The Bound of Varopoulos and Embeddings of $\ell^1(n)$

Gadadhar Misra

Indian Institute of Science, Bangalore

Ramakrishna Mission Vidyamandira
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Definition (Complex Grothendieck Constant)

Let $((a_{jk}))_{n\times n}$ *be a complex array satisfying*

$$\Big|\sum_{j,k=1}^{n} a_{jk}s_j t_k\Big| \leq \max\left\{|s_j|\,|t_k| : 1 \leq j,k \leq n\right\}, \tag{1}$$

where $s_j, t_k \in \mathbb{C}$. *Then there exists* $K > 0$ *such that for any choice of sequence of vectors* $(x_j)_1^n$, $(y_k)_1^n$ *in a complex Hilbert space* $\mathbb{H}$, *we have*

$$\Big|\sum_{j,k=1}^{n} a_{jk}\langle x_j, y_k\rangle\Big| \leq K \max\left\{\|x_j\|\|y_k\| : 1 \leq j,k \leq n\right\}. \tag{2}$$

The least constant K *satisfying inequality* (2) *is denoted by* $K_G^{\mathbb{C}}$ *and called complex Grothendieck constant.*

Overview

• The Grothendieck constant makes an unexpected appearance in the early work of Varopoulos. Setting

$$C_2(n) = \sup\{\|p(\boldsymbol{T})\| : \|p\|_{\mathbb{D}^n,\infty} \leq 1, \|\boldsymbol{T}\|_\infty \leq 1\},$$

where the supremum is taken over all complex polynomials p in n variables of degree at most 2 and commuting n - tuples $\boldsymbol{T} := (T_1, \ldots, T_n)$ of contractions, he shows that

$$\lim_{n\to\infty} C_2(n) \leq 2K_G^{\mathbb{C}},$$

where $K_G^{\mathbb{C}}$ is the complex Grothendieck constant.

• Rajeev Gupta in his PhD thesis shows that

$$\lim_{n\to\infty} C_2(n) \leq \frac{3\sqrt{3}}{4} K_G^{\mathbb{C}},$$

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- Although, the embedding of $\ell^\infty(n)$ as diagonal matrices in the normed space (with respect to the operator norm) of $n \times n$ matrices M_n is evidently isometric, Rajeev Gupta and Md. Ramiz Reza show that $\ell^1(n)$, $n > 1$, has no such isometric embedding into M_k for any $k \in \mathbb{N}$.
- Several isometric embeddings of $\ell^1(n), n \in \mathbb{N}$, into $\mathcal{B}(\mathbb{H})$ are discussed.
- All of these are shown to be completely isometric to the MIN structure. Adapting an example due to Parrott, an operator space structure for $\ell^1(n), n > 2$, is produced which is distinct from the MIN Structure.

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• All of these are shown to be completely isometric to the MIN structure. Adapting an example due to Parrott, an operator space structure for $\ell^1(n), n > 2$, is produced which is distinct from the MIN Structure.

Theorem (Varopoulos,1976)

Suppose $K_G^{\mathbb{C}}$ *denote the complex Grothendieck constant. Then*

$$K_G^{\mathbb{C}} \leq \sup \|p(T_1, \ldots, T_n)\| \leq 2K_G^{\mathbb{C}}$$

where supremum is over all $n \in \mathbb{N}$*, tuples of commuting contractions* $T = (T_1, \ldots, T_n)$ *and polynomial* p *of degree 2 with* $\|p\|_\infty \leq 1$*.*

The following theorem improves upon the upper bound of Varopolous.

Theorem

Suppose p is a polynomial of degree at most 2 in n variables and $T = (T_1, \ldots, T_n)$ be a tuple of commuting contractions on a Hilbert space $\mathbb{H}$. Then

$$\|p(T_1, \ldots, T_n)\| \leq \frac{3\sqrt{3}}{4} K_G^{\mathbb{C}} \|p\|_\infty.$$

Proof: • Let f be a complex valued analytic function on $\mathbb{D}^n$ with $\|f\|_{\mathbb{D}^n,\infty} \leq 1$.

•Let $a = (a_1, \ldots, a_n) \in \mathbb{D}^n$, $\Phi_j(z) = \frac{z+a_j}{1+\bar{a}_j z}$ and $\Phi(z_1, \ldots, z_n) = (\Phi_1(z_1), \ldots, \Phi_n(z_n))$.

• φ be the automorphism of the unit disc such that $\varphi(f(a)) = 0$.

• $D(\varphi \circ f \circ \Phi)(0) = \varphi'(f(a))Df(a)D\Phi(0)$.

• $g := \varphi \circ f \circ \Phi : \mathbb{D}^n \to \mathbb{D}$

• Applying the Schwarz lemma, we see that $Dg(0)$ is a contractive linear functional on $(\mathbb{C}^n, \|\cdot\|_{\mathbb{D}^n,\infty})$.

•$Df(a) = \varphi'(f(a))^{-1} \sum_{j=1}^n \frac{\partial_j g(0)}{1-|a_j|^2}$

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• $\|Df(a)\|_1 \leq (1-|f(a)|^2)\max_j \frac{1}{1-|a_j|^2}$.

• $\|Df(a)\|_1 \leq \frac{1}{1-r^2}$; where $|a_i| < r$.

• Df is a map from $r\mathbb{D}^n$ to $\frac{1}{1-r^2}(\mathbb{D}^n)^*$.

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$$\|(a_{ij})\|_{\ell^\infty(n) \to \ell^1(n)} \leq \frac{3\sqrt{3}}{4} \approx 1.3. \qquad (3)$$

• $p(z_1, \ldots, z_n) = a_0 + \sum_{j=1}^{n} a_j z_j + \sum_{j,k=1}^{n} a_{jk} z_j z_k$.

• Now applying Grothendieck inequality to

$B = \begin{pmatrix} a_0 & a_1/2 & a_2/2 & \cdots & a_n/2 \\ a_1/2 & a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_n/2 & a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ and using (3) we see that the proof is complete. □

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Operator space structures on $\ell^1(n)$

Definition

An abstract operator space is a normed linear space V together with a norm $\|\cdot\|_k$ defined on the linear space

$$M_k(V) := \{ (\!(v_{ij})\!) | v_{ij} \in V, 1 \leq i, j \leq k \}, \ \ k \in \mathbb{N},$$

with the understanding that $\|\cdot\|_1$ is the norm of V and the family of norms $\|\cdot\|_k$ satisfies the compatibility conditions:

1. $\|T \oplus S\|_{p+q} = \max \{ \|T\|_p, \|S\|_q \}$ *and*
2. $\|ASB\|_q \leq \|A\|_{op} \|S\|_p \|B\|_{op}$

for all $S \in M_q(V)$, $T \in M_p(V)$, $A \in M_{q \times p}(\mathbb{C})$ and $B \in M_{p \times q}(\mathbb{C})$.

• For two operator spaces $(V, \|\cdot\|_k)$ and $(W, \|\cdot\|_k)$, a linear bijection $T : V \to W$ is said to be a complete isometry if $T \otimes I_k : (M_k(V), \|\cdot\|_k) \to (M_k(W), \|\cdot\|_k)$ is an isometry for every $k \in \mathbb{N}$.

• Operator spaces $(V, \|\cdot\|_k)$ and $(W, \|\cdot\|_k)$ are said to be completely isometric if there is a linear complete isometry $T : V \to W$.

• There are two natural operator space structures on any normed linear space V.

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Definition (MIN Structure)

The MIN operator space structure denoted by MIN(V) on a normed linear space V is obtained by the isometric embedding of V in to the C^**-algebra* $C((V^*)_1)$*, the space of continuous functions on the unit ball* $(V^*)_1$ *of the dual space* V^**. Thus for* $\langle\!\langle v_{ij} \rangle\!\rangle$ *in* $M_k(V)$*, we set*

$$\left\| \langle\!\langle v_{ij} \rangle\!\rangle \right\|_{MIN} = \sup \left\{ \left\| \langle\!\langle f(v_{ij}) \rangle\!\rangle \right\| : f \in (V^*)_1 \right\},$$

where the norm of a scalar matrix $\langle\!\langle f(v_{ij}) \rangle\!\rangle$ *is the operator norm in* M_k*.*

Definition (MAX Structure)

Let V be a normed linear space and $(\!(v_{ij})\!) \in M_k(V)$. Define

$$\left\|(\!(v_{ij})\!)\right\|_{MAX} = \sup\left\{\left\|(\!(Tv_{ij})\!)\right\| : T : V \to \mathcal{B}(\mathbb{H})\right\},$$

where the supremum is taken over all isometries T and all Hilbert spaces $\mathbb{H}$. This operator space structure is denoted by MAX(V).

•The map $\phi : \ell^\infty(n) \to \mathcal{B}(\mathbb{C}^n)$, defined by $\phi(z_1, \ldots, z_n) = \mathrm{diag}(z_1, \ldots, z_n)$, is an isometric embedding of the normed linear space $\ell^\infty(n)$.

Lemma

For $m \in \mathbb{N}$ and $\theta_1, \ldots, \theta_m \in [0, 2\pi)$, there exists $a_1, a_2 \in \mathbb{C}$ such that

$$\max_{j=1,\ldots,m} \left| a_1 + e^{i\theta_j} a_2 \right| < |a_1| + |a_2|.$$

Consequently, there is no isometric embedding of $\ell^1(n)$, $n > 1$, into $\ell^\infty(k)$ for any $k \in \mathbb{N}$.

Proof `of the Consequence:`

• Suppose $S : \ell^1(2) \to \ell^\infty(k)$ defined by

$$S(z_1, z_2) := (a_1 z_1 + b_1 z_2, \ldots, a_k z_1 + b_k z_2)$$

is an isometry with smallest possible $k \in \mathbb{N}$. Then, due to the minimality of k, it follows that $|a_j| = |b_j| = 1$ for $j = 1, \ldots, k$. Without loss of generality, we can assume that $a_j = 1$ for $j = 1, \ldots, n$. From the Lemma, it follows that S cannot be an isometry.

• The converse is a restatement of the Lemma, namely, $S : \ell^1(2) \to \ell^\infty(k)$ defined by $S(z_1, z_2) := (z_1 + e^{i\theta_1} z_2, \ldots, z_1 + e^{i\theta_k} z_2)$ can not be an isometry.

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is an isometry with smallest possible $k \in \mathbb{N}$. Then, due to the minimality of k, it follows that $|a_j| = |b_j| = 1$ for $j = 1, \dots, k$. Without loss of generality, we can assume that $a_j = 1$ for $j = 1, \dots, n$. From the Lemma, it follows that S cannot be an isometry.

- The converse is a restatement of the Lemma, namely, $S : \ell^1(2) \to \ell^\infty(k)$ defined by $S(z_1, z_2) := (z_1 + e^{i\theta_1} z_2, \dots, z_1 + e^{i\theta_k} z_2)$ can not be an isometry.

Theorem

There is no finite dimensional embedding of $\ell^1(2)$.

Proof: • Let $\phi(a_1, a_2) = a_1 T_1 + a_2 T_2$ be an n-dimensional isometric embedding of $\ell^1(2)$.

$\bullet U_i := \begin{pmatrix} T_i & D_{T_i^*} \\ D_{T_i} & -T_i^*, \end{pmatrix}$ $i = 1, 2$, where D_{T_i} is the positive square root of the (positive) operator $I - T_i^* T_i$.

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$\bullet \psi(a_1, a_2) = a_1 U_1 + a_2 U_2$ is also an isometry.

• Since norms are preserved under unitary operations, without loss of generality, we assume $U_1 = I$ and U_2 to be a diagonal unitary, say, D.

•Applying the lemma, we obtain complex numbers a_1 and a_2 such that $\|\psi(a_1, a_2)\| < |a_1| + |a_2|$. □

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- Let $\mathbb{H}_1, \ldots, \mathbb{H}_n$ be Hilbert spaces and T_i be a contraction on $\mathbb{H}_i$ for $i = 1, \ldots, n$.
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Operator space structures different from the MIN structure

• Parrott's example shows that a linear contractive map on $\ell^1(3)$ may not be completely contractive.

• An explicit example for this, in a paper of G. Misra, explains that there are more than one operator space structure on $\ell^1(3)$. Using this example we give an explicit operator space structure on $\ell^1(3)$, which is different from the MIN structure.

• Consider the following 2×2 unitary operators:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ U := \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \text{ and } V := \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

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- Let $S_1 := \hat{T}_1 \oplus I$, $S_2 := \hat{T}_2 \oplus U$, $S_3 := \hat{T}_3 \oplus V$

- The computations done in this paper includes the following: $\|I \otimes I + U \otimes U + V \otimes V\| = 3.$ and

$$\sup_{z_1,z_2,z_3 \in \mathbb{D}} \|z_1 I + z_2 U + z_3 V\| < 3. \tag{4}$$

- Choose a diagonal operator D on $\ell^2(\mathbb{Z})$ such that $\|D\| \leq 1$ and $\mathbb{T} \subset \sigma(D)$.
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• Define $S : (\ell^1(3), \mathrm{MIN}) \longrightarrow B(\mathbb{K})$ by $S(e_1) = S_1,\ S(e_2) = S_2,\ S(e_3) = S_3$ and extend it linearly.

• From the previous theorem, $(z_1, z_2, z_3) \mapsto z_1 \hat{T}_1 + z_2 \hat{T}_2 + z_3 \hat{T}_3$ is an isometry and since h is of norm at most 1, it follows that the map $(z_1, z_2, z_3) \mapsto z_1 S_1 + z_2 S_2 + z_3 S_3$ is also an isometry.

• Also, we have

$$\|S_1 \otimes I + S_2 \otimes U + S_3 \otimes V\| \geq \|I \otimes I + U \otimes U + V \otimes V\| = 3.$$

• On the other hand from (4), we have

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Thank you