gamma) = \{\gamma h \gamma^{-1} : \ \gamma \in \Gamma\}$ be the conjugacy class of h in Γ. Then $\alpha_h(\Gamma) \subset C_h$. Now suppose that C_h is finite, hence Zariski-closed. Since a lattice Γ is Zariski-dense in G [Borel-Zimmer] this implies $\alpha_h(G) \subset C_h$. Therefore the centralizer

$$G_h^\circ := \{g \in G : \ gh = hg\}$$

is a closed subgroup of finite index in G. Since G is Zariski-connected, it follows that $G_h^\circ = G$. Hence h belongs to the center of G and therefore $h = e$. □

For a discrete series representation $\pi : G \to \mathrm{U}(H_\pi)$ the **formal dimension** $d_\pi \in \mathbf{R}_+$ is defined by Schur orthogonality

$$\int_G dg \ (\xi | g^\pi \eta)(g^\pi \sigma | \tau) = \frac{(\xi|\tau)(\sigma|\eta)}{d_\pi}$$

for all $\xi, \eta, \sigma, \tau \in H_\pi$. Equivalently, d_π is the Plancherel measure of the atom $\pi \in G^\sharp$.

Theorem 68. *Let G be a semi-simple Lie group with a discrete series representation $\pi : G \to U(\mathcal{H}_\pi)$. Let $\Gamma \subset G$ be a lattice subgroup. Then*

$$\dim_{W^*_\lambda(\Gamma)} \mathcal{H}_\pi = d_\pi \cdot |G/\Gamma|$$

where $|G/\Gamma|$ denotes the covolume of $\Gamma \subset G$.

Proof. We may assume that there is an isometry $u : H \to L^2(G)$ such that $u^* u = \mathrm{id}_H$ and $p := uu^* : L^2(G) \to H$ is the orthogonal projection. Identify

$$L^2(G) = \ell^2(\Gamma) \otimes L^2(D) = M^\tau \otimes L^2(D)$$

for a fundamental domain $D \subset G$. Then

$$a^\pi = a^\lambda \otimes \mathrm{id}$$

for all $a \in M$. The commutant $\mathcal{B}(L^2(G))^\circ_M$ of $M \subset \mathcal{B}(L^2(G))$ is a II_1-factor containing p. By definition,

$$\dim_M H = \mathrm{tr}_{\mathcal{B}(L^2_G)^\circ_M}(p)$$

for the normalized trace on $\mathcal{B}(L^2(G))^\circ_M$. The commutant is generated by finite sums $x = \sum\limits_{\gamma \in \Gamma} \rho_\gamma \otimes a_\gamma$, where $\rho_\gamma = J\lambda_\gamma J \in \mathrm{End}_M(M^\tau)$ and

$$a_\gamma = \sum_{m,n} a^{m,n}_\gamma e_m e^*_n \in \mathcal{F}(L^2_D)$$

are finite rank operators, for an orthonormal basis $e_n \in L^2_D$. It follows that

$$\mathrm{tr}_{\mathcal{B}(L^2_G)^\circ_M}(z) = \sum_{\gamma \in \Gamma} \mathrm{tr}_M(\lambda_\gamma) \mathrm{tr}_{L^2_D}(a_\gamma) = \mathrm{tr}_{L^2_D}(a_e) = \sum_n a^{n,n}_e.$$

The restriction map

$$q : L^2_G \to L^2_D, \; f \mapsto f|_D$$

is a co-isometry satisfying

$$\mathrm{tr}_{L^2_G(q^* y q)} = \mathrm{tr}_{L^2_D}(w)$$

for all $y \in \mathcal{B}(L^2_D)$ which are positive or have finite rank. It follows that

$$\mathrm{tr}_{\mathcal{B}(L^2_G)^\circ_M}(z) = \mathrm{tr}_{L^2_D}(a_e) = \mathrm{tr}_{L^2_G}(q^* a_e q) = \mathrm{tr}(\iota \otimes q)^* x (\iota \otimes q).$$

Since traces are normal functionals, (**??**) holds for positive $0 \leqslant z \in \mathcal{B}(L^2_G)^\circ_M$ since x is a monotone limit of elements of the form (**??**) with $a_e \geqslant 0$. In particular,

$$\dim_M H = \mathrm{tr}_{\mathcal{B}(L^2_G)^\circ_M}(p) = \mathrm{tr}_{L^2_G}(q^* p q) = \sum_n (e_n | q^* p q e_n) = \sum_n (q e_n | p q e_n) = \sum_n (e_n | p e_n) = \sum_n \|p e_n\|^2,$$

since $qe_n = e_n$. The isometry $q^* : L^2_D \to L^2_G$ associates to $f \in L^2_D$ its trivial extension to G which is zero outside of D. Consider the unitary transformation

$$\ell^2_\Gamma \otimes L^2_D \overset{\asymp}{\to} L^2_G,\ \delta_\gamma \otimes e_n \mapsto \gamma^\lambda(q^* e_n).$$

Now let $\zeta \in H_\pi$ be a unit vector. Then $p\xi = \xi$ and

$$1 = \|g^\lambda \zeta\|^2 = \|g^\lambda p\zeta\|^2 = \sum_{\gamma\in\Gamma}\sum_{n\in\mathbf{N}} |(\gamma^\lambda q^* e_n | g^\lambda p\xi)|^2$$

$$= \sum_{\gamma\in\Gamma}\sum_{n\in\mathbf{N}} |(q^* e_n | (\gamma^{-1}g)^\lambda p\xi)|^2$$

Therefore

$$|G/\Gamma| = \int\limits_D dg = \int\limits_D dg \|g^\lambda \zeta\|^2 = \sum_{\gamma\in\Gamma}\sum_{n\in\mathbf{N}} \int\limits_D dg\ |(q^* e_n | (\gamma^{-1}g)^\lambda p\xi)|^2$$

$$= \sum_{n\in\mathbf{N}} \int\limits_G dg\ |(q^* e_n | g^\lambda p\xi)|^2 = \sum_{n\in\mathbf{N}} \int\limits_G dg\ |(q^* e_n | p\ g^\pi \xi)|^2$$

$$= \sum_{n\in\mathbf{N}} \int\limits_G dg\ |(pq^* e_n | g^\pi \xi)|^2 = \sum_{n\in\mathbf{N}} \int\limits_G dg\ (pq^* e_n | g^\pi \xi)(g^\pi \xi | pq^* e_n)$$

$$= \sum_{n\in\mathbf{N}} \frac{\|pq^* e_n\|^2\ \|g^\pi \zeta\|^2}{d_\pi} = \sum_{n\in\mathbf{N}} \frac{\|pq^* e_n\|^2}{d_\pi} = \frac{1}{d_\pi} \dim_M H_\pi.$$

□

Let $\acute{Z}$ be the right half-space. For $\nu > p-1$ consider the discrete series Hilbert space $\mathcal{H}^2_s(\acute{Z})$. Then

$$\int\limits_{\acute{Z}_\bullet} dg\ |(\xi | g^\nu \eta)|^2 = \frac{\Gamma(\nu - \frac{d}{r})}{\pi^d} \|\zeta\|^2\ \|\eta\|^2$$

Therefore

$$\pi_s(\Gamma)' \cong W^*(\Gamma)_t$$

where

$$t = \frac{\nu - \frac{d}{r}}{\pi^d} \mathrm{Vol}(G/\Gamma)$$

3.2 Hecke operators

Let $\alpha \in G_\mathbf{Q} = PGL^\mathbf{Q}_2/\{\pm 1\}$. Put $\Gamma_\alpha := \Gamma \cap (\alpha\Gamma\alpha^{-1})$. Since Γ is an 'almost normal subgroup' of $G_\mathbf{Q}$ there exists a finite set $\gamma_i \in \Gamma,\ 1 \leqslant i \leqslant k$, such that the double cosets

$$\Gamma\alpha\Gamma = \bigcup_{i=1}^k \Gamma\alpha\gamma_i.$$

Now consider intertwining operators

$$\mathcal{L}^\Gamma(\mathcal{H}_s,\mathcal{H}_t) := \{A \in \mathcal{L}(\mathcal{H}_s,\mathcal{H}_t) :\ \gamma^t A = A\gamma^s \ \forall\ \gamma \in \Gamma\}.$$

Define

$$\mathcal{L}^\Gamma(\mathcal{H}_s,\mathcal{H}_t) \xleftarrow{\Phi_\alpha} \mathcal{L}^\Gamma(\mathcal{H}_s,\mathcal{H}_t)$$

by

$$\Phi_\alpha A := \frac{1}{k}\sum_{i=1}^k (\alpha\gamma_i)^{-t} A (\alpha\gamma_i)^s$$

for all $A \in \mathcal{L}^\Gamma(\mathcal{H}_s,\mathcal{H}_t)$. For $s = t$ we obtain the **commutant von Neumann algebra**

$$\mathcal{L}^\Gamma(\mathcal{H}_s) := \{A \in \mathcal{L}(\mathcal{H}_s) :\ [\Gamma^s, A] = 0\} = (\Gamma^s)' = W^*(\Gamma^s)'$$

and

$$\Phi_\alpha A := \frac{1}{k}\sum_{i=1}^k (\alpha\gamma_i)^{-s} A (\alpha\gamma_i)^s.$$

On the other hand, consider the von Neumann algebra $L^\infty(\acute{\mathbf{C}})^\Gamma$ of all bounded Γ-invariant functions f on $\acute{\mathbf{C}}$. For $f \in L^\infty(\acute{\mathbf{C}})^\Gamma$ let $f^{\mathbb{T}_s}$ denote the associated **Toeplitz operator** acting on $\mathcal{H}_s$. Then

Proposition 69.

$$(\Gamma^s)' = W^*\{f^{\mathbb{T}_s} : f \in L^\infty(\acute{\mathbf{C}})^\Gamma\}$$

On the level of symbols, Φ_α is given by

$$(\Phi_\alpha f)(z) = \frac{1}{k}\sum_{i=1}^k f(\alpha\gamma_i)^{-1} z)$$

Then the diagram

$$\begin{tikzcd} L^\infty(\acute{\mathbf{C}})^\Gamma \arrow[r, "\mathbb{T}_s"] \arrow[d, "\Phi_\alpha"'] & \mathcal{L}^\Gamma(\mathcal{H}_s) \arrow[d, "\Phi_\alpha"'] \\ L^\infty(\acute{\mathbf{C}})^\Gamma \arrow[r, "\mathbb{T}_s"] & \mathcal{L}^\Gamma(\mathcal{H}_s) \end{tikzcd}$$

commutes.

For $m > 0$ let

$$\mathbf{Z}_m^{2\times 2} := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{Z}^{2\times 2} :\ ad - bc = m\} \subset \mathrm{GL}_2(\mathbf{Q}) =: G_{\mathbf{Q}}.$$

Then $\mathbf{Z}_1^{2\times 2} := \mathrm{SL}_2(\mathbf{Z}) =: \Gamma$. The group Γ acts by left multiplication $(\gamma, \alpha) := \gamma\alpha$ on $\mathbf{Z}_m^{2\times 2}$. A system of (right) representatives is given by the matrices

$$\gamma_{d,b} := \begin{pmatrix} m/d & b \\ 0 & d \end{pmatrix} \in \mathbf{Z}^{2\times 2}, \quad 0 < d | m,\ 0 \leqslant b < d.$$

The Hecke operator, acting on k-automorphic forms, becomes

$$(T_m f)(u) = m^{k-1} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \backslash \mathbf{Z}_m^{2\times 2}} (cu+d)^{-k} \ f\Big(\frac{au+b}{cu+d}\Big)$$

$$= m^{k-1} \sum_{d|m} d^{-k} \sum_{b=0}^{d-1} \ f\Big(\frac{au+b}{d}\Big) = m^{k-1} \sum_{d|m} d^{-k} \sum_{b=0}^{d-1} \ f\Big(\frac{mu+bd}{d^2}\Big).$$

For a prime number $m = p$ this simplifies to

$$(T_p f)(u) = p^{k-1} \ f(pu) + \frac{1}{p} \sum_{b=0}^{p-1} f\Big(\frac{u+p}{p}\Big)$$

An Γ-intertwiner $A : \mathcal{H}_s \to \mathcal{H}_t$ has an integral kernel

$$A^\natural(z,w) = (\mathcal{K}_z^t | A \mathcal{K}_w^s)$$

which is sesqui-holomorphic and has the invariance property

$$A^\natural(\gamma z, \gamma w) = J(\gamma, z)^s \ A^\natural(z,w) \ \overline{J(\gamma, w)}$$

Consider the intertwiners

$$\begin{tikzcd} \mathcal{H}_{s+p} \arrow[dr, "g^{\Top_{s+p}}"'] & & \mathcal{H}_s \arrow[ll, "\overline{g}^{\Top_{s+p}} \ f^{\Top_s}"'] \arrow[dl, "f^{\Top_s}"] \\ & \mathcal{H}_{s+q} & \end{tikzcd} ,$$

where f is q-automorphic and g is $q-p$-automorphic. For $p = q$, g is constant. Then we have

Lemma 70. *The intertwiner* $\overline{g}^{\Top} \ f^{\Top}$ *has the integral kernel given by the Berezin transform* $(f\overline{g})^{\vDash}$

Proof.

$$(\mathcal{K}_z | \overline{g}^{\Top} \ f^{\Top} \mathcal{K}_w) = (g^{\Top} \mathcal{K}_z | f^{\Top} \mathcal{K}_w) = (g\mathcal{K}_z | f\mathcal{K}_w) = \int\limits_{\acute{Z}} \mu_s(d\zeta) \overline{g(\zeta) \mathcal{K}_z(\zeta)} f(\zeta) \mathcal{K}_w(\zeta)$$

$$= \int\limits_{\acute{Z}} \frac{\mu_0(d\zeta)}{\mathcal{K}(\zeta,\zeta)} \ \overline{g(\zeta)} \ \mathcal{K}(z,\zeta) \ f(\zeta) \ \mathcal{K}(\zeta, w)$$

$$= \mathcal{K}(z,w) \int\limits_{\acute{Z}} \frac{\mu_0(d\zeta)}{} \frac{\mathcal{K}(z,\zeta)\mathcal{K}(\zeta,w)}{\mathcal{K}(z,w)\mathcal{K}(\zeta,\zeta)} \ f(\zeta) \ \overline{g(\zeta)}$$

More directly,

$$(\mathcal{K}_z | \overline{g}^{\Top} \ f^{\Top} \mathcal{K}_z) = (\mathcal{K}_z | (\overline{g} f)^{\Top} \mathcal{K}_z) = \mathcal{K}_{z,z} ((\overline{g} f)^{\Top})_z^{\Perp} = \mathcal{K}_{z,z} (\overline{g} f)_z^{\vDash}$$

$\square$

3.3 Berezin quantization

For $f \in \mathcal{C}^\infty(\acute{Z})$ let $f^{\top\!\!\!\top} \in \mathcal{B}(\mathcal{H}_s^2(\acute{Z}))$ be the Toeplitz operator. Conversely, for $A \in \mathcal{B}(\mathcal{H}_s^2(\acute{Z}))$ let

$$A^{\perp\!\!\!\perp}(z, w) = \frac{(\mathcal{K}_z|A\ \mathcal{K}_w)}{\mathcal{K}_{z,w}}$$

be the Berezin symbol. Then

$$\mathrm{tr}(A\ f^{\top\!\!\!\top}) = \int\limits_{\acute{Z}} \acute{\mu}(dz) A^{\perp\!\!\!\perp}(z)\ f(z)$$

where $\acute{\mu}$ is a Haar measure on $\acute{Z}$. The adjoint operator A^* has Berezin symbol

$$A^{*\perp\!\!\!\perp}_{z,w} = \overline{A^{\perp\!\!\!\perp}_{w,z}}$$

Consider the unitary projective representation

$$(g^{-s}\phi)(z) = (\det g'_z)^{s/p}\ \phi(gz)$$

Then

$$(g^s A g^{-s})^{\perp\!\!\!\perp} = A^{\perp\!\!\!\perp} \circ g^{-1}$$

Lemma 71. *The restricted symbol* $A^{\perp\!\!\!\perp}_{|}$ *has sup-norm* $\|A^{\perp\!\!\!\perp}\|_\infty \leqslant \|A\|$, *i.e.,*

$$\sup_{z \in \acute{Z}} |A^{\perp\!\!\!\perp}_{z,z}| \leqslant \|A\|$$

Proof. By Cauchy-Schwarz we have $|(\mathcal{K}_z|A\mathcal{K}_z)| \leqslant \|\mathcal{K}_z\|\ \|A\mathcal{K}_z\| \leqslant \|A\|\ \|\mathcal{K}_z\|^2$ and therefore

$$|A^{\perp\!\!\!\perp}_{z,z}| = \frac{(\mathcal{K}_z|A\mathcal{K}_z)}{\|\mathcal{K}_z\|^2} \leqslant \|A\|.$$

□

Proposition 72. *With respect to the probability measure* μ_s, *a bounded operator* $A \in \mathcal{B}(\mathcal{H}_s^2(\acute{Z}))$ *has the integral kernel* $A^{\perp\!\!\!\perp}_{z,w}\mathcal{K}_{z,w}$. *Thus the Berezin symbol* $A^{\perp\!\!\!\perp}$ *determines the operator via*

$$(A\phi)(z) = \int\limits_{\acute{Z}} \mu_s(dw)\ \phi(w)\ A^{\perp\!\!\!\perp}_{z,w}\ \mathcal{K}_{z,w}$$

Proof. The reproducing property impliew

$$(A\phi)(z) = (\mathcal{K}_z|A\phi) = (\mathcal{K}_z|A \int\limits_{\acute{Z}} \mu_s(dw)\ \phi(w)\ \mathcal{K}_w)$$

$$= \int\limits_{\acute{Z}} \mu_s(dw)\ \phi(w)\ (\mathcal{K}_z|A\mathcal{K}_w) = \int\limits_{\acute{Z}} \mu_s(dw)\ \phi(w)\ A^{\perp\!\!\!\perp}_{z,w}\ \mathcal{K}_{z,w}$$

□

Proposition 73. *For each $w \in \acute{Z}$ the holomorphic function $\mathcal{K}_w\ A^{\perp\!\!\!\perp}_w$ belongs to $\mathcal{H}^2_s(\acute{Z})$ and has norm*

$$\|\mathcal{K}_w\ A^{\perp\!\!\!\perp}_w\|_s \leqslant \|A\|\ \mathcal{K}^{1/2}_{w,w}$$

Proof. For the first relation we have

$$\mathcal{K}_w\ A^{\perp\!\!\!\perp}_w(z) = \mathcal{K}_{z,w}\ A^{\perp\!\!\!\perp}_{z,w} = (\mathcal{K}_z|A\mathcal{K}_w)$$

and hence, by Cauchy-Schwarz,

$$|\mathcal{K}_w\ A^{\perp\!\!\!\perp}_w(z)| = |(\mathcal{K}_z|A\mathcal{K}_w)| \leqslant \|\mathcal{K}_z\|\ \|A\mathcal{K}_w\| = \|A\|\ \|\mathcal{K}_z\|\ \|A\mathcal{K}_w\| = \|A\|\ \mathcal{K}^{1/2}_{z,z}\ \mathcal{K}^{1/2}_{w,w}.$$

□

Proposition 74. *Let $A \geqslant 0$ be a positive operator. Then we have matrix inequalities*

$$(0)_{i,j} \leqslant \Big(A^{\perp\!\!\!\perp}_{z_i,z_j}\ \mathcal{K}_{z_i,z_j}\Big)_{i,j} \leqslant \Big(\|A\|\mathcal{K}_{z_i,z_j}\Big)_{i,j}$$

for all $z_1, \ldots, z_n \in \acute{Z}$.

Proof. Let $\lambda_i \in \mathbf{C}$. Putting $f = \sum\limits_i \lambda_i \mathcal{K}_{z_i} \in \mathcal{H}^2_s(\acute{Z})$ we have

$$\sum_{i,j} \overline{\lambda}_i\ \lambda_j\ A_{z_i,z_j}\mathcal{K}_{z_i,z_j} = \sum_{i,j} \overline{\lambda}_i\ \lambda_j\ (\mathcal{K}_{z_i}|A\mathcal{K}_{z_j}) = (\sum_i \lambda_i\mathcal{K}_{z_i}|A\sum_j \lambda_j\mathcal{K}_{z_j})$$

$$= (f|Af) \begin{cases} \geqslant 0 \\ \leqslant \|f\|\|Af\| \leqslant \|A\|(f|f) = \|A\| \sum\limits_{i,j} \overline{\lambda}_i\ \lambda_j\ \mathcal{K}_{z_i,z_j} \end{cases}$$

□

Proposition 75. *If A is positive, then*

$$|A^{\perp\!\!\!\perp}_{z,w}|\frac{|\mathcal{K}_{z,w}|}{\|\mathcal{K}_z\|\|\mathcal{K}_w\|} \leqslant \sup_{\zeta \in \acute{Z}} |A^{\perp\!\!\!\perp}_{\zeta,\zeta}| \leqslant \|A\|$$

If A is bounded, then

$$|A^{\perp\!\!\!\perp}_{z,w}|\frac{|\mathcal{K}_{z,w}|}{\|\mathcal{K}_z\|\|\mathcal{K}_w\|} \leqslant 4\|A\|$$

Proof. For $z, w \in \acute{Z}$ the positive matrix

$$\begin{pmatrix} A^{\perp\!\!\!\perp}_{z,z}\mathcal{K}_{z,z} & A^{\perp\!\!\!\perp}_{z,w}\mathcal{K}_{z,w} \\ \overline{A}^{\perp\!\!\!\perp}_{z,w}\overline{\mathcal{K}}_{z,w} & A^{\perp\!\!\!\perp}_{w,w}\mathcal{K}_{w,w} \end{pmatrix}$$

has determinant $\geqslant 0$ showing that

$$A^{\perp\!\!\!\perp}_{z,z}\mathcal{K}_{z,z}\ A^{\perp\!\!\!\perp}_{w,w}\mathcal{K}_{w,w} \geqslant |A^{\perp\!\!\!\perp}_{z,w}\overline{\mathcal{K}}_{z,w}|^2$$

Taking square roots, it follows that

$$|A^{\perp\!\!\!\perp}_{z,w}|\frac{|\mathcal{K}_{z,w}|}{\mathcal{K}^{1/2}_{z,z}\mathcal{K}^{1/2}_{w,w}} \leqslant \sqrt{A^{\perp\!\!\!\perp}_{z,z}}\sqrt{A^{\perp\!\!\!\perp}_{w,w}} \leqslant \sup_{\zeta \in \acute{Z}} |A^{\perp\!\!\!\perp}_{\zeta,\zeta}| \leqslant \|A\|$$

Writing $A \in \mathcal{B}(\mathcal{H}^2_s(\acute{Z}))$ into real/imaginary and positive/negative parts, the second assertion follows. □

3.3.1 Fundamental domains

A closed subset $F \subset \acute{Z}$ is called a **fundamental domain** for a discrete subgroup $\Gamma \subset \acute{Z}_\bullet$ if $\partial F = F \sim \mathring{F}$ has measure zero, and

$$\acute{Z} = \bigcup_{\gamma \in \Gamma} \gamma(F)$$

$$\gamma(\mathring{F}) \cap \mathring{F} \neq \emptyset \Rightarrow \gamma = 1.$$

Then the disjoint union $\bigcup\limits_{\gamma \in \Gamma} \gamma(\mathring{F})$ is an open dense subset of $\acute{Z}$ whose complement is a zero-set. For each function $\phi : \mathring{F} \to \mathbf{C}$ denote by $\tilde{\phi} : \acute{Z} \to \mathbf{C}$ the **zero-extension** of ϕ. Conversely, for a function $\Phi : \acute{Z} \to \mathbf{C}$ let $\underset{\sim}{\Phi} : F \to \mathbf{C}$ denote the restriction of Φ to F. Then $\underset{\sim}{\Phi} \in \mathcal{C}_c(\mathring{F})$ if $\Phi \in \mathcal{C}_c(\bigcup\limits_{\gamma \in \Gamma} \gamma(\mathring{F}))$. We have

$$(\underset{\sim}{\Phi}|\psi) = (\Phi|\tilde{\psi})$$

for all $\Phi \in L^2_s(\acute{Z}),\ \psi \in L^2_s(F)$.

Let $e_\gamma(\sigma) := \delta^\gamma_\sigma$ be the standard basis of $\ell^2(\Gamma)$. Consider the left-regular representation

$$(\lambda_\gamma f)(\sigma) := f(\gamma^{-1}\sigma)$$

of Γ on $\ell^2(\Gamma)$. Then

$$\lambda_\gamma e_\sigma = e_{\gamma\sigma}$$

since

$$(\lambda_\gamma e_\sigma)(\tau) = e_\sigma(\gamma^{-1}\tau) = \delta^\sigma_{\gamma^{-1}\tau} = \delta^{\gamma\sigma}_\tau = e_{\gamma\sigma}(\tau)$$

Proposition 76. *Define a map* $V : \ell^2(\Gamma) \otimes L^2_s(F) \to L^2_s(\acute{Z})$ *by*

$$V(e_\sigma \otimes \phi) := \sigma^s \tilde{\phi}$$

Then V *is unitary, with adjoint*

$$U(\Phi) = \sum_{\sigma \in \Gamma} e_\sigma \otimes \underset{\sim}{\sigma^{-s}\Phi}$$

Proof. Since $(\tilde{\phi}|\gamma^s\tilde{\psi}) = 0$ if $1 \neq \gamma \in \Gamma$ we obtain

$$(V(e_\sigma \otimes \phi)|V(e_\tau \otimes \psi)) = (\sigma^s\tilde{\phi}|\tau^s\tilde{\psi}) = (\tilde{\phi}|\sigma^{-s}\tau^s\tilde{\psi}) = (\tilde{\phi}|(\sigma^{-1}\tau)^s\tilde{\psi})$$

$$= \delta^\tau_\sigma(\tilde{\phi}|\tilde{\psi}) = \delta^\tau_\sigma(\phi|\psi) = (e_\sigma|e_\tau)(\phi|\psi) = (e_\sigma \otimes \phi|e_\tau \otimes \psi)$$

Thus V is isometric. Moreover,

$$(U\Phi|e_\tau \otimes \psi) = \sum_{\sigma \in \Gamma}(e_\sigma \otimes \underset{\sim}{\sigma^{-s}\Phi}|e_\tau \otimes \psi) = \sum_{\sigma \in \Gamma}(e_\sigma|e_\tau)(\underset{\sim}{\sigma^{-s}\Phi}|\psi)$$

$$= \sum_{\sigma \in \Gamma} \delta^\sigma_\tau(\underset{\sim}{\sigma^{-s}\Phi}|\psi) = (\underset{\sim}{\tau^{-s}\Phi}|\psi) = (\tau^{-s}\Phi|\tilde{\psi}) = (\Phi|\tau^s\tilde{\psi}) = (\Phi|V(e_\tau \otimes \psi))$$

It follows that $U = V^*$ and therefore V is a unitary operator. □

Proposition 77.

$$\gamma^s \circ V = V \circ \lambda_\gamma$$

Proof.

$$\gamma^s V(e_\sigma \otimes \phi) = \gamma^s(\sigma^s \tilde{\phi}) = (\gamma\sigma)^s \tilde{\phi} = V(e_{\gamma\sigma} \otimes \phi) = V\lambda_\gamma(e_\sigma \otimes \phi)$$

$\square$

For any $f \in L^\infty(\acute{Z})$ let $f^{⫪} \in \mathcal{B}(\mathcal{H}^2_s(\acute{Z}))$ be the associated Toeplitz operator. Then

$$(\mathcal{K}_z|f^{⫪}\mathcal{K}_w) = (\mathcal{K}_z|f\mathcal{K}_w) = \int\limits_{\acute{Z}} \mu_s(d\zeta)\overline{\mathcal{K}_z(\zeta)}\ f(\zeta)\ \mathcal{K}_w(\zeta)$$

$$= \int\limits_{\acute{Z}} \mu_s(d\zeta)\mathcal{K}_{z,\zeta}\ f(\zeta)\ \mathcal{K}_{\zeta,w} = \int\limits_{\acute{Z}} \frac{\mu_0(d\zeta)}{\mathcal{K}_{\zeta,\zeta}}\mathcal{K}_{z,\zeta}\ f(\zeta)\ \mathcal{K}_{\zeta,w}$$

It follows that the **Berezin transform** $f^{\vDash} := (f^{⫪})^{⫫}$ is given by

$$f^{\vDash}_{z,w} = \frac{(\mathcal{K}_z|f^{⫪}\mathcal{K}_w)}{\mathcal{K}_{z,w}} = \frac{1}{\mathcal{K}_{z,w}} \int\limits_{\acute{Z}} \frac{\mu_0(d\zeta)}{\mathcal{K}_{\zeta,\zeta}}\mathcal{K}_{z,\zeta}\ f(\zeta)\ \mathcal{K}_{\zeta,w} = \int\limits_{\acute{Z}} \frac{\mu_0(d\zeta)}{}\ \vDash^\zeta_{z,w}\ \ f(\zeta),$$

where

$$\vDash^\zeta_{z,w} := \frac{\mathcal{K}_{z,\zeta}\ \mathcal{K}_{\zeta,w}}{\mathcal{K}_{z,w}\ \mathcal{K}_{\zeta,\zeta}}$$

In particular,

$$\vDash^\zeta_z := \vDash^\zeta_{z,z} = \frac{|\mathcal{K}_{z,\zeta}|^2}{\mathcal{K}_{z,z}\ \mathcal{K}_{\zeta,\zeta}} = \Big(\frac{|\mathcal{K}_{z,\zeta}|}{\|\mathcal{K}_z\|\|\mathcal{K}_z\|}\Big)^2$$

By Cauchy-Schwarz we have

$$|\vDash^\zeta_{z,w}| \leqslant 1 \quad (z,w \in \acute{Z})$$

3.3.2 Berezin star product

The (weak/passive) Berezin star product of two symbol functions $A^{⫫}, B^{⫫}$ is defined by

$$\mathcal{K}_{z,w}(A^{⫫}\underline{⫫}B^{⫫})_{z,w} = \int\limits_{\acute{Z}} \acute{\mu}_s(d\zeta)\mathcal{K}_{z,\zeta}A^{⫫}_{z,\zeta}\mathcal{K}_{\zeta,w}B^{⫫}_{\zeta,w}$$

Equivalently,

$$(A^{⫫}\underline{⫫}B^{⫫})_{z,w} = \int\limits_{\acute{Z}} \acute{\mu}_0(d\zeta)A^{⫫}_{z,\zeta}\frac{\mathcal{K}_{z,\zeta}\mathcal{K}_{\zeta,w}}{\mathcal{K}_{\zeta,\zeta}\mathcal{K}_{z,w}}B^{⫫}_{\zeta,w}$$

Proposition 78.

$$(AB)^{⫫} = A^{⫫}\underline{⫫}B^{⫫}$$

Proof. Applying the reproducing kernel identity to $B\mathcal{K}_w \in \mathcal{H}_s^2(\acute{Z})$ we obtain

$$\mathcal{K}_{z,w}(AB)^{\perp\!\!\!\perp}_{z,w} = (\mathcal{K}_z|AB\mathcal{K}_w) = (\mathcal{K}_z|A\int\limits_{\acute{Z}} \acute{\mu}_s(d\zeta)(B\mathcal{K}_w)(\zeta)\mathcal{K}_\zeta)$$

$$= \int\limits_{\acute{Z}} \acute{\mu}_s(d\zeta)(\mathcal{K}_z|A\mathcal{K}_\zeta)(\mathcal{K}_\zeta|B\mathcal{K}_w) = \int\limits_{\acute{Z}} \acute{\mu}_s(d\zeta)\mathcal{K}_{z,\zeta}A^{\perp\!\!\!\perp}_{z,\zeta}\ \mathcal{K}_{\zeta,w}B^{\perp\!\!\!\perp}_{\zeta,w}$$

□

Let f, g be automorphic functions of weight pk. Then $M_f : \mathcal{H}_s^2(\acute{Z}) \to \mathcal{H}_{s+pk}^2(\acute{Z})$ is a bounded operator, and $M_g M_f^* \in \mathcal{B}(\mathcal{H}_{s+pk}(\acute{Z}))$.

Lemma 79.

$$(M_g M_f^*)^{\perp\!\!\!\perp}_{z,w} = \frac{g(z)\overline{f(w)}}{\mathcal{K}^{pk}_{z,w}}$$

Proof. We have

$$M_f^* \mathcal{K}_z^{s+pk} = \overline{f(z)}\mathcal{K}_z^s$$

and therefore

$$(M_g M_f^*)^{\perp\!\!\!\perp}_{z,w} = \frac{(\mathcal{K}_z^{s+pk}|M_g M_f^* \mathcal{K}_w^{s+pk})_{s+pk}}{\mathcal{K}^{s+pk}_{z,w}} = \frac{(M_g^* \mathcal{K}_z^{s+pk}|M_f^* \mathcal{K}_w^{s+pk})_s}{\mathcal{K}^{s+pk}_{z,w}}$$

$$= \frac{(\overline{g(z)}\mathcal{K}_z^s|\overline{f(w)}\mathcal{K}_w^s)_s}{\mathcal{K}^{s+pk}_{z,w}} = g(z)\overline{f(w)}\frac{\mathcal{K}^s_{z,w}}{\mathcal{K}^{s+pk}_{z,w}} = \frac{g(z)\overline{f(w)}}{\mathcal{K}^{pk}_{z,w}}$$

□

3.3.3 trace

Let $A \in (\Gamma^s)'$ commute with Γ^s. Then $\gamma^s A \gamma^{-s} = A$ and therefore $A^{\perp\!\!\!\perp}_{\gamma z} = A^{\perp\!\!\!\perp}_z$. Polarization yields

$$A^{\perp\!\!\!\perp}_{\gamma z,\gamma w} = A^{\perp\!\!\!\perp}_{z,w} \quad (\gamma \in \Gamma).$$

Proposition 80.

$$\tau(A) = \frac{1}{|F|}\int\limits_F \mu_0(dz)A^{\perp\!\!\!\perp}_{z,z}$$

defines a positive faithful trace on $\Gamma^{s'} \subset \mathcal{B}(\mathcal{H}_s^2(\acute{Z}))$

Proof. We have

$$(\mathcal{K}_z|AB\mathcal{K}_w) = (\mathcal{K}_z|A\int\limits_{\acute{Z}} \mu_s(d\zeta)(B\mathcal{K}_w)(\zeta)\mathcal{K}_\zeta)$$

$$= \int\limits_{\acute{Z}} \mu_s(d\zeta)(B\mathcal{K}_w)(\zeta)(\mathcal{K}_z|A\mathcal{K}_\zeta) = \int\limits_{\acute{Z}} \mu_s(d\zeta)(\mathcal{K}_z|A\mathcal{K}_\zeta)(\mathcal{K}_\zeta|B\mathcal{K}_w)$$

and therefore

$$(AB)^{\perp\!\!\!\perp}_{z,w} = \frac{(\mathcal{K}_z|AB\mathcal{K}_w)}{\mathcal{K}_{z,w}} = \frac{1}{\mathcal{K}_{z,w}} \int\limits_{\acute{Z}} \frac{\mu_0(d\zeta)}{\mathcal{K}_{\zeta,\zeta}} A^{\perp\!\!\!\perp}_{z,\zeta}\mathcal{K}_{z,\zeta}B^{\perp\!\!\!\perp}_{\zeta,w}\mathcal{K}_{\zeta,w} = \int\limits_{\acute{Z}} \mu_0(d\zeta)A^{\perp\!\!\!\perp}_{z,\zeta}B^{\perp\!\!\!\perp}_{\zeta,w}\frac{\mathcal{K}_{z,\zeta}\mathcal{K}_{\zeta,w}}{\mathcal{K}_{z,w}\mathcal{K}_{\zeta,\zeta}}.$$

Put

$$f(z,w) := |A^{\perp\!\!\!\perp}_{z,w}|^2 \frac{|\mathcal{K}_{z,w}|^2}{\mathcal{K}_{z,z}\mathcal{K}_{w,w}}.$$

Then

$$(AA^*)^{\perp\!\!\!\perp}_{z,z} = \int\limits_{\acute{Z}} \mu_0(d\zeta)A^{\perp\!\!\!\perp}_{z,\zeta}\overline{A}^{\perp\!\!\!\perp}_{z,\zeta}\frac{\mathcal{K}_{z,\zeta}\mathcal{K}_{\zeta,z}}{\mathcal{K}_{z,z}\mathcal{K}_{\zeta,\zeta}} = \int\limits_{\acute{Z}} \mu_0(d\zeta)|A^{\perp\!\!\!\perp}_{z,\zeta}|^2\frac{|\mathcal{K}_{z,\zeta}|^2}{\mathcal{K}_{z,z}\mathcal{K}_{\zeta,\zeta}} = \int\limits_{\acute{Z}} \mu_0(d\zeta)f(z,\zeta)$$

and, similarly,

$$(A^*A)^{\perp\!\!\!\perp}_{z,z} = \int\limits_{\acute{Z}} \mu_0(d\zeta)\overline{A}^{\perp\!\!\!\perp}_{\zeta,z}A^{\perp\!\!\!\perp}_{\zeta,z}\frac{\mathcal{K}_{z,\zeta}\mathcal{K}_{\zeta,z}}{\mathcal{K}_{z,z}\mathcal{K}_{\zeta,\zeta}} = \int\limits_{\acute{Z}} \mu_0(d\zeta)|A^{\perp\!\!\!\perp}_{\zeta,z}|^2\frac{|\mathcal{K}_{\zeta,z}|^2}{\mathcal{K}_{z,z}\mathcal{K}_{\zeta,\zeta}} = \int\limits_{\acute{Z}} \mu_0(d\zeta)f(\zeta,z).$$

It follows that

$$\tau(AA^*) = \frac{1}{|F|}\int\limits_F \mu_0(dz)(AA^*)^{\perp\!\!\!\perp}_{z,z} = \frac{1}{|F|}\int\limits_F \mu_0(dz)\int\limits_{\acute{Z}} \mu_0(d\zeta)f(z,\zeta) = \frac{1}{|F|}\int\limits_F \mu_0(dz)\sum_{\gamma\in\Gamma}\int\limits_F \mu_0(d\zeta)f(z,\gamma\zeta)$$

and

$$\tau(A^*A) = \frac{1}{|F|}\int\limits_F \mu_0(dz)(A^*A)^{\perp\!\!\!\perp}_{z,z} = \frac{1}{|F|}\int\limits_F \mu_0(dz)\int\limits_{\acute{Z}} \mu_0(d\zeta)f(\zeta,z) = \frac{1}{|F|}\int\limits_F \mu_0(dz)\sum_{\gamma\in\Gamma}\int\limits_F \mu_0(d\zeta)f(\gamma\zeta,z).$$

Since $f(\gamma\zeta,z) = f(\zeta,\gamma^{-1}z)$, the assertion follows. □

Proposition 81.

For automorphic forms f,g of weight pk, the inner product

$$\tau_s(M_f^*M_g) = (f|g)_{2k}$$

agrees with the Petersson inner product.

Proof.

$$\tau_s(M_f^*M_g) = \tau_{s+pk}(M_gM_f^*) = \frac{1}{|F|}\int\limits_F \acute{\mu}_0(dz)(M_gM_f^*)^{\perp\!\!\!\perp}_{z,z}$$

$$= \frac{1}{|F|}\int\limits_F \acute{\mu}_0(dz)\frac{\overline{f(z)}g(z)}{\mathcal{K}^{pk}_{z,z}} = \frac{1}{|F|}\int\limits_F \acute{\mu}_{pk}(dz)\ \overline{f(z)}g(z) = (f|g)_{2k}$$

□

Proposition 82. *Put*

$$\vDash^{\Gamma}(z,\zeta) := \sum_{\gamma\in\Gamma} \frac{|\mathcal{K}_{z,\gamma\zeta}|^2}{\mathcal{K}_{z,z}\mathcal{K}_{\gamma\zeta,\gamma\zeta}}$$

Let $f \in L^{\infty}(\acute{Z})^{\Gamma}$. *Then*

$$f^{\vDash}_{z,z} = \int\limits_{F} \mu_0(d\zeta)\ \vDash^{\Gamma}(z,\zeta)\ \underset{\sim}{f}(\zeta)$$

Proof.

$$f^{\vDash}_{z,z} = \int\limits_{\acute{Z}} \mu_0(d\zeta)\ \frac{|\mathcal{K}_{z,\zeta}|^2}{\mathcal{K}_{z,z}\mathcal{K}_{\zeta,\zeta}}\ f(\zeta) = \sum_{\gamma\in\Gamma} \int\limits_{F} \mu_0(d\zeta)\ \frac{|\mathcal{K}_{z,\gamma\zeta}|^2}{\mathcal{K}_{z,z}\mathcal{K}_{\gamma\zeta,\gamma\zeta}}\ f(\gamma\zeta)$$

$$= \int\limits_{F} \mu_0(d\zeta)\ \Big(\sum_{\gamma\in\Gamma} \frac{|\mathcal{K}_{z,\gamma\zeta}|^2}{\mathcal{K}_{z,z}\mathcal{K}_{\gamma\zeta,\gamma\zeta}}\Big)\ f(\zeta) = \int\limits_{F} \mu_0(d\zeta)\ \vDash^{\Gamma}(z,\zeta)\ \underset{\sim}{f}(\zeta)$$

□

Chapter 4

Scattering theory

4.1 Abstract Theory

4.1.1 Abstract scattering

Let $U_t \in U(\mathcal{H})$ be a unitary representation of $\mathbf{R}$ on a Hilbert space $\mathcal{H}$. Suppose there is a subspace $\mathcal{H}_\sigma \subset \mathcal{H}$ such that

$$U_t \mathcal{H}_\sigma \subset \mathcal{H}_\sigma \quad (\sigma t > 0)$$

$$\bigcap_{\sigma t > 0} U_t \mathcal{H}_\sigma = \{0\}$$

$$\bigcup_{t \in \mathbf{R}} U_t \mathcal{H}_\sigma \underset{\text{dense}}{\subset} \mathcal{H}$$

Lemma 83. *For $t > 0$ we have*

$$P_+^{\perp\!\!\!\perp} U_t P_-^{\perp\!\!\!\perp} \mathcal{H} \subset \mathcal{H}_\pm^{\perp\!\!\!\perp}.$$

Proof. For $t > 0$ and $f \in \mathcal{H}_-^{\perp\!\!\!\perp}$ we have $(\mathcal{H}_- | U_t f) = (U_{-t} \mathcal{H}_- | f) = 0$ since $U_{-t} \mathcal{H}_- \subset \mathcal{H}_-$. It follows that

$$U_t \mathcal{H}_-^{\perp\!\!\!\perp} \subset \mathcal{H}_-^{\perp\!\!\!\perp}.$$

Also, for $f \in \mathcal{H}_-^{\perp\!\!\!\perp}$ we have $(\mathcal{H}_- | P_+^{\perp\!\!\!\perp} f) = (\mathcal{H}_- | f - P_+ f) = (\mathcal{H}_- | f) - (\mathcal{H}_- | P_+ f) = 0$ since $\mathcal{H}_+$ and $\mathcal{H}_-$ are orthogonal. Thus

$$P_+^{\perp\!\!\!\perp} \mathcal{H}_-^{\perp\!\!\!\perp} \subset \mathcal{H}_-^{\perp\!\!\!\perp}.$$

In summary, we obtain

$$P_+^{\perp\!\!\!\perp} U_t P_-^{\perp\!\!\!\perp} H \subset P_+^{\perp\!\!\!\perp} U_t \mathcal{H}_-^{\perp\!\!\!\perp} \subset P_+^{\perp\!\!\!\perp} \mathcal{H}_-^{\perp\!\!\!\perp} \subset \mathcal{H}_-^{\perp\!\!\!\perp}.$$

Since, trivially, $P_+^{\perp\!\!\!\perp} U_t P_-^{\perp\!\!\!\perp} \mathcal{H} \subset \mathcal{H}_+^{\perp\!\!\!\perp}$, we obtain

$$P_+^{\perp\!\!\!\perp} U_t P_-^{\perp\!\!\!\perp} \mathcal{H} \subset \mathcal{H}_\pm^{\perp\!\!\!\perp}.$$

□

Lemma 84. *For $t > 0$ we have a semigroup*

$$P_+^{\perp\!\!\!\perp} U_t P_-^{\perp\!\!\!\perp}$$

on $\mathcal{H}_\pm^{\perp\!\!\!\perp}$.

Proof. Let $s, t > 0$. Then $P_+^{\perp\!\!\!\perp} U_t P_t = 0$ since $U_t \mathcal{H}_t \subset \mathcal{H}_t$. Moreover, $P_+^{\perp\!\!\!\perp} U_s P_-^{\perp\!\!\!\perp} \subset \mathcal{H}_-^{\perp\!\!\!\perp}$ by Lemma **??**. It follows that $P_-^{\perp\!\!\!\perp}\ P_+^{\perp\!\!\!\perp} U_s P_-^{\perp\!\!\!\perp} = P_+^{\perp\!\!\!\perp} U_s P_-^{\perp\!\!\!\perp}$ and hence

$$P_+^{\perp\!\!\!\perp} U_t P_-^{\perp\!\!\!\perp}\ P_+^{\perp\!\!\!\perp} U_s P_-^{\perp\!\!\!\perp} = P_+^{\perp\!\!\!\perp} U_t P_+^{\perp\!\!\!\perp} U_s P_-^{\perp\!\!\!\perp} = P_+^{\perp\!\!\!\perp} U_t (I - P_+) U_s P_-^{\perp\!\!\!\perp} = P_+^{\perp\!\!\!\perp} U_t U_s P_-^{\perp\!\!\!\perp} = P_+^{\perp\!\!\!\perp} U_{t+s} P_-^{\perp\!\!\!\perp}$$

□

4.1.2 Scattering operator

There exists a unitary W_σ such that

$$\begin{tikzcd}
\mathcal{H} \arrow[d, "W_\sigma"'] & \mathcal{H} \arrow[l, "U_t"'] \arrow[d, "W_\sigma"'] & \mathcal{H}_\sigma \arrow[l] \arrow[d, "W_\sigma"] \\
L^2(\mathbf{R}, E) & L^2(\mathbf{R}, E) \arrow[l, "\lambda_t \otimes \mathrm{id}"'] & L^2(\mathbf{R}_\sigma, E) \arrow[l] \\
L^2(i\mathbf{R}, E) \arrow[u, "\mathcal{F}"] & L^2(i\mathbf{R}, E) \arrow[l, "e^{it\lambda} \otimes \mathrm{id}"'] \arrow[u, "\mathcal{F}"] & \mathcal{H}^2_\sigma(i\mathbf{R}, E) \arrow[l] \arrow[u, "\mathcal{F}"']
\end{tikzcd}$$

Define the unitary scattering operator

$$\begin{tikzcd}
 & L^2(\mathbf{R}, E) \arrow[dd, "S"'] & L^2(\mathbf{R}, E) \arrow[l, "\lambda_t \otimes \mathrm{id}"'] \arrow[dd, "S"] \\
\mathcal{H} \arrow[ur, "W_+"] \arrow[dr, "W_-"'] & & \\
 & L^2(\mathbf{R}, E) & L^2(\mathbf{R}, E) \arrow[l, "\lambda_t \otimes \mathrm{id}"']
\end{tikzcd}$$

Since S commutes with translations, we have

$$Sf(t) = (S_\sharp^\sharp\ f)(t) = \int_{\mathbf{R}} S_\sharp(ds)\ f(t-s)$$

for some distribution $S_\sharp \in \mathcal{D}'(\mathbf{R})$. Now suppose $(\mathcal{H}_+|\mathcal{H}_-) = 0$. Then

$$SL^2(\mathbf{R}_-, E) \subset L^2(\mathbf{R}_-, E), \quad S^* L^2(\mathbf{R}_+, E) \subset L^2(\mathbf{R}_+, E).$$

Therefore $S_\sharp \in \mathcal{H}'(\mathbf{R}_-)$, i.e., $S_\sharp(s) = 0$ for $s > 0$. Let

$$\hat{S}_\sharp(\sigma) := \int_{\mathbf{R}} S_\sharp(s) e^{i\sigma s} \in H_-(i\mathbf{R}, \mathcal{L}(Z))$$

be its Fourier transform, called the **scattering matrix**. Then

$$\hat{S}_\sharp(\sigma) \in U(Z) \quad \mathrm{Im}(\sigma) = 0$$

$$\|\hat{S}_\sharp(\sigma)\| \leqslant 1 \quad \mathrm{Im}(\sigma) < 0$$

and

$$\begin{tikzcd}
H^2_-(i\mathbf{R}, E) & H^2_+(i\mathbf{R}, E) \arrow[l, "\hat{S}^\circ_\sharp"'] \\
L^2(\mathbf{R}_-, E) \arrow[u, "\mathcal{F}"] & L^2(\mathbf{R}_-, E) \arrow[l, "S^\sharp_\sharp"] \arrow[u, "\mathcal{F}"']
\end{tikzcd}$$

where $\hat{S}^\circ_\sharp$ denotes the multiplication operator

$$(\hat{S}^\circ_\sharp \phi)(\sigma) = (\hat{S}_\sharp)(\sigma)\ \phi(\sigma)$$

for $\phi \in H^2_+(i\mathbf{R}, E)$. Explicitly, $\hat{S}_\sharp(z) \in \mathcal{L}(Z)$ is determined by

$$S(e^{-izs}n) = \int\limits_{\mathbf{R}} S_\sharp(dt)\ e^{-iz(s-t)} = e^{-izs} \int\limits_{\mathbf{R}} S_\sharp(dt)\ e^{izt}n = e^{-izs}\ \hat{S}_\sharp(z)n$$

for all $n \in Z$.

4.1.3 Wave equation

Let H be a Hilbert space, $L \in \mathcal{L}(H)$ a self-adjoint operator. Consider the abstract wave equation

$$\ddot{u} = u_{tt} = Lu$$

The space

$$\mathcal{H} = \begin{pmatrix} \mathrm{Dom}(|L|^{1/2}) \\ H \end{pmatrix}$$

is called the **data space**. Write

$$\phi = \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} \phi^0 \\ \phi^1 \end{pmatrix}$$

Then (**??**) is equivalent to the first order equation

$$\dot{\phi}(t) = \phi_t = \begin{pmatrix} 0 & 1 \\ L & 0 \end{pmatrix}\ \phi(t)$$

For $\phi \in \mathcal{H}$ we define the **energy form**

$$Z(\phi) = (\phi^1|\phi^1) - (\phi^0|L\phi^0)$$

Lemma 85. *The energy form is independent of t*

Proof. Let u_t solve the wave equation. Then

$$\frac{d}{dt}E(\tilde{u}) = \frac{d}{dt}\Big((\dot{u}|\dot{u}) - (u|Lu)\Big) = (\ddot{u}|\dot{u}) + (\dot{u}|\ddot{u}) - (\dot{u}|Lu) - (u|L\dot{u})$$

$$= (Lu|\dot{u}) + (\dot{u}|Lu) - (\dot{u}|Lu) - (u|L\dot{u}) = (Lu|\dot{u}) - (u|L\dot{u}) = 0$$

since L is self-adjoint. ☐

4.2 Scattering for rank 1 spaces

For symmetric spaces Ω of rank 1 (including euclidean space) consider the Laplace-Beltrami operator

$$L = \Delta + (\rho|\rho) = \Delta + \frac{1}{4}$$

on L^2_Ω. Then the second-order wave equation $\partial_t^2 u = Lu$ for a function $u(t,x)$ on $\mathbf{R} \times \Omega$ is equivalent to the first-order system

$$\partial_t w = \begin{pmatrix} 0 & I \\ L & 0 \end{pmatrix} w = \Lambda w$$

where $w(t,w) = \begin{pmatrix} u(t,x) \\ \partial_t u(t,x) \end{pmatrix}$ is a smooth map $w : \mathbf{R} \times \Omega \to \mathbf{R}^2$. If we have a discrete group $\Gamma \in Aut(\Omega)$ with fundamental domain F we have the diagram

$$\begin{tikzcd} L^2_D \arrow[r, bend left, "q"] & L^2_F \arrow[l, bend left, "q^*"] \end{tikzcd}$$

where $qf := f|_F$ is a co-isometry and its adjoint isometry q^*f is the zero-extension on f. Thus $qq^* = \mathrm{id}$ and $p = q^*q : L^2_D \to L^2_F$ is the orthogonal projection. Then A is self-adjoint on $\mathcal{H} := L^2(F, \mathbf{C}^2)$ for the boundary conditions imposed by (**??**). Consider the resolvent $(\lambda - A)^{-1}$. For the basic solutions

$$u_t^{\pm}(z) = w^{1/2}\ \phi(y\ e^{-\pm t})$$

it follows that $\mathcal{H}_{\pm}$ consists of all data

$$w_t^{\pm}(z) = \begin{pmatrix} w^{1/2}\ \phi(w) \\ \pm w^{-3/2}\ \phi'(w) \end{pmatrix}$$

with $\phi \in \mathcal{C}^\infty(F^a), a > 1$. The main result [LaPh] states that $(\lambda - A)^{-1}$ is a compact operator and hence has a discrete spectrum. How do we find eigenfunctions? For $\tau \in i\mathbf{R}$ put

$$h_\tau(w) := \begin{pmatrix} w^{1/2+\tau} \\ \tau \ w^{1/2+\tau} \end{pmatrix}.$$

Then

$$Ah_\tau = \tau \ h_\tau.$$

Since h_τ depends only on $y = \mathrm{Im}(z)$ it is invariant under the parabolic subgroup $\Gamma_\infty = \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix}$ consisting of all translations

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}(z) = z + n$$

by integers $n \in \mathbf{Z}$. To make it fully invariant under $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ we define the **Eisenstein series**

$$Z_\tau(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h_\tau(\gamma z).$$

Note that this is not holomorphic in z (discrete series) but instead belongs to the principal series. The translation representation of $T_\pm$ on $L^2(\mathbf{R})^2$ is

$$T_\pm f(s) = \frac{\pm}{\sqrt{2}}(\partial_s, -1)(e^s f \circ e^{-s})$$

where $f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$.

Theorem 86. *The multiplication realization $\hat{T}_\pm$ on $L^2(i\mathbf{R})^2$ is given by the Eisenstein series*

$$\Theta_\pm \phi(\tau) = (E_\tau^\pm | f)_Z$$

for the function

$$h_\tau^\pm(z) = \frac{\pm}{\sqrt{2}\tau}(w^{1/2+\tau}, -\tau w^{1/2-\tau}).$$

Define the scattering operator

$$ST_-(f) = T_+(f)$$

Then S has the Fourier coefficients

$$\hat{S}(\tau) = -\frac{e_\tau^+}{e_\tau^-} = \frac{\Gamma(\frac{1}{2})\Gamma(\tau)}{\Gamma(\tau + \frac{1}{2})} \frac{\zeta(2\tau)}{\zeta(1 + 2\tau)}.$$

4.3 Harmonic analysis on Symmetric Spaces

4.3.1 Structure of Lie groups

For a Riemannian symmetric space G/K let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, and let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. The finite Weyl group

$$W := K_A^\bullet / K_A^\circ$$

Choose a **vertreter system** $k_w \in K_A^\bullet$ of $w \in W = \mathcal{N}_A(K)/\mathcal{Z}_A(K)$. W acts linearly on $\mathfrak{a}$ and hence on the symmetric algebra $\mathcal{S}(\mathfrak{a}) \approx \mathcal{P}(\mathfrak{a}^\sharp)$. The invariant part

$$\mathcal{S}(\mathfrak{a})^W \approx \mathcal{D}(G/K)$$

yields the algebra of invariant differential operators on G/K, while the skew-invariant part

$$\mathcal{S}(\mathfrak{a})_-^W$$

is spanned by a single operator π. Its square π^2 is an invariant differential operator. The generalized **Poincaré** inequality is

$$(f|\pi^2 f) \geqslant \text{const.}(f|f)$$

for all smooth functions f on G/K. Now consider the Iwasawa decomposition

$$G = KA\overline{N}$$

and let $g_A,\ g_K$ denote the Iwasawa components of $g \in G$. By [He/432] we have

$$(an_K)_A = a\ (ana^{-1})_A\ n_A^{-1}$$

for all $n \in N$. The Kostant convexity theorem says

$$\log(e^t K)_A = \text{conv} W t$$

In general, a **Harish-Chandra isomorphism** identifies a subalgebra $A \subset \mathcal{U}(\mathfrak{g})$ with $\mathcal{U}(\mathfrak{a})$ for a sub Liegebra $\mathfrak{a} \subset \mathfrak{g}$.

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, "\subset"] \arrow[d] & \mathfrak{g} \arrow[d] \\
\mathcal{U}(\mathfrak{h}) \arrow[r] & \mathcal{U}(\mathfrak{g})
\end{tikzcd}$$

For example, for a complex Liegebra $\mathfrak{g}$ the center $Z(\mathcal{U}(\mathfrak{g}))$ is a commutative subalgebra identified with $\mathcal{U}(\mathfrak{h}) = \mathcal{S}(\mathfrak{h}) = \mathcal{P}(\mathfrak{h}^\sharp)$ for a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. In the real case we have $\mathcal{D}(G/K)^G \approx \mathcal{U}(\mathfrak{a}) = \mathcal{P}(\mathfrak{a}^\sharp)$ for a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}$.

4.3.2 Geodesics

For $\Theta \subset \Sigma$ put

$$\acute{\mathfrak{a}} := \langle H_\alpha : \ \alpha \in \Theta \rangle, \quad \grave{\mathfrak{a}} := \{t \in \mathfrak{a} : \ t|\Theta = 0\}$$

$$\grave{\mathfrak{a}}_+ := \{t \in \grave{\mathfrak{a}} : \ t|\alpha > 0 \ \forall \ \alpha \in \Sigma \sim \langle \Theta \rangle\}$$

$$\acute{\mathfrak{g}} := [\mathfrak{g}^o_{\grave{\mathfrak{a}}} c, \mathfrak{g}^o_{\grave{\mathfrak{a}}} c]$$

$$\acute{\mathfrak{k}} := \mathfrak{k} \cap \acute{\mathfrak{g}}$$

$$\acute{\mathfrak{n}}^+ := \sum_{\alpha \in \langle \Theta \rangle_+} \mathfrak{g}^\alpha$$

For $t \in \overline{\mathfrak{a}}_+$ and $k \in K$ we obtain a geodesic

$$\gamma_{t,k}(\alpha) := k \ e^{\alpha t} K \in G/K$$

The geodesic is regular for $t \in \mathfrak{a}_+$, and singular for $t \in \mathfrak{a}_+^\Theta$ for some $\Theta \neq \emptyset$. Since $K' \cap K_A^\circ = (K')_{A'}^\circ$ we have an injection

$$K'/(K')_{A'}^\circ \subset K/K_A^\circ$$

The inclusion $K_A^\circ \subset K_{\mathfrak{a}''}^\circ$ induces an exact sequence

$$K_{A''}^\circ / K_A^\circ \to K/K_A^\circ \to K/K_{A''}^\circ$$

with typical fibre

$$K_{A''}^\circ / K_A^\circ = K'/(K')_{A'}^\circ$$

4.4 geodesics

An element $\gamma \in \mathrm{SL}^2_{\mathbf{Z}} \subset G := \mathrm{SL}^2_{\mathbf{R}}$ is called **hyperbolic/elliptic/parabolic, resp.** if the trace $\mathrm{tr}(\gamma)$ has absolute value $> 2/ < 2/ = 2, \mathit{resp.}$ These properties are invariant under conjugation. In the elliptic case we have eigenvalues in $\mathbf{T}$ and γ is conjugate to an element in K. In the hyperbolic case, we have real eigenvalues and γ is conjugate to an element in the centralizer

$$G_{\mathfrak{a}}^\circ = K_{\mathfrak{a}}^\circ \ A,$$

$$A = \{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : \ a > 0 \}.$$

Writing $g\gamma g = k_\gamma \ a$ for some $a \in A$ and $k_\gamma \in K_{\mathfrak{a}}^\circ$ we have

$$o \cdot a = o \cdot k_\gamma a = o \cdot g^{-1} \gamma g.$$

Then

$$\gamma \sim \begin{pmatrix} N^{1/2} & 0 \\ 0 & N^{-1/2} \end{pmatrix}$$

for some $N > 1$, and the centralizer

$$\Gamma^\circ_\gamma = \gamma_0^{\mathbf{Z}}$$

is a cyclic group with generator

$$\gamma_0 \sim \begin{pmatrix} N_0^{1/2} & 0 \\ 0 & N_0^{-1/2} \end{pmatrix}$$

By [TS/546] the group $K^\circ_{\mathfrak{a}''}$ normalizes both K' and G'. Thus

$$m^\Theta k_\Theta m^{-\Theta} = k'_\Theta,\ m^\Theta g m^{-\Theta} = g'$$

for all $m^\Theta \in K^\circ_{\mathfrak{a}''}$. Define an action of $K^\circ_{\mathfrak{a}''}$ on G'/K' by

$$m^\Theta \cdot gK' := g'\ K'.$$

This way $\mathcal{L}_\Theta$ becomes a $K^\circ_{A''}$-module. Consider the homogeneous vector bundle

$$K \times^\circ_{K_{A''}} \mathcal{L}(\mathfrak{a}' \times G'/K')$$

and the L^2-sections

$$\Gamma^2(K \times^\circ_{K_{A''}} \mathcal{L}(\mathfrak{a}' \times G'/K'))$$

4.4.1 Root analysis

Let m_α be the restricted root multiplicity. Put

$$\sinh_\alpha(t) := \sinh(\alpha|t)^{m_\alpha}$$

and

$$\sinh_\Pi(t) := \Pi_{\alpha\in\Pi} \sinh_\alpha(t).$$

For $\alpha \in \Sigma_+$ define [Wa/325]

$$B^\tau_\alpha = B\Big(\frac{m_\alpha}{2}, \frac{m_{\alpha/2}}{4} + \tau|\alpha_0\Big) = B\Big(\frac{m_\alpha}{2}, \frac{m_{\alpha/2}}{4} + \frac{\tau|\alpha}{\alpha|\alpha}\Big)$$

and $B^\tau_\Pi := \prod\limits_{\alpha\in\Pi} B^\tau_\alpha$.

4.4.2 Invariant measures

By [HeJo/2] and [He/458] the Haar measure on G is given by

$$\int\limits_G dg\ f(g) = \int\limits_K dk \int\limits_A a^{2\rho} \int\limits_N dn\ f(ka\overline{n}) = \int\limits_K dk \int\limits_A da\ a^{2\rho} \int\limits_N dn\ f(ka\overline{n})$$

$$= c \int\limits_K dk \int\limits_{\mathfrak{a}_+} dt \ \sinh_{\Sigma_+}(t) \int\limits_K dk' \ f(ke^t k')$$

For $\mathrm{Re}(\tau) > 0$ define the c-**function**

$$c^\tau := \int\limits_N dn \ n_A^{-\tau-\rho}$$

and normalize the Haar measure dn on N by

$$c^\rho = \int\limits_N dn \ \overline{n}_A^{-2\rho} = 1.$$

Then Harish-Chandra's formula

$$\int\limits_{K/K_A^\circ} dk \ \phi[k] = \int\limits_N dn \ n_A^{-2\rho} \ \phi[n_K]$$

holds. Normalize the Haar measure on $N \cap (k_w^{-1} \overline{N} k_w)$ by

$$\int\limits_{N \cap (k_w^{-1} \overline{N} k_w)} dn \ n_A^{-2\rho} = 1$$

Proposition 87.

$$a^\rho \int\limits_N dn \ f(a^{-1}na) = a^{-\rho} \int\limits_N dn \ f(n)$$

Proof.

$$a^{\rho-\tau} \int\limits_N dn \ n_A^{-\tau-\rho} \ (a^{-1}na)_A^{\tau-\rho} = \phi^\tau(a^{-1}) = \phi^{-\tau}(a) = a^{-\tau-\rho} \int\limits_N dn \ n_A^{\tau-\rho} \ (ana^{-1})_A^{-\tau-\rho}$$

implies

$$a^\rho \int\limits_N dn \ n_A^{-\tau-\rho} \ (a^{-1}na)_A^{\tau-\rho} = a^{-\rho} \int\limits_N dn \ n_A^{\tau-\rho} \ (ana^{-1})_A^{-\tau-\rho}$$

□

Theorem 88 (Wa/325)**.** *Let* Re $\tau > 0$.*Then*

$$c^\tau = \frac{B_{\Sigma_+}^\tau}{B_{\Sigma_+}^\rho}$$

Proof. For rank 1 let $\Sigma_+^0 = \{\beta\}$. Then $\Sigma_+ = \{\beta, 2\beta\}$ and hence

$$\rho = \frac{m_\beta}{2}\beta + m_{2\beta}\beta = (\frac{m_\beta}{2} + m_{2\beta})\beta,$$

$$\rho|\beta_0 = \frac{\rho|\beta}{\beta|\beta} = \frac{m_\beta}{2} + m_{2\beta}$$

Write $\tau = iz\rho$. Then $(\tau|\beta_0) = iz(\rho|\beta_0)$ and hence $iz = \frac{\tau|\beta_0}{\rho|\beta_0}$. It follows that

$$2P(1+iz) - m_{2\beta} = (\rho|\beta_0)(1 + \frac{\tau|\beta_0}{\rho|\beta_0}) - m_{2\beta} = (\rho + \tau|\beta_0) - m_{2\beta} = \frac{m_\beta}{2} + \tau|\beta_0$$

and hence $P(1+iz) = \frac{1}{2}(\frac{m_\beta}{2} + m_{2\beta} + \tau|\beta_0)$. By [He/437] the unnormalized integral is

$$\int\limits_0^\infty \frac{dr}{r} \; \frac{r^{m_\beta/2}}{(1+r)^{m_\beta/2+\tau|\beta_0}} \int\limits_0^\infty \frac{dt}{t} \; \frac{t^{m_{2\beta}/2}}{(1+t)^{\frac{1}{2}(m_\beta/2+m_{2\beta}+\tau|\beta_0)}}$$

$$= B\Big(\frac{m_\beta}{2}, \tau|\beta_0\Big) \; B\Big(\frac{m_{2\beta}}{2}, \frac{m_\beta}{4} + \frac{\tau|\beta_0}{2}\Big) = B_\beta^\tau \; B_{2\beta}^\tau$$

since $m_{\beta/2} = 0$ and $\tau|(2\beta)_0 = \frac{\tau|\beta_0}{2}$. The duplication formula

$$\frac{\Gamma(2z)}{\Gamma(z)} = \frac{1}{2\sqrt{\pi}} 4^z \; \Gamma(z + \frac{1}{2})$$

implies

$$\frac{\Gamma(\frac{\boldsymbol{m}_\beta}{4} + \frac{\tau|\beta_0}{2})}{\Gamma(\frac{\boldsymbol{m}_\beta}{2} + \tau|\beta_0)} = c \cdot \frac{2^{-\tau|\beta_0}}{\Gamma(\frac{\boldsymbol{m}_\beta}{4} + \frac{\tau|\beta_0}{2} + \frac{1}{2})}$$

Therefore

$$B\Big(\frac{m_\beta}{2}, \tau|\beta_0\Big) \; B\Big(\frac{m_{2\beta}}{2}, \frac{m_\beta}{4} + \frac{\tau|\beta_0}{2}\Big) = c' \cdot \frac{2^{-\tau|\beta_0} \; \Gamma(\tau|\beta_0)}{\Gamma(\frac{\boldsymbol{m}_\beta}{4} + \frac{\tau|\beta_0}{2} + \frac{1}{2}) \; \Gamma(\frac{m_\beta}{4} + \frac{m_{2\beta}}{2} + \frac{\tau|\beta_0}{2})}$$

By [He/447] we have

$$c(\tau) = c_0 \prod_{\beta \in \Sigma_+^0} \frac{2^{-\tau|\beta_0} \; \Gamma(\tau|\beta_0)}{\Gamma(\frac{\boldsymbol{m}_\beta}{4} + \frac{\tau|\beta_0}{2} + \frac{1}{2}) \; \Gamma(\frac{m_\beta}{4} + \frac{m_{2\beta}}{2} + \frac{\tau|\beta_0}{2})}$$

$$= c_1 \prod_{\beta \in \Sigma_+^0} B\Big(\frac{m_\beta}{2}, \tau|\beta_0\Big) \; B\Big(\frac{m_{2\beta}}{2}, \frac{m_\beta}{4} + \frac{\tau|\beta_0}{2}\Big)$$

We claim that

$$\prod_{\beta \in \Sigma_+^0} B\Big(\frac{m_\beta}{2}, \tau|\beta_0\Big) \; B\Big(\frac{m_{2\beta}}{2}, \frac{m_\beta}{4} + \frac{\tau|\beta_0}{2}\Big) = \prod_{\alpha \in \Sigma_+} B\Big(\frac{m_\alpha}{2}, \frac{m_{\alpha/2}}{4} + \tau|\alpha_0\Big)$$

For $\beta \in \Sigma_+^0$ we have $m_{\beta/2} = 0$. Therefore the claim is equivalent to

$$\prod_{\beta \in \Sigma_+^0} B\Big(\frac{m_{2\beta}}{2}, \frac{m_\beta}{4} + \frac{\tau|\beta_0}{2}\Big) = \prod_{\alpha \in \Sigma_+ \sim \Sigma_+^0} B\Big(\frac{m_\alpha}{2}, \frac{m_{\alpha/2}}{4} + \tau|\alpha_0\Big)$$

Since every $\alpha \in \Sigma_+ \sim \Sigma_+^0$ has the form $\alpha = 2\beta$ for a unique $\beta \in \Sigma_+^0$, the assertion follows with $\tau|\alpha_0 = \frac{\tau|\beta_0}{2}$. □

4.4.3 Principal series representations and intertwiners

Let ρ be the half-sum of positive restricted roots. For $\tau = i\lambda \in \mathfrak{a}^\sharp_{\mathbf{C}}$ define the **principal series representation** of G on $H := L^2(K/K^\circ_A)$ by

$$(g^\tau f)[k] = (g^{-1}k)_A^{\tau-\rho}\ f[(g^{-1}k)_K]$$

Here $[k] := kK^\circ_A)$. For $\tau \in i\mathfrak{a}^\sharp$ this is unitary and irreducible. Define an operator ${}^wS_\tau \in \mathcal{L}(H)$ by

$$\frac{1}{B^\rho_{\Sigma_+\cap w^{-1}\Delta_-}}({}^wS_\tau f)[k] = \frac{1}{B^\tau_{\Sigma_+\cap w^{-1}\Delta_-}} \int\limits_{\overline{N}\cap(k_w^{-1}Nk_w)} dm\ m_A^{-(\tau+\rho)}\ f[kk_w m_K]$$

Then

$$\begin{tikzcd} H & H \arrow[l, "g^{w\cdot\tau}"'] \arrow[d, "{}^wS_\tau"] \\ H \arrow[from=u, "{}^wS_\tau"'] & L^2(K/K^\circ_A) \arrow[l, "g^\tau"'] \end{tikzcd} .$$

Put

$$\widehat{K/K^\circ_A} := \{\varkappa \in \hat{K} :\ V_\varkappa^{K^\circ_A} \neq \{0\}\}.$$

For $F \in L^2(i\mathfrak{a}^\sharp, H)$ and $\varkappa \in \widehat{K/K^\circ_A}$ define the Fourier coefficient

$$f^\varkappa_\tau := \int\limits_K dk\ f(\tau, kK^\circ_A)\ k^{-\varkappa} \in \mathcal{L}(V_\varkappa)$$

Then we have the inversion formula

$$f(\tau, K^\circ_A) = \sum_{\varkappa\in\widehat{K/K^\circ_A}} d_\varkappa\ \mathrm{tr}(f^\varkappa_\tau|_{V_\varkappa^{K^\circ_A}})$$

Then we have the diagonalization

$$({}^wS_\tau f_\tau)^\varkappa = {}^wS^\varkappa_\tau f^\varkappa_\tau$$

for endomorphisms ${}^wS^\varkappa_\tau \in \mathcal{L}(V_\varkappa)$ which preserve and are unitary on $V_\varkappa^{K^\circ_A}$. By Harish-Chandra there exist meromorphic functions $\Theta^\varkappa \in \mathcal{L}(V_\varkappa^{K^\circ_A})$ such that for $t \in \mathfrak{a}_+$ we have

$$\int\limits_K dk\ (e^tk)_A^{\tau\rho}(e^tk)_K^\varkappa = \sum_{w\in W}\sum_{\mu\in L} e^{-t|(w\tau+\rho+\mu)}\ c(w\tau)\ \Gamma^\varkappa_\mu(w\tau)^w S^{\varkappa *}_{\overline{\tau}}$$

where $\Gamma^\varkappa_\mu \in \mathcal{L}(V_\varkappa^{K^\circ_A})$ are rational and holomorphic on $\mathfrak{a}^\sharp_+ + i\mathfrak{a}^\sharp$.

4.4.4 Spherical functions

By [HeJo/1] a spherical function is a continuous bi-invariant function $\phi : G \to \mathbf{C}$ satisfying

$$\int\limits_K dk\ \phi(xky) = \phi(z)\ \phi(w)$$

The non-zero bounded spherical function characterize the maximal ideals of the convolution algebra $L^1(K\backslash G/K)$ via

$$\phi^{\perp\!\!\perp} := \{f \in L^1(K\backslash G/K) : \int\limits_G dg\ f(g)\phi(g) = 0\}$$

For a nc symmetric space G/K the **spherical function** has the form [7, p. 435]

$$\phi^\tau(g) = \int\limits_K dk\ (gk)_A^{\tau-\rho} = \int\limits_K dk\ (g^{-\tau}1)[k]$$

where $\tau \in \mathfrak{a}_{\mathbf{C}}^\sharp$ and $[k] := kK_A^\circ$. By [7, p. 419] we have

$$\phi^\tau(g^{-1}) = \phi^{-\tau}(g)$$

By [HeJo] we have

$$\phi^\tau \text{ bounded} \Leftrightarrow -\mathrm{Re}(\tau) \in \mathrm{conv}(W\rho).$$

Proposition 89. *On A the spherical function is given by*

$$\phi^\tau(a) = a^{\tau-\rho} \int\limits_N dn\ n_A^{-\tau-\rho}\ (ana^{-1})_A^{\tau-\rho}$$

Proof. We have

$$\phi^\tau(a) = \int\limits_K dk\ (ak)_A^{\tau-\rho} = \int\limits_N dn\ n_A^{-2\rho}\ (an_K)_A^{\tau-\rho} = \int\limits_N dn\ n_A^{-2\rho}\ (\mathfrak{t}_a(ana^{-1})_A \cdot n_A^{-1})^{\tau-\rho}$$

$$= a^{\tau-\rho} \int\limits_N dn\ n_A^{-2\rho} n_A^{\rho-\tau}\ (ana^{-1})_A^{\tau-\rho} = a^{\tau-\rho} \int\limits_N dn\ n_A^{-\tau-\rho}\ (ana^{-1})_A^{\tau-\rho}$$

□

Now let $t \in \mathfrak{a}_+$. Since $\lim_{t\to\infty}(e^t\overline{n}e^{-t})_A = 1$, the dominated convergence theorem implies

$$\lim_{t\to\infty} e^{t|\rho-\tau}\ \phi^\tau(e^t) = \lim_{t\to\infty} \int\limits_N dn\ \overline{n}_A^{-\tau-\rho}\ (e^t\overline{n}e^{-t})_A^{\tau-\rho}$$

$$= \int\limits_N dn\ \overline{n}_A^{-\tau-\rho}\ \lim_{t\to\infty}(e^t\overline{n}e^{-t})_A^{\tau-\rho} = \int\limits_N dn\ \overline{n}_A^{-\tau-\rho} = c(\tau),$$

If τ is regular ($s\tau \neq \tau$ for all $s \neq e$) it follows that $c(\tau) \neq 0$. Thus there is an asymptotic expansion

$$\lim_{A^+ \ni a \to \infty} a^{\rho-\tau} \ \Phi_\tau(a) = c(\tau).$$

Joint eigenfunctions $\Phi_\tau(a)$ for radial parts [7, p. 429]

$$\tilde{D}\Phi_\tau = \Gamma_\tau(D) \ \Phi_\tau$$

of $D \in \mathbf{D}(G/K)$. In particular, by [He/427]

$$\tilde{L}_X \Phi_\tau = (\tau\tau - \rho|\rho)\Phi_\tau.$$

Let $\beta = \{\beta_1, \ldots, \beta_r\}$ be positive simple roots. Recursion formula

$$\Phi_\tau(a) = a^{\tau-\rho} \sum_{\boldsymbol{m} \in \mathbf{N}^r} \Gamma_{\boldsymbol{m}}(\tau) a^{-\boldsymbol{m}\beta}$$

with $\Gamma_0(\tau) = 1$. Then we have $|W|$ linearly independent solutions $\Phi_{w\tau}, \ w \in W$. Hence

$$\phi^\tau = \sum_{w \in W} c(w\tau)\Phi_{w\tau}$$

with $c_w(\tau) = c(w\tau), \ c(\tau) = c_e(\tau)$. In summary,

$$\phi^\tau = \sum_{w \in W} c(w\tau) a^{w\tau-\rho} \sum_{\boldsymbol{m} \in \mathbf{N}^r} \Gamma_{\boldsymbol{m}}(w\tau) a^{-\boldsymbol{m}\beta}$$

Example 90. *In the complex case we have [7, p. 432]*

$$\phi^\tau(a) = c^\tau \frac{\sum\limits_w \det w \ a^{w\tau}}{\sum\limits_w \det w \ a^{w\rho}}$$

with

$$c_\tau = \frac{(\Sigma_+|\rho)}{(\Sigma_+|\tau)}$$

For $w = [k_w] \in W$ with $k_w \in K_A^\bullet$ put $N_w := N \cap k_w^{-1} \overline{N} k_w$ and normalize by

$$\int\limits_{N_w} dn \ \overline{n}_A^{-2\rho} = 1$$

For $\varkappa \in K^\sharp$ let [Wa/319]

$$T^w_{\tau,\varkappa} := k^\varkappa_w \int\limits_{N_w} dn \ \overline{n}_A^{-\tau+\rho} \ \overline{n}_K^\varkappa \in \mathcal{L}(V_\varkappa)$$

The integral converges on

$$C_w := \{\tau \in \mathfrak{a}^\sharp_{\mathbf{C}} : \int\limits_{N_w} dn \ \overline{n}_A^{-(\mathrm{Re}(\tau)+\rho)} < \infty\}$$

$$= \{\tau \in \mathfrak{a}^\sharp_{\mathbf{C}} :\ \mathrm{Re}\tau|Q_\alpha > 0\ \forall\ \alpha \in \Sigma^0_+(w)\}$$

Here $\Sigma^0_+(w)$ consists of all indivisible positive roots α such that $w\alpha < 0$. For $w_* \in W$ satisfying $w_*\Sigma_+ = -\Sigma_+$ we have $N_w = N$ since $k_{w_*}^{-1}\overline{N}k_{w_*} = N$. Therefore

$$T^{w_*}_{\tau,\varkappa} := k^{\varkappa}_{w_*} \int\limits_N dn\ \overline{n}_A^{-\tau+\rho}\ \overline{n}_K^{\varkappa}$$

and

$$C_{w_*} = \{\tau \in \mathfrak{a}^\sharp_{\mathbf{C}} :\ \mathrm{Re}\ \tau \in \mathfrak{a}^\sharp_+\}.$$

By [Wa/320] we have the cocycle formula

$$T^{w'w}_{\tau,\varkappa} = T^{w'}_{w\tau,\varkappa}\ T^{w}_{\tau,\varkappa}$$

for $k_{w'w} = k_{w'}k_w$ and $\ell_{w'w} = \ell_{w'} + \ell_w$. By [7, p. 458] for each $\tau \in \mathfrak{a}^\sharp_{\mathbf{C}}$ and $[k] \in K/K^\circ_A$ we obtain an eigenfunction

$$\phi_{\tau,[k]}[g] := (g^{-1}k)_A^{\tau+\rho}$$

of $\mathbf{D}(G/K)$. Therefore

$$D\phi_{\tau,[k]} = \Gamma_\tau(D)\phi_{\tau,[k]}$$

for the Harish-Chandra homomorphism

$$\mathbf{D}(G/K) \xrightarrow{\Gamma} \mathcal{P}(\mathfrak{a})^W$$

By [7, p. 463] we have

$$c_w(\tau) := \int\limits_{N_w} dn\ \overline{n}_A^{-(\tau+\rho)} = \tilde{B}_{\Sigma^0_+ \cap w^{-1}\Delta^0_-}(\tau)$$

where, putting $\underline{\alpha} := \alpha/(\alpha|\alpha)$, we define

$$\tilde{B}_\alpha(\tau) = \frac{\Gamma(\tau|\underline{\alpha})}{2^{\tau|\underline{\alpha}}\ \Gamma(\frac{1}{2}(\frac{m_\alpha}{2} + 1 + \tau|\underline{\alpha}))\ \Gamma(\frac{1}{2}(\frac{m_\alpha}{2} + m_{2\alpha} + \tau|\underline{\alpha}))}$$

4.4.5 Harmonic Analysis on Lie groups

Let G be a locally compact unimodular 2nd countable group of type I. For any choice of Haar measure dg on G there exists a positive Plancherel measure $d\gamma$ on the unitary dual $G^\sharp$ such that

$$\int\limits_G dg\ |f(g)|^2 = \int\limits_{G^\sharp} d\gamma\ \|f_\gamma\|^2_{HS}$$

for all $f \in L^1(G) \cap L^2(G)$. Here

$$f_\gamma = \int\limits_G dg\ f(g)\ g^\gamma \in \mathcal{L}^2(V_\gamma).$$

Now let G be a connected semi-simple Lie group with finite center. Then G is of type I and for every $f \in \mathcal{C}_0^\infty(G)$ we have $f(\gamma) \in \mathcal{L}^1(V_\gamma)$. More over, the assignment

$$\mathrm{tr}_\gamma f := \mathrm{tr}(f(\gamma))$$

is a distribution on G. Consider the **spherical Fourier transform**

$$\tilde{f}_{\tau,[k]} = \int\limits_{G/K} dg\ (g^{-1}k)_A^{\tau-\rho}\ f[g] = \int\limits_{G/K} dg\ f[g]\ (g^{-\tau}1)[k]$$

for $\tau \in i\mathfrak{a}^\sharp$. In short,

$$\tilde{f}_\tau = \int\limits_{G/K} dg\ f[g]\ g^{-\tau}1 \in L^2(K/K_A^o c).$$

Then

$$\frac{1}{|B_{\Sigma_+}^\rho|^2} \int\limits_{G/K} dz\ |f(z)|^2 = \int\limits_{i\mathfrak{a}_+^\sharp} \frac{d\tau}{|B_{\Sigma_+}^\tau|^2} \int\limits_{K/K_A^\circ} dk\ |\tilde{f}_{\tau,[k]}|^2$$

and [He/459] the Fourier transform has the inverse

$$f[g] = \frac{|B_{\Sigma_+}^\rho|^2}{|W|} \int\limits_{i\mathfrak{a}^\sharp} \frac{d\tau}{|B_{\Sigma_+}^\tau|^2} \int\limits_{K/K_A^\circ} dk\ (g^{-1}k)_A^{-(\tau+\rho)}\ \tilde{f}_{\tau,[k]}.$$

4.5 Scattering on symmetric spaces

4.5.1 Extension of rings and fields

If $K \subset K'$ is a finite field extension, then $\mathrm{tr}_K^{K'} : K' \to K$ is defined by

$$\mathrm{tr}_K^{K'} \alpha = \mathrm{tr} M_\alpha,$$

where $M_\alpha : K' \to K'$ denotes multiplication by $\alpha \in K'$. Let

$$m_\alpha(X) = \prod_{i=1}^n (X - \lambda_i(\alpha)) \in K[X]$$

denote the minimal polynomial of $\alpha \in K'$ over K. Then

$$\frac{1}{[K' : K[\alpha]]}\ \mathrm{tr}_K^{K'} \alpha = \sum_{i=1}^n \lambda_i(\alpha)$$

For $\alpha \in K$ we have

$$\mathrm{tr}_K^{K'} \alpha = [K' : K]\alpha$$

If $K \subset K'$ is a (finite) Galois extension, then

$$\operatorname{tr}_K^{K'} \alpha = \sum_{w \in \operatorname{Gal}_K^{K'}} w\alpha$$

If $K \subset K' \subset K''$ is a tower of finite field extensions, then

$$\operatorname{tr}_K^{K''}(\alpha) = \operatorname{tr}_{K'}^{K''}(\operatorname{tr}_K^{K'}(\alpha))$$

$$\begin{tikzcd} K & K' \arrow[l, "\operatorname{tr}_K^{K'}"'] & K'' \arrow[l, "\operatorname{tr}_{K'}^{K''}"'] \arrow[ll, bend left, "\operatorname{tr}_K^{K''}"] \end{tikzcd}$$

Define the **trace form** $K' \times K' \to K$ by

$$(x|y) := \operatorname{tr}_K^{K'}(xy)$$

Then $(|)$ is non-degenerate if K'/K is separable, and identically zero if K'/K is purely inseparable.

Example 91. *Let*

$$\mathbf{Q}' = \mathbf{Q}(\sqrt{d}) = \{a + b\sqrt{d} : \ a, b \in \mathbf{Q}\} = \mathbf{Q}\langle 1, \sqrt{d} \rangle$$

be a quadratic extension field. Then $M_{a+b\sqrt{d}}$ *has the basis*

$$(M_{a+b\sqrt{d}}) = \begin{pmatrix} a & bd \\ b & a \end{pmatrix}$$

and the minimal polynomial

$$m_{a+b\sqrt{d}}(X) = X^2 - 2aX + (a^2 - db^2)$$

with roots $\lambda_1 = a + b\sqrt{d}$ *and* $\lambda_2 = \overline{\lambda}_1 = a - b\sqrt{d}$. *Hence*

$$\operatorname{tr}_{\mathbf{Q}}^{\mathbf{Q}(\sqrt{d})}(a + b\sqrt{d}) = 2a$$

Example 92. *Let* $\mathbf{F}_q \subset \mathbf{F}_q^n$ *be a finite extension of finite fields. Then*

$$\operatorname{tr}_{\mathbf{F}_q}^{\mathbf{F}_q^n} \alpha = \alpha + \alpha^q + \alpha^{q^2} + \ldots + \alpha^{q^{n-1}} = \sum_{i \in \mathbf{Z}/n\mathbf{Z}} \alpha^{q^i}$$

as a sum over the cyclic Galois group $\operatorname{Gal}_{\mathbf{F}_q}^{\mathbf{F}_q^n} = \mathbf{Z}/n\mathbf{Z}$. *For* $\alpha \in \mathbf{F}_q$ *we have* $\alpha^q = \alpha$ *and therefore* $\operatorname{tr}_{\mathbf{F}_q}^{\mathbf{F}_q^n} \alpha = n\ \alpha$.

By Chevalley, the ring extension $\mathcal{P}^W \subset \mathcal{P}$ has the following properties:

$$\mathcal{P} = \mathcal{P}^W[h_1, \ldots, h_r]$$

is a free polynomial algebra. There exists a graded W-invariant subspace $\mathcal{H} \subset \mathcal{P}$ (harmonic polynomials) such that

$$\mathcal{P} = \mathcal{P}^W \otimes \mathcal{H} = (\mathcal{P}_+^W \cdot \mathcal{P}) \oplus \mathcal{H}$$

Consider the quotients fields $\underline{\mathcal{P}}$ and $\underline{\mathcal{P}}^W$. Then

$$\underline{\mathcal{P}}^W = (\underline{\mathcal{P}})^W$$

and the field extension $\underline{\mathcal{P}}^W \subset \underline{\mathcal{P}}$ is Galois with Galois group W. It follows that

$$\mathrm{tr}_{\underline{\mathcal{P}}^W}^{\underline{\mathcal{P}}} p = \sum_{w \in W} w \cdot p := p^W$$

for all $p \in \mathcal{P}$.

By [HC] we have restriction maps

$$\begin{tikzcd}
\mathcal{P}(\mathfrak{a}) & \mathcal{P}(\mathfrak{p}) \arrow[l] \\
\mathcal{P}(\mathfrak{a})_W^\circ \arrow[u] & \mathcal{P}(\mathfrak{p})_K^\circ \arrow[l, "\asymp"'] \arrow[u]
\end{tikzcd}$$

Here

$$W = K_A^\bullet / K_A^\circ = G_A^\bullet / G_A^\circ$$

since $G_A^\odot = A\ K_A^\odot$. For $\alpha \in \Sigma$ choose $t_\alpha \in \mathfrak{a}$ such that

$$B(t, t_\alpha) = t\alpha$$

for all $t \in \mathfrak{a}$. Then W is generated by

$$s_\alpha(t) = t - \frac{2t_\alpha}{t_\alpha \alpha} t_\alpha.$$

By [HC2/251] there exist homogeneous $u_w \in \mathcal{P}(\mathfrak{a})$ such that

$$\mathcal{P} = \sum_{w \in W} u_w \cdot \mathcal{P}_W^+$$

and

$$\underline{\mathcal{P}} = \sum_{w \in W} u_w \cdot \underline{\mathcal{P}}_W^+$$

with u_w free over the subfield $\underline{\mathcal{P}}_W^+ \subset \underline{\mathcal{P}}$. Let $u^w \in \underline{\mathcal{P}}$ denote the dual basis satisfying

$$(u_\sigma u^\tau)_W^0 = \delta_\sigma^\tau$$

for all $\sigma, \tau \in W$. Then $u^\sigma \in \frac{\mathcal{P}}{\pi}$ and

$$\mathcal{D} = \sum_{w \in W} u^w \ \mathcal{P}_W^+$$

On the other hand, by [HC/254] we have

$$\mathcal{P}(\mathfrak{a})_W^\circ = \mathbf{C}[i_1, \ldots, i_r]$$

where $r = \dim \mathfrak{a}$.

Consider the polynomial algebra $\mathcal{P}$ as a (free) $\mathcal{P}_W^+$-module, and denote by

$$\mathcal{P}^\sharp = \mathrm{Hom}_{\mathcal{P}_W^+}(\mathcal{P}, \mathcal{P}_W^+)$$

the dual $\mathcal{P}_W^+$-module. The quotient fields $\underline{\mathcal{P}}_W^+ \subset \underline{\mathcal{P}}$ form a Galois extension with Galois group W. Therefore there exists a **conditional expectation** $\mathrm{tr} : \underline{\mathcal{P}} \to \underline{\mathcal{P}}_W^+$ defined by

$$\mathrm{tr} z = z_W^\circ := \sum_{w \in W} w \cdot z$$

for all $z \in \underline{\mathcal{P}}$. This induces a non-degenerate bilinear **trace form**

$$\underline{\mathcal{P}} \times \underline{\mathcal{P}} \ni (x, y) \mapsto (xy)_W^\circ.$$

Hence the dual $\underline{\mathcal{P}}_W^+$-vector space $\underline{\mathcal{P}}^\sharp$ of all $\underline{\mathcal{P}}_W^+$-linear functionals $\underline{\mathcal{P}} \to \underline{\mathcal{P}}_W^+$ has the form

$$\underline{\mathcal{P}}^\sharp = \{x^* : \ z \in \underline{\mathcal{P}}\},$$

where

$$y \mapsto x^* y := (x|y) = (xy)_W^\circ.$$

By [HC/251] there exist homogeneous polynomials $p_w, \ w \in W$ such that

$$\begin{tikzcd}
\mathcal{P} \arrow[r, phantom, "="] & \mathcal{P}_W^+\langle p_w : \ w \in W\rangle \\
\underline{\mathcal{P}} \arrow[u] \arrow[r, phantom, "="] & \underline{\mathcal{P}}_W^+\langle p_w : \ w \in W\rangle \arrow[u]
\end{tikzcd}$$

and the p_w are linearly independent over $\underline{\mathcal{P}}_W^+$. Let $\tilde{p}^w \in \underline{\mathcal{P}}$ denote the **dual basis**, determined by

$$(p_s|\tilde{p}^t) = \mathrm{tr}(p_s \tilde{p}^t) = (p_s \tilde{p}^t)_W^\circ = \delta_s^t$$

for all $s, t \in W$.

Example 93. *For rank $r = 1$ we have $\mathfrak{a} = \mathbf{R}$ and $W = \mathcal{Z}/2\mathbf{Z} = \{(-1)^\epsilon : \ \epsilon = 0, 1\}$. Hence $\mathcal{P} = \mathbf{R}[\tau]$ and $\mathcal{P}_W^+ = \mathbf{R}[\tau^2]$. Since*

$$\mathcal{P} = \mathcal{P}_W^+\langle 1, \tau\rangle,$$

the basis is $p_0(\tau) := 1, \ p_1(\tau) := \tau$. The dual basis in $\underline{\mathcal{P}} = \mathbf{R}(\tau)$ is $p^0(\tau) = \frac{1}{2}, \ p^1(\tau) = \frac{1}{2\tau}$.

Now consider the $\mathcal{P}_W^+$-submodule

$$\mathcal{P}^\sharp = \{\lambda \in \underline{\mathcal{P}}^\sharp : \ \lambda\mathcal{P} \subset \mathcal{P}_W^+\} = \{x^* : \ z \in \underline{\mathcal{P}}, \ (x|\mathcal{P}) \subset \mathcal{P}_W^+\}$$

of $\underline{\mathcal{P}}^\sharp$. Put

$$\underset{\sim}{\mathcal{P}} := \{z \in \underline{\mathcal{P}} : \ (x|\mathcal{P}) \subset \mathcal{P}_W^+\}.$$

Then

$$\mathcal{P}^\sharp = \{x^* : \ z \in \underset{\sim}{\mathcal{P}}\}.$$

Lemma 94.

$$\underset{\sim}{\mathcal{P}} = \mathcal{P}_W^+\langle \tilde{p}^w : \ w \in W\rangle.$$

Proof. Let $p = \sum_{w \in W} i^w \ p_w \in \mathcal{P}$ where $i^w \in \mathcal{P}_W^+$. The identity

$$(\tilde{p}^s|p) = (\tilde{p}^s|\sum_{w\in W} i^w p_w) = \sum_{w\in W}(\tilde{p}^s|i^w p_w) = \sum_{w\in W} i^w(\tilde{p}^s|p_w) = i^s \in \mathcal{P}_W^+$$

shows that $\tilde{p}^s \in \underset{\sim}{\mathcal{P}}$ for each $s \in W$. Conversely, let $x = \sum_{w\in W} j_w \tilde{p}^w \in \underset{\sim}{\mathcal{P}}$, with $j_w \in \underline{\mathcal{P}}_W^+$. Then for each $s \in W$ we have

$$\mathcal{P}_W^+ \ni (x|p_s) = \sum_{w\in W}(j_w\tilde{p}^w|p_s) = \sum_{w\in W} j_w(\tilde{p}^w|p_s) = j_s$$

showing that $z \in \mathcal{P}_W^+\langle \tilde{p}^w : \ w \in W\rangle$. Thus (**??**) is proved. ☐

In order to describe $\underset{\sim}{\mathcal{P}}$ more explicitly, let

$$\mathcal{P}_W^- := \{p \in \mathcal{P} : \ w \cdot p = (-1)^w \ p \ \forall \ w \in W\}$$

be the $\mathcal{P}_W^+$-submodule of skew-polynomials. By [HC/253] we have

$$\mathcal{P}_W^- = \mathcal{P}_W^+\pi.$$

where

$$\pi(\tau) := \prod_{\alpha \in \Sigma_+^0} (\tau|\alpha) \in \mathcal{P}_W^-.$$

Harish-Chandra [5, p. 251] has proved

Proposition 95.

$$\underset{\sim}{\mathcal{P}} = \mathcal{P}/\pi.$$

Proof. For $p \in \mathcal{P}$ we have $p_W^- = \sum_{w\in W}(-1)^w \ w \cdot p \in \mathcal{P}_W^-$. By Lemma **??** it follows that $p_W^- = \pi \cdot q$ for some $q \in \mathcal{P}_W^+$. Therefore

$$\left(\frac{1}{\pi}|p\right) = (p/\pi)_W^\circ = \sum_{w\in W} w\cdot\frac{p}{\pi} = \sum_{w\in W}\frac{w\cdot p}{w\pi} = \sum_{w\in W}(-1)^w\frac{w\cdot p}{\pi} = \frac{1}{\pi}\sum_{w\in W}(-1)^w \ w\cdot p = \frac{p_W^-}{\pi} = q.$$

This shows that $\left(\frac{1}{\pi}|\mathcal{P}\right) \subset \mathcal{P}$ and hence $\mathcal{P}/\pi \subset \underset{\sim}{\mathcal{P}}$. For the (deeper) converse inclusion one shows that

$$\underset{\sim}{p}^w \in \frac{\mathcal{P}}{\pi}$$

for all $w \in W$. Then the assertion follows with Lemma **??**. □

4.5.2 Cauchy data space

On the other hand, define the **jet space**

$$\mathcal{H} = \{\mathcal{P}^\sharp(\mathfrak{a}^\sharp) \xrightarrow[\text{lin}]{f} \mathcal{C}_c^\infty(G/K)] : \ (pq)|f_x = p_x^\Delta(q|f) \ \forall\, p \in \mathcal{P}(\mathfrak{a}^\sharp)^W,\ q \in \mathcal{P}(\mathfrak{a}^\sharp)\}.$$

The space $\mathcal{C}_c^\infty(G/K)$ is a $\mathcal{P}(\mathfrak{a}^\sharp)^W$-module via

$$(p, \phi) \mapsto p^\Delta \phi,$$

where for $p \in \mathcal{P}(\mathfrak{a}^\sharp)^W$ the G-invariant differential operator $p^\Delta \in \mathcal{D}(G/K)^G$ is defined by

$$p^\Delta \Phi_\tau = p(\tau)\Phi_\tau \quad (\lambda \in \mathfrak{a}^\sharp).$$

The **Cauchy data space** is defined by

$$\mathcal{D} := \{\mathcal{P} \xrightarrow[\text{lin}]{\lambda} \mathcal{C}_c^\infty(G/K) : \ (ip)\lambda = i_x^\Delta(p\lambda) \ \forall\, i \in \mathcal{P}_W^+\}$$
$$= \mathcal{P}^\sharp \otimes_{\mathcal{P}_W^+} \mathcal{C}_c^\infty(G/K) = (\mathcal{P}^\sharp \otimes_{\mathbf{R}} \mathcal{C}_c^\infty(G/K)/\mathcal{N}$$

for the submodule $\mathcal{N}$ generated by $(ix)^* \otimes \phi - x^* \otimes i^\Delta \phi$ for all $i \in \mathcal{P}_W^+$. By Proposition **??** an alternative expression is

$$\mathcal{D} = \underset{\sim}{\mathcal{P}} \otimes_{\mathcal{P}_W^+} \mathcal{C}_c^\infty(G/K)$$

via the identification

$$(z \otimes \phi|p) := (x|p)^\Delta \phi$$

for all $z \in \underset{\sim}{\mathcal{P}},\ p \in \mathcal{P}$ and $\phi \in \mathcal{C}_c^\infty(G/K)$. This makes sense, since $(x|p) = x^* p \subset \mathcal{P}_W^+$ by assumption, and satisfies the module property

$$(ix)^* \otimes \phi = x^* \otimes i^\Delta \phi$$

for all $i \in \mathcal{P}_W^+$. The **energy form** on $\mathcal{D}$ is now defined by

$$(z \otimes \phi|y \otimes \psi)_Z := \int\limits_{G/K} d\mu_0\ \overline{\phi}\ (\pi^2(x|y))^\Delta \psi = \int\limits_{G/K} d\mu_0\ \overline{(\pi^2(x|y))^\Delta \phi}\ \psi$$

since operators in $\mathcal{D}(G/K)^K$ are symmetric. Using Proposition **??** we can also write

$$(\frac{p}{\pi} \otimes \phi|(\frac{q}{\pi} \otimes \psi)_Z = \int\limits_{G/K} dz\ \overline{((pq)_W^\circ)_x^\Delta \phi}\ \psi(z) = \int\limits_{G/K} dz\ \overline{\phi(z)}\ ((pq)_W^\circ)_x^\Delta \psi.$$

For the basis $p_s(\tau)$ and dual basis $\tilde{p}^t(\tau)$ we have

$$(\underset{\sim}{p}^s \otimes \phi_s|\underset{\sim}{p}^t\phi_t)_Z = \int\limits_{G/K} dz\overline{\phi(z)}\ ((\pi\underset{\sim}{p}^s\ \pi\underset{\sim}{p}^t)^0_W)^\Delta_x\psi$$

Example 96. *For rank $r = 1$ we have $\pi(\tau) = \tau$. Since*

$$(p^0p^0)^0_W = 1,\ (p^0p^1)^0_W = \frac{1}{4}(\frac{1}{\tau})^0_W = 0,\ (p^1p^1)^0_W = \frac{1}{4}(\frac{1}{\tau^2})^\circ_W = \frac{1}{4}\frac{1}{\tau^2}$$

we have

$$\pi^2(p^0p^0)^0_W = \tau^2,\ \pi^2(p^0p^1)^0_W = 0,\ \pi^2(p^1p^1)^0_W = \frac{1}{4}$$

and hence

$$\Big(\pi^2(p^0p^0)^0_W\Big)^\Delta_x = -\Delta_1 - (\rho|\rho),\ \Big(\pi^2(p^0p^1)^0_W\Big)^\Delta_x = 0,\ \Big(\pi^2(p^1p^1)^0_W\Big)^\Delta_x = \frac{1}{4}$$

Therefore

$$(p^0 \otimes \phi_0 + p^1 \otimes \phi_1|p^0 \otimes \phi_0 + p^1 \otimes \pi_1)_Z = \int\limits_{G/K} dz\Big(\overline{(\Delta - \rho|\rho)\phi_0(z)}\pi_0(z) + \overline{\phi_1(z)}\pi_1(z)\Big)$$

For each $p \in \mathcal{P}(\mathfrak{a}^\sharp)$ we obtain an operator L_p on $\mathcal{H}$ by putting

$$q|(L_p\psi) := (qp)|\psi$$

Lemma 97. *For each $p \in \mathcal{P}$ there is an operator $L_p \in \mathcal{L}(\mathcal{D})$ defined by*

$$q(L_pj) := (qp)|j$$

for all $p \in \mathcal{P}$.

Proof. Let $i \in \mathcal{P}^W$. Then

$$(iq)(L_pj) = ((iq)p)|j = (i(qp))|j = i^\Delta(qp|j) = i^\Delta(q(L_pj))$$

Therefore the linear functional L_qj on $\mathcal{P}$ is $\mathcal{P}^W$-linear and hence belongs to $\mathcal{D}$. □

4.5.3 solution space

The space $\mathcal{C}^\infty(\mathfrak{a})$ is a $\mathcal{P}(\mathfrak{a}^\sharp)$-module via

$$(p, q) \mapsto p^\partial q,$$

where for $p \in \mathcal{P}(\mathfrak{a}^\sharp)$ the constant coefficient differential operator $p^\partial \in \mathcal{D}(\mathfrak{a})$ is defined by

$$p^\partial e^{(\cdot|\lambda)} = \overline{p(\lambda)}\ e^{(\cdot|\lambda)} \quad (\lambda \in \mathfrak{a}^\sharp).$$

The **solution space** $\mathcal{L}$ consists of all smooth functions $u : \mathfrak{a} \times G/K \to \mathbf{C}$, more precisely, $\mathfrak{a} \overset{u}{\to} \mathcal{C}_c^\infty(G/K)$, which satisfy the **(hyperbolic) wave equation**

$$i^{\partial}_{t}u(t,x) = i^{\Delta}_{x}u(t,x)$$

for all $i \in \mathcal{P}(\mathfrak{a}^\sharp)^W$. Here $t \in \mathfrak{a}$ is regarded as multi-variable ’time’. Thus

$$\mathcal{L} = \{\mathfrak{a} \xrightarrow[\text{smooth}]{u} \mathcal{C}_c^\infty(G/K) :\ p^{\partial}u(t,x) = p^{\Delta}u(t,x)\ \forall\ p \in \mathcal{P}(\mathfrak{a}^\sharp)^W\}.$$

For a smooth function $u : \mathfrak{a} \times G/K \to \mathbf{C}$ a **jet** in the variable $t \in \mathfrak{a}$ is the linear functional ${}^{\partial}_{t}u : \mathcal{P} \to \mathcal{C}_c^\infty(G/K)$ defined by

$$p({}^{\partial}_{t}u) := p^{\partial}_{t}u$$

In order to express the wave equation we require that the jet j_t satisfies the covariance condition

$$(ip)j_t = i^{\Delta}_{x}(pj_t)$$

for all $i \in \mathcal{P}^W$.

Proposition 98. *There is a smooth map*

$$\mathfrak{a} \times \mathcal{L} \to \mathcal{D},\ (t,u) \mapsto {}^{\partial}_{t}u$$

defined by

$$q({}^{\partial}_{t}u) := p^{\partial}_{t}u.$$

This map satisfies the Cauchy problem

$$p^{\partial}_{t}({}^{\partial}_{t}u) = L_p({}^{\partial}_{t}u) \quad \forall\ p \in \mathcal{P}.$$

Proof. For each $i \in \mathcal{P}^W$ we have

$$(iq)({}^{\partial}_{t}u) = (iq)^{\partial}_{t}u = p^{\partial}_{t}(i^{\partial}_{t}u) = p^{\partial}_{t}(i^{\Delta}_{x}u) = i^{\Delta}_{x}(p^{\partial}_{t}u) = i^{\Delta}_{x}(q|{}^{\partial}_{t}u)$$

Therefore ${}^{\partial}_{t}u \in \mathcal{D}$. Moreover,

$$q(p^{\partial}_{t}({}^{\partial}_{t}u)) = p^{\partial}_{t}p^{\partial}u = (qp)^{\partial}_{t}u = (qp)(j_t u) = q(L_p({}^{\partial}_{t}u))$$

□

Let $\Phi : \mathfrak{a} \times G/K \to \mathbf{C}$ be a solution Define $\Phi_0 : \mathcal{P}(\mathfrak{a}^\sharp) \to \mathbf{C}$ by

$$q|\Phi_0 := p^{\partial}_{0}\Phi$$

Then

$$(jq)|\Phi_0 = (jq)^{\partial}_{0}\Phi = p^{\partial}_{0}\ j^{\partial}_{0}\Phi = p^{\partial}_{0}(j^{\Delta}\Phi) = j^{\Delta}(p^{\partial}_{0}\Phi) = j^{\Delta}(q|lF_0)$$

It follows that $\Phi_0 \in \mathcal{H}$.

In different notation,

$$j_t^{\partial}\Phi = j_x^{\Delta}\Phi$$

If $\Phi \in \tilde{\mathcal{H}}$, then $q^{\partial}\Phi \in \tilde{\mathcal{H}}$ for all $q \in \mathcal{P}(\mathfrak{a}^{\sharp})$ since

$$j^{\partial}(q^{\partial}\Phi) = q^{\partial}(j^{\partial}\Phi) = q^{\partial}(j^{\Delta}\Phi) = j^{\Delta}(q^{\partial}\Phi).$$

Thus $\tilde{\mathcal{H}}$ is a $\mathcal{P}(\mathfrak{a}^{\sharp})$-module. On the other hand, put

Consider the following Cauchy problem for smooth maps $\Psi : \mathfrak{a} \to \mathcal{H}$

$$(p_t^{\partial}\Psi)(q) = \Psi_t(pq), \quad \Psi_0 = \psi$$

Lemma 99. *The assignment*

$$\Psi(t,x)q := p_t^{\partial}\Phi(t,x)$$

defines an isomorphism $\mathcal{L} \to \mathcal{H}$.

Proof. We have for $j \in \mathcal{P}(\mathfrak{a}^{\sharp})^W$

$$\Psi(jq) = (jq)^{\partial}\Phi = q^{\partial}(j^{\partial}\Phi) = q^{\partial}(j^{\Delta}\Phi) = j^{\Delta}(q^{\partial}\Phi) = j^{\Delta}\Psi(q)$$

It follows that $\Psi_t \in \mathcal{H}$ for every $t \in \mathfrak{a}$. □

The geodesic flow $T^*X \times A \to T^*X$ defines **plane wave solutions**

$$\Phi^w_{-\tau,[k]}(t,[g]) = e^{t\tau}(g^{-1}k)_A^{w\tau-\rho} = e^{t\tau}(g^{w\tau}1)[k]$$

for each fixed $\tau \in i\mathfrak{a}^{\sharp}$ and $k \in K$. Thus $\Phi^w_{-\tau,[k]} \in \mathcal{L}$ and we may form the jet

$$^{\partial}_{0}\Phi^w_{-\tau,[k]} \in \mathcal{H}$$

at $0 \in \mathfrak{a}$.

Lemma 100. *For each $\tau \in i\mathfrak{a}^{\sharp}$ and $k \in K$ we have*

$$^{\partial}_{0}\Phi^w_{-\tau,[k]}[g] = \sum_{s\in W} p_s(\tau)\ \tilde{p}^{s*} \otimes (g^{w\tau}1)[k]$$

Proof. Let $p = \sum_s i^s\ p_s \in \mathcal{P}$. Then $(p|\tilde{p}^s) = i^s$ and therefore

$$p\Big({}^{\partial}_{0}\Phi^w_{-\tau,[k]}[g]\Big) = p_0^{\partial}\Big(\Phi^w_{-\tau,[k]}[g]\Big) = p_0^{\partial}e^{t\tau}(g^{w\tau}1)[k] = p(\tau)(g^{w\tau}1)[k] = \sum_{s\in W} p_s(\tau)i^s(\tau)(g^{w\tau}1)[k]$$

$$= \sum_{s\in W} p_s(\tau)(i^s)^{\Delta}(g^{w\tau}1)[k] = \sum_{s\in W} p_s(\tau)\ (p|\tilde{p}^s)^{\Delta}(g^{w\tau}1)[k] = p\Big(\sum_{s\in W} p_s(\tau)\ \tilde{p}^{s*} \otimes (g^{w\tau}1)[k]\Big)$$

□

Corollary 101. *Let* $y = \sum_{t\in W} \tilde{p}^t\ w_t$*, with* $w_t \in \mathcal{P}_W^+$*, and* $\psi \in \mathcal{C}_c^\infty(G/K)$*. Then*

$$\left({}_0^{\partial}\Phi^w_{-\tau,[k]} | w^* \otimes \psi\right)_Z = \sum_{s,t\in W} p_s(\tau) w_t(\tau)\ \ (\tilde{p}^s|\tilde{p}^t)^\Delta (g^{w\tau}1)[k] \otimes \psi[g]$$

Proof. Since $w_t^\Delta\ (g^{w\tau}1)[k] = w_t(\tau)(g^{w\tau}1)[k]$

$$\left({}_0^{\partial}\Phi^w_{-\tau,[k]} | w^* \otimes \psi\right)_Z = \sum_{s,t\in W} p_s(\tau)\ (\tilde{p}^s|\tilde{p}^t w_t)^\Delta\ (g^{w\tau}1)[k] \otimes \psi[g]$$

$$= \sum_{s,t\in W} p_s(\tau)\ (\tilde{p}^s|\tilde{p}^t)^\Delta w_t^\Delta\ (g^{w\tau}1)[k] \otimes \psi[g]$$

$$= \sum_{s,t\in W} p_s(\tau) w_t(\tau)\ \ (\tilde{p}^s|\tilde{p}^t)^\Delta (g^{w\tau}1)[k] \otimes \psi[g]$$

□

Define a **plane wave transform** $F_w : \mathcal{H} \to L^2(i\mathfrak{a}^\sharp, H)$ by

$$(F_w f)_{\tau,[k]} := (\iota_0\ \Phi^w_{\tau,[k]} | f)_Z$$

Then [TiSh/539] asserts for all $f \in \mathcal{H}$

$$\|f\|_E^2 = \int_{i\mathfrak{a}^\sharp} \frac{d\tau}{|\pi_\tau\ c_\tau|^2} \int_K dk\ |(F_w f)_{\tau,[k]}|^2.$$

Lemma 102. *For* $\psi \in L^2(K/K_A^\circ)$ *we have*

$$(1|a^{-\tau}\psi) = a^{\tau-\rho} \int_N dn\ (ana^{-1})_A^{\tau-\rho}\ n_A^{-\tau-\rho}\ \psi[(ana^{-1})_K]$$

Proof.

$$(1|g^{-\tau}\psi) = (g^\tau 1|\psi) = \int_{K/K_A^\circ} dk\ \overline{(g^\tau 1)[k]}\ \psi[k]$$

$$= \int_{K/K_A^\circ} dk\ \overline{(g^{-1}k)_A^{\tau-\rho}}\ \psi[k] = \int_{K/K_A^\circ} dk\ (g^{-1}k)_A^{-\tau-\rho}\ \psi[k].$$

In particular,

$$(1|a^{-\tau}\psi) = \int_{K/K_A^\circ} dk\ (a^{-1}k)_A^{-\tau-\rho}\ \psi[k] = \int_N dn\ n_A^{-2\rho}\ (a^{-1}n_K)_A^{-\tau-\rho}\ \psi[n_K]$$

$$= \int_N dn\ n_A^{-2\rho}\ (a^{-1}\cdot(a^{-1}na)_A \cdot n_A^{-1})^{-\tau-\rho}\ \psi[n_K] = a^{\tau+\rho}\int_N dn\ n_A^{\tau-\rho}\ (a^{-1}na)_A^{-\tau-\rho}\ \psi[n_K]$$

$$= a^{\tau-\rho}\int_N dn\ (ana^{-1})_A^{\tau-\rho}\ n_A^{-\tau-\rho}\ \psi[(ana^{-1})_K]$$

□

By [TiSh/542] we have the wave solutions

$$\tilde{\Psi}_t[g] = \int\limits_{i\mathfrak{a}^\sharp} \frac{d\tau}{|b(\tau)|^2}\ e^{-t\tau}\ (1|g^{-\tau}\Psi_\tau)$$

where $\Psi_\tau \in \mathcal{K}_R \subset \mathcal{C}_c^\infty(i\mathfrak{a}^\sharp \times K/K_A^\circ)$.

4.5.4 Outgoing representation and scattering operators

Proposition 103. *Let $g \in G'$. Then*

$$\lim_{s\to\infty} e^{s\rho}\ \tilde{\Psi}_{s+t}[kge^s] = \int\limits_{i\mathfrak{a}^\sharp} \frac{d\tau}{b_\Theta^\tau\ d_\Theta^\tau}\ e^{-t\tau}\Big(1|i^*((kg)^{-\tau}\frac{\Psi_\tau^+}{b(-\tau)})\Big)_{H_\Theta}$$

and, more generally,

$$\lim_{s\to\infty} e^{s\rho}\ p^\partial\tilde{\Psi}_{s+t}[kge^s] = \int\limits_{i\mathfrak{a}^\sharp} \frac{d\tau}{b_\Theta^\tau\ d_\Theta^\tau}\ p(\tau)\ e^{-t\tau}\Big(1|i^*((kg)^{-\tau}\frac{\Psi_\tau}{b(-\tau)})\Big)_{H_\Theta}$$

Proof. Let $N = \acute{N}\backslash N$ and write $n = \acute{n}\ \backslash n$. Then $e^s n e^{-s} = \acute{n}(e^s\backslash n e^{-s})$ and hence

$$(e^s n e^{-s})_A^{\tau-\rho}\ n_A^{-\tau-\rho} = \acute{n}_A^{\tau-\rho}(e^s\backslash n e^{-s})_A^{\tau-\rho}\acute{n}_A^{-\tau-\rho}\backslash n_A^{-\tau-\rho}$$

$$= \acute{n}_A^{-2\rho}(e^s\backslash n e^{-s})_A^{\tau-\rho}\backslash n_A^{-\tau-\rho} = \acute{n}_{\acute{A}}^{-2\acute{\rho}}(e^s\backslash n e^{-s})_A^{\tau-\rho}\backslash n_A^{-\tau-\rho},$$

where

$$\acute{\rho} := \frac{1}{2}\sum_{\alpha\in\langle\Theta\rangle_+} \alpha\ m_\alpha.$$

Thus Lemma **??** implies

$$e^{s\rho}\ e^{-(s+t)\tau}(1|(e^s)^{-\tau}\psi) = e^{s\rho}\ e^{-(s+t)\tau}\ e^{s(\tau-\rho)}\int\limits_N \frac{dn}{n_A^{\tau+\rho}}\ (e^s n e^{-s})_A^{\tau-\rho}\ \psi[(e^s n e^{-s})_K]$$

$$= e^{-t\tau}\int\limits_N \frac{dn}{n_A^{\tau+\rho}}\ (e^s n e^{-s})_A^{\tau-\rho}\ \psi[(e^s n e^{-s})_K] = e^{-t\tau}\int\limits_{\backslash N} \frac{d\backslash n}{\backslash n_A^{\tau+\rho}}\ (e^s\backslash n e^{-s})_A^{\tau-\rho}\int\limits_{\acute{N}} \frac{d\acute{n}}{\acute{n}_{\acute{A}}^{2\acute{\rho}}}\ \psi[(\acute{n}e^s\backslash n e^{-s})_K]$$

Let $w_* \in W$ satisfy $w_*\langle\Theta\rangle_+ = \langle\Theta\rangle_-$. Then $\backslash N = N \cap (k_{w_*}^{-1}\overline{N}k_{w_*})$. Hence

$$\int\limits_{\backslash N} \frac{d\backslash n}{\backslash n_A^{\tau+\rho}} = c_{w_*}^\tau = \frac{b(\tau)}{d_\Theta^\tau\ b_\Theta^\tau}$$

since $b(\tau) = \pi(\tau)c(\tau) = d_\Theta^\tau\ b_\Theta^\tau\ c_{w_*}^\tau$. Since $(\acute{n})_K = (\acute{n})_{\acute{K}}$ we have

$$\int\limits_{\acute{N}} \frac{d\acute{n}}{\acute{n}_{\acute{A}}^{2\acute{\rho}}}\ \psi[(\acute{n})_K] = \int\limits_{\acute{N}} \frac{d\acute{n}}{\acute{n}_{\acute{A}}^{2\acute{\rho}}}\ \psi[(\acute{n})_{\acute{K}}] = \int\limits_{\acute{K}/(\acute{K})_{\acute{A}}^\circ} d\acute{k}\ \psi[\acute{k}] = (1|i^*\psi)_{H_\Theta}.$$

Since $e^s \backslash n e^{-s} \to e$ it follows that

$$\lim_{s\to\infty} e^{s\rho}\ e^{-(s+t)\tau}(1|(e^s)^{-\tau}\psi) = e^{-t\tau} \int\limits_{\backslash N} \frac{d\backslash n}{\backslash n_A^{\tau+\rho}} \int\limits_{\acute{N}} \frac{d\acute{n}}{\acute{n}_{\acute{A}}^{2\acute{\rho}}}\ \psi[(\acute{n})_K] = \frac{b(\tau)}{d_\Theta^\tau\ b_\Theta^\tau}\ e^{-t\tau}\ (1|i^*\psi)_{H_\Theta}$$

Putting $\psi := (kg)^{-\tau}\frac{\Psi_\tau}{b(-\tau)}$ we obtain, using the dominated convergence theorem,

$$\lim_{s\to\infty} e^{s\rho}\ \tilde{\Psi}_{s+t}[kge^s] = \lim_{s\to\infty} e^{s\rho} \int\limits_{i\mathfrak{a}^\sharp} \frac{d\tau}{b(\tau)}\ e^{-(s+t)\tau}\Big(1|(kge^s)^{-\tau}\frac{\Psi_\tau}{b(-\tau)}\Big)_H$$

$$= \int\limits_{i\mathfrak{a}^\sharp} \frac{d\tau}{b(\tau)} \lim_{s\to\infty} e^{s\rho}\, e^{-(s+t)\tau}\Big(1|(e^s)^{-\tau}(kg)^{-\tau}\frac{\Psi_\tau}{b(-\tau)}\Big)_H = \int\limits_{i\mathfrak{a}^\sharp} \frac{d\tau}{d_\Theta^\tau\ b_\Theta^\tau}\ e^{-t\tau}\Big(1|i^*(kg)^{-\tau}\frac{\Psi_\tau}{b(-\tau)}\Big)_{H_\Theta}$$

$$= \int\limits_{i\grave{\mathfrak{a}}^\sharp} d\grave{\tau}\ e^{-\backslash t\grave{\tau}} \int\limits_{i\acute{\mathfrak{a}}^\sharp} \frac{d\acute{\tau}}{d_\Theta^\tau\ b_\Theta^\tau}\ e^{-\acute{t}\acute{\tau}}\Big(1|i^*g^{-\tau}k^{-\tau}\frac{\Psi_\tau}{b(-\tau)}\Big)_{H_\Theta}$$

$$= \int\limits_{i\grave{\mathfrak{a}}^\sharp} d\grave{\tau}\ e^{-\backslash t\grave{\tau}} \int\limits_{i\acute{\mathfrak{a}}^\sharp} \frac{d\acute{\tau}}{d_\Theta^\tau\ b_\Theta^\tau}\ e^{-\acute{t}\acute{\tau}}\Big(1|g^{-\tau'}i^*k^{-\tau}\frac{\Psi_\tau}{b(-\tau)}\Big)_{H_\Theta}$$

□

For $u \in \mathcal{L}$ consider the limit

$$(\overset{w}{\hat{W}}_\Theta u)_t^k[g'] = \lim_{s''\to\infty} e^{s''\rho} d_\Theta^\partial\ u_{w^{-1}(s''+t)}[kg'e^{s''}]$$

Then

$$(\hat{W}_\Theta\tilde{\Psi})_t^k[g'] = \lim_{s''\to\infty} e^{s''\rho} d_\Theta^\partial\ \tilde{\Psi}_{s''+t}[kg'e^{s''}]$$

$$= \int\limits_{i\mathfrak{a}^{\Theta\sharp}} d\tau''\ e^{-t''\tau''} \int\limits_{i\mathfrak{a}'^\sharp} \frac{d\tau'}{b_\Theta^\tau}\ e^{-t'\tau'}\Big(1|g^{-\tau'}i^*k^{-\tau}\frac{\Psi_\tau}{b(-\tau)}\Big)_{H_\Theta}$$

Consider the homogeneous vector bundle

$$K \times_{K_{\backslash A}^\circ} \mathcal{L}(\acute{\mathfrak{a}} \times \acute{G}/\acute{K}) = \{[k,u'] = [km'', (m'')^{-1}u'] :\ m'' \in K_{\backslash A}^\circ\}$$

Then

$$(\overset{w}{\hat{W}}_\Theta u)_t^{km''}[g'] = (\overset{w}{\hat{W}}_\Theta u)_t^k(m''[g'])$$

for all $m'' \in K_{\backslash A}^\circ$. Therefore

$$k \mapsto (\overset{w}{\hat{W}}_\Theta u)_t^k$$

induces a section of $K \times_{K_{\backslash A}^\circ} \mathcal{L}(\acute{\mathfrak{a}} \times \acute{G}/\acute{K})$.

By [T-S/546] we have the Plancherel formula

$$\|u\|_E^2 = \int\limits_{\grave{\mathfrak{a}}} d\backslash t \int\limits_{K/K_{\backslash A}^\circ} dk\ \|(\overset{w}{\hat{W}}_\Theta u)_{(-,\backslash t)}^k\|_{E_\Theta}^2$$

Thereofore

$$\mathcal{L}(\mathfrak{a} \times G/K) = L^2(\grave{\mathfrak{a}}) \otimes \Gamma^2(K \times_{K_{\backslash A}^\circ} \mathcal{L}(\acute{\mathfrak{a}} \times \acute{G}/\acute{K})$$

4.5.5 Paley-Wiener theorem

Put $H := L^2(K/K_A^\circ)$. The **Paley-Wiener theorem** states that

$$L^2(i\mathfrak{a}^\sharp, H) \xleftarrow[\text{isom}]{\Phi} L^2(\mathfrak{a}, H)$$

$$L^2_-(i\mathfrak{a}^\sharp, H) \xleftarrow[\text{isom}]{\Phi} L^2(\mathfrak{a}_-, H)$$

where

$$L^2_-(i\mathfrak{a}^\sharp, H) := \{\mathfrak{a}^\sharp_+ + i\mathfrak{a}^\sharp \xrightarrow[\text{hol}]{h} H : \sup_{\sigma \in \mathfrak{a}^\sharp_+} \int\limits_{i\mathfrak{a}^\sharp} d\tau \ \|h(\sigma+\tau)\|_H^2 < \infty\}$$

is the H-valued **Hardy space**.

Then

$$\|f\|^2 = \int\limits_{i\mathfrak{a}^\sharp} \frac{d\tau}{|b(\tau)|^2} \int\limits_{K/K_A^\circ} dk \ |F_w f(\tau, kK_A^\circ)|^2$$

Define an operator $\hat{W}^\Theta_w : \mathcal{H} \to \Gamma(\mathcal{H}_\Theta)$

$$\hat{W}^\Theta_w(t, gK', k) := \lim_{\mathfrak{a}^\Theta_+ \ni s \to +\infty} e^{\rho|s} \ d_\Theta(\partial_t) u(w^{-1}(s+t), kg \ e^s \ K)$$

This solves the wave equation for $(t', z_\Theta) \in \mathfrak{a}' \times G'/K'$.

$$u(t, gK) = \int\limits_{i\mathfrak{a}^\sharp} \frac{d\tau}{|b(\tau)|^2} \ e^{-t\tau} (1|g^{-\tau} a_\tau^+)_H$$

$$\begin{aligned}
&\lim_{\alpha \to +\infty} e^{\alpha s} p(\frac{\partial}{\partial s}) u(s+t, g \ e^{\alpha s} K) \\
&= \lim_{\alpha \to +\infty} e^{\alpha s} p(\frac{\partial}{\partial s}) \int\limits_{i\mathfrak{a}^\sharp} \frac{d\tau}{|b(\tau)|^2} \ e^{-(s+t)|\tau} \Big(1|(kge^{\alpha s})^{-\tau} a_\tau^+\Big)_H \\
&= \int\limits_{i\mathfrak{a}^\sharp} \frac{d\tau}{b_\Theta(\tau)} \frac{p(\tau)}{d_\Theta(\tau)} \ e^{-t\tau} \Big(1|i^* g^{-\tau} k^{-\tau} \frac{a_\tau^+}{b(-\tau)}\Big)_{H_\Theta}
\end{aligned}$$

Then there is a commuting diagram

$$\begin{tikzcd}
L^2(\mathfrak{a}, H) \arrow[r, "\Phi"'] & L^2(i\mathfrak{a}^\sharp, H) & L^2(i\mathfrak{a}^\sharp, H) \arrow[l, "b(-\lambda)^{-1}"] & \mathcal{H} \arrow[l, "F_1"] \arrow[lll, bend right, "\hat{W}_1"'] \\
L^2(\mathfrak{a}_-, H) \arrow[r, "\Phi"] \arrow[u] & L^2_-(i\mathfrak{a}^\sharp, H) \arrow[u] & L^2_-(i\mathfrak{a}^\sharp) \arrow[l, "b(-\lambda)^{-1}"'] & \mathcal{H}_- \arrow[l, "F_1"'] \arrow[u] \arrow[lll, bend left, "\hat{W}_1"]
\end{tikzcd}$$

Define

$$W_w := R_w^{-1} \circ \hat{W}_w$$

$$\mathcal{S}^w := W_+ W_w^{-1} = R_w^{-1} \hat{W}_+ \hat{W}_w^{-1} R_w$$

Then

$$\mathbf{S}^w := \Phi \mathcal{S}^w \Phi^{-1}$$

is a multiplication operator with multiplier

$$\frac{b(-w\lambda)}{b(-\lambda)} S_\lambda^w$$

For $s \in W_\Theta$ we have

$$\mathcal{S}^s = W_+ W_s^{-1} = {}^\Theta W_+ \ \hat{W}_+^\Theta \ (\hat{W}_+^\Theta)^{-1} \ W_s^{-1} = {}^\Theta W_+ \ {}^\Theta W_s^{-1}$$

4.5.6 Coordinates

For a finite linear group $W \subset \mathrm{GL}(V)$ the Shephard-Todd theorem states that W is generated by pseudo-reflections if and only if

$$K_W[V] = K[freefin]$$

is a finitely generated free polynomial algebra, if and only if

$$K[V] = K_W[V] < freefin >$$

is a finitely generated free module of rank $|W|$. Now consider a basis

$$\mathcal{P}(\mathfrak{a}^\sharp) = \mathcal{P}_W(\mathfrak{a}^\sharp) < h^\sigma : \ \sigma \in W >$$

of homogeneous harmonic polynomials $h^\sigma(\lambda)$ on $\mathfrak{a}^\sharp$. Then any $p(\lambda) \in \mathcal{P}(\mathfrak{a}^\sharp)$ has a unique decomposition

$$p(\lambda) = p_\sigma(\lambda) \ h^\sigma(\lambda)$$

where $p_\sigma \in \mathcal{P}_W(\mathfrak{a}^\sharp)$. Multiplying by the linear functional $t^*(\lambda) := (\lambda|t)$ for any $t \in \mathfrak{a}$ we also have

$$(\lambda|t)p(\lambda) = (t^*p)^\sigma(\lambda) \ h_\sigma(\lambda)$$

where

$$(t^*p)^\sigma = t_\rho^\sigma p^\rho$$

is a matrix of G-invariant differential operators on G/K. There is a matrix A_ρ^σ of invariant differential operators on G/K such that

$$(\phi|\psi) := \sum_{\sigma,\rho} \int\limits_{G/K} \overline{\phi}^\rho A_\rho^\sigma \psi^\sigma$$

defines an inner product on the Schwartz space $\mathcal{S}(G/K)^W \ni (\phi^\sigma), (\psi^\sigma)$. The Hilbert completion carries a unitary representation $\mathfrak{a} \ni t \mapsto U(t)$ of $\mathfrak{a}$. By definition,

$$(U(a)\phi^\sigma)(z) := \Phi(a,x)$$

for uniquely determined functions $\Phi^\sigma : \mathfrak{a} \times G/K \to \mathbf{C}$ satisfying the initial value problem

$$t^\partial \Phi^\sigma(a,x) = t^\sigma_\rho \Phi^\rho(a,x)$$

and the initial condition

$$\Phi^\sigma(0,x) = \phi^\sigma(z)$$

for all $a \in \mathfrak{a},\ z \in G/K$ and $\sigma \in W$. For each $\sigma \in W$ define

$$\mathfrak{a}_\sigma := \{t \in \mathfrak{a} :\ \alpha(\sigma t) > 0\ \forall\ t \in \mathfrak{a}_\sigma\}$$

and consider the σ-light cone

$$C_\sigma := \{(t, gK) \in \mathfrak{a} \times G/K :\ t - W(g^{-1} \subset \mathfrak{a}_\sigma\}.$$

Then

$$\mathcal{H}_\sigma := \{\phi \in \mathcal{S}(G/K)^{|W|} :\ \Phi|_{C_\sigma} = 0\}$$

are closed subspaces are pairwise orthogonal and satisfy the Laz-Phillips axioms

$$U(t)\mathcal{H}_\sigma \subset \mathcal{H}_\sigma \quad (t \in \mathfrak{a}_\sigma)$$

$$\bigcap_{t\in\mathfrak{a}} U(t)\mathcal{H}_\sigma = \{0\}$$

$$\bigcup_{t\in\mathfrak{a}} U(t)\mathcal{H}_\sigma \underset{\text{dense}}{\subset} \mathcal{S}(G/K)^{|W|}.$$

***

Proof. The conventional Fourier transform $L^2(V_{\mathbf{R}}) \to L^2(V_{\mathbf{R}})$, defined by

$$\hat{g}(\xi) = \int_{V_{\mathbf{R}}} dz\ e^{-2\pi i(x|\xi)}\ g(z),$$

satisfies

$$g(z) = \int_{V_{\mathbf{R}}} d\zeta\ e^{2\pi i(x|\xi)}\ \hat{g}(\xi).$$

Putting $x := y/(2\pi)$ and $g(z) := f(2\pi i x)$ we have

$$f^\bullet(\xi) = \frac{1}{(2\pi)^d} \int_{V_{\mathbf{R}}} dy\ e^{-iy\xi}\ f(iy) = \int_{V_{\mathbf{R}}} dz\ e^{-2\pi i z\xi}\ f(2\pi i x) = \int_{V_{\mathbf{R}}} dz\ e^{-2\pi i z\xi}\ g(z) = \hat{g}(\xi).$$

Therefore, using (**??**),

$$f(ix) = g\Big(\frac{x}{2\pi}\Big) = \int_{V_{\mathbf{R}}} d\zeta\ e^{2\pi i(x/(2\pi)|\xi)}\ \hat{g}(\xi) = \int_{V_{\mathbf{R}}} d\zeta\ e^{i(x|\xi)}\ \hat{g}(\xi) = \int_{V_{\mathbf{R}}} d\zeta\ e^{i(x|\xi)}\ f^\bullet(\xi).$$

Identifying $V_{\mathbf{R}} \approx V^\sharp_{\mathbf{R}}$ via an inner product, the assertion follows. $\square$

Consider the dual lattice

$$L^\sharp := \{\lambda \in V_{\mathbf{R}}^\sharp : \ (L|\lambda) \subset 2\pi i \mathbf{Z}\}.$$

Proposition 104. *We have the Poisson summation formula*

$$(2\pi)^{d/2}\ |L|^{1/2} \sum_{\ell \in L} f(\ell) = |L^\sharp|^{1/2} \sum_{\lambda \in L^\sharp} f^\bullet(\lambda)$$

Proof. The lattice $L/(2\pi i) \subset V_{\mathbf{R}}$ has the dual lattice $L^\sharp \subset V_{\mathbf{R}} \approx V_{\mathbf{R}}^\sharp$ in the usual sense. Moreover,

$$|L/2\pi i| = \operatorname{Vol}(V_{\mathbf{R}}/(L/(2\pi i))) = (2\pi)^d \operatorname{Vol}(iV_{\mathbf{R}}/L) = (2\pi)^d\ |L|.$$

Therefore the usual Poisson summation formula, applied to $g(z) := f(2\pi i x)$, yields

$$(2\pi)^{d/2}\ |L|^{1/2} \sum_{\ell \in L} f(\ell) = |L/(2\pi i)|^{1/2} \sum_{\ell \in L} g\Big(\frac{\ell}{2\pi i}\Big) = |L^\sharp|^{1/2} \sum_{\lambda \in L^\sharp} \hat{g}(\lambda) = |L^\sharp|^{1/2} \sum_{\lambda \in L^\sharp} f^\bullet(\lambda)$$

□

Remark 105. *Different authors have different conventions.*

- *Kudla, Bump:*

$$\Gamma\backslash G,\ G_{\mathbf{Q}}\backslash G_{\mathbf{A}}/K_{\mathbf{A}},\ D = G_{\mathbf{R}}/K_{\mathbf{R}}, G = NAK$$

- *Baily:*

$$G/\Gamma,\ K_{\mathbf{A}}\backslash G_{\mathbf{A}}/G_{\mathbf{Q}},\ D = K\backslash G, G = KAN$$