Contractive and completely contractive homomorphisms over function algebras

Gadadhar Misra

Indian Institute of Science
Bangalore
(joint with Avijit Pal and Cherian Varughese)

Functional Analysis Seminar
University of Leipzig
July 21, 2015

the homomorphisms that we study

- Let $\|\cdot\|_{\mathbf{A}}$ be a norm on $\mathbb{C}^m$ given by the formula

$$\|(z_1,\ldots,z_m)\|_{\mathbf{A}} = \|z_1A_1+\cdots+z_mA_m\|_{\mathrm{op}}$$

  for some choice of m matrices $\mathbf{A}=(A_1,\ldots,A_m)$. Let $\Omega_{\mathbf{A}}$ be the corresponding unit ball. Let $\mathscr{O}(\Omega_{\mathbf{A}})$ denote the algebra of all functions holomorphic on any open set U containing the closed unit ball $\bar{\Omega}_{\mathbf{A}}$.

- Given $p\times q$ matrices $V_1,\ldots,V_m$ and a function $f\in\mathscr{O}(\Omega_{\mathbf{A}})$, define, for a fixed $w\in\Omega_{\mathbf{A}}$, the homomorphism

$$\rho_{\mathbf{V}}(f) := \begin{pmatrix} f(0)I_p & \sum_{i=1}^m \partial_i f(0)\ V_i \\ 0 & f(0)I_q \end{pmatrix}$$

We study contractivity and complete contractivity of such homomorphisms.

the homomorphisms that we study

- Let $\|\cdot\|_{\mathbf{A}}$ be a norm on $\mathbb{C}^m$ given by the formula

$$\|(z_1,\dots,z_m)\|_{\mathbf{A}} = \|z_1A_1+\cdots+z_mA_m\|_{\mathrm{op}}$$

  for some choice of m matrices $\mathbf{A}=(A_1,\dots,A_m)$. Let $\Omega_{\mathbf{A}}$ be the corresponding unit ball. Let $\mathscr{O}(\Omega_{\mathbf{A}})$ denote the algebra of all functions holomorphic on any open set U containing the closed unit ball $\bar{\Omega}_{\mathbf{A}}$.
- Given $p\times q$ matrices $V_1,\dots,V_m$ and a function $f\in\mathscr{O}(\Omega_{\mathbf{A}})$, define, for a fixed $w\in\Omega_{\mathbf{A}}$, the homomorphism

$$\rho_{\mathbf{V}}(f) := \begin{pmatrix} f(0)I_p & \sum_{i=1}^m \partial_i f(0)\ V_i \\ 0 & f(0)I_q \end{pmatrix}$$

We study contractivity and complete contractivity of such homomorphisms.

the Parrott examples

- Consider the linear map $L_{\mathbf{V}} : (\mathbb{C}^m, \|\cdot\|_{\mathbf{A}}^*) \to \mathscr{M}_{p\times q}(\mathbb{C}),$ given by the formula
$$L_{\mathbf{V}}(z) = z_1 V_1 + \cdots + z_m V_m$$
induced by the homomorphism $\rho_{\mathbf{V}}$.
- The contractivity (resp. complete contractivity) of the homomorphism $\rho_{\mathbf{V}}$ determines the contractivity (resp. complete contractivity) of the linear map $L_{\mathbf{V}}$ and vice-versa.
- It is known that contractive homomorphisms of the disc and the bi-disc algebras are completely contractive, thanks to the dilation theorems of B. Sz.-Nagy and Ando respectively.

the Parrott examples

- Consider the linear map $L_{\mathbf{V}} : (\mathbb{C}^m, \|\cdot\|^*_{\mathbf{A}}) \to \mathscr{M}_{p\times q}(\mathbb{C}),$ given by the formula

$$L_{\mathbf{V}}(z) = z_1 V_1 + \cdots + z_m V_m$$

  induced by the homomorphism $\rho_{\mathbf{V}}$.
- The contractivity (resp. complete contractivity) of the homomorphism $\rho_{\mathbf{V}}$ determines the contractivity (resp. complete contractivity) of the linear map $L_{\mathbf{V}}$ and vice-versa.
- It is known that contractive homomorphisms of the disc and the bi-disc algebras are completely contractive, thanks to the dilation theorems of B. Sz.-Nagy and Ando respectively.

the Parrott examples

- Consider the linear map $L_{\mathbf{V}} : (\mathbb{C}^m, \|\cdot\|_{\mathbf{A}}^*) \to \mathscr{M}_{p\times q}(\mathbb{C}),$ given by the formula

$$L_{\mathbf{V}}(z) = z_1 V_1 + \cdots + z_m V_m$$

  induced by the homomorphism $\rho_{\mathbf{V}}.$
- The contractivity (resp. complete contractivity) of the homomorphism $\rho_{\mathbf{V}}$ determines the contractivity (resp. complete contractivity) of the linear map $L_{\mathbf{V}}$ and vice-versa.
- It is known that contractive homomorphisms of the disc and the bi-disc algebras are completely contractive, thanks to the dilation theorems of B. Sz.-Nagy and Ando respectively.

examples

- However, examples of contractive homomorphisms $\rho_{\mathbf{V}}$ of the tri-disc algebra that are not completely contractive were soon found by Parrott. The homomorphisms $\rho_{\mathbf{V}}$ are modelled on the examples of Parrott. Homomorphsims of this form also provide examples of contractive homomorphisms of the (Euclidean) ball algebra which are not completely contractive.
- From the work of V. Paulsen and E. Ricard, it follows that if $m \geq 3$ and $\mathbb{B}$ is any ball in $\mathbb{C}^m$ with respect to some norm, say $\|\cdot\|_{\mathbb{B}}$, then there exists a contractive linear map $L : (\mathbb{C}^m, \|\cdot\|^*_{\mathbb{B}}) \to \mathscr{B}(\mathscr{H})$ which is not complete contractive. The characterization of those balls in $\mathbb{C}^2$ for which contractive linear maps are always completely contractive remained open. We answer this question for balls of the form $\Omega_{\mathbf{A}}$ in $\mathbb{C}^2$.

examples

- However, examples of contractive homomorphisms $\rho_{\mathbf{V}}$ of the tri-disc algebra that are not completely contractive were soon found by Parrott. The homomorphisms $\rho_{\mathbf{V}}$ are modelled on the examples of Parrott. Homomorphsims of this form also provide examples of contractive homomorphisms of the (Euclidean) ball algebra which are not completely contractive.
- From the work of V. Paulsen and E. Ricard, it follows that if $m \geq 3$ and $\mathbb{B}$ is any ball in $\mathbb{C}^m$ with respect to some norm, say $\|\cdot\|_{\mathbb{B}}$, then there exists a contractive linear map $L : (\mathbb{C}^m, \|\cdot\|_{\mathbb{B}}^*) \to \mathscr{B}(\mathscr{H})$ which is not complete contractive. The characterization of those balls in $\mathbb{C}^2$ for which contractive linear maps are always completely contractive remained open. We answer this question for balls of the form $\Omega_{\mathbf{A}}$ in $\mathbb{C}^2$.

linear maps on the dual unit ball

- A straightforward application of the vonNeumann inequality shows that $\sup_{\|f\|_\infty=1}\{\|\rho_{\mathbf{V}}(f)\|_{\rm op} : f \in \mathscr{O}(\Omega_{\mathbf{A}})\} \leq 1$ if and only if $\sup_{\|g\|_\infty=1}\{\|\rho_{\mathbf{V}}(g)\|_{\rm op} : g \in \mathscr{O}(\Omega_{\mathbf{A}}), g(0)=0\} \leq 1.$ Thus $\rho_{\mathbf{V}}$ is contractive on $\mathscr{O}(\Omega_{\mathbf{A}})$ if and only if it is contractive on the subset of functions which vanish at $0.$
- Let Ω_A^* denote the unit ball of the normed linear space $(\mathbb{C}^m, \|\cdot\|_A)^*.$ An easy application of the Schwarz lemma then shows that

$$\Omega_{\mathbf{A}}^* = \big\{\big(\partial_1 f(0), \partial_2 f(0), \cdots, \partial_m f(0)\big) : f \in \mathrm{Hol}(\Omega_{\mathbf{A}}, \mathbb{D}), f(0)=0\big\}.$$

- Hence $\|\rho_{\mathbf{V}}\| \leq 1$ iff $\sup_{\|f\|_\infty=1, f(0)=0} \|\sum_{i=1}^m \partial_i f(0)\, V_i\|_{\rm op} \leq 1.$ Thus the induced linear map $L_{\mathbf{V}}(w) = z_1 V_1 + \cdots + z_m V_m$ is contractive if and only if the homomorphism $\rho_{\mathbf{V}}$ is contractive.

linear maps on the dual unit ball

- A straightforward application of the vonNeumann inequality shows that $\sup_{\|f\|_\infty=1}\{\|\rho_{\mathbf{V}}(f)\|_{\text{op}} : f \in \mathscr{O}(\Omega_{\mathbf{A}})\} \leq 1$ if and only if $\sup_{\|g\|_\infty=1}\{\|\rho_{\mathbf{V}}(g)\|_{\text{op}} : g \in \mathscr{O}(\Omega_{\mathbf{A}}), g(0)=0\} \leq 1.$ Thus $\rho_{\mathbf{V}}$ is contractive on $\mathscr{O}(\Omega_{\mathbf{A}})$ if and only if it is contractive on the subset of functions which vanish at $0.$
- Let Ω_A^* denote the unit ball of the normed linear space $(\mathbb{C}^m, \|\cdot\|_A)^*.$ An easy application of the Schwarz lemma then shows that

$$\Omega_{\mathbf{A}}^* = \{\big(\partial_1 f(0), \partial_2 f(0), \cdots, \partial_m f(0)\big) : f \in \text{Hol}(\Omega_{\mathbf{A}}, \mathbb{D}),\, f(0)=0\}.$$

- Hence $\|\rho_{\mathbf{V}}\| \leq 1$ iff $\sup_{\|f\|_\infty=1, f(0)=0} \|\sum_{i=1}^m \partial_i f(0)\, V_i\|_{\text{op}} \leq 1.$ Thus the induced linear map $L_{\mathbf{V}}(w) = z_1 V_1 + \cdots + z_m V_m$ is contractive if and only if the homomorphism $\rho_{\mathbf{V}}$ is contractive.

linear maps on the dual unit ball

- A straightforward application of the vonNeumann inequality shows that $\sup_{\|f\|_\infty=1}\{\|\rho_{\mathbf{V}}(f)\|_{\rm op} : f \in \mathscr{O}(\Omega_{\mathbf{A}})\} \le 1$ if and only if $\sup_{\|g\|_\infty=1}\{\|\rho_{\mathbf{V}}(g)\|_{\rm op} : g \in \mathscr{O}(\Omega_{\mathbf{A}}), g(0)=0\} \le 1.$ Thus $\rho_{\mathbf{V}}$ is contractive on $\mathscr{O}(\Omega_{\mathbf{A}})$ if and only if it is contractive on the subset of functions which vanish at $0.$
- Let Ω_A^* denote the unit ball of the normed linear space $(\mathbb{C}^m, \|\cdot\|_A)^*.$ An easy application of the Schwarz lemma then shows that

$$\Omega_{\mathbf{A}}^* = \big\{\big(\partial_1 f(0), \partial_2 f(0), \cdots, \partial_m f(0)\big) : f \in \mathrm{Hol}(\Omega_{\mathbf{A}}, \mathbb{D}),\ f(0)=0\big\}.$$

- Hence $\|\rho_{\mathbf{V}}\| \le 1$ iff $\sup_{\|f\|_\infty=1, f(0)=0} \ \|\sum_{i=1}^m \partial_i f(0)\ V_i\|_{\rm op} \le 1.$ Thus the induced linear map $L_{\mathbf{V}}(w) = z_1 V_1 + \cdots + z_m V_m$ is contractive if and only if the homomorphism $\rho_{\mathbf{V}}$ is contractive.

complete contractivity

- For a holomorphic function $F:\Omega_{\mathbf{A}}\to\mathscr{M}_k,$ define

$$\rho_{\mathbf{V}}^{(k)}(F):=(\rho_{\mathbf{V}}(F_{ij}))_{i,j=1}^m=\begin{pmatrix} F(0)\otimes I & \sum_{i=1}^m(\partial_i F(0))\otimes V_i\\ 0 & F(0)\otimes I\end{pmatrix}.$$

Using a method similar to that used for $\rho_{\mathbf{V}}$ it can be shown that

$$\|\rho_{\mathbf{V}}^{(k)}\|\leq 1 \text{ if and only if } \sup_F\{\|\sum_{i=1}^m(\partial_i F(0))\otimes V_i\|\}\leq 1,$$

where the supremum is taken over all holomorphic functions $F:\Omega_A\to(\mathscr{M}_k)_1,\ \ F(0)=0.$ That is, by repeating the argument used for $\rho_{\mathbf{V}},$ we have

$$\|\rho_{\mathbf{V}}^{(k)}\|\leq 1 \text{ if and only if } \|L_{\mathbf{V}}^{(k)}\|\leq 1,$$

where $L_{\mathbf{V}}^{(k)}:(\mathbb{C}^m\otimes\mathscr{M}_k,\|\cdot\|^*_{\Omega_{\mathbf{A}},k})\to(\mathscr{M}_k\otimes\mathscr{M}_{p,q},\|\cdot\|_{\mathrm{op}})$ is the map $L_{\mathbf{V}}^{(k)}(\Theta_1,\Theta_2,\cdots,\Theta_m)=\Theta_1\otimes V_1+\Theta_2\otimes V_2+\cdots+\Theta_m\otimes V_m$ for $(\Theta_1,\Theta_2,\cdots,\Theta_m)\in\mathscr{M}_k$

the polynomial $P_{\mathbf{A}}$

- A very useful construct for our analysis is the matrix valued polynomial $P_{\mathbf{A}} : \Omega_{\mathbf{A}} \to (\mathscr{M}_n, \|\cdot\|_{\text{op}})_1$ defined by

$$P_{\mathbf{A}}(z_1, z_2, \cdots, z_m) = z_1 A_1 + z_2 A_2 + \cdots + z_m A_m,$$

that is, $\|P_{\mathbf{A}}\|_\infty := \sup_{(z_1,\cdots,z_m)\in\Omega_{\mathbf{A}}} \|P_{\mathbf{A}}(z)\|_{\text{op}} = 1$ by definition.

- The typical procedure used to show the existence of a homomorphism which is contractive but not completely contractive is to construct a contractive homomorphism $\rho_{\mathbf{V}}$ (by making a suitable choice of $\mathbf{V}$) and to then show that its evaluation on $P_{\mathbf{A}}$, that is, $\rho_{\mathbf{V}}^{(n)}(P_{\mathbf{A}})$, has norm greater than 1.

the polynomial $P_{\mathbf{A}}$

- A very useful construct for our analysis is the matrix valued polynomial $P_{\mathbf{A}} : \Omega_{\mathbf{A}} \to (\mathscr{M}_n, \|\cdot\|_{\mathrm{op}})_1$ defined by

$$P_{\mathbf{A}}(z_1, z_2, \cdots, z_m) = z_1 A_1 + z_2 A_2 + \cdots + z_m A_m,$$

that is, $\|P_{\mathbf{A}}\|_\infty := \sup_{(z_1,\cdots,z_m)\in\Omega_{\mathbf{A}}} \|P_{\mathbf{A}}(z)\|_{\mathrm{op}} = 1$ by definition.

- The typical procedure used to show the existence of a homomorphism which is contractive but not completely contractive is to construct a contractive homomorphism $\rho_{\mathbf{V}}$ (by making a suitable choice of $\mathbf{V}$) and to then show that its evaluation on $P_{\mathbf{A}}$, that is, $\rho_{\mathbf{V}}^{(n)}(P_{\mathbf{A}})$, has norm greater than $1.$

defining function and test functions

- For $(\alpha, \beta) \in \mathbb{B}^2 \times \mathbb{B}^2$, define $p_{\mathbf{A}}^{(\alpha,\beta)} : \Omega_{\mathbf{A}} \to \mathbb{C}$ to be the map $p_{\mathbf{A}}^{(\alpha,\beta)}(z_1, z_2) = \langle P_{\mathbf{A}}(z_1, z_2)\alpha, \beta\rangle = z_1\langle A_1\alpha, \beta\rangle + z_2\langle A_2\alpha, \beta\rangle$, which is linear. The sup norm $\|p_{\mathbf{A}}^{(\alpha,\beta)}\|_\infty \leq 1$ by definition.
- Let $\mathscr{P}_{\mathbf{A}}$ denote the collection of linear functions $\{p_{\mathbf{A}}^{(\alpha,\beta)} : (\alpha, \beta) \in \mathbb{B}^2 \times \mathbb{B}^2\}$.
- The map $P_{\mathbf{A}}$, which we call the defining function of the domain and the collection of functions $\mathscr{P}_{\mathbf{A}}$, which we call a family of test functions encode a significant amount of information relevant to our purpose about the homomorphism $\rho_{\mathbf{V}}$. For instance, $\rho_{\mathbf{V}}$ is contractive if its restriction to $\mathscr{P}_{\mathbf{A}}$ is contractive. By evaluating $\rho_{\mathbf{V}}^{(2)}$ on $P_{\mathbf{A}}$, one may often detect the lack of complete contractivity – $\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_{\mathbf{V}}(p_{\mathbf{A}}^{(\alpha,\beta)})\| \leq \|\rho_{\mathbf{V}}^{(2)}(P_{\mathbf{A}})\|$.

defining function and test functions

- For $(\alpha,\beta) \in \mathbb{B}^2 \times \mathbb{B}^2$, define $p_{\mathbf{A}}^{(\alpha,\beta)} : \Omega_{\mathbf{A}} \to \mathbb{C}$ to be the map $p_{\mathbf{A}}^{(\alpha,\beta)}(z_1,z_2) = \langle P_{\mathbf{A}}(z_1,z_2)\alpha,\beta\rangle = z_1\langle A_1\alpha,\beta\rangle + z_2\langle A_2\alpha,\beta\rangle$, which is linear. The sup norm $\|p_{\mathbf{A}}^{(\alpha,\beta)}\|_\infty \leq 1$ by definition.
- Let $\mathscr{P}_{\mathbf{A}}$ denote the collection of linear functions $\{p_{\mathbf{A}}^{(\alpha,\beta)} : (\alpha,\beta) \in \mathbb{B}^2 \times \mathbb{B}^2\}$.
- The map $P_{\mathbf{A}}$, which we call the defining function of the domain and the collection of functions $\mathscr{P}_{\mathbf{A}}$, which we call a family of test functions encode a significant amount of information relevant to our purpose about the homomorphism $\rho_{\mathbf{V}}$. For instance, $\rho_{\mathbf{V}}$ is contractive if its restriction to $\mathscr{P}_{\mathbf{A}}$ is contractive. By evaluating $\rho_{\mathbf{V}}^{(2)}$ on $P_{\mathbf{A}}$, one may often detect the lack of complete contractivity – $\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_{\mathbf{V}}(p_{\mathbf{A}}^{(\alpha,\beta)})\| \leq \|\rho_{\mathbf{V}}^{(2)}(P_{\mathbf{A}})\|$.

defining function and test functions

- For $(\alpha,\beta) \in \mathbb{B}^2 \times \mathbb{B}^2$, define $p_{\mathbf{A}}^{(\alpha,\beta)} : \Omega_{\mathbf{A}} \to \mathbb{C}$ to be the map $p_{\mathbf{A}}^{(\alpha,\beta)}(z_1,z_2) = \langle P_{\mathbf{A}}(z_1,z_2)\alpha,\beta\rangle = z_1\langle A_1\alpha,\beta\rangle + z_2\langle A_2\alpha,\beta\rangle$, which is linear. The sup norm $\|p_{\mathbf{A}}^{(\alpha,\beta)}\|_\infty \leq 1$ by definition.
- Let $\mathscr{P}_{\mathbf{A}}$ denote the collection of linear functions $\{p_{\mathbf{A}}^{(\alpha,\beta)} : (\alpha,\beta) \in \mathbb{B}^2 \times \mathbb{B}^2\}$.
- The map $P_{\mathbf{A}}$, which we call the defining function of the domain and the collection of functions $\mathscr{P}_{\mathbf{A}}$, which we call a family of test functions encode a significant amount of information relevant to our purpose about the homomorphism $\rho_{\mathbf{V}}$. For instance, $\rho_{\mathbf{V}}$ is contractive if its restriction to $\mathscr{P}_{\mathbf{A}}$ is contractive. By evaluating $\rho_{\mathbf{V}}^{(2)}$ on $P_{\mathbf{A}}$, one may often detect the lack of complete contractivity – $\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_{\mathbf{V}}(p_{\mathbf{A}}^{(\alpha,\beta)})\| \leq \|\rho_{\mathbf{V}}^{(2)}(P_{\mathbf{A}})\|$.

- Choosing $\mathbf{A} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$, we see that $\Omega_{\mathbf{A}}$ defines the Euclidean ball $\mathbb{B}^2$ in $\mathbb{C}^2$.

Theorem

For any pair $V_1 = \begin{pmatrix} v_{11} & v_{12} \end{pmatrix}, V_2 = \begin{pmatrix} v_{21} & v_{22} \end{pmatrix}$ *and* $\Omega_{\mathbf{A}} = \mathbb{B}^2$, *we have*

(i) $\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_{\mathbf{V}}(p_{\mathbf{A}}^{(\alpha,\beta)})\|^2 = \|\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}\|_{op}^2$

(ii) $\|\rho_{\mathbf{V}}^{(2)}(P_{\mathbf{A}})\|_{op}^2 = \|\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}\|_2^2$

Consequently, $\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_{\mathbf{V}}(p_{\mathbf{A}}^{(\alpha,\beta)})\| < \|\rho_{\mathbf{V}}^{(2)}(P_{\mathbf{A}})\|_{op}$ *if* V_1 *and* V_2 *are linearly independent.*

- Choosing $\mathbf{A} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$, we see that $\Omega_{\mathbf{A}}$ defines the Euclidean ball $\mathbb{B}^2$ in $\mathbb{C}^2$.

Theorem
For any pair $V_1 = \begin{pmatrix} v_{11} & v_{12} \end{pmatrix}, V_2 = \begin{pmatrix} v_{21} & v_{22} \end{pmatrix}$ *and* $\Omega_{\mathbf{A}} = \mathbb{B}^2$, *we have*

(i) $\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_{\mathbf{V}}(p_{\mathbf{A}}^{(\alpha,\beta)})\|^2 = \|\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}\|_{\mathrm{op}}^2$

(ii) $\|\rho_{\mathbf{V}}^{(2)}(P_{\mathbf{A}})\|_{\mathrm{op}}^2 = \|\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}\|_2^2$.

Consequently, $\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_{\mathbf{V}}(p_{\mathbf{A}}^{(\alpha,\beta)})\| < \|\rho_{\mathbf{V}}^{(2)}(P_{\mathbf{A}})\|_{\mathrm{op}}$ *if* V_1 *and* V_2 *are linearly independent.*

the example of the (Euclidean) ball algebra

- Choosing $\mathbf{A} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \right),$ we see that $\Omega_{\mathbf{A}}$ defines the Euclidean ball $\mathbb{B}^2$ in $\mathbb{C}^2.$

Theorem

For any pair $V_1 = \begin{pmatrix} v_{11} & v_{12} \end{pmatrix}, V_2 = \begin{pmatrix} v_{21} & v_{22} \end{pmatrix}$ *and* $\Omega_{\mathbf{A}} = \mathbb{B}^2,$ *we have*

(i) $\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_{\mathbf{V}}(p_{\mathbf{A}}^{(\alpha,\beta)})\|^2 = \| \left(\begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix}\right) \|_{\mathrm{op}}^2,$

(ii) $\|\rho_{\mathbf{V}}^{(2)}(P_{\mathbf{A}})\|_{\mathrm{op}}^2 = \| \left(\begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix}\right) \|_2^2.$

Consequently, $\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_{\mathbf{V}}(p_{\mathbf{A}}^{(\alpha,\beta)})\| < \|\rho_{\mathbf{V}}^{(2)}(P_{\mathbf{A}})\|_{\mathrm{op}}$ *if* V_1 *and* V_2 *are linearly independent.*

the example of the (Euclidean) ball algebra

- Choosing $\mathbf{A} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \right)$, we see that $\Omega_{\mathbf{A}}$ defines the Euclidean ball $\mathbb{B}^2$ in $\mathbb{C}^2$.

Theorem

For any pair $V_1 = \begin{pmatrix} v_{11} & v_{12} \end{pmatrix}, V_2 = \begin{pmatrix} v_{21} & v_{22} \end{pmatrix}$ *and* $\Omega_{\mathbf{A}} = \mathbb{B}^2$, *we have*

(i) $\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_{\mathbf{V}}(p_{\mathbf{A}}^{(\alpha,\beta)})\|^2 = \| \left(\begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix}\right) \|_{\mathrm{op}}^2,$

(ii) $\|\rho_{\mathbf{V}}^{(2)}(P_{\mathbf{A}})\|_{\mathrm{op}}^2 = \| \left(\begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix}\right) \|_2^2.$

Consequently, $\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_{\mathbf{V}}(p_{\mathbf{A}}^{(\alpha,\beta)})\| < \|\rho_{\mathbf{V}}^{(2)}(P_{\mathbf{A}})\|_{\mathrm{op}}$ *if* V_1 *and* V_2 *are linearly independent.*

the example of the (Euclidean) ball algebra

- Choosing $\mathbf{A} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right)$, we see that $\Omega_{\mathbf{A}}$ defines the Euclidean ball $\mathbb{B}^2$ in $\mathbb{C}^2$.

Theorem

For any pair $V_1 = \begin{pmatrix} v_{11} & v_{12} \end{pmatrix}, V_2 = \begin{pmatrix} v_{21} & v_{22} \end{pmatrix}$ *and* $\Omega_{\mathbf{A}} = \mathbb{B}^2$, *we have*

(i) $\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_{\mathbf{V}}(p_{\mathbf{A}}^{(\alpha,\beta)})\|^2 = \| \left(\begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix}\right) \|_{\text{op}}^2$,

(ii) $\|\rho_{\mathbf{V}}^{(2)}(P_{\mathbf{A}})\|_{\text{op}}^2 = \| \left(\begin{smallmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{smallmatrix}\right) \|_2^2$.

Consequently, $\sup_{\|\alpha\|=\|\beta\|=1} \|\rho_{\mathbf{V}}(p_{\mathbf{A}}^{(\alpha,\beta)})\| < \|\rho_{\mathbf{V}}^{(2)}(P_{\mathbf{A}})\|_{\text{op}}$ *if* V_1 *and* V_2 *are linearly independent.*

unitary equivalence and linear equivalence

- Set $\widetilde{\mathbf{A}} = (UA_1W, UA_2W)$ for any pair of 2×2 unitary matrices U and W. Then
$$\|(z_1, z_2)\|_{\mathbf{A}} = \|z_1(UA_1W) + z_2(UA_2W)\|_{\text{op}} = \|(z_1, z_2)\|_{\widetilde{\mathbf{A}}}.$$
There are therefore various choices of the pairs (A_1, A_2), related as above, which give rise to the same norm which may be used to ensure A_1 is diagonal.
- For $\mathbf{z} = (z_1, z_2)$ in $(\mathbb{C}^2, \|\cdot\|_{\mathbf{A}})$, let T be the linear transformation
$$\tilde{z}_1 = pz_1 + qz_2, \tilde{z}_2 = rz_1 + sz_2,$$
where $p, q, r, s \in \mathbb{C}$. Then $\|T\mathbf{z}\|_{\mathbf{A}} = \|\mathbf{z}\|_{\tilde{\mathbf{A}}}, \tilde{\mathbf{A}} = T \otimes I$
- In our study of the existence of contractive homomorphisms which are not completely contractive, two sets of matrices $\mathbf{A} = (A_1, A_2)$ and $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2)$, which are related through linear combinations as above, yield the same result.

unitary equivalence and linear equivalence

- Set $\widetilde{\mathbf{A}} = (UA_1W, UA_2W)$ for any pair of 2×2 unitary matrices U and W. Then
$$\|(z_1,z_2)\|_{\mathbf{A}} = \|z_1(UA_1W)+z_2(UA_2W)\|_{\text{op}} = \|(z_1,z_2)\|_{\widetilde{\mathbf{A}}}.$$
There are therefore various choices of the pairs (A_1,A_2), related as above, which give rise to the same norm which may be used to ensure A_1 is diagonal.
- For $\mathbf{z} = (z_1,z_2)$ in $(\mathbb{C}^2, \|\cdot\|_{\mathbf{A}})$, let T be the linear transformation
$$\tilde{z}_1 = pz_1 + qz_2, \tilde{z}_2 = rz_1 + sz_2,$$
where $p,q,r,s \in \mathbb{C}$. Then $\|T\mathbf{z}\|_{\mathbf{A}} = \|\mathbf{z}\|_{\tilde{\mathbf{A}}}, \tilde{\mathbf{A}} = T\otimes I$
- In our study of the existence of contractive homomorphisms which are not completely contractive, two sets of matrices $\mathbf{A} = (A_1,A_2)$ and $\widetilde{\mathbf{A}} = (\widetilde{A}_1,\widetilde{A}_2)$, which are related through linear combinations as above, yield the same result.

unitary equivalence and linear equivalence

- Set $\widetilde{\mathbf{A}} = (UA_1W, UA_2W)$ for any pair of 2×2 unitary matrices U and W. Then
$$\|(z_1, z_2)\|_{\mathbf{A}} = \|z_1(UA_1W) + z_2(UA_2W)\|_{\text{op}} = \|(z_1, z_2)\|_{\widetilde{\mathbf{A}}}.$$
There are therefore various choices of the pairs (A_1, A_2), related as above, which give rise to the same norm which may be used to ensure A_1 is diagonal.
- For $\mathbf{z} = (z_1, z_2)$ in $(\mathbb{C}^2, \|\cdot\|_{\mathbf{A}})$, let T be the linear transformation
$$\tilde{z}_1 = pz_1 + qz_2, \tilde{z}_2 = rz_1 + sz_2,$$
where $p, q, r, s \in \mathbb{C}$. Then $\|T\mathbf{z}\|_{\mathbf{A}} = \|\mathbf{z}\|_{\tilde{\mathbf{A}}}, \tilde{\mathbf{A}} = T \otimes I$
- In our study of the existence of contractive homomorphisms which are not completely contractive, two sets of matrices $\mathbf{A} = (A_1, A_2)$ and $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2)$, which are related through linear combinations as above, yield the same result.

a reduction

- Since A_1 has already been chosen to be diagonal, we consider transformations as above with $q = 0$ to preserve the diagonal structure of A_1. By further conjugating with a diagonal unitary and a permutation matrix it follows that we need to consider only the following three families of matrices:

A_1	A_2
$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$ $d \in \mathbb{C}$	$\begin{pmatrix} 1 & b \\ c & 0 \end{pmatrix} c \in \mathbb{C}, b \in \mathbb{R}_+$
$\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$ $d \in \mathbb{C}$	$\begin{pmatrix} 1 & b \\ c & 0 \end{pmatrix} c \in \mathbb{C}, b \in \mathbb{R}_+$
$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$ $d \in \mathbb{C}$	$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} c \in \mathbb{C}, b \in \mathbb{R}_+$

the dual space

- Fix $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case, the dual norm $\|\cdot\|^*_{\Omega_{\mathbf{A}}}$ is given by the formula:

$$\|(\omega_1, \omega_2)\|^*_{\Omega_{\mathbf{A}}} = \begin{cases} \frac{|\omega_1|^2 + 4|\omega_2|^2}{4|\omega_2|} & \text{if } |\omega_2| \geq \frac{|\omega_1|}{2}; \\ |\omega_1| & \text{if } |\omega_2| \leq \frac{|\omega_1|}{2}. \end{cases}$$

Equipped with the information about the dual norm we can directly construct a pair $\mathbf{V} = (V_1, V_2)$ such that $\|L_{\mathbf{V}}\| \leq 1$ and $\|L^{(2)}_{\mathbf{V}}(P_{\mathbf{A}})\| > 1$.

Theorem
Picking $V_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, V_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$, *we have*

(i) $\|L_{\mathbf{V}}\|_{(\mathbb{C}^2, \|\cdot\|^*_{\Omega_{\mathbf{A}}}) \to (\mathbb{C}^2, \|\cdot\|_2)} = 1$,

(ii) $\|L^{(2)}_{\mathbf{V}}(P_{\mathbf{A}})\| = \sqrt{\frac{3}{2}}$.

Consequently $\rho_{\mathbf{V}}$, *for this choice of* $\mathbf{V} = (V_1, V_2)$, *is contractive on* $\mathscr{O}(\Omega_{\mathbf{A}})$ *but not completely contractive.*

the dual space

- Fix $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case, the dual norm $\|\cdot\|^*_{\Omega_{\mathbf{A}}}$ is given by the formula:

$$\|(\omega_1, \omega_2)\|^*_{\Omega_{\mathbf{A}}} = \begin{cases} \frac{|\omega_1|^2+4|\omega_2|^2}{4|\omega_2|} & \text{if } |\omega_2| \geq \frac{|\omega_1|}{2}; \\ |\omega_1| & \text{if } |\omega_2| \leq \frac{|\omega_1|}{2}. \end{cases}$$

Equipped with the information about the dual norm we can directly construct a pair $\mathbf{V} = (V_1, V_2)$ such that $\|L_{\mathbf{V}}\| \leq 1$ and $\|L^{(2)}_{\mathbf{V}}(P_{\mathbf{A}})\| > 1$.

the dual space

- Fix $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case, the dual norm $\|\cdot\|^*_{\Omega_{\mathbf{A}}}$ is given by the formula:

$$\|(\omega_1,\omega_2)\|^*_{\Omega_{\mathbf{A}}} = \begin{cases} \frac{|\omega_1|^2+4|\omega_2|^2}{4|\omega_2|} & \text{if } |\omega_2| \geq \frac{|\omega_1|}{2}; \\ |\omega_1| & \text{if } |\omega_2| \leq \frac{|\omega_1|}{2}. \end{cases}$$

Equipped with the information about the dual norm we can directly construct a pair $\mathbf{V} = (V_1, V_2)$ such that $\|L_{\mathbf{V}}\| \leq 1$ and $\|L^{(2)}_{\mathbf{V}}(P_{\mathbf{A}})\| > 1$.

Theorem

Picking $V_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, V_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$, *we have*

(i) $\|L_{\mathbf{V}}\|_{(\mathbb{C}^2, \|\cdot\|^*_{\Omega_{\mathbf{A}}}) \to (\mathbb{C}^2, \|\cdot\|_2)} = 1$,

(ii) $\|L^{(2)}_{\mathbf{V}}(P_{\mathbf{A}})\| = \sqrt{\frac{3}{2}}$.

Consequently $\rho_{\mathbf{V}}$, *for this choice of* $\mathbf{V} = (V_1, V_2)$, *is contractive on* $\mathscr{O}(\Omega_{\mathbf{A}})$ *but not completely contractive.*

the dual space

- Fix $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case, the dual norm $\|\cdot\|^*_{\Omega_{\mathbf{A}}}$ is given by the formula:

$$\|(\omega_1, \omega_2)\|^*_{\Omega_{\mathbf{A}}} = \begin{cases} \frac{|\omega_1|^2 + 4|\omega_2|^2}{4|\omega_2|} & \text{if } |\omega_2| \geq \frac{|\omega_1|}{2}; \\ |\omega_1| & \text{if } |\omega_2| \leq \frac{|\omega_1|}{2}. \end{cases}$$

Equipped with the information about the dual norm we can directly construct a pair $\mathbf{V} = (V_1, V_2)$ such that $\|L_{\mathbf{V}}\| \leq 1$ and $\|L^{(2)}_{\mathbf{V}}(P_{\mathbf{A}})\| > 1$.

Theorem

Picking $V_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, V_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$, *we have*

(i) $\|L_{\mathbf{V}}\|_{(\mathbb{C}^2, \|\cdot\|^*_{\Omega_{\mathbf{A}}}) \to (\mathbb{C}^2, \|\cdot\|_2)} = 1$,

(ii) $\|L^{(2)}_{\mathbf{V}}(P_{\mathbf{A}})\| = \sqrt{\frac{3}{2}}$.

Consequently $\rho_{\mathbf{V}}$, *for this choice of* $\mathbf{V} = (V_1, V_2)$, *is contractive on* $\mathscr{O}(\Omega_{\mathbf{A}})$ *but not completely contractive.*

the dual space

- Fix $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case, the dual norm $\|\cdot\|^*_{\Omega_\mathbf{A}}$ is given by the formula:

$$\|(\omega_1, \omega_2)\|^*_{\Omega_\mathbf{A}} = \begin{cases} \frac{|\omega_1|^2 + 4|\omega_2|^2}{4|\omega_2|} & \text{if } |\omega_2| \geq \frac{|\omega_1|}{2}; \\ |\omega_1| & \text{if } |\omega_2| \leq \frac{|\omega_1|}{2}. \end{cases}$$

Equipped with the information about the dual norm we can directly construct a pair $\mathbf{V} = (V_1, V_2)$ such that $\|L_\mathbf{V}\| \leq 1$ and $\|L^{(2)}_\mathbf{V}(P_\mathbf{A})\| > 1$.

Theorem
Picking $V_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, V_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$, *we have*

(i) $\|L_\mathbf{V}\|_{(\mathbb{C}^2, \|\cdot\|^*_{\Omega_\mathbf{A}}) \to (\mathbb{C}^2, \|\cdot\|_2)} = 1$,

(ii) $\|L^{(2)}_\mathbf{V}(P_\mathbf{A})\| = \sqrt{\frac{3}{2}}$.

Consequently $\rho_\mathbf{V}$, *for this choice of* $\mathbf{V} = (V_1, V_2)$, *is contractive on* $\mathscr{O}(\Omega_\mathbf{A})$ *but not completely contractive.*

- The existence of contractive homomorphisms which are not completely contractive, in many cases, may by established by comparing different isometric embeddings of the space $(\mathbb{C}^2, \|\cdot\|_{\mathbf{A}})$ into $(\mathscr{M}_2, \|\cdot\|_{\rm op})$ which lead to distinct operator space structures. For instance, the two embeddings $(z_1, z_2) \mapsto z_1 A_1 + z_2 A_2$ and $(z_1, z_2) \mapsto z_1 A_1^{\rm t} + z_2 A_2^{\rm t}$ give rise to distinct operator space structures on $(\mathbb{C}^2, \|\cdot\|_2)$ and for many others.
- The opposite phenomenon also occurs, namely, many distinct isometric embeddings of $(\mathbb{C}^2, \|\cdot\|_{\mathbf{A}})$ into $(\mathscr{M}_n, \|\cdot\|_{\rm op})$ yield (completely isometric) operator space structures. This is seen easily by means of the lemma that follows.

operator space structure

- The existence of contractive homomorphisms which are not completely contractive, in many cases, may by established by comparing different isometric embeddings of the space $(\mathbb{C}^2, \|\cdot\|_{\mathbf{A}})$ into $(\mathscr{M}_2, \|\cdot\|_{\rm op})$ which lead to distinct operator space structures. For instance, the two embeddings $(z_1, z_2) \mapsto z_1 A_1 + z_2 A_2$ and $(z_1, z_2) \mapsto z_1 A_1^{\rm t} + z_2 A_2^{\rm t}$ give rise to distinct operator space structures on $(\mathbb{C}^2, \|\cdot\|_2)$ and for many others.
- The opposite phenomenon also occurs, namely, many distinct isometric embeddings of $(\mathbb{C}^2, \|\cdot\|_{\mathbf{A}})$ into $(\mathscr{M}_n, \|\cdot\|_{\rm op})$ yield (completely isometric) operator space structures. This is seen easily by means of the lemma that follows.

operator space structure, contd.

Lemma

For $B \in \mathscr{M}_{m,n}$ *and* $\alpha_1, \alpha_2 \in \mathbb{C}$, *we have* $\left\| \begin{pmatrix} \alpha_1 I_m & B \\ 0 & \alpha_2 I_n \end{pmatrix} \right\| = \left\| \begin{pmatrix} \alpha_1 & \|B\| \\ 0 & \alpha_2 \end{pmatrix} \right\|$.

- Now consider the pair $\mathbf{A} = (A_1, A_2)$ with $A_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$. Given any $m \times n$ matrix B with $\|B\| = |\beta|$ we have the following isometric embedding of $(\mathbb{C}^2, \|\cdot\|_{\mathbf{A}})$ into $(\mathscr{M}_{m+n}, \|\cdot\|_{\text{op}})$

$$(z_1, z_2) \mapsto \begin{pmatrix} z_1 \alpha_1 I_m & z_2 B \\ 0 & z_1 \alpha_2 I_n \end{pmatrix}.$$

For various choices of m, n and the matrix B this represents a large collection of isometric embeddings, all of which give the same operator space structure on $(\mathbb{C}^2, \|\cdot\|_{\mathbf{A}})$!

operator space structure, contd.

Lemma

For $B \in \mathscr{M}_{m,n}$ *and* $\alpha_1, \alpha_2 \in \mathbb{C}$, *we have* $\left\| \begin{pmatrix} \alpha_1 I_m & B \\ 0 & \alpha_2 I_n \end{pmatrix} \right\| = \left\| \begin{pmatrix} \alpha_1 & \|B\| \\ 0 & \alpha_2 \end{pmatrix} \right\|$.

- Now consider the pair $\mathbf{A} = (A_1, A_2)$ with $A_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$. Given any $m \times n$ matrix B with $\|B\| = |\beta|$ we have the following isometric embedding of $(\mathbb{C}^2, \|\cdot\|_{\mathbf{A}})$ into $(\mathscr{M}_{m+n}, \|\cdot\|_{\text{op}})$

$$(z_1, z_2) \mapsto \begin{pmatrix} z_1 \alpha_1 I_m & z_2 B \\ 0 & z_1 \alpha_2 I_n \end{pmatrix}.$$

For various choices of m, n and the matrix B this represents a large collection of isometric embeddings, all of which give the same operator space structure on $(\mathbb{C}^2, \|\cdot\|_{\mathbf{A}})$!

Thank you!