Homogeneous bundles and operators in the Cowen-Douglas class

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Dedicated to the memory of

Professor Ronald G. Douglas

bounded symmetric domains

A domain $\mathscr{D} \subseteq \mathbb{C}^n$ is said to be symmetric if it has an involutive holomorphic automorphism s_z having z as an isolated fixed point for each $z \in \mathscr{D}$.

The typical examples are the unit ball in matrices $(\mathbb{C}^{n\times m})_1$ of size $n \times m$. These include the Euclidean ball $\mathbb{B}_n$, that is, $m = 1$.

Let $G := \mathrm{Aut}(\mathscr{D})$ be the bi-holomorphic automorphism group of $\mathscr{D}$. For the matrix unit ball, $G := \mathrm{SU}(n,m)$, which consists of all linear automorphisms leaving the form $\begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}$ on $\mathbb{C}^{n+m}$ invariant.

Thus $g \in \mathrm{SU}(n,m)$ is of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The group $\mathrm{SU}(n,m)$ acts on $(\mathbb{C}^{n\times m})_1$ via the map

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto (az+bz)(cz+dz)^{-1},\ z \in (\mathbb{C}^{n\times m})_1.$$

This action is transitive. Indeed $(\mathbb{C}^{n\times m})_1 \cong \mathrm{SU}(n,m)/\mathbf{K}$, where $\mathbf{K}$ is the stabilizer of $\mathbf{0}$ in $(\mathbb{C}^{n\times m})_1$.

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homogeneous n-tuple

When $\mathscr{D}$ is a bounded symmetric domain and H is any Hilbert space, call an n-tuple $T = (T_1, \ldots, T_n)$ of commuting bounded operators homogeneous if their joint Taylor spectrum is contained in $\overline{\mathscr{D}}$ and for every holomorphic automorphism g of $\mathscr{D}$, there exists a unitary operator U_g such that

$$g(T_1, \ldots, T_n) = (U_g^{-1} T_1 U_g, \ldots, U_g^{-1} T_n U_g),$$

or more briefly

$$g(T)_i = U_g^{-1} T_i U_g \ \ (1 \leq i \leq n). \tag{1}$$

If a homogeneous n-tuple of operators T is irreducible, then it is possible to choose U_g so that the map $g \mapsto U_g$ is a projective unitary representation. This follows from a powerful selection theorem of Kenugi and Novikov.

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Over the past years, some progress has been made to answer these two questions, at least, when the n-tuple of homogeneous operators is in the Cowen-Douglas class.

A parametrization of all homogeneous holomorphic Hermitian vector bundles over a bounded symmetric domain $\mathscr{D}$ was obtained in 1992 by David Wilkins. However, his differential geometric proofs give a realization of the corresponding homogeneous operator only in the Cowen-Douglas class of rank 1 and 2 over the unit disk.

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the Cowen-Douglas class

Let V be a finite dimensional inner product space, $\dim V = k$ and $\mathscr{H} \subset \mathrm{Hol}(\mathscr{D}, V)$ be a Hilbert space containing all the V - valued polynomials as a dense set.

Suppose also that the operators M_j, defined by $(M_j)f(z) = z_j f(z)$ preserve $\mathscr{H}$ and are bounded on it.

The Cowen-Douglas class $\hat{B}_k(\mathscr{D})$ consists of these commuting n-tuple of operators $M^* := (M_1^*, \ldots, M_n^*)$. The original defintion of Cowen and Douglas is somewhat different and is more intrinsic.

It is not hard to see that the map

$$\gamma : w \mapsto \cap_{i=1}^n \ker(M_i - w_i)^*,\ w \in \Omega,$$

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Cowen and Douglas show that

- $E \subseteq \Omega \times \mathscr{H}$ with fiber $E_w = \cap_{i=1}^{n} \ker(M_i - w_i)^*$ is a holomorphic Hermitian vector bundle,
- isomorphism classes of E correspond to unitary equivalence classes of T,
- E is irreducible as a holomorphic Hermitian vector bundle if and only if T is irreducible.

Say that a vector bundle is homogeneous if the action of the group $\text{Aut}(\mathscr{D})$ lifts to an isometric action on the bundle E.

Theorem

An n-tuple of operators T in the Cowen-Douglas class is homogeneous if and only if the corresponding holomorphic Hermitian vector bundle E is homogeneous under $\tilde{G}$, the universal covering group of the group G.

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Reproducing Kernel

It is important to note here that E has a reproducing kernel. Indeed, $\mathrm{ev}_w : \mathscr{H} \to E_w^*$ induced by the map $f \mapsto \langle f, \cdot \rangle$ is continuous and hence $K(z,w) = \mathrm{ev}_z \circ \mathrm{ev}_w^*$ is a reproducing kernel for E.

Here we will always use trivialization of the bundles with standard Euclidean inner product. The Hilbert space $\mathscr{H} \subseteq \mathrm{Hol}(\Omega, \mathbb{C}^n)$ has a reproducing kernel $K_w(z) : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\langle f, K_w \xi \rangle = \langle f(w), \xi \rangle, f \in \mathscr{H}, \xi \in \mathbb{C}^n.$$

Cowen and Douglas determine intrinsic conditions on an operator T on a Hilbert space $\mathscr{H}$ to ensure that the map $w \mapsto \ker(T - w) \subseteq \mathscr{H}$ is holomorphic. Thus ensuring the existence of a vector bundle E_T and establishing an equivalence of categories.

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known results

When $\mathscr{D}$ is the unit disc in $\mathbb{C}$, a complete description of all homogeneous operators in $B_k(\mathscr{D})$ is now known (joint with A. Korányi).

A description of all homogeneous n- tuples in $B_1(\mathscr{D})$, when $\mathscr{D}$ is a domain of tube type is also known (joint with B. Bagchi). Arazy and Zhang have obtained similar results for general domains of classical type.

For a large subclass of $B_k(\mathscr{D})$ for any bounded symmetric domain $\mathscr{D}$, there are precise results in a recent paper (joint with H. Upmeier).

The "classification" of all the homogeneous commuting n - tuple of bounded operators in the class $B_k(\mathscr{D})$ has been now completed (joint with A. Korányi).

Of course, there are many questions that remain unanswered.

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The simply connected universal covering group $\tilde{G}$ with Lie algebra $\mathfrak{g}$ acts on $\mathscr{D}$ by holomorphic automorphisms; one has $\mathscr{D} \cong \tilde{G}/\tilde{K}$ with $\tilde{K}$ corresponding to $\mathfrak{k}$.

The complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ has a vector space direct sum decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$.

In this realization $\mathscr{D}$ appears as a balanced convex domain in $\mathfrak{p}^+ \cong \mathbb{C}^n$.

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preliminaries

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methods

It is known that all the $\tilde{G}$ - homogeneous Hermitian holomorphic vector bundles can be obtained by holomorphic induction from representations of (ρ, V) of the parabolic Lie algebra $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^{-}$ on finite dimensional inner product spaces.

The Hermitian holomorphic homogeneous vector bundles (meaning homogeneous as Hermitian bundles) come from (ρ, V) such that V has a $\tilde{K}$- invariant inner product.

The representations, and the induced bundles, have composition series with irreducible factors.

The main result is the construction of an explicit differential operator intertwining the isometric action of the group $\tilde{G}$ on the bundle with the the action on the direct sum of its factors.

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Still, the highly non-trivial more general representations of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^{-}$ and the corresponding holomorphic homogeneous vector bundles exist and correspond to homogeneous irreducible n-tuples of operators in the Cowen-Douglas class of $\mathscr{D}$.

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quasi-invariant kernels

Suppose that the kernel function K transforms according to the rule

$$J_g(z)K(g(z),g(w))J_g(w)^* = K(z,w),\ g \in G,\ z,w \in \mathscr{D},$$

for some holomorphic function $J_g : \mathscr{D} \to \mathbb{C}$.
Then the kernel K is said to be quasi-invariant, which is equivalent to saying that the map $U_g : f \to J_g\,(f \circ g^{-1}),\ g \in G$, is unitary.
If we further assume that the $J_g : \mathscr{D} \to \mathbb{C}$ is a cocycle, then U is a homomorphism.

The kernel K is quasi-invariant if and only if the coreesponding n-tuple M of multiplication by the coordinate functions is homogeneous.

Therefore, a characterization of all the quasi-invariant kernels defined on $\mathscr{D}$, is equivalent to finding all the holomorphic cocycles, which is also the same as finding all the holomorphic Hermitian homogeneous vector bundles over $\mathscr{D}$.

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trivialization

Given a representation (ρ, V) of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$, the holomorphically induced bundle has a canonical trivialization such that the sections are the elements of $\mathrm{Hol}(\mathscr{D}, V)$, and $\tilde{G}$ acts via the multiplier

$$\rho(\tilde{b}(g,z)) = \rho^0(\tilde{k}(g,z))\rho^-(\exp Y(g,z)),$$

where ρ^0 and ρ^- are the restrictions of (ρ, V) to $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{p}^-$ respectively.

The representation (ρ, V) is a direct sum of subspaces $V_J := V_{\lambda - j}$ carrying an irreducible representation ρ_j^0 of $\mathfrak{k}^{\mathbb{C}}$ $(0 \le j \le m)$.

Also, we have non-zero $\mathfrak{k}^{\mathbb{C}}$-equivariant maps $\rho_j^- : \mathfrak{p}^- \to \mathrm{Hom}(V_{j-1}, V_j)$.

The space of such maps is 1-dimensional: This is an equivalent restatement of the known fact that $\mathfrak{p}^- \otimes V_{j-1}$ as a representation of $\mathfrak{k}^{\mathbb{C}}$ is multiplicity free.

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Let P_j be the orthogonal projection from $\mathfrak{p}^- \otimes V_{j-1}$ to V_j. We define for $Y \in \mathfrak{p}^-$, $v \in V_{j-1}$,

$$\tilde{\rho}_j(Y)v = P_j(Y \otimes v).$$

Then $\tilde{\rho}_j$ has the $\mathfrak{k}^{\mathbb{C}}$-equivariant property, and it follows that $\rho_j^- = y_j \tilde{\rho}_j$ with some $y_j \neq 0$.

We write $y = (y_1, \ldots, y_m)$ and denote by E^y the induced vector bundle. We observe here that the vector bundle E^y is uniquely determined by $\rho_0^0, P_1, \ldots, P_m$ and y.

This data cannot be arbitrarily chosen: The $\tilde{\rho}_j$ $(1 \leq j \leq m)$ together must give a representation of the abelian Lie algebra $\mathfrak{p}^-$.

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the intertwining operator

Theorem

There exists positive constants c_{jk}, the operator $\Gamma : \mathrm{Hol}(\mathscr{D}, V) \to \mathrm{Hol}(\mathscr{D}, V)$ given by

$$(\Gamma f_j)_\ell = \begin{cases} c_{\ell j} y_\ell \cdots y_{j+1} (P_\ell \iota D) \cdots (P_{j+1} \iota D) f_j & \text{if } \ell > j, \\ f_j \ \text{if } \ell = j, \\ 0 \ \text{if } \ell < j \end{cases}$$

intertwines the actions of $\tilde{G}$ on the trivialized sections of E^0 and E^y.

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For a bounded symmetric $\mathscr{D}$, we call a n-tuple T in $\hat{\mathrm{B}}_k(\mathscr{D})$ and its corresponding bundle E basic if E is induced by an irreducible ρ.

When $\mathscr{D}$ is the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, E is basic if and only if it is induced by some $\chi_\lambda \otimes \sigma$ with $\lambda < \sigma_\lambda$.

Theorem

If $\mathscr{D}$ is the unit ball in $\mathbb{C}^n$, all homogenous n - tuples in $\hat{\mathrm{B}}_k(\mathscr{D})$ are similar to direct sums of basic homogeneous n - tuples.

the Cowen-Douglas operators, similarity, ...

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Thank you!