Homogeneous Vector Bundles and intertwining Operators for Symmetric Domains

Gadadhar Misra

Indian Institute of Science
Bangalore
(joint with A. Korányi and H. Upmeier)

Geometry of Banach Spaces and Operator Theory

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what is a Homogeneous operator?

Let $\mathbb{D} \subset \mathbb{C}$, be the unit disc. The group

$$G_0 := SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |\, a \,|^2 - |\, b \,|^2 = 1 \right\}$$

acts on $\mathbb{D}$: $z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$. The Möbius Group G is the group $G_0/\{\pm I\}$. It is the group of holomorphic automorphism of $\mathbb{D}$.

Definition

A bounded operator T on a Hilbert space $\mathcal{H}$ is homogeneous if the spectrum $\sigma(T)$ is contained in the closed unit disc $\bar{\mathbb{D}}$ and for every $g \in G$, there exists a unitary U_g such that

$$g(T) = U_g^{-1} T U_g.$$

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kernel function

All Hilbert spaces $\mathscr{H}$ are assumed to be spaces of holomorphic functions $f : \mathbb{D} \to V$ taking their values in a finite dimensional Hilbert space V and possessing a reproducing kernel K.

A reproducing kernel is a function $K : \mathbb{D} \times \mathbb{D} \to \mathrm{Hom}(V, V)$ holomorphic in the first variable and anti-holomorphic in the second, such that $K_w\zeta$ defined by $(K_w\zeta)(z) := K(z,w)\zeta$ is in $\mathscr{H}$ for each $w \in \mathbb{D}$, $\zeta \in V$, and

$$\langle f, K_w\zeta\rangle_{\mathscr{H}} = \langle f(w), \zeta\rangle_V$$

for all $f \in \mathscr{H}$.

As is well known, if $\{e_n\}_{n=0}^{\infty}$ is any orthonormal basis of $\mathscr{H}$, then we have

$$K(z,w) = \sum_{n=0}^{\infty} e_n(z)e_n(w)^*$$

with the sum converging pointwise.

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cocycle

We will be concerned with multiplier representations of the universal cover $\tilde{G}$ on the Hilbert space $\mathscr{H}$. A cocycle is a continuous function $J : \tilde{G} \times \mathbb{D} \to \mathrm{Hom}(V,V)$, holomorphic on $\mathbb{D}$, such that

$$J(gh,z) = J(h,z)J(g,hz)$$

for all $g, h \in \tilde{G}$ and $z \in \mathbb{D}$. For $g \in \tilde{G}$, we define $U(g)$ on $\mathrm{Hol}(\mathbb{D},V)$ by

$$(U(g)f)(z) = J(g^{-1},z)f(g^{-1}(z)).$$

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multiplier representations

Suppose that the action $g \mapsto U(g),\ g \in \tilde{G},$ preserves $\mathscr{H}$ and is unitary on it, then we say that U is a multiplier representation of $\tilde{G}$.

Also, if the reproducing kernel K transforms according to the rule

$$J(g,z)K(g(z),g(w))J(g,w)^* = K(z,w)$$

for all $g \in \tilde{G};\ z,w \in \mathbb{D}$, then we say that K is quasi-invariant.

Proposition

Suppose $\mathscr{H}$ has a reproducing kernel K. Then the multiplier representation U defined using J is unitary if and only if K is quasi-invariant.

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Let $\mathscr{H}$ be a space of functions, say, on the unit disc or the unit circle. Suppose that

the operator T defined by the rule $(Tf)(z) = zf(z), f \in \mathscr{H}$ is bounded and that

there is a multiplier representation, say $U,$ of the group G on the Hilbert space $\mathscr{H}$

then the operator T is homogeneous and U is the associated representation.

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discrete series representations of Möb

Holomorphic Discrete Series: Fix a real $\lambda > 0$. The group $\tilde{G}$ acts on the Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{D})$, usually called the weighted Bergman space, which is a space of holomorphic functions on $\mathbb{D}$ with reproducing kernel $(1 - z\bar{w})^{-2\lambda}$ via the cocycle $(g')^{\lambda}$. This action is the Discrete representation D_{λ}^{+} of the group $\tilde{G}$.

The operator $M^{(\lambda)}$ of multiplication by the coordinate function z on the Hilbert space $\mathbb{A}^{(\lambda)}(\mathbb{D})$ is homogeneous with the associated representation D_{λ}^{+}.

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A bounded operator T on a Hilbert space $\mathscr{H}$ is said to be in the Cowen - Douglas class of the domain $\Omega \subseteq \mathbb{C}$ if its eigenspaces $E_w, w \in \Omega$ are of constant finite dimension.

Cowen and Douglas show that $E \subseteq \Omega \times \mathscr{H}$ with fiber E_w is a holomorphic Hermitian vector bundle,
isomorphism classes of E correspond to unitary equivalence classes of T,
E is irreducible as a holomorphic Hermitian vector bundle if and only if T is irreducible.

Important to note here is that E has a reproducing kernel. Indeed, $\mathrm{ev}_w : \mathscr{H} \to E_w^*$ induced by the map $f \mapsto \langle f, \cdot \rangle$ is continuous and hence $K(z,w) = \mathrm{ev}_w^* \circ \mathrm{ev}_z$ is a reproducing kernel for E^*.

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trivialization

Here we will always use trivialization of the bundles with standard Euclidean inner product. The Hilbert space $\mathscr{H} \subseteq \mathrm{Hol}(\Omega, \mathbb{C}^n)$ has a reproducing kernel $K_w(z) : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\langle f, K_w \xi \rangle = \langle f(w), \xi \rangle, f \in \mathscr{H}, \xi \in \mathbb{C}^n.$$

The operators in the Cowen-Douglas class can be realized as the adjoint of the multiplication operator M defined by $(Mf)(z) = zf(z)$ on a Hilbert space with holomorphic functions possessing a reproducing kernel.

Theorem

An operator T in the Cowen-Douglas class is homogeneous if and only if the corresponding holomorphic Hermitian bundle E is homogeneous under $\tilde{G}$.

Goal: Describe all homogeneous holomorphic Hermitian vector bundles!

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Given $\eta > 0$ and $Y = (Y_1, \ldots, Y_m)$, where Y_j is a $d_j \times d_j$ complex matrix, define

$$\left(\Gamma^{(Y,\eta)} f_j\right)_\ell = \begin{cases} \frac{1}{(\ell-j)!}\frac{1}{(2\eta+2j)_{\ell-j}} Y_\ell \cdots Y_{j+1} D^{\ell-j} f_j & \text{if } \ell \geq j \\ 0 & \text{if } \ell < j \end{cases}.$$

Let $\mathscr{H}^{(Y,\eta)}$ denote the image of $\Gamma^{(Y,\eta)}$ in the space of holomorphic functions $\mathrm{Hol}(\mathbb{D},\mathbb{C}^n)$. Define a Hilbert space structure on $\mathscr{H}^{(Y,\eta)}$ by stipulating $\Gamma^{(Y,\eta)}$ to be unitary. We thus have a reproducing kernel Hilbert space.

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Transfer the natural $\tilde{G}$ - action on $\mathbf{A}^{(\eta)} = \oplus_{j=0}^{m} n_j \mathbb{A}^{(\eta+j)}$ to $\mathscr{H}^{(Y,\eta)}$. This actions lifts to a multiplier representation on $\mathscr{H}^{(Y,\eta)}$ with multiplier $J_g^{(Y,\eta)}(z) = D_g(z)\exp(-cY)D_g(z)$, where $D_g(z)$ is the diagonal matrix with $D_g(z)_{j,j} = (cz+d)^{-\frac{j}{2}} I_{d_j}$.

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