The Bergman kernel

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Let $\mathscr{D}$ be a bounded open connected subset of $\mathbb{C}^m$ and $\mathbb{A}^2(\mathscr{D})$ be the Hilbert space of square integrable (with respect to volume measure) holomorphic functions on $\mathscr{D}$. The Bergman kernel $B : \mathscr{D} \times \mathscr{D} \to \mathbb{C}$ is uniquely defined by the two requirements:

$$\begin{aligned} B_w &\in \mathbb{A}^2(\mathscr{D}) && \text{for all } w \in \mathscr{D} \\ \langle f, B_w \rangle &= f(w) && \text{for all } f \in A^2(\mathscr{D}). \end{aligned}$$

The existence of B_w is guaranteed as long as the evaluation functional $f \to f(w)$ is bounded.

We have $B_w(z) = \langle B_w, B_z \rangle$. Consequently, for any choice of $n \in \mathbb{N}$ and an arbitrary subset $\{w_1, \ldots, w_n\}$ of $\mathscr{D}$, the $n \times n$ matrix $\left(\!\left(B_{w_i}(w_j)\right)\!\right)_{i,j=1}^{n}$ must be positive definite.

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Fourier series

Notice first that if $e_n(z), \quad n \geq 0$ is an orthonormal basis for the Bergman space $\mathbb{A}^2(\mathscr{D}),$ then any $f \in \mathbb{A}^2(\mathscr{D})$ has the Fourier series expansion $f(z) = \sum_{n=0}^{\infty} a_n e_n(z).$ Assuming that the sum

$$B_w(z) := \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)},$$

is in $\mathbb{A}^2(\mathscr{D})$ for each $w \in \mathscr{D},$ we see that

$$\langle f(z), B_w(z) \rangle = f(w), w \in \mathscr{D}.$$

example

For the Bergman space $\mathbb{A}^2(\mathbb{D}^m)$, of the polydisc $\mathbb{D}^m$, the orthonormal basis is $\{\sqrt{\prod_{i=1}^m (n_i+1)}z^I : I=(i_1,\ldots,i_m)\}$. Clearly, we have

$$B_{\mathbb{D}^m}(z,w)=\sum_{|I|=0}^{\infty}\Big(\prod_{i=1}^{m}(n_i+1)\Big)z^I\bar{w}^I=\prod_{i=1}^{m}(1-z_i\bar{w}_i)^{-2}.$$

Similarly, for the Bergman space of the ball $\mathbb{A}^2(\mathbb{B}^m)$, the orthonormal basis is $\{\sqrt{\binom{-m-1}{|I|}\binom{|I|}{I}}z^I : I=(i_1,\ldots,i_m)\}$. Again, it follows that

$$B_{\mathbb{B}^m}(z,w)=\sum_{|I|=0}^{\infty}\binom{-m-1}{\ell}\Big(\sum_{|I|=\ell}\binom{|I|}{I}z^I\bar{w}^I\Big)=(1-\langle z,w\rangle)^{-m-1}.$$

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For the Bergman space $\mathbb{A}^2(\mathbb{D}^m)$, of the polydisc $\mathbb{D}^m$, the orthonormal basis is $\{\sqrt{\prod_{i=1}^m (n_i+1)} z^I : I = (i_1, \ldots, i_m)\}$. Clearly, we have

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quasi-invariance of B

Any bi-holomorphic map $\varphi : \mathscr{D} \to \tilde{\mathscr{D}}$ induces a unitary operator $U_\varphi : \mathbb{A}^2(\tilde{\mathscr{D}}) \to \mathbb{A}^2(\mathscr{D})$ defined by the formula

$$(U_\varphi f)(z) = J(\varphi, z)\big(f \circ \varphi\big)(z),\ f \in \mathbb{A}^2(\tilde{\mathscr{D}}),\ z \in \mathscr{D}.$$

This is an immediate consequence of the change of variable formula for the volume measure on $\mathbb{C}^n$:

$$\int_{\tilde{\mathscr{D}}} f\, dV = \int_{\mathscr{D}} (f \circ \varphi)\, |J_{\mathbb{C}}\varphi|^2 dV.$$

Consequently, if $\{\tilde{e}_n\}_{n\geq 0}$ is any orthonormal basis for $\mathbb{A}^2(\tilde{\mathscr{D}})$, then $\{e_n\}_{n\geq 0}$, where $\tilde{e}_n = J(\varphi,\cdot)(\tilde{e}_n \circ \varphi)$ is an orthonormal basis for the Bergman space $\mathbb{A}^2(\tilde{\mathscr{D}})$.

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quasi-invariance of B

Expressing the Bergman kernel $B_{\mathscr{D}}$ of the domains $\mathscr{D}$ as the infinite sum $\sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$ using the orthonormal basis in $\mathbb{A}^2(\mathscr{D})$, we see that the Bergman Kernel B is *quasi-invariant*, that is, If $\varphi : \mathscr{D} \to \widetilde{\mathscr{D}}$ is holomorphic then we have the transformation rule

$$J(\varphi, z)B_{\tilde{\mathscr{D}}}(\varphi(z), \varphi(w))\overline{J(\varphi, w)} = B_{\mathscr{D}}(z, w),$$

where $J(\varphi, w)$ is the Jacobian determinant of the map φ at w.

If $\mathscr{D}$ admits a transitive group of bi-holomorphic automorphisms, then this transformation rule gives an effective way of computing the Bergman kernel. Thus

$$B_{\mathscr{D}}(z, z) = |J(\varphi_z, z)|^2 B_{\mathscr{D}}(0, 0),\ z \in \mathscr{D},$$

where φ_z is the automorphism of $\mathscr{D}$ with the property $\varphi_z(z) = 0$.

quasi-invariance of B

Expressing the Bergman kernel $B_{\mathscr{D}}$ of the domains $\mathscr{D}$ as the infinite sum $\sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$ using the orthonormal basis in $\mathbb{A}^2(\mathscr{D})$, we see that the Bergman Kernel B is *quasi-invariant*, that is, If $\varphi : \mathscr{D} \to \widetilde{\mathscr{D}}$ is holomorphic then we have the transformation rule

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the invariant

The quasi-invariance of the Bergman kernel $B_{\mathscr{D}}(z;w)$ also leads to a bi-holomorphic invariant for the domain $\mathscr{D}$. Setting

$$\mathscr{K}_{B_{\mathscr{D}}}(z) = \big(\tfrac{\partial^2}{\partial z_i \partial \bar{z}_j} \log B_{\mathscr{D}}\big)(z)$$

to be the curvature of the metric $B_{\mathscr{D}}(z,z)$, the function

$$\mathbb{I}_{\mathscr{D}}(z) := \frac{\det \mathscr{K}_{B_{\mathscr{D}}}(z)}{B_{\mathscr{D}}(z)}, \; z \in \mathscr{D}$$

is a bi-holomorphic invariant for the domain $\mathscr{D}$.

the multiplier

Consider the special case, where $\varphi : \mathscr{D} \to \mathscr{D}$ is an automorphism. Clearly, in this case, U_φ is unitary on $\mathbb{A}^2(\mathscr{D})$ for all $\varphi \in \mathrm{Aut}(\mathscr{D})$.

The map $J : \mathrm{Aut}(\mathscr{D}) \times \mathscr{D} \to \mathbb{C}$ satisfies the cocycle property, namely

$$J(\psi\varphi, z) = J(\varphi, \psi(z))J(\psi, z),\ \varphi, \psi \in \mathrm{Aut}(\mathscr{D}),\ z \in \mathscr{D}.$$

This makes the map $\varphi \to U_\varphi$ a homomorphism.

Thus we have a unitary representation of the Lie group $\mathrm{Aut}(\mathscr{D})$ on $\mathbb{A}^2(\mathscr{D})$.

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the proof that $\mathbb{I}_{\mathscr{D}}$ is an invariant

Let $\varphi : \mathscr{D} \to \tilde{\mathscr{D}}$ be a bi-holomorphic map. Applying the change of variable formula twice to the function $\log B_{\tilde{\mathscr{D}}}(\varphi(z), \varphi(w)),$ we have

$$\Big(\!\Big(\frac{\partial^2}{\partial z_i \partial \bar{w}_j} \log B_{\tilde{\mathscr{D}}}(\varphi(z), \varphi(w))\Big)\!\Big)_{ij} = \Big(\!\Big(\frac{\partial \varphi_\ell}{\partial z_i}\Big)\!\Big)_{i\ell} \Big(\!\Big(\Big(\frac{\partial^2}{\partial z_\ell \partial \bar{w}_k} \log B_{\mathscr{D}}\Big)(\varphi(z), \varphi(w))\Big)\!\Big)_{\ell k} \Big(\!\Big(\frac{\partial \bar{\varphi}_k}{\partial \bar{z}_j}\Big)\!\Big)_{kj}.$$

Now, the Bergman kernel $B_{\mathscr{D}}$ transforms according to the rule:

$$\det{}_{\mathbb{C}} D\varphi(w) B_{\tilde{\mathscr{D}}}(\varphi(w), \varphi(w)) \overline{\det{}_{\mathbb{C}} D\varphi(w)} = B_{\mathscr{D}}(w, w),$$

Thus $\mathscr{K}_{B_{\tilde{\mathscr{D}}} \circ (\varphi, \varphi)}(w, w)$ equals $\mathscr{K}_{B_{\mathscr{D}}}(w, w).$ Hence we conclude that $\mathscr{K}_{B_{\mathscr{D}}}$ is quasi-invariant under a bi-holomorphic map φ , namely,

$$D\varphi(w)^{\sharp} \mathscr{K}_{\tilde{\mathscr{D}}}(\varphi(w), \varphi(w)) \overline{D\varphi(w)} = \mathscr{K}_{\mathscr{D}}(w, w), \; w \in \mathscr{D}.$$

the proof cntd.

Taking determinants on both sides we get

$$\det \mathscr{K}_{\mathscr{D}}(w,w) = J_C \varphi f(z) \det \mathscr{K}_{\tilde{\mathscr{D}}}(\varphi(w),\varphi(w)).$$

Thus we get the invariance of $\mathbb{I}_{\mathscr{D}}$:

$$\begin{aligned}
\frac{\det \mathscr{K}_{\mathscr{D}}(w,w)}{B_{\mathscr{D}}(w,w)} &= \frac{|J_{\mathbb{C}}\varphi(z)|^2 \det \mathscr{K}_{\tilde{\mathscr{D}}}(\varphi(w),\varphi(w))}{B_{\mathscr{D}}(w,w)} \\
&= \frac{|J_{\mathbb{C}}\varphi(z)|^2 \det \mathscr{K}_{\tilde{\mathscr{D}}}(\varphi(w),\varphi(w))}{|J_{\mathbb{C}}\varphi(w)|^2 B_{\tilde{\mathscr{D}}}(\varphi(w),\varphi(w))} \\
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Theorem

For any homogeneous domain $\mathscr{D}$ in $\mathbb{C}^n$, the function $\mathbb{I}_{\mathscr{D}}(z)$ is constant.

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Theorem

For any homogeneous domain $\mathscr{D}$ *in* $\mathbb{C}^n$, *the function* $\mathbb{I}_{\mathscr{D}}(z)$ *is constant.*

proof of the theorem

Since $\mathscr{D} \subseteq \mathbb{C}^n$ is homogeneous, it follows that there exists a bi-holomorphic map φ_u of $\mathscr{D}$ for each $u \in \mathscr{D}$ such that $\varphi_u(0) = u$. Applying the transformation rule for $\mathbb{I}$, we have

$$\begin{aligned}\mathbb{I}_{\mathscr{D}}(0) &= \frac{\det \mathscr{K}_{\mathscr{D}}(0,0)}{B_{\mathscr{D}}(0,0)} \\ &= \frac{\det \mathscr{K}_{\mathscr{D}}(\varphi_u(0),\varphi_u(0))}{B_{\mathscr{D}}(\varphi_u(0),\varphi_u(0))} \\ &= \frac{\det \mathscr{K}_{\mathscr{D}}(u,u)}{B_{\mathscr{D}}(u,u)} = \mathbb{I}_{\mathscr{D}}(u),\ u \in \mathscr{D}\end{aligned}$$

It is easy to compute $\mathbb{I}_{\mathscr{D}}(0)$ when $\mathscr{D}$ is the bi-disc and the Euclidean ball in $\mathbb{C}^2$. For these two domains, it has the value 4 and 9 respectively. We conclude that these domains therefore can't be bi-holomorphically equivalent!

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new kernels?

Let K be a complex valued positive definite kernel on $\mathscr{D}$. For w in $\mathscr{D}$, and p in the set $\{1,\ldots,d\}$, let $e_p:\Omega\to\mathscr{H}$ be the antiholomorphic function:

$$e_p(w) := K_w(\cdot)\otimes\frac{\partial}{\partial\bar{w}_p}K_w(\cdot) - \frac{\partial}{\partial\bar{w}_p}K_w(\cdot)\otimes K_w(\cdot).$$

Setting $G(z,w)_{p,q}=\langle e_p(w),e_q(z)\rangle$, we have

$$\frac{1}{2}G(z,w)_{p,q}{}^{\sharp} = K(z,w)\frac{\partial^2}{\partial z_q\partial\bar{w}_p}K(z,w) - \frac{\partial}{\partial\bar{w}_p}K(z,w)\frac{\partial}{\partial z_q}K(z,w)).$$

The curvature K of the metric K is given by the $(1,1)$ - form $\sum\frac{\partial^2}{\partial w_q\partial\bar{w}_p}\log K(w,w)dw_q\wedge d\bar{w}_p$. Set

$$\mathscr{K}_K(z,w) := \Big(\!\Big(\frac{\partial^2}{\partial z_q\partial\bar{w}_p}\log K(z,w)\Big)\!\Big)_{qp}.$$

We note that $K(z,w)^2\mathscr{K}(z,w)=\frac{1}{2}G(z,w)^{\sharp}$. Hence $K(z,w)^2\mathscr{K}(z,w)$ defines a positive definite kernel on $\mathscr{D}$ taking values in $\mathrm{Hom}(V,V)$.

rewrite the transformation rule

Or equivalently,

$$\begin{aligned}\mathscr{K}(\varphi(z),\varphi(w)) &= D\varphi(z)^{\sharp^{-1}}\mathscr{K}(z,w)\overline{D\varphi(z)}^{-1}\\ &= D\varphi(z)^{\sharp^{-1}}\mathscr{K}(z,w)\left(D\varphi(w)^{\sharp^{-1}}\right)^{*}\\ &= m_0(\varphi,z)\mathscr{K}(z,w)m_0(\varphi,w)^{*},\end{aligned}$$

where $m_0(\varphi,z)=D\varphi(z)^{\sharp^{-1}}$ and multiplying both sides by $K^2,$ we have

$$K(\varphi(z),\varphi(w))^2\mathscr{K}(\varphi(z),\varphi(w)) = m_2(\varphi,z)K(z,w)^2\mathscr{K}(z,w)m_2(\varphi,w)^{*},$$

where $m_2(\varphi,z)=\left(\det_{\mathbb{C}}D\varphi(w)^2 D\varphi(z)^{\sharp}\right)^{-1}$ is a multiplier. Of course, we now have that

(i) $K^{2+\lambda}(z,w)\mathscr{K}(z,w)$, $\lambda>0,$ is a positive definite kernel and

(ii) it transforms with the co-cycle $m_\lambda(\varphi,z)=\left(\det_{\mathbb{C}}D\varphi(z)^{2+\lambda}D\varphi(z)^{\dagger}\right)^{-1}$ in place of $m_2(\varphi,z)$.

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Thank you!