Reproducing kernel for a class of weighted Bergman spaces on the symmetrized polydisc

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Fall Eastern Sectional Meeting of the AMS
Ithaca, New York
September 09, 2011

Indian Institute of Science
Bangalore
(joint with S. Shyam Roy and G. Zhang)

The Bergman space on the symmetrized polydisc

The symmetrization map

The elementary symmetric function φ_i of degree $i(\geq 0)$ is the sum of all products of i distinct variables z_i so that $\varphi_0 = 1$ and

$$\varphi_i(z_1, \ldots, z_n) = \sum_{1 \leq k_1 < k_2 < \ldots < k_i \leq n} z_{k_1} \cdots z_{k_i}.$$

For $n \geq 1$, let $\mathbf{s} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the function of symmetrization given by the formula

$$\mathbf{s}(z_1, \ldots, z_n) = \big(\varphi_1(z_1, \ldots, z_n), \ldots, \varphi_n(z_1, \ldots, z_n)\big).$$

The image $\mathbb{G}_n := \mathbf{s}(\mathbb{D}^n)$ under the map $\mathbf{s}$ of the polydisc $\mathbb{D}^n := \{\boldsymbol{z} \in \mathbb{C}^n : \|\boldsymbol{z}\|_\infty < 1\}$ is the symmetrized polydisc. The restriction map $\mathbf{s}_{|\mathrm{res}\,\mathbb{D}^n} : \mathbb{D}^n \to \mathbb{G}_n$ is known to be a proper holomorphic map.

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Bergman kernel

The Bergman space $\mathbb{A}^2(\Omega)$ on any bounded domain $\Omega \subseteq \mathbb{C}^n$ is the Hilbert space of square integrable holomorphic functions on Ω. The Bergman kernel of the domain Ω is the reproducing kernel function of the Bergman space $\mathbb{A}^2(\Omega)$.

For the symmetrized polydisc $\mathbb{G}_n$, the Bergman kernel function can be computed explicitly using the formula available for the polydisc along with the transformation rule for the Bergman kernel under a proper holomorphic mapping. Here is an alternative approach:

Realize the Bergman space $\mathbb{A}^2(\mathbb{G}_n)$ of the symmetrized polydisc as a subspace of the Bergman space $\mathbb{A}^2(\mathbb{D}^n)$ on the polydisc using the symmetrization map $\mathbf{s}$.

Find a natural orthonormal basis for this subspace.

Compute the kernel function for the subspace (in closed form) as an infinite sum.

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the embedding

The map $\Gamma : \mathbb{A}^2(\mathbb{G}_n) \to \mathbb{A}^2(\mathbb{D}^n)$ defined by the formula

$$(\Gamma f)(\boldsymbol{z}) = (f \circ \mathbf{s})(\boldsymbol{z}) J_{\mathbf{s}}(\boldsymbol{z}),\ \boldsymbol{z} \in \mathbb{D}^n,$$

where $J_{\mathbf{s}}$ is the complex Jacobian of the map $\mathbf{s},$ is an isometry. Let $\mathbb{A}^2_{\text{anti}}(\mathbb{D}^n) \subseteq \mathbb{A}^2(\mathbb{D}^n)$ be the image $\operatorname{ran}\Gamma \subseteq \mathbb{A}^2(\mathbb{D}^n).$ It consists of anti-symmetric functions:

$$\operatorname{ran}\Gamma := \{f : f(\boldsymbol{z}_\sigma) = \operatorname{sgn}(\sigma) f(\boldsymbol{z}), \sigma \in \Sigma_n\,, f \in \mathbb{A}^2(\mathbb{D}^n)\},$$

where Σ_n is the symmetric group on n symbols. An orthonormal basis of $\mathbb{A}^2_{\text{anti}}(\mathbb{D}^n)$ may then be transformed in to an orthonormal basis of the $\mathbb{A}^2(\mathbb{G}_n)$ via the unitary map $\Gamma^*.$ Evaluating the sum

$$\sum_{k \geq 0} e_k(\boldsymbol{z})\overline{e_k(\boldsymbol{w})},\ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{G}_n,$$

for some choice of an orthonormal basis in $\mathbb{A}^2(\mathbb{G}_n),$ we obtain the Bergman kernel for $\mathbb{G}_n.$

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weighted Bergman spaces

This scheme works equally well for a class of weighted Bergman spaces $\mathbb{A}^{(\lambda)}(\mathbb{D}^n),\ \lambda > 1,$ determined by the kernel function

$$\mathbf{B}^{(\lambda)}_{\mathbb{D}^n}(\boldsymbol{z},\boldsymbol{w}) = \prod_{i=1}^{n}(1-z_i\bar{w}_i)^{-\lambda},\ \boldsymbol{z} = (z_1,\ldots,z_n),\ \boldsymbol{w} = (w_1,\ldots,w_n) \in \mathbb{D}^n,$$

defined on the polydisc and the corresponding weighted Bergman spaces $\mathbb{A}^{(\lambda)}(\mathbb{G}^n)$ on the symmetrized polydisc.

The limiting case of $\lambda = 1,$ as is well-known, is the Hardy space on the polydisc. We show that the Szeegö kernel for the symmetrized polydisc is of the form

$$\mathbb{S}^{(1)}_{\mathbb{G}_n}(\mathbf{s}(\boldsymbol{z}),\mathbf{s}(\boldsymbol{w})) = \prod_{i,j=1}^{n}(1-z_i\bar{w}_j)^{-1}, \boldsymbol{z},\boldsymbol{w} \in \mathbb{D}^n.$$

This shows that the Hardy kernel is not a power of the Bergman kernel unlike the case of bounded symmetric domains.

the weighted volume measure

For $\lambda > 1$, let $dV^{(\lambda)} := \left(\frac{\lambda-1}{\pi}\right)^n \Big(\prod_{i=1}^n (1-r_i^2)^{\lambda-2} r_i dr_i d\theta_i \Big)$ be a measure on the polydisc. Let $dV_{\mathbf{s}}^{(\lambda)}$ be the measure on the symmetrized polydisc $\mathbb{G}_n$ obtained by the change of variable formula:

$$\int_{\mathbb{G}_n} f\, dV_{\mathbf{s}}^{(\lambda)} = \int_{\mathbb{D}^n} (f \circ \mathbf{s})\, |J_{\mathbf{s}}|^2 dV^{(\lambda)},\ \lambda > 1$$

where $J_{\mathbf{s}}(\boldsymbol{z}) = \prod_{1 \le i < j \le n} (z_i - z_j)$ is the complex jacobian determinant of the symmetrization map $\mathbf{s}$.

The weighted Bergman space $\mathbb{A}^{(\lambda)}(\mathbb{G}_n), \lambda > 1,$ on the symmetrized polydisc $\mathbb{G}_n$ is the subspace of the Hilbert space $L^2(\mathbb{G}_n, dV_{\mathbf{s}}^{(\lambda)})$ consisting of holomorphic functions.

Here $dV_{\mathbf{s}}^{(\lambda)}$ is the measure $\|J_s\|_\lambda^{-2} dV_{\mathbf{s}}^{(\lambda)}$ and $\|J_s\|_\lambda$ denotes the norm of the function J_s in the Hilbert space $L^2(\mathbb{D}^n, dV^{(\lambda)})$. The norm of $f \in \mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ is given by $\|f\|^2 = \int_{\mathbb{G}_n} |f|^2 dV_{\mathbf{s}}^{(\lambda)}$.

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natural unitary map

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The operator Γ is an isometry.

Since $J_{\mathbf{s}}(\boldsymbol{z}_\sigma) = \operatorname{sgn}(\sigma) J_{\mathbf{s}}(\boldsymbol{z}), \sigma \in \Sigma_n$, the image of $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ under the isometry Γ in $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$ is a subspace of $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$ which is the space of anti-symmetric functions.

Pick g in $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$. Take $h = J_{\mathbf{s}}^{-1} g$ on the open set $\{(z_1, \ldots, z_n) \in \mathbb{D}^n : z_i \neq z_j,\, i \neq j\}$. It follows that $g = J_{\mathbf{s}}(f \circ \mathbf{s})$ for some function f defined on $\mathbb{G}_n$.

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module isomorphism

The subspace $\mathbb{A}^{(\lambda)}_{\mathrm{anti}}(\mathbb{D}^n)$ is invariant under the multiplication by the elementary symmetric functions. It therefore admits a module action via the map

$$(p, f) \mapsto p(\varphi_1, \ldots, \varphi_n) f, \; f \in \mathbb{A}^{(\lambda)}_{\mathrm{anti}}(\mathbb{D}^n), \; p \in \mathbb{C}[\boldsymbol{z}]$$

over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$.

The polynomial ring also acts naturally via point-wise multiplication on the Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$.

The unitary operator Γ intertwines the multiplication by the elementary symmetric functions on the Hilbert space $\mathbb{A}^{(\lambda)}_{\mathrm{anti}}(\mathbb{D}^n)$ with the multiplication by the co-ordinate functions on $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$.

Thus the two modules $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ and $\mathbb{A}^{(\lambda)}_{\mathrm{anti}}(\mathbb{D}^n)$ are isomorphic via the unitary map Γ. Moreover, $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ is contractive.

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Thus the two modules $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ and $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$ are isomorphic via the unitary map Γ. Moreover, $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ is contractive.

module isomorphism

The subspace $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$ is invariant under the multiplication by the elementary symmetric functions. It therefore admits a module action via the map

$$(p, f) \mapsto p(\varphi_1, \ldots, \varphi_n)f,\ f \in \mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n),\ p \in \mathbb{C}[\boldsymbol{z}]$$

over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$.

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Partitions

A partition $\boldsymbol{p}$ is any finite sequence $\boldsymbol{p} := (p_1, \ldots, p_n)$ of non-negative integers in decreasing order, that is,

$$p_1 \geq p_2 \geq \cdots \geq p_n.$$

Let $[n]$ denote the set of all partitions of size n. If a partition $\boldsymbol{p}$ also has the the property $p_1 > \cdots > p_n \geq 0$, then we may write $\boldsymbol{p} = \boldsymbol{m} + \boldsymbol{\delta}$, where $\boldsymbol{m}$ is some partition in $[n]$ and $\boldsymbol{\delta} = (n-1, n-2, \ldots, 1, 0)$. Let $[\![n]\!]$ be the set of all partitions of the form $\boldsymbol{m} + \boldsymbol{\delta}$ for $\boldsymbol{m} \in [n]$.

monomials

Let $\boldsymbol{z}^{\boldsymbol{m}} := z_1^{m_1} \cdots z_n^{m_n}$, $m \in [n]$, be a monomial. Consider the polynomial $a_{\boldsymbol{m}}$ obtained by anti-symmetrizing the monomial $\boldsymbol{z}^{\boldsymbol{m}}$:

$$a_{\boldsymbol{m}}(\boldsymbol{z}) := \sum_{\sigma \in \sum_n} \operatorname{sgn}(\sigma)\, \boldsymbol{z}^{\boldsymbol{m}_\sigma},$$

where $\boldsymbol{z}^{\boldsymbol{m}_\sigma} = z_1^{m_{\sigma(1)}} \cdots z_n^{m_{\sigma(n)}}$. Thus for any $\boldsymbol{p} \in \llbracket n \rrbracket$, we have

$$a_{\boldsymbol{p}}(\boldsymbol{z}) = a_{\boldsymbol{m}+\boldsymbol{\delta}}(\boldsymbol{z}) = \sum_{\sigma \in \sum_n} \operatorname{sgn}(\sigma)\, \boldsymbol{z}^{(\boldsymbol{m}+\boldsymbol{\delta})_\sigma},$$

$\boldsymbol{m} \in [n]$ and it follows that

$$a_{\boldsymbol{p}}(\boldsymbol{z}) = a_{\boldsymbol{m}+\boldsymbol{\delta}}(\boldsymbol{z}) = \det\left(\left(\!\left(z_i^{p_j} \right)\!\right)_{i,j=1}^{n} \right),\ \boldsymbol{p} \in \llbracket n \rrbracket.$$

orthogonality

Lemma. The set S $\{m_{\sigma(k)} - m'_{\nu(k)} : \sigma, \nu \in \Sigma_n, m_i > m_j, m'_i > m'_j$ for $i < j, m_1 \neq m'_1, 1 \leq k \leq n\}$ is not $\{0\}$.

It follows that the functions $a_{\boldsymbol{p}}, \boldsymbol{p} \in [\![n]\!]$ are orthogonal in the Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{D}^n)$. The norm of the vector $a_{\boldsymbol{p}}$ is easily calculated:

$$\begin{aligned} c_{\boldsymbol{p}}^{-1} := \|a_{\boldsymbol{p}}\|_{\mathbb{A}^{(\lambda)}(\mathbb{D}^n)} &= \left\| \det\left(\left((z_i^{p_j}) \right)_{i,j=1}^n \right) \right\|_{\mathbb{A}^{\lambda}(\mathbb{D}^n)} \\ &= \Big\| \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) \prod_{k=1}^n z_k^{p_{\sigma(k)}} \Big\|_{\mathbb{A}^{(\lambda)}(\mathbb{D}^n)} = \sqrt{\frac{n!\boldsymbol{p}!}{(\lambda)_{\boldsymbol{p}}}}. \end{aligned}$$

The vectors $a_{\boldsymbol{p}}$ span the subspace $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$ and therefore $\{e_{\boldsymbol{p}} = c_{\boldsymbol{p}}\, a_{\boldsymbol{p}} : \boldsymbol{p} \in [\![n]\!]\}$ is an orthonormal basis for $\mathbb{A}^{(\lambda)}_{\text{anti}}(\mathbb{D}^n)$

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the reproducing kernel

So the reproducing kernel $K_{\text{anti}}^{(\lambda)}$ for $\mathbb{A}_{\text{anti}}^{(\lambda)}(\mathbb{D}^n)$ is given by

$$K_{\text{anti}}^{(\lambda)}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\boldsymbol{p} \in [\![n]\!]} e_{\boldsymbol{p}}(\boldsymbol{z})\overline{e_{\boldsymbol{p}}(\boldsymbol{w})}, \text{ for } \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n.$$

For all $\sigma \in \Sigma_n$, we have $e_{\sigma(\boldsymbol{p})}(\boldsymbol{z})\overline{e_{\sigma(\boldsymbol{p})}(\boldsymbol{w})} = e_{\boldsymbol{p}}(\boldsymbol{z})\overline{e_{\boldsymbol{p}}(\boldsymbol{w})}, \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n.$ Therefore, it follows that

$$K_{\text{anti}}^{(\lambda)}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\boldsymbol{p} \in [\![n]\!]} e_{\boldsymbol{p}}(\boldsymbol{z})\overline{e_{\boldsymbol{p}}(\boldsymbol{w})} = \frac{1}{n!}\sum_{\boldsymbol{p} \geq 0} e_{\boldsymbol{p}}(\boldsymbol{z})\overline{e_{\boldsymbol{p}}(\boldsymbol{w})}, \tag{1}$$

where $\boldsymbol{p} \geq 0$ stands for all multi-indices $\boldsymbol{p} = (p_1, \ldots, p_n)$ with the property that each $p_i \geq 0$ for $1 \leq i \leq n$.

Proposition. The reproducing kernel $K_{\text{anti}}^{(\lambda)}$ is given explicitly by the formula:

$$K_{\text{anti}}^{(\lambda)}(\boldsymbol{z}, \boldsymbol{w}) = \frac{1}{n!}\det\left(\left((1 - z_j\bar{w}_k)^{-\lambda}\right)\right)_{j,k=1}^{n}, \ \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n.$$

Schur functions

The determinant function $a_{\boldsymbol{m}+\boldsymbol{\delta}}$ is divisible by each of the difference $z_i - z_j, \quad 1 \leq i < j \leq n$ and hence by the product $\prod_{1 \leq i < j \leq n}(z_i - z_j) = \det\left(\left(\!\left(z_i^{n-j}\right)\!\right)_{i,j=1}^{n}\right) = a_{\boldsymbol{\delta}}(\boldsymbol{z})$. The quotient $S_{\boldsymbol{p}} := a_{\boldsymbol{m}+\boldsymbol{\delta}}/a_{\boldsymbol{\delta}}, \ \boldsymbol{p} = \boldsymbol{m} + \boldsymbol{\delta}$ is therefore well-defined and is called the Schur function. The Schur function $S_{\boldsymbol{p}}$ is symmetric and defines a function on the symmetrized polydisc $\mathbb{G}_n$. Since the Jacobian of the map $\mathbf{s} : \mathbb{D}^n \to \mathbb{G}_n$ coincides with a_{δ}, it follows that the Schur functions $\{S_{\boldsymbol{p}} := a_{\boldsymbol{m}+\delta}/a_{\delta} : \boldsymbol{p} \in [\![n]\!]\}$ is a set of mutually orthogonal vectors in $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$. The linear span of these vectors is dense in $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$.

Also, the norms of these vectors coincide with those of $a_{\boldsymbol{p}}$ in $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$, modulo the normalizing constant $\|J_{\mathbf{s}}\|_{\lambda}$, via the unitary map Γ. Hence the set $\{\hat{e}_{\boldsymbol{p}} = c_{\boldsymbol{p}} S_{\boldsymbol{p}} : \boldsymbol{p} \in [\![n]\!]\}$ is an orthonormal basis for $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$, where $c_{\boldsymbol{p}} = \sqrt{\frac{\|J_{\mathbf{s}}\|_{\lambda} (\lambda)_{\boldsymbol{p}}}{n!\boldsymbol{p}!}}$.

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the kernel function for $\mathbb{G}_n$

Theorem. The reproducing kernel $\mathbf{B}^{(\lambda)}_{\mathbb{G}_n}$ for the weighted Bergman space $\mathbb{A}^{(\lambda)}(\mathbb{G}_n)$ on the symmetrized poly-disc is given by the formula:

$$
\begin{aligned}
\mathbf{B}^{(\lambda)}_{\mathbb{G}_n}(\mathbf{s}(\boldsymbol{z}), \mathbf{s}(\boldsymbol{w})) &= \boldsymbol{p} \in [\![n]\!] c_{\boldsymbol{p}}^2 \, S_{\boldsymbol{p}}(\boldsymbol{z}) \overline{S_{\boldsymbol{p}}(\boldsymbol{w})} \\
&= \frac{\|J_{\mathbf{s}}\|^2_\lambda}{n!} \frac{\det\big(\big((1 - z_j \bar{w}_k)^{-\lambda}\big)\big)_{j,k=1}^n}{a_{\boldsymbol{\delta}}(\boldsymbol{z}) \overline{a_{\boldsymbol{\delta}}(\boldsymbol{w})}}
\end{aligned}
$$

for $\boldsymbol{z}, \boldsymbol{w}$ in $\mathbb{D}^n$.

The Hardy space

Let $d\Theta$ be the normalized Lebesgue measure on the torus $\mathbb{T}^n$. The Hardy space $H^2(\mathbb{G}_n)$ on the symmetrized polydisc $\mathbb{G}_n$ consists of holomorphic functions on $\mathbb{G}_n$ with the property:

$$\|f\| = \|J_{\boldsymbol{s}}\|^{-1}\Big\{\sup_{0<r<1}\int_{\mathbb{T}^n}|f\circ \mathbf{s}(r\,e^{i\Theta})|^2|J_{\mathbf{s}}(r\,e^{i\Theta})|^2 d\Theta\Big\} < \infty,$$

where $\|J_{\boldsymbol{s}}\|^2 = \int_{\mathbb{T}^n}|J_{\mathbf{s}}|^2 d\Theta$ ensuring $\|1\| = 1$.

As before, the operator $\Gamma : H^2(\mathbb{G}_n) \longrightarrow H^2(\mathbb{D}^n)$ given by $\Gamma(f) = \|J_{\boldsymbol{s}}\|^{-1} J_{\mathbf{s}}\,(f\circ \mathbf{s})$ for $f \in H^2(\mathbb{G}_n)$ is an isometry.

The subspace of anti-symmetric functions $H^2_{\text{anti}}(\mathbb{D}^n)$ in the Hardy space $H^2(\mathbb{D}^n)$ coincides with the image of $H^2(\mathbb{G}_n)$ under the isometry Γ. Thus the operator $\Gamma : H^2(\mathbb{G}_n) \longrightarrow H^2_{\text{anti}}(\mathbb{D}^n)$ is onto and therefore unitary.

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orthonormal basis

The functions $a_{\boldsymbol{p}}, \boldsymbol{p} \in [\![n]\!]$ continue to be an orthogonal spanning set for the subspace $H^2_{\text{anti}}(\mathbb{D}^n)$. Now, all of the vectors $a_{\boldsymbol{p}}$ have the same norm, namely, $\sqrt{n!}$.

Consequently, the set of vectors $\{e_{\boldsymbol{p}}(\boldsymbol{z}) := \frac{1}{\sqrt{n!}} a_{\boldsymbol{p}}(\boldsymbol{z}) : \boldsymbol{p} \in [\![n]\!]\}$ is an orthonormal basis for the subspace $H^2_{\text{anti}}(\mathbb{D}^n)$ of the Hardy space on the polydisc, while the set $\{\hat{e}_{\boldsymbol{p}} := \frac{\|J_{\boldsymbol{s}}\|}{\sqrt{n!}} S_{\boldsymbol{p}} : \boldsymbol{p} \in [\![n]\!]\}$ forms an orthonormal basis for the Hardy space $H^2(\mathbb{G}_n)$ of the symmetrized polydisc $\mathbb{G}_n$ via the unitary map Γ.

However, $\|J_{\boldsymbol{s}}\| = \sqrt{n!}$ and therefore,]hl$\hat{e}_{\boldsymbol{p}} = S_{\boldsymbol{p}}$. Thus computations similar to the case $\lambda > 1$ yields an explicit formula for the reproducing kernel $K^{(1)}_{\text{anti}}(\boldsymbol{z}, \boldsymbol{w})$ of the subspace $H^2_{\text{anti}}(\mathbb{D}^n)$. Indeed,

$$K^{(1)}_{\text{anti}}(\boldsymbol{z}, \boldsymbol{w}) = \frac{1}{n!} \det\left(\left((1 - z_j \bar{w}_k)^{-1}\right)\right)^n_{j,k=1}.$$

This is the limiting case, as $\lambda \to 1$.

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the Szegö Kernel

Let $\mathbb{S}_{\mathbb{G}_n}$ be the Szegö kernel for the symmetrized polydisc $\mathbb{G}_n$. Clearly,

$$\mathbb{S}_{\mathbb{G}_n}(\mathbf{s}(\boldsymbol{z}), \mathbf{s}(\boldsymbol{w})) = \frac{\det\left(\left((1 - z_j \bar{w}_k)^{-1}\right)\right)_{j,k=1}^{n}}{J_{\mathbf{s}}(\boldsymbol{z})\overline{J_{\mathbf{s}}(\boldsymbol{w})}}, \; \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n.$$

Now, using the well-known identity due to Cauchy, we have

$$\begin{aligned} \mathbb{S}_{\mathbb{G}_n}(\mathbf{s}(\boldsymbol{z}), \mathbf{s}(\boldsymbol{w})) &= \sum_{\boldsymbol{p} \in [\![n]\!]} S_{\boldsymbol{p}}(\boldsymbol{z})\overline{S_{\boldsymbol{p}}(\boldsymbol{w})} \\ &= \prod_{j,k=1}^{n} (1 - z_j \bar{w}_k)^{-1}, \; \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n. \end{aligned}$$

Therefore, we have a formula for the Szegö kernel of the symmetrized polydisc $\mathbb{G}_n$.

Theorem. The Szegö kernel $\mathbb{S}_{\mathbb{G}_n}$ of the symmetrized polydisc $\mathbb{G}_n$ is given by the formula

$$\mathbb{S}_{\mathbb{G}_n}(\mathbf{s}(\boldsymbol{z}), \mathbf{s}(\boldsymbol{w})) = \prod_{j,k=1}^{n} (1 - z_j \bar{w}_k)^{-1}, \boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n.$$

Thank you!