Determinant of an operator

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Examples of identities are so ubiquitous that they often escape our notice. Here is a list of some of these.

1. The identity $x^2 - 1 = (x+1)(x-1)$ used by the ancient Babylonians.
2. The fundamental identity $\sin^2 x + \cos^2 x = 1$ of trigonometry.
3. (Newton's forumale) If $e_m = \sum_{1 \le i_1 < \cdots < i_m \le n} \lambda_{i_1} \cdots \lambda_{i_m}$ and $p_k = \lambda_1^k + \cdots + \lambda_n^k$, then for $m = 1, 2, \ldots,$

$$me_m = \textstyle\sum_{k=1}^{m} (-1)^{k-1} p_k e_{m-k}.$$

4. The Jacobi identity: $[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$ for Lie structures.

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Hamilton-Cayley polynomial

Any $n \times n$ matrix A is the root of the Hamilton-Cayley polynomial

$$\det(\lambda - A) = \lambda^n + \sum_{i=1}^n \gamma_i(A)\lambda^{n-i},$$

where $\gamma_1(A) = -\mathrm{tr}(A), \ldots, \gamma_n(A) = (-1)^n \det(A).$

For a $2 \times 2,$ matrix $A,$ this means that $A^2 - \mathrm{tr}(A)A + \det(A)I = 0.$

Now, if $\mathrm{tr}(A) = 0$, then $A^2 = -\det(A)I.$ Therefore, $[A^2, B] = A^2B - BA^2 = 0.$

It follows that for three 2×2 matrices $A, B, C,$ we have the Wagner identity

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polynomial identity

We see that the non-commutative polynomial $[[X,Y]^2,Z] = (XY - YX)^2 Z - Z(XY - YX)^2$ is zero when evaluated on any three 2×2 matrices A,B,C since $\mathrm{tr}(AB - BA)$ is always zero.

However, Wagner's identity is not true for 3×3 matrices.

It is therefore natural to say that a non-commutative polynomial P in the ring $F[X_1, \dots, X_m]$ is a polynomial identity for an algebra $\mathscr{R}$ if it vanishes identically when evaluated on any m elements $A_1, \dots, A_m$ from the algebra in $\mathscr{R}$.

We have seen that taking $m = 3$, $\mathscr{R} = \mathscr{M}_2(\mathbb{C})$, the non-commutative polynomial $P[X,Y,Z] := [[X,Y]^2,Z]$ serves as a polynomial identity in $\mathscr{M}_2(\mathbb{C})$.

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Which algebras satisfy polynomial identity

What are the implications of the existence of a polynomial identity in an algebra?

What are the polynomial identities of a given algebra? What happens in the particular example of the matrix algebra?

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Let $\mathfrak{S}_h$ be the permutation group on h symbols and S_h be the standard polynomial

$$S_h(X_1,...,X_h) := \sum_{\sigma \in \mathfrak{S}_h} \mathrm{Sgn}(\sigma) X_{\sigma(1)} \cdots X_{\sigma(h)}$$

in non-commuting variables $X_1, \ldots, X_h.$ For any set of $2n$ element $A_1, \ldots, A_{2n}$ in the algebra $\mathscr{M}_n(\mathscr{R})$ of $n \times n$ matrices over a commutative ring $\mathscr{R},$ the Amitsur-Levitzki theorem asserts that $S_{2n}(A_1, \ldots, A_{2n}) = 0,$

We can multiply more than two matrices. We can write $A \times B \times C$ for the product of A, B and $C.$ The order of the matrices is important, but the order in which we perform the multiplication is not. This is because multiplication of matrices is associative, that is

$$(A \times B) \times C = A \times (B \times C).$$

2×2

Here is the Amitsur Levitzki Theorem for 2×2 matrices: For every four 2×2 matrices $A, B, C,$ and D ,

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The standard polynomial S_h is easily seen to be multi-linear and alternating. Thus

$$S_h(X_1,\ldots,X_i+X_i',\ldots,X_h) = S_h(X_1,\ldots,X_i,\ldots,X_h) + S_h(X_1,\ldots,X_i',\ldots,X_h)$$

and vanishes if two of the arguments are equal, that is, $S_h(X_1,\ldots,X,\ldots,X,\ldots,X_h) = 0$.

Hence to prove the Amitsur-Levitzki Theorem, it suffices to prove that $S_h(B_1,\ldots,B_{2h})$, where $B_1,\ldots,B_{2h}$ is chosen from any (vector space) basis of the algebra $\mathscr{M}_h(\mathbb{C})$. In particular, it is enough to choose them from the set of elementary matrices $E_{i,j}$, where $E_{i,j}$ is the matrix with 1 at the position (i,j) and 0 elsewhere.

The proof of the Amitsur-Levitzki Theorem by Rosset

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Thus we can verify the validity of the Amitsur-Levitzki theorem by checking that $S_4(E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2})$. The proof in the general case can be given based on this idea.

A very short proof due to Rosset is based on a very clever use of the Cayley-Hamilton theorem.

He uses a particular instance of the Hamilton-Cayley trace identity which is of the form

$$A^k + \sum_{j=1}^{k} \left(\sum_{j_1+\cdots+j_u=j} \alpha_{(j_1,\ldots,j_u)} \mathrm{tr} A^{j_1} \cdots \mathrm{tr} A^{j_u} \right) A^{k-j} = 0,$$

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An operator T on a Hilbert space $\mathscr{H}$ is said to be hyponormal if the commutator $[T^*, T] := T^*T - TT^*$ is positive.

The Berger-Shaw theorem says that if T is a m-cyclic hyponormal operator, then the commutator $[T^*, T]$ is trace class and

$$\mathrm{tr}[T^*, T] \leq \frac{m}{\pi} A(\sigma(T))$$

There has been some attempt to show that if a commuting n-tuple of bounded linear operators T is hyponormal and cyclic, then the cross commutators must be trace class. The first of these is due to Athavale and the other is due to Douglas and Yan.

Douglas and Yan using techniques from commutative algebra reduce their proof to the case of a single operator.

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$$\left[\left[T^*, T\right]\right] := \left(\left([T_j^*, T_i]\right)\right)_{i,j=1}^n : \bigoplus_n \mathscr{H} \longrightarrow \bigoplus_n \mathscr{H}$$

is positive, that is, for each $x \in \bigoplus_n \mathscr{H}$, $\left\langle \left[\left[T^*, T\right]\right] x, x \right\rangle \geq 0,$ and it is said to be weakly hyponormal if for each vector $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n,$ the sum $\sum_{i=1}^n \alpha_i T_i$ is a hyponormal operator on $\mathscr{H}$.

Question: If the n -tuple T is strongly hyponormal and cyclic, then does it follow that the commutators $[T_j^*, T_i]$, $1 \leq i,j \leq n$ is necessarily trace class?

It is easy to verify that the answer is “no”, in general. Take for instance, the example of the Hardy space $H^2(\mathbb{D}^2)$ and the pair of operators to be the multiplication by the coordinate functions (M_1, M_2) . Here the operators $M_j^* M_i - M_i M_j^* = 0$, $j \neq i$. However, the commutators $M_j^* M_j - M_j M_j^*$ are of infinite multiplicity and they are not even compact.

What might be a possible generalization of the Berger-Shaw theorem to the case of commuting tuples of operators?

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Instead of asking for the trace of the commutators to be finite, we only ask that the trace of a “certain” determinant (or, in the language of Helton and Howe, the generalized commutator) is finite.

One may argue that it is not asking for much. But then to arrive at this conclusion, we don’t assume much either.

As in the Berger-Shaw theorem, we assume finite multiplicity but instead of either strong or weak hyponormality, we only assume that the determinant is positive. In many ways, it is a mild condition and this gives us the finiteness of the trace, what is more, we can even get an explicit bound.

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what is the determinant

Let $B := ((B_{ij}))$ be an $n \times n$ block matrix with entries from $\mathscr{L}(\mathscr{H})$. The determinant of B is the operator

$$\mathrm{Det}(B) := \sum_{\sigma,\tau} \mathrm{sgn}(\sigma) B_{\tau(1),\sigma(\tau(1))} B_{\tau(2),\sigma(\tau(2))} \cdots B_{\tau(n),\sigma(\tau(n))}.$$

The map $\mathrm{Det} : \mathscr{L}(\mathscr{H})^n \times \ldots \times \mathscr{L}(\mathscr{H})^n \mapsto \mathscr{L}(\mathscr{H})$ is clearly an alternating multi-linear map.

Let $T = (T_1, T_2, \ldots, T_n)$ be a n-tuple of commuting operators. Let us say that the determinant of the n-tuple T is the operator $\mathrm{Det}(\llbracket T^*, T \rrbracket)$.

For operators of the form $\llbracket T^*, T \rrbracket$, Helton and Howe define the generalized commutator of $A = (A_1, A_2, \ldots, A_{2n})$:

$$\mathrm{GC}(A) := S_{2n}(A_1, \ldots, A_{2n}).$$

The generalized commutator of the n-tuple of commuting operators T is the operator $\mathrm{GC}(A)$ if we choose $A_1 = T_1^*, A_2 = T_1, \ldots, A_{2n-1} = T_n^*, A_{2n} = T_n$.

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Det and GC are the same

Thanks to Cherian Varughese, we see that $\mathrm{Det}(T)$ and $\mathrm{GC}(T)$ are equal, which is perhaps implicit in the paper of Helton and Howe.

Recall the example of the pair of multiplication operators on the Hardy space, $H^2(\mathbb{D}^2)$. In this case,

$$[[M^*, M]] = \begin{pmatrix} [(M_z \otimes I)^*, (M_z \otimes I)] & [(I \otimes M_z)^*, (M_z \otimes I)] \\ [(M_z \otimes I)^*, (I \otimes M_z)] & [(I \otimes M_z)^*, (I \otimes M_z)] \end{pmatrix}$$

$$= \begin{pmatrix} P \otimes I & 0 \\ 0 & I \otimes P \end{pmatrix} \geq 0.$$

It now follows that $\mathrm{Det}\big([[M^*, M]]\big) = 2(P \otimes P)$.

Thus $\mathrm{Det}\big([[M^*, M]]\big)$ is positive and trace class.

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Thank You!