Alfors' Schwarz Lemma

Mathematical Sciences Institute Belagavi
in association with
The(Indian) Mathematics Consortium
Lecture Series On Operator Theory

Gadadhar Misra
Indian Statistical Institute
Bangalore
And
Indian Institute of Technology
Gandhinagar

February 25-27, 2022

the Schwarz Lemma

- Let $B(0,r)=\{z\in\mathbb{C}:|z|<r\}$ be tht open disc of radius r and $B[0,r]$ be closed disc of radius r.

the Schwarz Lemma

- Let $B(0,r) = \{z \in \mathbb{C} : |z| < r\}$ be tht open disc of radius r and $B[0,r]$ be closed disc of radius $r.$
- Schwarz Lcmma: Suppose that $f : B(0,1) \to B(0,1)$ is a holomorphic function with $f(0) = 0.$ Then

the Schwarz Lemma

- Let $B(0,r)=\{z\in\mathbb{C}:|z|<r\}$ be tht open disc of radius r and $B[0,r]$ be closed disc of radius $r.$
- Schwarz Lcmma: Suppose that $f:B(0,1)\to B(0,1)$ is a holomorphic function with $f(0)=0.$ Then

  (i) $|f(z)|\leqslant|z|, z\in B(0,1)$

the Schwarz Lemma

- Let $B(0,r)=\{z\in\mathbb{C}:|z|<r\}$ be tht open disc of radius r and $B[0,r]$ be closed disc of radius r.
- Schwarz Lcmma: Suppose that $f:B(0,1)\to B(0,1)$ is a holomorphic function with $f(0)=0$. Then

  (i) $|f(z)|\leqslant|z|, z\in B(0,1)$

  (ii) $|f'(0)|\leqslant 1$

the Schwarz Lemma

- Let $B(0,r)=\{z\in\mathbb{C}:|z|<r\}$ be tht open disc of radius r and $B[0,r]$ be closed disc of radius r.
- Schwarz Lcmma: Suppose that $f:B(0,1)\to B(0,1)$ is a holomorphic function with $f(0)=0$. Then

  (i) $|f(z)|\leqslant|z|, z\in B(0,1)$

  (ii) $|f'(0)|\leqslant 1$

  (iii) If these exists $z_0\in B(0,1), z_0\neq 0$ such that $|f(z_0)|=|z_0|$ or $|f'(0)|=1$, then f must be of the form $f(z)=cz$, $z\in B(0,1)$, for Some c with $|c|=1$.

the Schwarz Lemma

- Let $B(0,r) = \{z \in \mathbb{C} : |z| < r\}$ be tht open disc of radius r and $B[0,r]$ be closed disc of radius r.
- Schwarz Lcmma: Suppose that $f : B(0,1) \to B(0,1)$ is a holomorphic function with $f(0) = 0$. Then

  (i) $|f(z)| \leqslant |z|, z \in B(0,1)$

  (ii) $|f'(0)| \leqslant 1$

  (iii) If these exists $z_0 \in B(0,1), z_0 \neq 0$ such that $|f(z_0)| = |z_0|$ or $|f'(0)| = 1$, then f must be of the form $f(z) = cz$, $z \in B(0,1)$, for Some c with $|c| = 1$.
- The Proof of the Schwarz lemma is an in immediate Consequence of the Maximum modulus principle.

maximum modulus principle

- The maximun modulus principle: Suppose that Ω is a bounded domain (open and connected) in $\mathbb{C}$, f is a holomorplic function on Ω, and $B[a,R] \subseteq \Omega$. Then

$$|f(a)| \leqslant \sup\{|f(a+re^{i\theta})| : \theta \in \mathbb{R}\}.$$

  Equality occurs if and only of f is constant.

maximum modulus principle

- The maximun modulus principle: Suppose that Ω is a bounded domain (open and connected) in $\mathbb{C}$, f is a holomorplic function on Ω, and $B[a,R] \subseteq \Omega$. Then

$$|f(a)| \leqslant \sup\{|f(a+re^{i\theta})| : \theta \in \mathbb{R}\}.$$

Equality occurs if and only of f is constant.

- Proof: Assume that $|f(a+re^{1\theta})| \leqslant |f(a)|$, $\theta \in \mathbb{R}$. Then the holomorplic function f has a power series expansion

$$f(z) = \sum c_n (z-a)^n, \quad z \in B(a,R),$$

and If $0 < r < R$, we have (Parsevel's formula)

$$\sum |c_n|^2 r^{2n} = 1/2\pi \int_{-\pi}^{\pi} \left| f\left(a + re^{i\theta}\right) \right|^2 d\theta.$$

proof of the Schwarz lemma

- It follows that

$$\sum |\, c_n \,|^2 \, r^{2n} \leqslant |f(a)|^2 = |c_0|^2.$$

Hence $0 = c_1 = c_2 = \cdots,$ implying $f(z) = f(a), \quad z \in B(a, r).$

Since Ω is connected, it follows f must be a constant.

proof of the Schwarz lemma

- It follows that

$$\sum | c_n |^2 r^{2n} \leqslant |f(a)|^2 = |c_0|^2.$$

Hence $0 = c_1 = c_2 = \cdots,$ implying $f(z) = f(a), \quad z \in B(a,r).$ Since Ω is connected, it follows f must be a constant.

- Proof of the Schwarz Lemma: Since $f(0)=0,$ we have $a_0 = 0$, $f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad z \in B(0,1).$ Let $h(z) = \sum_{n=1}^{\infty} a_n z^{n-1}, \quad z \in B(0,1).$ Then h is holomorphic on $B(0,1)$ and $f(z) = zh(z), \quad z \in B(0,1).$ By the maximum modulus theorem

$$\sup\{|h(z)| : |z| \leq r\} = \sup\{|h(z)| : |z| = r\} = \frac{1}{r}\sup\{|f(z)| : |z| \leq r\}$$

for all $r, \quad 0 < r < 1.$ Since $|f(z)| \leq 1$ for all $z \in B(0,1)$ we get, on letting $r \to 1$, that $\sup\{|h(z)|; z \in B(0,1)\} \leq 1.$ Hence $|f(z)| \leq |z|$ for all $z \in B(0,1).$ This completes the proof of (i).

proof of the Schwarz lemma

- Moreover, if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $|h(z_0)| = 1$ and, by the maximum modulus theorem, h is a constant function of modulus 1, that is, there exists $\theta \in \mathbb{R}$ such that $f(z) = zh(z) = e^{i\theta} z$, $z \in B(0,1)$.

proof of the Schwarz lemma

- Moreover, if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $|h(z_0)| = 1$ and, by the maximum modulus theorem, h is a constant function of modulus 1, that is, there exists $\theta \in \mathbb{R}$ such that $f(z) = zh(z) = e^{i\theta} z$, $z \in B(0,1)$.
- Since $\frac{f(z)}{z} = h(z)$ for $z \in B(0,1) \setminus \{0\}$ and $f(0) = 0$ it follows that

$$|f'(0)| = \lim_{\substack{z \to 0 \\ z \neq 0}} \frac{|f(z)|}{|z|} = \lim_{z \to 0} |h(z)| = |h(0)| \leq 1.$$

If $|f'(0)| = |h(0)| = 1$, then by the maximum modulus theorem, h is a constant function of modulus 1 and as before $f(z) = e^{i\theta} z, \quad z \in B(0,1)$.

automorphisms of the disc

- Theorem: For a fixed $\alpha \in B(0,1)$, $\varphi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha}z}$ is a rational function mapping $B(0,1)$ onto $B(0,1)$ and also $\partial B(0,1)$ onto $\partial B(0,1)$. It is one to one on $B[0,1]$. The inverse of φ_α is $\varphi_{-\alpha}$.

automorphisms of the disc

- Theorem: For a fixed $\alpha \in B(0,1)$, $\varphi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha}z}$ is a rational function mapping $B(0,1)$ onto $B(0,1)$ and also $\partial B(0,1)$ onto $\partial B(0,1)$. It is one to one on $B[0,1]$. The inverse of φ_α is $\varphi_{-\alpha}$.
- Proof: The function φ_α is holomorphic in the whole plane except for $z = 1/\bar{\alpha}$ which is outside $B[0,1]$. We see that $\varphi_{-\alpha}(\varphi_\alpha(z)) = z$. Thus φ_α is one-one and $\varphi_{-\alpha}$ is its inverse. If $t \in \mathbb{R}$, then

$$\left|\frac{e^{it}-\alpha}{1-\bar{\alpha}e^{it}}\right| = \left|\frac{e^{it}-\alpha}{e^{-it}-\bar{\alpha}}\right| = 1,$$

and we see that φ_α maps $\partial B(0,1)$ into itself. The same is true of $\varphi_{-\alpha}$ hence $\varphi_\alpha(\partial B[0,1]) = \partial B[0,1]$. Applying the maximum modulus principle, we conclude that $\varphi_\alpha(B(0,1)) \subseteq B(0,1)$. This is equally true of $\varphi_{-\alpha}$.

Schwarz lemma, in general

- Suppose that $\alpha,\ \beta$ are complex numbers; $|\alpha|, |\beta| < 1$
  Question: How large can $|f'(\alpha)|$ be if
  $f : B(0,1) \to B(0,1)$ and $f(\alpha) = \beta$?

Schwarz lemma, in general

- Suppose that $\alpha,\ \beta$ are complex numbers; $|\alpha|, |\beta| < 1$
  Question: How large can $|f'(\alpha)|$ be if
  $f : B(0,1) \to B(0,1)$ and $f(\alpha) = \beta$?

- Answer: $|f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}$. To verify this, put $g = \varphi_\beta \circ f \circ \varphi_{-\alpha}$. Since $\varphi_\beta,\ \varphi_\alpha : B(0,1) \to B(0,1)$, it follows that $g : B(0,1) \to B(0,1)$. Also, $g(0) = 0$. Thus $|g'(0)| \leq 1$ by the Schwarz lemma. Differentiating g using the chain rule, we have

$$g'(0) = \varphi'_\beta(\beta)\ \ f'(\alpha)\ \ \varphi'_{-\alpha}(0).$$

This verifies the correctness of our answer since

$$\varphi'_\alpha(0) = 1 - |\alpha|^2,\ \ \varphi'_\alpha(\alpha) = (1 - |\alpha|^2)^{-1}.$$

Schwarz lemma, in general

- Suppose that $\alpha,\ \beta$ are complex numbers; $|\alpha|, |\beta| < 1$
  Question: How large can $|f'(\alpha)|$ be if
  $f : B(0,1) \to B(0,1)$ and $f(\alpha) = \beta$?

- Answer: $|f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}$. To verify this, put
  $g = \varphi_\beta \circ f \circ \varphi_{-\alpha}$. Since $\varphi_\beta,\ \varphi_\alpha : B(0,1) \to B(0,1)$, it follows that $g : B(0,1) \to B(0,1)$. Also, $g(0) = 0$. Thus $|g'(0)| \leq 1$ by the Schwarz lemma. Differentiating g using the chain rule, we have

$$g'(0) = \varphi_\beta'(\beta)\ \ f'(\alpha)\ \ \varphi_{-\alpha}'(0).$$

  This verifies the correctness of our answer since

$$\varphi_\alpha'(0) = 1 - |\alpha|^2,\ \ \varphi_\alpha'(\alpha) = (1 - |\alpha|^2)^{-1}.$$

- Equality occurs if and only if $g(z) = cz$, for some $c : |c| = 1$. Thus $f(z) = \varphi_{-\beta}(c\varphi_\alpha(z)),\ z \in B(0,1)$.

the automorphism group

- A remarkable feature: f is a rational function, although no continuity assumption was made on f near the boundary.

the automorphism group

- A remarkable feature: f is a rational function, although no continuity assumption was made on f near the boundary.

- Theorm: Suppose that f is a bijective holomorphic function on $B(0,1)$ and that $f(\alpha)=0$. Then there exists a constant $c:|c|=1$ such that

$$f(z)=c\ \varphi_{\alpha}(z),\ \ z\in B(0,1).$$

Proof: Let g be the inverse of f, defined by $g(f(z))=z,\ z\in B(0,1)$. Since f is one to one, f' has no zero in $B(0,1)$, so g defines a holomorphic function on $B(0,1)$. We have $|f'(\alpha)|\le\frac{1}{1-|\alpha|^2}$, $|g'(0)|\le 1-|\alpha|^2$. By the chain rule, $g'(0)\,f'(\alpha)=1$. Since $g'(0)\ \ f'(\alpha)=1$, therefore we must have $|f'(\alpha)|=\frac{1}{1-|\alpha|^2}$, $|g'(0)|=1-|\alpha|^2$. Hence with $\beta=0$, f must be of the form $c\ \varphi_{\alpha}$.

distance decreasing

- Let $f: B(0,1) \to B(0,1)$ be holomorphic. Then for any $a, b \in B(0,1)$, $\left|\frac{f(a)-f(b)}{1-\overline{f(a)}f(b)}\right| \leq \left|\frac{a-b}{1-\bar{a}b}\right|$.

distance decreasing

- Let $f : B(0,1) \to B(0,1)$ be holomorphic. Then for any $a, b \in B(0,1)$, $\left|\frac{f(a)-f(b)}{1-\overline{f(a)}f(b)}\right| \leq \left|\frac{a-b}{1-\bar{a}b}\right|$.
- In particular, $\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}$ for all $z \in B(0,1)$.

distance decreasing

- Let $f: B(0,1) \to B(0,1)$ be holomorphic. Then for any $a, b \in B(0,1)$, $\left|\frac{f(a)-f(b)}{1-\overline{f(a)}f(b)}\right| \le \left|\frac{a-b}{1-\bar{a}b}\right|$.
- In particular, $\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}$ for all $z \in B(0,1)$.
- Riemannian Metric: A C^2 function $\varphi: \Omega \to \mathbb{R}_+$ defined on an open connected subset Ω of $\mathbb{C}$, is said to be a Riemannian metric. If $f: B(0,1) \to B(0,1)$ is holomorphic, then it is distance decreasing with respect to the Poincare metric: $\varrho(z) := \frac{1}{1-|z|^2}$ defined on $B(0,1)$, that is, $f^*(\varrho) \le \varrho$.

distance decreasing

- Let $f : B(0,1) \to B(0,1)$ be holomorphic. Then for any $a, b \in B(0,1)$, $\left|\frac{f(a)-f(b)}{1-\overline{f(a)}f(b)}\right| \leq \left|\frac{a-b}{1-\bar{a}b}\right|$.
- In particular, $\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}$ for all $z \in B(0,1)$.
- Riemannian Metric: A C^2 function $\varphi : \Omega \to \mathbb{R}_+$ defined on an open connected subset Ω of $\mathbb{C}$, is said to be a Riemannian metric. If $f : B(0,1) \to B(0,1)$ is holomorphic, then it is distance decreasing with respect to the Poincare metric: $\varrho(z) := \frac{1}{1-|z|^2}$ defined on $B(0,1)$, that is, $f^*(\varrho) \leq \varrho$.
- Here for any metric $\varphi : \Omega \to \mathbb{R}_+$ on Ω, and any C^2-function $f : \tilde{\Omega} \to \Omega$, the pull-back $f^*\varphi$ is

$$(f^*\varphi)(z) := |f'(z)|\varphi(f(z))$$

and it defines a metric on $\tilde{\Omega}$.

Ahlfors' Schwarz lemma

- The Gaussian curvature of a Riemannian metric φ^2 is defined to be

$$K_\varphi(z) = -\varphi(z)^{-2}\ \Delta \log \varphi(z).$$

Ahlfors' Schwarz lemma

- The Gaussian curvature of a Riemannian metric φ^2 is defined to be

$$K_\varphi(z) = -\varphi(z)^{-2}\ \Delta \log \varphi(z).$$

- Claim: We will verify that any holomorphic function $f : \Omega \to B(0,r)$ defines a metric $f^* p_r$ of constant negative curvature on $\Omega \backslash \{f' = 0\}$, where $p_r(z) := \frac{r}{r^2 - |z|^2}$ is the Poincare metric of $B(0,r)$ and

$$(f^* p_r)(z) := \frac{r|f'(z)|}{r^2 - |f(z)|^2},\ z \in \Omega \backslash \{f' = 0\}.$$

Ahlfors' Schwarz lemma

- The Gaussian curvature of a Riemannian metric φ^2 is defined to be

$$K_\varphi(z) = -\varphi(z)^{-2}\ \Delta \log \varphi(z).$$

- Claim: We will verify that any holomorphic function $f:\Omega \to B(0,r)$ defines a metric f^*p_r of constant negative curvature on $\Omega\backslash\{f'=0\}$, where $p_r(z) := \frac{r}{r^2-|z|^2}$ is the Poincare metric of $B(0,r)$ and

$$(f^*p_r)(z) := \frac{r|f'(z)|}{r^2-|f(z)|^2},\ z \in \Omega\backslash\{f'=0\}.$$

- Ahlfors' Lemma: Let $\varphi \geq 0$ be a continuous function on $B(0,1)$. Assume that φ is C^2 on the open set $D_\varphi := \{\varphi > 0\}$. Suppose $K_\varphi \leq -\eta$, on D_φ for some $\eta > 0$. Then

$$f^*(\varphi)(z) \leq \frac{4}{\eta}\frac{1}{1-|z|^2},\ z \in B(0,1).$$

- Proof of the claim: For a holomorphic function f defined on an open connected set $\Omega \subseteq \mathbb{C}$ with $|f(z)| \leq r$, we have

$$\begin{aligned}
\Delta \log(r^2 - |f|^2)^{-1} &= -4\frac{\partial^2}{\partial \bar{z}\partial z}\log(r^2 - |f|^2) \\
&= 4\frac{\partial}{\partial \bar{z}}\left(\frac{\bar{f}f'}{r^2 - |f|^2}\right) \\
&= 4f'\left(\frac{\bar{f}'}{r^2 - |f|^2} + \frac{\bar{f}f\bar{f}'}{(r^2 - |f|^2)^2}\right) \\
&= 4|f'|^2\left(\frac{r^2 - |f|^2 + |f|^2}{(r^2 - |f|^2)^2}\right) \\
&= 4\left(\frac{r|f'|}{r^2 - |f|^2}\right)^2
\end{aligned}$$

- Proof of the claim: For a holomorphic function f defined on an open connected set $\Omega \subseteq \mathbb{C}$ with $|f(z)| \leq r$, we have

$$\begin{aligned}
\Delta \log(r^2 - |f|^2)^{-1} &= -4\frac{\partial^2}{\partial \bar{z}\partial z}\log(r^2-|f|^2)\\
&= 4\frac{\partial}{\partial \bar{z}}\left(\frac{\bar{f}f'}{r^2-|f|^2}\right)\\
&= 4f'\left(\frac{\bar{f}'}{r^2-|f|^2}+\frac{\bar{f}f\bar{f}'}{(r^2-|f|^2)^2}\right)\\
&= 4|f'|^2\left(\frac{r^2-|f|^2+|f|^2}{(r^2-|f|^2)^2}\right)\\
&= 4\left(\frac{r|f'|}{r^2-|f|^2}\right)^2
\end{aligned}$$

- In particular,

$$\Delta \log \frac{r}{r^2-|z|^2} = 4\left(\frac{r}{r^2-|z|^2}\right)^2, \ |z| < r.$$

proof of Ahlfors' lemma

- Fix $\zeta \in \mathbb{D}$, and let $r \in (|\zeta|, 1)$. Put $p_r(z) = \frac{r}{r^2 - |z|^2}$ on $B(0,r)$. Since $p_r(z) \to \infty$ as $|z| \to r$ and $f^*\varphi$ is continuous on $B[0,r]$, it is clear that the function $\psi := \frac{f^*\varphi}{p_r}$ attain its maximum on $B(0,r)$ at some $\xi \in B(0,r)$. If $(f^*\varphi)(\xi) = 0$, then $\varphi \equiv 0$.

proof of Ahlfors' lemma

- Fix $\zeta \in \mathbb{D}$, and let $r \in (|\zeta|, 1)$. Put $p_r(z) = \frac{r}{r^2 - |z|^2}$ on $B(0,r)$. Since $p_r(z) \to \infty$ as $|z| \to r$ and $f^*\varphi$ is continuous on $B[0,r]$, it is clear that the function $\psi := \frac{f^*\varphi}{p_r}$ attain its maximum on $B(0,r)$ at some $\xi \in B(0,r)$. If $(f^*\varphi)(\xi) = 0$, then $\varphi \equiv 0$.
- Hence we may assume that $\xi \in D_\varphi$. Then ξ is also a local maximum of $\log \psi$, and it follows that $\Delta \log \psi \leq 0$ at q. Now, at ξ:

$$\begin{aligned} 0 \geq \Delta \log \psi &= \Delta \log f^*\varphi - \Delta \log p_r \\ &\geq 4(f^*\varphi^2 - p_r^2), \end{aligned}$$

that is, $\psi(\xi) \leq 1$. Thus $f^*\varphi \leq p_r$ on $B(0,r)$. Letting $r \uparrow 1$, we conclude that $(f^*\varphi)(z) \leq \frac{1}{1-|p|^2}, z \in B(0,1)$, as required.

applications

- Definition: For any open connected set $\Omega \subseteq \mathbb{C}$, let $N\mathfrak{C}(\Omega)$ denote the set of continuous functions $\varphi \geq 0$ on Ω such that φ is C^2 on $\{\varphi > 0\}$ and $\Delta \log \varphi \geq 4\varphi^2$ there.

applications

- Definition: For any open connected set $\Omega \subseteq \mathbb{C}$, let $N\mathfrak{C}(\Omega)$ denote the set of continuous functions $\varphi \geq 0$ on Ω such that φ is C^2 on $\{\varphi > 0\}$ and $\Delta \log \varphi \geq 4\varphi^2$ there.

- It is easy, using the chain rule, to verify:
  Proposition: Suppose that $f : \Omega \to \tilde{\Omega}$ is a holomorphic maps of open connected sets in $\mathbb{C}$. Then $\varphi \in N\mathfrak{C}(\tilde{\Omega})$ implies $f^*\varphi = |f'|(\varphi \circ f)$ is in $N\mathfrak{C}(\Omega)$.

applications

- Definition: For any open connected set $\Omega \subseteq \mathbb{C}$, let $N\mathfrak{C}(\Omega)$ denote the set of continuous functions $\varphi \geq 0$ on Ω such that φ is C^2 on $\{\varphi > 0\}$ and $\Delta \log \varphi \geq 4\varphi^2$ there.

- It is easy, using the chain rule, to verify:
  Proposition: Suppose that $f : \Omega \to \tilde{\Omega}$ is a holomorphic maps of open connected sets in $\mathbb{C}$. Then $\varphi \in N\mathfrak{C}(\tilde{\Omega})$ implies $f^*\varphi = |f'|(\varphi \circ f)$ is in $N\mathfrak{C}(\Omega)$.

- It follows from the Ahlfors lemma that $N\mathfrak{C}(\mathbb{C}) = \{0\}$.

applications

- Definition: For any open connected set $\Omega \subseteq \mathbb{C}$, let $N\mathfrak{C}(\Omega)$ denote the set of continuous functions $\varphi \geq 0$ on Ω such that φ is C^2 on $\{\varphi > 0\}$ and $\Delta \log \varphi \geq 4\varphi^2$ there.
- It is easy, using the chain rule, to verify:
  Proposition: Suppose that $f : \Omega \to \tilde{\Omega}$ is a holomorphic maps of open connected sets in $\mathbb{C}$. Then $\varphi \in N\mathfrak{C}(\tilde{\Omega})$ implies $f^*\varphi = |f'|(\varphi \circ f)$ is in $N\mathfrak{C}(\Omega)$.
- It follows from the Ahlfors lemma that $N\mathfrak{C}(\mathbb{C}) = \{0\}$.
- Verification: Pick φ in $N\mathfrak{C}(\mathbb{C})$. Fix $a \in \mathbb{C}$. For any $r > |a|$, taking $f : B(0,r) \to B(0,r)$, $f(z) = z$, Ahlfors Lemma yields $(f^*\varphi)(a) = \varphi(a) \leq \frac{r}{r^2 - |a|^2}$. As $r \to \infty$, we see that $\varphi(a) = 0$.

applications

- Corollary: Let $f : \mathbb{C} \to \Omega$, $\Omega \subseteq \mathbb{C}$, be a holomophic function. If $N\mathfrak{C}(\Omega) \neq \{0\}$, then f must be constant.

applications

- Corollary: Let $f : \mathbb{C} \to \Omega, \quad \Omega \subseteq \mathbb{C},$ be a holomophic function. If $N\mathfrak{C}(\Omega) \neq \{0\},$ then f must be constant.
- Liouville's theorem As a corollary, taking $\Omega = B(0, M),$ we see that every bounded entire function must be a constant.

applications

- Corollary: Let $f:\mathbb{C}\to\Omega,\ \ \Omega\subseteq\mathbb{C},$ be a holomophic function. If $N\mathfrak{C}(\Omega)\neq\{0\},$ then f must be constant.
- Liouville's theorem As a corollary, taking $\Omega=B(0,M),$ we see that every bounded entire function must be a constant.
- Picard's little theorem Similarly, if $f:\mathbb{C}\to\mathbb{C}_{\{0,1\}},$ where $\mathbb{C}_{\{0,1\}}:=\mathbb{C}\backslash\{0,1\}$ is holomophic, then f is constant.

applications

- Corollary: Let $f:\mathbb{C}\to\Omega,\ \ \Omega\subseteq\mathbb{C},$ be a holomophic function. If $N\mathfrak{C}(\Omega)\neq\{0\},$ then f must be constant.
- Liouville's theorem As a corollary, taking $\Omega=B(0,M),$ we see that every bounded entire function must be a constant.
- Picard's little theorem Similarly, if $f:\mathbb{C}\to\mathbb{C}_{\{0,1\}},$ where $\mathbb{C}_{\{0,1\}}:=\mathbb{C}\backslash\{0,1\}$ is holomophic, then f is constant.
- proof: To verify this, all we need to do is show that $N\mathfrak{C}(\mathbb{C}_{\{0,1\}})\neq\{0\}.$ The non-zero function

$$\varphi(z)=|z|^{\beta/2-1}|1-z|^{\beta/2-1}(1+|z|^{\beta})(1+|z-1|)^{\beta},\ \beta>0,$$

is in $N\mathfrak{C}(\mathbb{C}_{\{0,1\}})$ for $0<\beta<2/7.$

Thank You!