Curvature inequalities for operators in the Cowen-Douglas class

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Gadadhar Misra
joint with
S. Biswas, D. Keshari, A. Pal, Md. R. Reza and S. Ghara

Indian Statistical Institute
Bangalore
And
Indian Institute of Technology
Gandhinagar

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in conclusion ...

- Suppose the restiction of a bounded operator T on a Hilbert space $\mathscr{H}$ to "all" the two dimensional subspaces is contractive. Then it does not necessarily follow that the operator T is contractive.
- Suppose that the operator T possesses an eigenvector $\gamma(w)$ for w in some open set in $U \subseteq \mathbb{C}$ and that the map $w \mapsto \gamma(w)$ is holomorphic. Then the restriction of the operator $T - w$ to the two dimensional subspaces $\{\gamma(w), \gamma'(w)\}$, $w \in U$ is nilpotent and encodes important information about the operator T. Indeed, in some instances, "as we have seen", this information is enough to determine the unitary equivalence class of the operator T.
- While the norm bound for the operator T is not related to those of the two dimensional restrictions directly, it (metric inequalities) can be extracted from these (curvature inequalities)!
- Without any additional effort, may work with commuting tuples of bounded operators on a Hilbert space possessing an open set of joint eigenvalues w in some open set $U \subseteq \mathbb{C}^m$.

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holomorphic functions

- Let $\mathscr{H}$ be a Hilbert space and $\mathbb{D}$ be the unit disc. Suppose that there exists a map $\gamma : \mathbb{D} \to \mathscr{H}$ which is holomorphic, that is, the complex valued function

$$w \to \langle \gamma(w), \zeta \rangle,\ w \in \mathbb{D},$$

is holomorphic for every vector ζ in $\mathscr{H}$.

- The derivative $\gamma'(w) : \mathbb{C} \to \mathscr{H}$ of the map γ at w may therefore be thought of as a vector in $\mathscr{H}$.
- Let $\Gamma(w) \subseteq \mathscr{H},\ w \in \mathbb{D},$ be the subspace consisting of the two linearly independent vectors $\gamma(w)$ and $\gamma'(w)$.

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Nilpotent action on $\Gamma(w)$

- There is a natural nilpotent action $N(w)$ on the space $\Gamma(w)$ determined by the rule

$$\gamma'(w) \xrightarrow{N(w)} \gamma(w) \xrightarrow{N(w)} 0.$$

- Let $e_0(w), e_1(w)$ be the orthonormal basis for $\Gamma(w)$ obtained from $\gamma(w), \gamma'(w)$ by the Gram-Schmidt orthonormalization. The matrix representation of $N(w)$ with respect to this orthonormal basis is of the form $\begin{pmatrix} 0 & h(w) \\ 0 & 0 \end{pmatrix}$.
- It is easy to compute $h(w)$. Indeed, we have

$$h(w) = \frac{\|\gamma(w)\|^2}{(\|\gamma'(w)\|^2 \|\gamma(w)\|^2 - |\langle \gamma'(w), \gamma(w) \rangle|^2)^{\frac{1}{2}}}.$$

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contraction

- Now, the operator $wI + N(w) = \begin{pmatrix} w & h(w) \\ 0 & w \end{pmatrix}$ defined on $\Gamma(w)$ is contractive if and only if $h(w) \le 1 - |w|^2$.
- Let $\mathscr{H}$ be the Hilbert space $\ell^2(\mathbb{N})$ and $\gamma_0(w) = (1, w, w^2, \ldots, w^n, \ldots)$. Clearly, $\langle \gamma_0(w), \zeta \rangle = \zeta_0 + w\bar{\zeta}_1 + \cdots + w^n \bar{\zeta}_n + \cdots$ is holomorphic for every choice of $\zeta = (\zeta_0, \zeta_1 \ldots \zeta_n, \ldots)$ in $\ell^2(\mathbb{N})$.
- Now, $\gamma_0'(w) = (0, 1, 2w, \ldots, nw^{n-1}, \ldots)$. A simple computation gives $h_0(w) = 1 - |w|^2$ and thus $\left\| \begin{pmatrix} w & h_0(w) \\ 0 & w \end{pmatrix} \right\| = 1$.
- This is the restriction of the unilateral backward shift operator to the invariant subspace $\Gamma(w) \subseteq \ell^2(\mathbb{N})$.

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curvature

- The holomorphic function γ admits a power series expansion in some small neighborhood of 0, say, $\gamma(w) = \sum_{k=0}^{\infty} \zeta_k w^k$, $\zeta_k \in \mathcal{H}$. Then we have
$$\|\gamma(w)\|^2 = \langle \gamma(w), \gamma(w) \rangle = \sum_{j,k} \langle \zeta_j, \zeta_k \rangle w^j \bar{w}^k.$$

- Using the linearity of differentiation, we then find that

$$\begin{aligned}
\mathcal{K}(w) &:= -\frac{\partial^2}{\partial \bar{w} \partial w} \log \langle \gamma(w), \gamma(w) \rangle \\
&= -\frac{\partial}{\partial \bar{w}} \frac{\langle \frac{\partial}{\partial w} \gamma(w), \gamma(w) \rangle}{\langle \gamma(w), \gamma(w) \rangle} \\
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negative curvature

- The Cauchy - Schwarz inequality implies that

$$\|\frac{\partial}{\partial w}\gamma(w)\|^2\|\gamma(w)\|^2 - |\langle\frac{\partial}{\partial w}\gamma(w),\gamma(w)\rangle|^2 \geq 0.$$

It therefore follows that the curvature $\mathscr{K}(w)$ is negative.

- Since $h(w)^2 = -\frac{1}{\mathscr{K}(w)}$, setting

$$\mathscr{K}_0(w) := -\frac{1}{h_0(w)^2} = -\frac{1}{(1-|w|^2)^2},$$

we conclude that the inequality $h(w) \leq (1-|w|^2)$ is equivalent to the curvature inequality $\mathscr{K}(w) \leq \mathscr{K}_0(w)$.

- Let $\mathscr{L}$ be the trivial holomorphic line bundle over the unit disc $\mathbb{D}$. We can think of γ as a frame for $\mathscr{L}$ with the induced metric given by $g(w) := \|\gamma(w)\|^2$, $w \in \mathbb{D}$. Then $\mathscr{K}$ is the curvature of the line bundle $\mathscr{L}$.

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a class of operators

- Let $T : \mathscr{H} \to \mathscr{H}$ be a bounded linear operator for which
  a) each $w \in \mathbb{D}$ is an eigenvalue,
  b) the $w \mapsto \gamma(w)$, where $\gamma(w)$ is the eigenvector with eigenvalue w is holomorphic.
  c) the dimension of the eigenspace is 1.
- The class of operators $B_1(\mathbb{D})$ was introduced by Cowen and Douglas. They showed, among other things, that the unitary equivalence class of the operator T and the equivalence class of holomorphic Hermitian bundle $\mathscr{L}$ determined by the holomorphic frame γ determine each other.
- As a result, the curvature function $\mathscr{K}$ is a complete invariant for the operator T.
- Also, they show that an operator T in this class is unitarily equivalent to the adjoint M^* of the multiplication operator M by the co-ordinate function on some Hilbert space $\mathscr{H}$ of holomorphic functions on $\Omega^* := \{z \in \mathbb{C} : \bar{z} \in \Omega\}$ possessing a reproducing kernel

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kernel function

- Let $\Omega^* := \{\bar{w} : w \in \Omega\}$. A kernel function $K : \Omega^* \times \Omega^* \to \mathbb{C}$ is holomorphic in the first and anti-holomorphic in the second variable. Therefore, the map $w \to K(\cdot, \bar{w}), w \in \Omega$ is holomorphic.
- It is Hermitian, $K(z, w) = \overline{K(w, z)}$, and positive definite, that is, $\big(\!\big(K(w_i, w_j)\big)\!\big)_{i,j=1}^{n}$ is positive definite for every subset $\{w_1, \ldots, w_n\}$ of Ω^*, $n \in \mathbb{N}$.
- For any fixed $w \in \Omega^*$, the holomorphic function $K(\cdot, w)$ belongs to $\mathscr{H}$ and

$$f(w) = \langle f, K(\cdot, w)\rangle, \; f \in \mathscr{H}, \; w \in \Omega^*.$$

- The reproducing property of K ensures that $M^* K(\cdot, w) = \bar{w} K(\cdot, w)$. Therefore, for $w \in \Omega$, $\gamma(\bar{w}) := K(\cdot, \bar{w})$ defines a natural holomorphic frame for the operator M^*.

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- The reproducing property of K ensures that $M^* K(\cdot, w) = \bar{w} K(\cdot, w)$. Therefore, for $w \in \Omega$, $\gamma(\bar{w}) := K(\cdot, \bar{w})$ defines a natural holomorphic frame for the operator M^*.

curvature inequality

- For any operator T in the calsss $B_1(\Omega)$, we have $(T - wI)\gamma(w) = 0$. Differentiating with respect to w, we see that

$$T\gamma'(w) = \gamma(w) + w\gamma'(w).$$

Thus the restriction of $T - wI$ to the subspace $\Gamma(w)$ is nilpotent of order 2. We therefore set $N_T(w) := (T - wI)_{|\Gamma(w)}$. We assign the natural meaning to h_T and $\mathscr{K}_T$.

- The backward shift S_- acting on the space $\ell^2(\mathbb{N})$ is easily seen to satisfy all of a), b) and c) with $\gamma(w) = (1, w, w^2, \ldots, w^n, \ldots)$. The curvature $\mathscr{K}_{S_-}(w)$ coincides with $\mathscr{K}_0(w) = -(1 - |w|^2)^{-2}$.

PROPOSITION

If T is a contraction in $B_1(\mathbb{D})$, then $\mathscr{K}_T(w) \le \mathscr{K}_{S_-}(w)$.

Proof. If T is a contraction, then clearly so is the operator $wI + N_T(w)$ and the contractivity of $wI + N_T(w)$ is equivalent to $\mathscr{K}_T(w) \le \mathscr{K}_{S_-}(w)$.

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weighted shifts

- Let $\mathscr{H}$ be the space $\ell^2(\mathbb{N})$, as before. Now, let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be a weighted shift, that is, $T(a_0, a_1, \ldots, a_n, \ldots) = (a_1 w_0, \ldots, a_n w_{n-1}, \ldots)$ for some choice of $w_0, \ldots, w_1, \ldots \in \mathbb{C}$. For $w \in \mathbb{C}$ with $|w|$ small, it is possible to find complex numbers $\alpha_0, \alpha_1, \ldots$ such that

$$T(\alpha_0, \alpha_1 w, \alpha_2 w^2, \ldots) = w(\alpha_0, \alpha_1 w, \alpha_2 w^2, \ldots)$$

  and having the additional property that the dimension of this eigenspace is 1.

- Now, the operator T is contractive if and only if $\sup_n w_n \le 1$. Here

$$\begin{aligned} \|\gamma(w)\|^2 &= \|(\alpha_0, \alpha_1 w, \ldots, \alpha_n w^n, \ldots)\|^2 \\ &= \sum_{n=0}^{\infty} |\alpha_n|^2 |w|^{2n} \end{aligned}$$

- Thus $\mathscr{K}_T(w) = -\frac{\partial^2}{\partial \bar{w} \partial w} \log \|\gamma(w)\|^2 \le \mathscr{K}_{S_-}(w)$, assuming only that $\sup_n w_n \le 1$.

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an alternative description

- The curvature inequality for a contraction becomes evident after we make the following observations.
- Verify, using the two properties:

$$M^* K(\cdot, w) = \bar{w} K(\cdot, w) \text{ the closed linear span of } \{K(\cdot, w) : w \in \mathbb{D}\} = \mathscr{H},$$

that

$$\|M^*\| \le 1 \text{ if and only if } K_0(z, w) := (1 - \bar{w}z) K(z, w)$$

is positive definite. But the curvature of the metric $K_0(w, w)$ is always negative, that is,

$$\begin{aligned} 0 &\ge -\frac{\partial^2}{\partial \bar{w} \partial w} \log K_0(w, w) \\ &= -\frac{\partial^2}{\partial \bar{w} \partial w} \log K(w, w) + (1 - |w|^2)^{-2}, \end{aligned}$$

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a counter example

- What about the converse? We give an example to show that the converse is false in general.
- Let W be the weighted shift operator with the weight sequence $\{\sqrt{\frac{1}{2}}, \sqrt{\frac{16}{15}}, 1, 1, \ldots\}$. Evidently, it is not a contraction. However, in this case, we can pick $\gamma(w)$ with $\|\gamma(w)\|^2 = \frac{8+8|w|^2-|w|^4}{1-|w|^2}$. Clearly, we have

$$-\frac{\partial^2}{\partial w \partial \bar{w}} \log(8+8|w|^2-|w|^4) = -\frac{8(8-4|w|^2-|w|^4)}{(8+8|w|^2-|w|^4)^2}, \; w \in \mathbb{D},$$

which is negative for $|w|<1$. In this example, we therefore find that $\mathcal{K}_W(w) \leq \mathcal{K}_{S_-}(w)$, although the operator W is not a contraction.

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polarization

- If γ is holomorphic and admits the power series expansion $\gamma(w) = \zeta_0 + \zeta_1 w + \zeta_2 w^2 + \cdots,$ then the norm $\|\gamma(w)\|^2$ is a function of w and $\bar{w}$. It has the form

$$\sum_{j,k=0}^{\infty} \langle \zeta_j, \zeta_k \rangle w^j \bar{w}^k, \; \zeta_0, \zeta_2, \ldots \in \mathscr{H}.$$

  Polarizing $\|\gamma(w)\|^2$, we obtain a new function $\tilde{\gamma}(z,w) := \langle \gamma(z), \gamma(w) \rangle$.

- Thus $\big(\!\big(\tilde{\gamma}(z_i, z_j)\big)\!\big)$ is non negative definite for all choices of $z_1, \ldots z_n$ in $\mathbb{D}$. This is just the positive-definiteness of the kernel function $K(z,w) = \langle \gamma(z), \gamma(w) \rangle$!
- The curvature $\mathscr{K}$ is a real analytic function and we have shown that $-\mathscr{K}$ is positive.
- Let $\widetilde{\mathscr{K}}(z,w) := \frac{\partial^2}{\partial \bar{w} \partial z} \log \tilde{\gamma}(z,w)$ denote the function obtained from polarization of the curvature $\mathscr{K}$.
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a secret

- Refining the computation that established the positivity of $\mathscr{K}$, we obtain a stronger inequality. Set

$$\varphi(w) := K(\cdot, w) \otimes \bar{\partial} K(\cdot, w) - \bar{\partial} K(\cdot, w) \otimes K(\cdot, w).$$

Note that $\varphi(w) \in \mathscr{H}$, $w \in \mathbb{D}$.

- Moreover, a straightforward computation using the reproducing property of K shows that

$$\begin{aligned} \langle \varphi(z), \varphi(w) \rangle &= \| \frac{\partial}{\partial w} \gamma(w) \|^2 \| \gamma(w) \|^2 - | \langle \frac{\partial}{\partial w} \gamma(w), \gamma(w) \rangle |^2 \\ &= \| \varphi(w) \|^4 \frac{\partial^2}{\partial \bar{w} \partial w} \log \| \varphi(w) \|^{-2} \\ &= -\mathscr{G}_{K^{-2}}(w), \end{aligned}$$

where $\gamma(w) = K(\cdot, w)$, as before and $\mathscr{G}_{K^{-2}}$ is the Gaussian curvature of the metric $K(w,w)^{-2}$.

- Thus the Gaussian curvature $\mathscr{G}_{K^{-2}}$ is a non-negative definite kernel.

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PROPOSITION

Let $T \in B_1(\mathbb{D})$ *be a contraction. Assume that* T *is unitarily equivalent to the operator* M^* *on* $(\mathcal{H}, K)$ *for some non-negative definite kernel* K *on the unit disc. Then the following inequality holds:*

$$K^2(z,w) \preceq \mathbb{S}_{\mathbb{D}}^{-2}(z,w)\mathcal{G}_{K^{-1}}(z,w),$$

that is, the matrix

$$\left(\!\left(\mathbb{S}_{\mathbb{D}}^{-2}(w_i,w_j)\mathcal{G}_{K^{-1}}(w_i,w_j) - K^2(w_i,w_j)\right)\!\right)_{i,j=1}^{n}$$

is non-negative definite for every subset $\{w_1,\ldots,w_n\}$ *of* $\mathbb{D}$ *and* $n \in \mathbb{N}$.

the proof

- Setting $G(z,w) = (1 - z\bar{w})K(z,w),$ we see that

$$
\begin{aligned}
&-G(z,w)^2 \partial\bar{\partial} \log G(z,w) \\
&\qquad = (1 - z\bar{w})^2 K^2(z,w)\big(-\partial\bar{\partial}\log K(z,w) + \partial\bar{\partial}\log \mathbb{S}_{\mathbb{D}}(z,w)\big),
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  $z, w \in \mathbb{D}.$ Since $G(z,w)$ is non-negative definite on $\mathbb{D} \times \mathbb{D},$ it follows that

$$(1 - z\bar{w})^2 K(z,w)^2 \big(-\partial\bar{\partial}\log K(z,w) + \partial\bar{\partial}\log \mathbb{S}_{\mathbb{D}}(z,w)\big) \preceq 0.$$

  Also, $\mathbb{S}_{\mathbb{D}}(z,w)^{-2}\partial\bar{\partial}\log \mathbb{S}_{\mathbb{D}}(z,w) = 1,$ therefore the proof is complete.

- The inequality for the Gaussian curvature is stronger than the ordinary curvature inequality. For instance, this stronger form of the inequality does not hold for the example $\|\gamma(w)\|^2 = \frac{8+8|w|^2-|w|^4}{1-|w|^2}.$

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contractivity and infinite divisibility

- Say that a positive definite kernel K is infinitely divisible if K^t is positive definite for all $t > 0$. Ask if assuming that the kernel $K(z, w)$ is is both necessary and sufficient for positive definiteness of the curvature function $-\widetilde{\mathscr{K}}$.
- The answer is affirmative!
- Putting all this together we have the following theorem:

Theorem

Let $T : \mathscr{H} \to \mathscr{H}$ be a bounded linear operator satisfying a), b) and c) admitting a holomorphic frame $\gamma : \mathbb{D} \to \mathscr{H}$. Assume that $(1 - z\bar{w})\tilde{\gamma}(z, w)$ is infinitely divisible. Then T is contractive if and only if the function

$$-\widetilde{\mathscr{K}}_T(z, w) + \widetilde{\mathscr{K}}_{S_-^*}(z, w)$$

is positive definite.

contractivity and infinite divisibility

- Say that a positive definite kernel K is infinitely divisible if K^t is positive definite for all $t > 0$. Ask if assuming that the kernel $K(z, w)$ is is both necessary and sufficient for positive definiteness of the curvature function $-\widetilde{\mathcal{K}}$.
- The answer is affirmative!
- Putting all this together we have the following theorem:

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the proof

Proof.

- If the kernel K is infinitely divisible then $\log K$ must be conditionally positive definite. This is the same as

$$K_0(z,w) := \log K(z,w) - \log K(z,w_0) - \log K(w_0,w) + \log K(w_0,w_0)$$

is a positive definite kernel for a fixed but arbitrary $w_0 \in \Omega$. After differentiating K_0 twice, we obtain $\widetilde{\mathscr{K}}$ which is positive definite.

- Conversely, anti-differentiating $\widetilde{\mathscr{K}}_0$, determines $\log K_0$ up to addition of a holomorphic function φ and its complex conjugate. Recall that if $\log K_0$ is positive definite then K_0 is infinitely divisible.

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curvature inequality, strong form

Definition

If K is a non negative definite kernel such that $(1 - z\bar{w})K(z, w)$ is infinitely divisible then we say that M on $\mathscr{H}_K$ is infinitely divisible contraction.

Corollary

Let K be a positive definite kernel on the open unit disc. Assume that the the adjoint M^* of the multiplication operator M on the reproducing kernel Hilbert space $(\mathscr{H}, K)$ belongs to $\mathrm{B}_1(\mathbb{D})$. Then the polarization of the function $\frac{\partial^2}{\partial w \partial \bar{w}} \log\big((1 - w\bar{w})K(w, w)\big)$ is positive definite if and only if the multiplication operator M is an infinitely divisible contraction.

Thank You!