A sheaf model for semi-Fredholm Hilbert modules

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Motivation

The Cowen - Douglsa class

A Hilbert module over the polynomial ring $\mathbb{C}[\underline{z}] := \mathbb{C}[z_1, \ldots, z_m]$ is a Hilbert space $\mathcal{H}$ which is a $\mathbb{C}[\underline{z}]$ -module with the assumption

$$\|p \cdot f\| \leq C_p \|f\|, \quad f \in, \quad p \in \mathbb{C}[\underline{z}],$$

for some $C_p > 0$.

The multiplication M_j by the complex variable z_j, $M_j f = z_j \cdot f$, $1 \leq j \leq m$, then defines a commutative tuple $M = (M_1, ..., M_m)$ of linear bounded operators acting on ″ and vice-versa.

A Hilbert module $\mathcal{H}$ over the polynomial ring $\mathbb{C}[\underline{z}]$ is said to be in the Cowen-Douglas class $B_n(\Omega)$, $n \in \mathbb{N}$, if

$\dim \mathcal{H}/m_w\mathcal{H} = n < \infty$ for all $w \in \Omega$

$\cap_{w \in \Omega} m_w \mathcal{H} = \{0\}$, where m_w denotes the maximal ideal in

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Cowen and Douglas prove that isomorphic Hilbert modules correspond to equivalent vector bundles and vice-versa.

Also, they provide a model for the Hilbert modules in $B_n(\Omega)$. Cowen and Douglas (Curto and Salinas, in general) show that these modules can be realized as a Hilbert space consisting of holomorphic functions on Ω possessing a reproducing kernel. The module action is then simply the pointwise multiplication.

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Not an example!

However, many natural examples of Hilbert modules fail to be in the class $B_n(\Omega)$.

For instance, $H^2_0(\mathbb{D}^2) := \{f \in H^2(\mathbb{D}^2) : f(0) = 0\}$ is not in $B_n(\mathbb{D}^2)$.

The problem is that the dimension of the joint kernel

$$\mathcal{H}/\mathfrak{m}_w\mathcal{H} \cong \cap_{j=0}^{m}\mathrm{Ker}(M_j - w_j)^*$$

is no longer a constant.

Indeed, we have (an easy calculation)

$$\dim\left(\mathcal{H}/\mathfrak{m}_w\mathcal{H}\right) = \begin{cases} 1 & \text{if } w \neq (0,0) \\ 2 & \text{if } w = (0,0). \end{cases}$$

We outline an attempt to systematically study examples like the one given above using methods of complex analytic geometry.

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What about the kernel?

The computation of the dimension of the joint kernel for the module $\mathrm{H}_0^2(\mathbb{D}^2)$ serves another purpose as well.

It shows that the module $\mathrm{H}_0^2(\mathbb{D}^2)$ is not equivalent to the usual Hardy module. The dimension of the joint kernel for the Hardy module is 1 everywhere on the bi-disc.

This is a gennuine multi-variate phenomenon – for the unit disc, the Hardy module is equivalent to all its sub-modules.

Clearly, the dimension of the joint kernel is an important unitary invaraint for a module. However, in many instances, calculating this dimension, or other numerical invariants is possible only after determining the kernel itself.

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Definitions

A Hilbert module $\mathcal{M} \subset \mathcal{O}(\Omega)$ is said to be in the class $\mathfrak{B}_1(\Omega)$ if

it possesses a reproducing kernel K (we don't rule out the possibility: $K(w, w) = 0$ for w in some closed subset X of Ω) and

The dimension of $\mathcal{M}/\mathfrak{m}_w\mathcal{M}$ is finite for all $w \in \Omega$.

Most of the examples in $\mathfrak{B}_1(\Omega)$ arises in the form of a submodule of some Hilbert module $\mathcal{H}(\subseteq \mathcal{O}(\Omega))$ in the Cowen-Douglas class $B_1(\Omega)$.

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A couple of questions

Let $\mathcal{M} \in \mathfrak{B}_1(\Omega)$ be a Hilbert module and $\mathcal{I} \subseteq \mathcal{M}$ be a polynomial ideal. Assume without loss of generality that $0 \in V(\mathcal{I})$. Now, we ask

if there exists a set of polynomials $p_1, \ldots, p_t$ such that

$$p_i(\tfrac{\partial}{\partial \bar{w}_1}, \ldots, \tfrac{\partial}{\partial \bar{w}_m})K_{[\mathcal{I}]}(z, w)|_{w=0}, \; i = 1, \ldots, t,$$

spans the joint kernel of $[\mathcal{I}]$;

what conditions, if any, will ensure that the polynomials $p_1, \ldots, p_t$, as above, is a generating set for $\mathcal{I}$?

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Relation between $\mathfrak{B}_1(\Omega)$ and $\mathrm{B}_1(\Omega)$

The following Lemma isolates a very large class of elements from $\mathfrak{B}_1(\Omega)$ which belong to $\mathrm{B}_1(\Omega_0)$ for some open subset $\Omega_0 \subseteq \Omega$.

Lemma. Suppose $\mathcal{M} \in \mathfrak{B}_1(\Omega)$ is the closure of a polynomial ideal $\mathcal{I}$. Then $\mathcal{M}$ is in $\mathrm{B}_1(\Omega)$ if the ideal $\mathcal{I}$ is singly generated while if it is generated by the polynomials $p_1, p_2, \ldots, p_t$, then $\mathcal{M}$ is in $\mathrm{B}_1(\Omega \setminus X)$ for $X = \{z : p_1(z) = \ldots = p_t(z) = 0\}$.

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The sheaf model

Construction of the sheaf model

Following the correspondence of a vector bundle with a locally free sheaf, we construct a sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ for the Hilbert module $\mathcal{M}$.

The sheaf $\mathcal{S}^{\mathcal{M}}$ is the subsheaf of the sheaf of holomorphic functions $\mathcal{O}(\Omega)$ whose stalk $\mathcal{S}^{\mathcal{M}}_w$ at $w \in \Omega$ is

$$\{(f_1)_w \mathcal{O}_w + \cdots + (f_n)_w \mathcal{O}_w : f_1, \ldots, f_n \in \mathcal{M}\}$$

For any Hilbert module $\mathcal{M}$ in $\mathfrak{B}_1(\Omega)$, the sheaf $\mathcal{S}^{\mathcal{M}}$ is coherent.

This is essentially Noether's stationary lemma!

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Theorem. Suppose $g_i^0, 1 \leq i \leq d,$ be a minimal set of generators for the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$. Then there exists a open neighborhood Ω_0 of w_0 such that

$$K(\cdot, w) := K_w = g_1^0(w)K_w^{(1)} + \cdots + g_n^0(w)K_w^{(d)}, \; w \in \Omega_0$$

for some choice of anti-holomorphic functions $K^{(1)}, \ldots, K^{(d)} : \Omega_0 \to \mathcal{M}$,

the vectors $K_w^{(i)}, 1 \leq i \leq d$, are linearly independent in $\mathcal{M}$ for w in Ω_0

the vectors $\{K_{w_0}^{(i)} \mid 1 \leq i \leq d\}$ are uniquely determined by these generators $g_1^0, \ldots, g_d^0$,

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We point out that the linear span of the set of vectors $\{K_{w_0}^{(i)} \mid 1 \le i \le d\}$ in $\mathcal{M}$ is independent of the generators $g_1^0, \ldots, g_d^0$,

and that the vectors $K_{w_0}^{(i)}, 1 \le i \le d$, are eigenvectors for the adjoint of the action of $\mathbb{C}[z]$ on the Hilbert module $\mathcal{M}$ at w_0.

Key ingredients in the proof are the following observations.

There is a decomposition for a function in any submodule of $\mathcal{O}_{w_0}$ in terms of its generators valid over a small neighbourhood of w_0.

The coefficients in this decomposition satisfy uniform norm bounds in a even smaller compact neighbourhood of w_0.

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We point out that the linear span of the set of vectors $\{K_{w_0}^{(i)} \mid 1 \leq i \leq d\}$ in $\mathcal{M}$ is independent of the generators $g_1^0, \ldots, g_d^0$,

and that the vectors $K_{w_0}^{(i)}, 1 \leq i \leq d$, are eigenvectors for the adjoint of the action of $\mathbb{C}[\underline{z}]$ on the Hilbert module $\mathcal{M}$ at w_0.

Key ingredients in the proof are the following observations.

There is a decomposition for a function in any submodule of $\mathcal{O}_{w_0}$ in terms of its generators valid over a small neighbourhood of w_0.

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One easy consequence of the decomposition theorem is the inequality

$$\begin{aligned}\dim\ker \mathrm{D}_{(\mathrm{M}-\mathrm{w}_0)^*} &\geq \sharp\{\text{minimal generators for } \mathrm{S}^{\mathcal{M}}_{\mathrm{w}_0}\}\\ &\geq \dim \mathcal{S}^{\mathcal{M}}_{\mathrm{w}_0}/\mathfrak{m}(\mathcal{O}_{\mathrm{w}_0})\mathcal{S}^{\mathcal{M}}_{\mathrm{w}_0}.\end{aligned}$$

One of the basic question is to ask if we have equality under additional hypothesis on the Hilbert module $\mathcal{M}$.

Thus assuming $\mathcal{M}$ to be an analytic Hilbert module then Chen and Guo have shown that equality is forced.

We show that this property continues to hold for submodules of analytic Hilbert modules.

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The $H_0^2(\mathbb{D}^2)$ example, again!

In the example of the module $H_0^2(\mathbb{D}^2)$, we have

$$\mathcal{S}_w^{H_0^2(\mathbb{D}^2)} = \begin{cases} \mathcal{O}_w & \text{if } w \neq (0,0) \\ \mathfrak{m}_{(0,0)}\mathcal{O}_{(0,0)} & \text{if } w = (0,0). \end{cases}$$

While the germs of holomorphic function $\mathcal{O}_w$ at $w \in \mathbb{D}^2$ is singly genarated (even if $w = (0,0)$), the ideal $\mathfrak{m}_{(0,0)}\mathcal{O}_{(0,0)} \subseteq \mathcal{O}_{(0,0)}$ is 2 - generated.

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A Corollary

Corollary. If $\mathcal{M} = [\mathcal{I}]$ be a submodule of an analytic Hilbert module over $\mathbb{C}[\underline{z}]$, where $\mathcal{I}$ is an ideal in the polynomial ring $\mathbb{C}[\underline{z}]$ and $\mathrm{w} \in V(\mathcal{I})$ is a smooth point, then

$$\dim \ker D_{(M-w)^*} = \begin{cases} 1 & \text{for } w \notin V(\mathcal{I}) \cap \Omega; \\ \text{codimension of } V(\mathcal{I}) & \text{for } w \in V(\mathcal{I}) \cap \Omega. \end{cases}$$

The joint kernel of a Hilbert module

Let $\mathcal{J}$ be an ideal in the polynomial ring $\mathbb{C}[\underline{z}]$.

The characteristic space of an ideal $\mathcal{J}$ in $\mathbb{C}[\underline{z}]$ at the point w is the vector space

$$\mathbb{V}_w(\mathcal{J}) := \{q \in \mathbb{C}[\underline{z}] : q(D)p|_w = 0,\ p \in \mathcal{J}\}.$$

The envolope $\mathcal{J}^e_w$ of the ideal $\mathcal{J}$ is

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If the zero set of the ideal $\mathcal{J}$ is $\{w\}$ then $\mathcal{J}^e_w = \mathbb{V}_w(\mathcal{J})$.

This describes an ideal by prescribing conditions on derivatives. We stretch this a little more.

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An auxiliary space

Let $\tilde{\mathbb{V}}_w(\mathcal{I})$ be the auxiliary space $\mathbb{V}_w(\mathfrak{m}_w\mathcal{I})$. Then we have

$$\dim \cap \mathrm{Ker}(M_j - w_j)^* = \dim \tilde{\mathbb{V}}_w(\mathcal{I})/\mathbb{V}_w(\mathcal{I}).$$

Actually, we have something much more substantial.

Lemma. Fix $w_0 \in \Omega$ and polynomials $q_1, \ldots, q_t$. Let $\mathcal{I}$ be a polynomial ideal and K be the reproducing kernel corresponding the Hilbert module $[\mathcal{I}]$, which is assumed to be in $\mathfrak{B}_1(\Omega)$. Then the vectors

$$q_1(\bar{D})K(\cdot, w)|_{w=w_0}, \ldots, q_t(\bar{D})K(\cdot, w)|_{w=w_0}$$

form a basis of the joint kernel $\cap_{j=1}^m \ker(M_j - w_{0j})^*$ if and only if the classes $[q_1^*], \ldots, [q_t^*]$ form a basis of $\tilde{\mathbb{V}}_{w_0}(\mathcal{I})/\mathbb{V}_{w_0}(\mathcal{I})$.

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A canonical set of generators

Theorem. Let $\mathcal{I} \subset \mathbb{C}[\underline{z}]$ be a homogeneous ideal and $\{p_1, \ldots, p_v\}$ be a minimal set of generators for $\mathcal{I}$ consisting of homogeneous polynomials. Let K be the reproducing kernel corresponding to the Hilbert module $[\mathcal{I}]$, which is assumed to be in $\mathfrak{B}_1(\Omega)$. Then there exists a set of generators $q_1, ..., q_v$ for the ideal $\mathcal{I}$ such that the set

$$\{q_i(\bar{D})K(\cdot, w)|_{w=0} : 1 \leq i \leq v\}$$

is a basis for $\cap_{j=1}^{m} \ker M_j^*$.

We note that the new set $\{q_1, \ldots, q_v\}$ of generators for $\mathcal{I}$ is more or less "canonical". It is uniquely determined modulo a linear transformation as shown below.

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An Example

Let $\mathcal{I} \subset \mathbb{C}[z_1, z_2]$ be the ideal generated by $z_1 + z_2$ and z_2^2. We have $V(\mathcal{I}) = \{0\}$. The reproducing kernel K for $[\mathcal{I}] \subseteq H^2(\mathbb{D}^2)$ is

$$\begin{aligned} K_{[\mathcal{I}]}(z, w) &= \frac{1}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} - \frac{(z_1 - z_2)(\bar{w}_1 - \bar{w}_2)}{2} - 1 \\ &= \frac{(z_1 + z_2)(\bar{w}_1 + \bar{w}_2)}{2} + i + j \geq 2^{\infty} z_1^i z_2^j \bar{w}_1^i \bar{w}_2^j. \end{aligned}$$

The vector $\bar{\partial}_2^2 K_{[\mathcal{I}]}(z, w)|_0 = 2z_2^2$ is not in the joint kernel of $P_{[\mathcal{I}]}(M_1^*, M_2^*)|_{[\mathcal{I}]}$ since $M_2^*(z_2^2) = z_2$ and $P_{[\mathcal{I}]} z_2 = (z_1 + z_2)/2 \neq 0$.

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Let $\mathcal{I}$ be the ideal generated by $z_1 + z_2$ and z_2^2 and $\tilde{\mathcal{I}}$ be the ideal generated by z_1 and z_2^2. Since z_1 is not a linear combination of $z_1 + z_2$ and z_2^2, it follows that $\mathcal{I} \neq \tilde{\mathcal{I}}$.

Indeed, our Theorem provides an effective tool for deciding when an ideal is a monomial ideal.

Let $\{q_1, \ldots, q_v\}$ be a canonical set of generators for $\mathcal{I}$. Let Λ be the collection of monomials in the expressions of $\{q_1, \ldots, q_v\}$ that are in $\mathcal{I}$. If the number of algebraically independent monomials in Λ is v, then $\mathcal{I}$ is a monomial ideal.

However, we have $q_1 = z_1 + z_2$ and $q_2 = (z_1 - z_2)^2$ and they generate the ideal $\mathcal{J}$ as well. Moreover, $\{(\bar{\partial}_1 + \bar{\partial}_2)K(\cdot, w)|_0, (\bar{\partial}_1 - \bar{\partial}_2)^2 K(\cdot, w)|_0\}$ forms a basis of the joint kernel.

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New Invariants

Let $\mathbb{P}_0$ be the orthogonal projection onto the joint kernel $\mathcal{M}/\mathfrak{m}_{w_0}\mathcal{M}$

Lemma. The dimension of $\ker \mathbb{P}_0\big(\mathcal{M}/\mathfrak{m}_w\mathcal{M}\big)$ is constant in a suitably small neighbourhood Ω_0 of $w_0 \in \Omega$.

Thus

$$\mathcal{P}^{\mathcal{M}}_{w_0} := \{(w, f) \in \Omega \times \mathcal{M} : f \in \ker \mathbb{P}_0 D_{(M-w)^*}\} \text{ and } \pi(w, f) = w$$

may possibly define a holomorphic Hermitian vector bundle on the open set Ω_0.

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Existence of holomorphic structure

Existence of the operator $R_M(w)$ satisfying

$$\begin{aligned} R_M(w) D_{(M-w)^*} &= I - P_{\ker D_{(M-w)^*}} \\ D_{(M-w)^*} R_M(w) &= P_{\operatorname{ran} D_{(M-w)^*}} \end{aligned}$$

on Ω_0 is established.

(Here, $D_{(M-w)^*} : \mathcal{M} \to \mathcal{M} \oplus \cdots \oplus \mathcal{M}$ is the operator $f \mapsto ((M_1 - w_1)^* f, \ldots, (M_m - w_m)^* f)$)

Then the operator

$$P(\bar{w}, \bar{w}_0) = I - \{I - R_M(w_0) D_{\bar{w} - \bar{w}_0}\}^{-1} R_M(w_0) D_{(M-w)^*},$$

is clearly seen to be well-defined and holomorphic for $w \in B(w_0; \| R(w_0) \|^{-1})$

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Theorem. If any two Hilbert modules $\mathcal{M}$ and $\tilde{\mathcal{M}}$ from $\mathfrak{B}_1(\Omega)$ are equivalent, then the corresponding holomorphic Hermitian vector bundles $\mathcal{P}_{w_0}^{\mathcal{M}}$ and $\mathcal{P}_{w_0}^{\tilde{\mathcal{M}}}$, they determine on Ω_0 are equivalent.

Examples, calculation of the invariant

For $\lambda, \mu > 0$, let $K^{(\lambda,\mu)}$ denote the positive definite kernel $\frac{1}{(1-z_1\bar{w}_1)^{\lambda}(1-z_2\bar{w}_2)^{\mu}}$, $z, w \in \mathbb{D}^2$ on the bi-disc. Let $H_0^{(\lambda,\mu)}(\mathbb{D}^2) := \{f \in H^{(\lambda,\mu)}(\mathbb{D}^2) : f(0,0) = 0\}$ be the corresponding Hilbert module in $\mathfrak{B}_1(\mathbb{D}^2)$. The normalized metric $h_0(w, w)$, which is real analytic, is of the form

$$h_0(w,w) = I + \begin{pmatrix} \frac{\lambda+1}{2}|w_1|^2 + \frac{\lambda^2\mu}{(\lambda+\mu)^2}|w_2|^2 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 w_1\bar{w}_2 \\ \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 w_2\bar{w}_1 & \frac{\lambda\mu^2}{(\lambda+\mu)^2}|w_1|^2 + \frac{\mu+1}{2}|w_2|^2 \end{pmatrix}$$

$$+O(|w|^3),$$

where $O(|w|^3)_{i,j}$ is of degree ≥ 3.

The final outcome of these calculations

The curvature for $\mathcal{P}$ at $(0,0)$ is given by the 2×2 matrices

$$\begin{pmatrix} \frac{\lambda+1}{2} & 0 \\ 0 & \frac{\lambda\mu^2}{(\lambda+\mu)^2} \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 \\ 0 & 0 \end{pmatrix},$$

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$\mathrm{H}_0^{(\lambda,\mu)}(\mathbb{D}^2)$ and $\mathrm{H}_0^{(\lambda',\mu')}(\mathbb{D}^2)$ are equivalent if and only if $\lambda = \lambda'$ and $\mu = \mu'$.

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Thank you!