Invariants for a class of Cowen-Douglas operators

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February 12, 2014

the Cowen-Douglas class

The class of operators which has come to be known as the "Cowen-Douglas class" consists of those bounded linear operators T on a complex separable Hilbert space $\mathcal{H}$ which

- possess an open set $\Omega \subset \mathbb{C}$ of eigenvalues of constant multiplicity, say n and
- admit a holomorphic choice of eigenvectors: $s_1(w), \ldots, s_n(w)$, $w \in \Omega$.

In other words, there exists holomorphic functions $s_1, \ldots, s_n : \Omega \to \mathcal{H}$ which span the eigenspace of T at w.

The holomorphic choice of eigenvectors $s_1, \ldots, s_n$ defines a holomorphic Hermitian vector bundle E_T via the map

$$s : \Omega \to \mathrm{Gr}(n, \mathcal{H}), \ \ s(w) = \ker(T - w) \subseteq \mathcal{H}.$$

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the Cowen-Douglas theorem

One of the striking results from the late seventies due to Cowen and Douglas says:
There is a one to one correspondence between the unitary equivalence class of the operators T and the equivalence classes of the holomorphic Hermitian vector bundles E_T determined by them.

Furthermore, they find a set of complete invariants, not very tractable unless $n = 1$, for this equivalence. For $n = 1$, as is well-known, the curvature

$$\mathsf{K}(w) = -\frac{\partial^2}{\partial w \bar{\partial} w} \log \|s(w)\|^2 dw \wedge d\bar{w}$$

of the line bundle L_T is a complete invariant of L_T, or equivalently, that of the operator T.

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proof that the curvature is a complete invariant

Pick a holomorphic frame $s_i(w)$ for the line bundle E_i and let $\Gamma_i(w) = \langle s_i(w), s_i(w)\rangle$ be the Hermitian metric, $i = 1, 2$. Suppose that the two curvatures K_E and K_F are equal on some open (simply connected) subset $\Omega_0 \subseteq \Omega$. It then follows that $u = \log(\Gamma_1/\Gamma_2)$ is harmonic ensuring the existence of a harmonic conjugate v of u on Ω_0. Define $\tilde{s}_2(w) = e^{(u(w)+iv(w))/2} s_2(w)$. Then clearly, $\tilde{s}_2(w)$ is a new holomorphic frame for F. Consequently, we have

$$\begin{aligned}
\tilde{\Gamma}_2(w) &= \langle \tilde{s}_2(w), \tilde{s}_2(w)\rangle \\
&= \langle e^{(u(w)+iv(w))/2} s_2(w), e^{(u(w)+iv(w))/2} s_2(w)\rangle \\
&= e^{u(w)} \langle s_2(w), s_2(w)\rangle \\
&= \Gamma_1(w).
\end{aligned}$$

it is not a complete invariant if rank is > 1

If the rank of the (holomorphic Hermitian) vector bundle E is $> 1,$ then the the holomorphic frame

$$s_1, \ldots, s_n : \Omega \to \mathcal{H}$$

defines a Hermitian metric on $E,$ namely,

$$\Gamma_s(w) = \left(\!\left(\langle s_i(w), s_j(w)\rangle\right)\!\right)$$

and the curvature

$$\mathsf{K}_E(w) = \bar{\partial}\big(G_s^{-1}(\partial G_s)\big)(w)$$

clearly depends on the choice of the frame $s.$ It is easily seen that while the eigenvalues of the curvature provide a set of invariants for the vector bundle $E,$ they are not complete except in the case where the vector bundle E is the direct sum of line bundles!

the problem

The splitting of a holomorphic Hermitian vector bundle into a direct sum is determined by the vanishing of the second fundamental form.

We isolate those irreducible holomorphic Hermitian vector bundles, namely, the ones possessing a flag structure, for which the curvature together with the second fundamental form (relative to the flag) is a complete set of invariants.

Among these, we describe in detail the ones that correspond to irreducible operators in the Cowen-Douglas class $B_2(\Omega)$. All irreducible homogeneous operators in $B_2(\mathbb{D})$ are in this class. We obtain a description of all these operators.

This classification was given earlier by D. Wilkins using a sophisticated mix of Riemannian geometry and operator theory.

We also describe the case of $n > 2$, where the answer is much more complicated.

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the new class

Definition

We let $\mathcal{F}B_2(\Omega)$ denote the set of operators $T \in B_2(\Omega)$ which admit a decomposition of the form $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ for some choice of operators $T_0, T_1 \in \mathcal{B}_1(\Omega)$ and an intertwiner S between T_0 and T_1, that is, $T_0 S = S T_1$.

An operator T in $B_2(\Omega)$ admits a decomposition of the form $\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ for some pair of operators T_0 and T_1 in $B_1(\Omega)$. In defining the new class $\mathcal{F}B_2(\Omega)$, we are merely imposing one additional condition, namely that $T_0 S = S T_1$.

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holomorphic and orthogonal frames

We show that T is in the class $\mathcal{F}B_2(\Omega)$ if and only if there exist a frame $\{\gamma_0, \gamma_1\}$ of the vector bundle E_T such that $\gamma_0(w)$ and

$$t_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - \gamma_1(w)$$

are orthogonal for all w in Ω. This is also equivalent to the existence of a frame $\{\gamma_0, \gamma_1\}$ of the vector bundle E_T such that

$$\frac{\partial}{\partial w}\|\gamma_0(w)\|^2 = \langle \gamma_1(w), \gamma_0(w)\rangle,\ w \in \Omega.$$

Our main point is that it is often easier to work with the orthogonal frame $\{\gamma_0, t_1\}$. Of course, the operator action on this frame is more complicated.

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main theorem

Theorem

Let $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ *and* $\tilde{T} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}$ *be two operators in* $\mathcal{F}B_2(\Omega)$.

Also let t_1 *and* $\tilde{t}_1$ *be non-zero sections of the holomorphic Hermitian line bundles* E_{T_1} *and* $E_{\tilde{T}_1}$ *respectively.*

The operators T *and* $\tilde{T}$ *are equivalent if and only if*

$$\mathcal{K}_{T_0} = \mathcal{K}_{\tilde{T}_0}, \quad \frac{\|S(t_1)\|^2}{\|t_1\|^2} = \frac{\|\tilde{S}(\tilde{t}_1)\|^2}{\|\tilde{t}_1\|^2}.$$

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second fundamental form, flag structure

In any decomposition $\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$, of an operator $T \in \mathcal{F}B_2(\Omega)$, let t_1 be a non zero section of holomorphic Hermitian vector bundle E_{T_1}. The intertwining property ensures that $S(t_1)$ is a non zero section of E_{T_0} on some open subset of Ω. Following the methods of Douglas-M, the second fundamental form of E_{T_0} in E_T is easy to compute:

It is the $(1,0)$ -form $\dfrac{-\mathcal{K}_{T_0}(z)}{\left(-\mathcal{K}_{T_0}(z)+\frac{\|t_1(z)\|^2}{\left\|S\left(t_1(z)\right)\right\|^2}\right)^{1/2}} d\bar{z}$, where

$-\mathcal{K}_{T_0}(z) = \frac{\partial^2}{\partial z \partial \bar{z}} \log \|\gamma_0(z)\|^2$ is the co-efficient of the curvature $(1,1)$ -form. Thus the second fundamental form of E_{T_0} in E_T together with the curvature of E_{T_0} is a complete invariant for the operator T. The inclusion of the line bundle E_{T_0} in the vector bundle E_T of rank 2 is the flag structure of E_T.

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irreducibility

Cowen and Douglas point out that an operator in $B_1(\Omega)$ must be irreducible. However, determining which operators in $B_n(\Omega),\ n > 1,$ are irreducible is a formidable task. It turns out that the operators in $\mathcal{F}B_2(\Omega)$ are always irreducible. Indeed, if we assume S is invertible, then T is strongly irreducible.

model

An operator in the Cowen-Douglas class $B_n(\Omega)$, up to unitary equivalence, is the adjoint of the multiplication operator on a Hilbert space consisting of holomorphic functions on $\Omega^* := \{\bar{w} : w \in \Omega\}$ possessing a reproducing kernel. What about operators in $\mathcal{F}B_n(\Omega)$?

Let $\gamma = (\gamma_0, \gamma_1)$ be a holomorphic frame for the vector bundle E_T, $T \in \mathcal{F}B_2(\Omega)$. Then the operator T is unitarily equivalent to the adjoint of the multiplication operator M on a reproducing kernel Hilbert space $\mathcal{H}_\Gamma \subseteq \mathrm{Hol}(\Omega^*, \mathbb{C}^2)$ possessing a reproducing kernel $K_\Gamma : \Omega^* \times \Omega^* \to \mathbb{C}^{2\times 2}$, of the form:

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the kernel

$$K_\Gamma(z,w) = \begin{pmatrix} \langle \gamma_0(\bar w), \gamma_0(\bar z)\rangle & \langle \gamma_1(\bar w), \gamma_0(\bar z)\rangle \\ \langle \gamma_0(\bar w), \gamma_1(\bar z)\rangle & \langle \gamma_1(\bar w), \gamma_1(\bar z)\rangle \end{pmatrix}$$

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$z, w \in \Omega$, where t_1 and $\gamma_0 := S(t_1)$ are frames of the line bundles E_{T_1} and E_{T_0} respectively.

It follows that $\gamma_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - t_1(w)$ and that $t_1(w)$ is orthogonal to $\gamma_0(w)$, $w \in \Omega$.

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examples

We now give examples of natural classes of operators that belong to $\mathcal{F}B_2(\Omega)$. Indeed, we were led to the definition of this new class $\mathcal{F}B_2(\Omega)$ of operators by trying to understand these examples better.

An operator T is called homogeneous if $\phi(T)$ is unitarily equivalent to T for all ϕ in Möb which are analytic on the spectrum of T.

If an operator T is in $\mathcal{B}_1(\mathbb{D})$, then T is homogeneous if and only if $\mathcal{K}_T(w) = -\lambda(1 - |w|^2)^{-2}$, for some $\lambda > 0$.

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homogeneous operators

A model for all homogeneous operators in $B_n(\mathbb{D})$ has been obtained in a recent paper (joint with Koranyi).
Specializing to $n = 2:$ For $\lambda > 1$ and $\mu > 0,$ set $K_0(z, w) = (1 - z\bar{w})^{-\lambda}$ and $K_1(z, w) = \mu(1 - z\bar{w})^{-\lambda-2}.$

An irreducible operator T in $B_2(\mathbb{D})$ is homogeneous if and only if it is unitarily equivalent to the adjoint of the multiplication operator on the Hilbert space $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{D}, \mathbb{C}^2)$ determined by the positive definite kernel of the form $K_\Gamma.$

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A model for all homogeneous operators in $B_n(\mathbb{D})$ has been obtained in a recent paper (joint with Koranyi).
Specializing to $n = 2$: For $\lambda > 1$ and $\mu > 0$, set $K_0(z, w) = (1 - z\bar{w})^{-\lambda}$ and $K_1(z, w) = \mu(1 - z\bar{w})^{-\lambda-2}$.

An irreducible operator T in $B_2(\mathbb{D})$ is homogeneous if and only if it is unitarily equivalent to the adjoint of the multiplication operator on the Hilbert space $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{D}, \mathbb{C}^2)$ determined by the positive definite kernel of the form K_Γ.

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Hilbert modules, localization

An operator T in $B_1(\Omega)$ acting on a Hilbert space $\mathcal{H}$ makes it a module over the polynomial ring via the usual point-wise multiplication. An important tool in the study of these modules is the localization.

This is the Hilbert module $J\mathcal{H}^{(k)}_{\text{loc}}$ corresponding to the spectral sheaf $J\mathcal{H} \otimes_{\mathcal{P}} \mathbb{C}^k_w$, where $\mathcal{P}$ is the polynomial ring and

- $J : \mathcal{H} \to \text{Hol}(\Omega, \mathbb{C}^k)$ is the jet map, namely, $Jf = \sum_{\ell=0}^{k-1} \partial^\ell f \otimes \varepsilon_{\ell+1}$, $\varepsilon_1, \ldots, \varepsilon_k$ are the standard unit vectors in $\mathbb{C}^k$.
- $\mathbb{C}^k_w$ is a k - dimensional module over the polynomial ring,
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$$(\mathfrak{J}f)(w) = \begin{pmatrix} f(w) & 0 & \cdots & 0 \\ \binom{2}{1}\partial f(w) & f(w) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k}{1}\partial^{k-1} f(w) & \binom{k-1}{1}\partial^{k-2} f(w) & \cdots & f(w) \end{pmatrix},$$

that is, $(f, v) \mapsto (\mathfrak{J}f)(w)v$, $f \in \mathcal{P}$, $v \in \mathbb{C}^k$.

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realization

We now consider the localization with $k = 2$. If we assume that the operator T has been realized as the adjoint of the multiplication operator on a Hilbert space of holomorphc function possessing a kernel function, say K, then the kernel $JK^{(2)}_{\text{loc}}$ for the localization (of rank 2) given in in the work of Douglas-M-Varughese coincides with K_Γ. In this case, we have $K_1 = K = K_0$.

The operator T, in this case, has the form $\begin{pmatrix} T_0 & \binom{2}{1} I \\ 0 & T_1 \end{pmatrix}$.

As is to be expected, using the complete set of unitary invariants given in the main Theorem, we see that the unitary equivalence class of the Hilbert module $\mathcal{H}$ is in one to one correspondence with that of $J\mathcal{H}^{(2)}_{\text{loc}}$.

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conclusion

Thus the class $\mathcal{F}B_2(\Omega)$ contains two very interesting classes of operators. For $n > 2,$ we find that there are competing definitions. One of these contains the homogeneous operators and the other contains the Hilbert modules obtained from the localization.

the class $\mathcal{F}B_n(\Omega)$, $n \geq 2$

Let $\mathcal{F}B_n(\Omega)$ be the set of all operators T in the Cowen-Douglas class $B_n(\Omega)$ for which we can find operators $T_0, T_1, \ldots, T_{n-1}$ in $B_1(\Omega)$ and a decomposition of the form

$$T = \begin{pmatrix} T_0 & S_{0\,1} & S_{0\,2} & \ldots & S_{0\,n-1} \\ 0 & T_1 & S_{1\,2} & \ldots & S_{1\,n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & T_{n-2} & S_{n-2\,n-1} \\ 0 & \ldots & \ldots & 0 & T_{n-1} \end{pmatrix}$$

such that none of the operators $S_{i\,i+1}$ are zero and $T_i S_{i\,i+1} = S_{i\,i+1} T_{i+1}, \quad i = 0, \ldots, n-1.$

unitary intertwining operators

If there exists a invertible bounded linear operator X intertwining any two operators, say $T, \tilde{T}$ in $\mathcal{F}B_n(\Omega)$ ($XT = \tilde{T}X$), then we prove that X must be upper triangular with respect to the decomposition mandated in the definition of the class $\mathcal{F}B_n(\Omega)$. It then follows that any unitary operator intertwining these two operators must be diagonal.

Thus we see that they are unitarily equivalent if and only there exists unitary operators $U_i : \mathcal{H}_i \to \tilde{\mathcal{H}}_i$ such that $U_i^* \widetilde{T}_i U_i = T_i, \quad i = 0, 1, \cdots n-1,$ and $U_i S_{i,j} = \widetilde{S}_{i,j} U_j, i < j.$

The first of these conditions immediately translates into a condition on the curvature of the line bundles E_{T_i}. The second condition is somewhat more mysterious and is related to a finite number of second fundamental forms inherent in our description of the operator T.

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Jordan form

Let T be an operator acting on a Hilbert space $\mathcal{H}$. Assume that there exists a representation of the form

$$T = \begin{pmatrix} T_0 & S_{0\,1} & 0 & \dots & 0 \\ 0 & T_1 & S_{1\,2} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & T_{n-2} & S_{n-2\,n-1} \\ 0 & \dots & 0 & 0 & T_{n-1} \end{pmatrix}$$

for the operator T with respect to some orthogonal decomposition $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{n-1}$.

Suppose also that the operator T_i is in $B_1(\Omega)$, $0 \leq i \leq n-1$, the operator $S_{i-1,i}$ is non zero and $T_{i-1}S_{i-1,i} = S_{i-1,i}T_i$, $1 \leq i \leq n-1$. Then we show that the operator T must be in the Cowen-Douglas class $B_n(\Omega)$.

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orthogonal vs. holomorphic frames

We can also relate the frame of the vector bundle E_T to those of the line bundles E_{T_i}, $i = 0, 1, \ldots, n-1$. Indeed, we show that there is a frame $\{\gamma_0, \gamma_1, \cdots, \gamma_{n-1}\}$ of E_T such that

$$t_i(w) := \gamma_i(w) + \cdots + \binom{i}{j}\gamma_{i-j}^{(j)}(w) + \cdots + \gamma_0^{(i)}(w)$$

is a non-vanishing section of the line bundle E_{T_i} and it is orthogonal to $\gamma_i(w)$, $i = 0, 1, 2, \ldots, i-1$.

We also have $t_{i-1} := S_{i-1\,i}(t_i)$, $1 \leq i \leq n-1$.

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complete invariants

Theorem

Pick two operators T and $\tilde{T}$ which admit a Jordan form. Find an orthogonal frame $\{\gamma_0, t_1, \cdots, t_{n-1}\}$ (resp. $\{\tilde{\gamma}_0, \tilde{t}_1, \cdots, \tilde{t}_{n-1}\}$) for the vector bundle $\bigoplus_{i=0}^{n} E_{T_i}$ (resp. $\bigoplus_{i=0}^{n} E_{\tilde{T}_i}$) as above. Then the operators T and $\tilde{T}$ are unitarily equivalent if and only if

$$\mathcal{K}_{T_0} = \mathcal{K}_{\tilde{T}_0} \text{ and } \frac{\|S_{i-1\,i}(t_i)\|^2}{\|t_i\|^2} = \frac{\|\tilde{S}_{i-1\,i}(\tilde{t}_i)\|^2}{\|\tilde{t}_i\|^2},\ 1 \leq i \leq n-1.$$

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Thank you!