Homogeneous operators on Hilbert spaces of holomorphic functions

Gadadhar Misra

joint with

Adam Korányi

Indian Statistical Institute

R. V. College Post

Bangalore 560 059

Abstract

- New examples of homogeneous operators involving infinitely many parameters are constructed.
- They are realized on Hilbert spaces of holomorphic functions with reproducing kernels.
- These kernels are computed explicitly.
- All the examples are irreducible and belong to the Cowen - Douglas class.

Relationship with homogeneous bundles

Even though the construction is completely explicit, it is based on certain facts about Hermitian holomorphic homogeneous vector bundles. These facts also make possible a description of all homogeneous Cowen - Douglas operators, in a somewhat less explicit way.

Related work

1. D. Wilkins – classification of holomorphic hermitian homogeneous bundles over symmetric Riemannian manifolds
2. I. Biswas – classification of holomorphic hermitian homogeneous bundles over the unit disc
3. T. Stigger and M. K. Vemuri – Inductive algebras
4. J. Arazy and G. Zhang – holomorphic hermmitians homogeneous line bundles over bounded symmetric domains
5. S. S. Roy – Further study of the Wilkins examples
6. Joint work with B. Bagchi, D. N. Clark and N. S. N. Sastry

Preliminaries

Definition 1. *An operator* T *on a Hilbert space is said to be homogeneous if its* spectrum *is contained in the closure of the unit disc* $\mathbb{D}$ *in* $\mathbb{C}$ *and if* $g(T)$ *is* unitarily equivalent *to* T *for every element* g *of the holomorphic automorphism group* G *of* $\mathbb{D}$*.*

There are general results about such operators, but relatively few examples are known. In this note a large family of examples is constructed and a step is made towards the description of all such operators in the Cowen - Douglas class of $\mathbb{D}$.

Construction of the examples

Let $\tilde{G}$ denote the universal covering group of G. For $\lambda > 0$, let $A^{(\lambda)}$ denote the Hilbert space of holomorphic functions on $\mathbb{D}$ with reproducing kernel $(1 - z\bar{w})^{-2\lambda}$. The well-known discrete series D_λ^+ of unitary representation of $\tilde{G}$ acts on $A^{(\lambda)}$ by

$$(D_\lambda^+(g)f)(z) = \left(\frac{\partial(g^{-1}z)}{\partial z}\right)^\lambda f(g^{-1}(z))$$

with the power defined to ensure continuity on $\tilde{G} \times \mathbb{D}$.

The jet map

Let $m \geq 1$ be an integer and let $\lambda > \frac{m}{2}$. For $0 \leq j \leq m$ we write $\lambda_j = \lambda - \frac{m}{2} + j$ and define the operator $\Gamma_j : A^{(\lambda_j)} \to \text{Hol}(\mathbb{D}, \mathbb{C}^{m+1})$ by

$$(\Gamma_j f)_\ell = \begin{cases} 0 & \text{if } \ell < j \\ \binom{\ell}{j} \frac{1}{(2\lambda_j)_{\ell - j}} f^{(\ell - j)} & \text{if } \ell \geq j \end{cases}.$$

Denote by $\mathbf{A}^{(\lambda_j)}$ the image of Γ_j. The algebraic sum of the spaces $\mathbf{A}^{(\lambda_j)}$ can be shown to be direct. Hence, for any $\mu_1, \ldots, \mu_m$ and $\mu_0 = 1$ we can define a norm on the direct sum by

$$\langle \sum^m \Gamma_j f_j, \sum^m \Gamma_j g_j \rangle = \sum^m \mu_j^2 \langle f_j, g_j \rangle, \;\; f_j, g_j \in A^{(\lambda_j)}.$$

The Hilbert space

We denote the Hilbert space obtained in this way by $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}$ (we write $\boldsymbol{\mu} = (\mu_0, \mu_1, \ldots, \mu_m)$).
The direct sum of maps $\Gamma := \oplus \mu_j \, \Gamma_j$ is then a Hilbert space isomorphism of $\oplus A^{(\lambda_j)}$ onto $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}$. This isomorphism transfers the representation $\oplus D^+_{\lambda_j}$ to $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}$; we denote its image by U.
For $g \in \tilde{G}$, we have $g''(z) = -2c \, g'(z)^{3/2}$ with a constant c depending on g. We use c with this meaning in the following theorem.

Theorem 1

Theorem 1. *U is a multiplier representation of $\tilde{G}$, that is, $(U(g)f)(z) = J(g^{-1}, z)f(g^{-1}(z))$ for $g \in \tilde{G}$, $z \in \mathbb{D}$. The matrix entries of the multiplier are given by*

$$J(g,z)_{p,\ell} = \begin{cases} 0 & \text{if } p < \ell \\ \binom{p}{\ell}(-c)^{p-\ell} g'(z)^{\lambda - \frac{m}{2} + \frac{p+\ell}{2}} & \text{if } p \geq \ell. \end{cases}$$

The theorem is proved by direct computation.
It is well known that in each $A^{(\lambda_j)}$ the monomials, appropriately normalized, form an orthonormal basis. Using the isomorphism Γ we obtain an orthonormal basis of $\mathbf{A}^{(\lambda, \boldsymbol{\mu})}$.

The orthonormal basis

Explicitly, the basis is $\{\mu_j\, e_n^j : 0 \le j \le m,\ n \ge 0\}$, where the ℓ'th component of e_n^j is

$$e_n^j(z)_\ell = \begin{cases} 0 & \text{if } \ell < j \text{ or } \ell > n+j \\ \binom{\ell}{j} \frac{\sqrt{n!}}{(n-\ell+j)!} \frac{\sqrt{(2\lambda_j)_n}}{(2\lambda_j)_{\ell-j}} z^{(n-\ell+j)} & \text{if } j \le \ell \le n+j. \end{cases}$$

Here $(x)_p := x(x+1)\cdots(x+p-1)$ is the Pochhammer symbol. It follows that the polynomials form a dense set in $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}$. Therefore, the linear operator $M = M^{(\lambda,\boldsymbol{\mu})}$ on $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}$ defined by

$$(Mf)(z) = zf(z)$$

is densely defined.

The matrix representation

To find the expression of M in terms of our orthonormal basis, we define the subspace $H(n)$ as the linear span of the vectors $\{e^j_{n-j} : 0 \leq j \leq \min(n,m)\}$ for each $n \geq 0$. Clearly M maps $H(n)$ to $H(n+1)$. We have

$$M\mu_j\, e^j_{n-j} = \sum_{n=0}^{m} M(n)_{k,j}\, \mu_k\, e^k_{n+1-k}.$$

We define the number $E(n)_{\ell j}$ by $(e^j_{n-j}(z))_\ell = E(n)_{\ell j} z^{n-\ell}$ and write $E(n)$ for the matrix $E(n)_{\ell j}$. We define the diagonal matrix $D(\mu)$ by $D(\mu)_{\ell j} = \mu_\ell \delta_{\ell j}$.

Stirling's formula

Now we have for $n \geq m$

$$M(n) = D(\mu)^{-1} E(n+1)^{-1} E(n) D(\mu),$$

(with a small modification for $0 \leq n < m$). Using Stirling's formula one verifies that, as $n \to \infty$,

$$M(n) \sim I_{m+1} + O\left(\tfrac{1}{n}\right), \tag{1}$$

where $O\left(\frac{1}{n}\right)$ stands for an $(m+1) \times (m+1)$ matrix whose entries are $O(\frac{1}{n})$.

Cowen-Douglas class

Theorem 2. *The operator* $M = M^{(\lambda,\boldsymbol{\mu})}$ *defined on* $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}$ *by* $(Mf)(z) = zf(z)$ *belongs to the Cowen - Douglas class and is homogeneous.*

For the proof one remarks that by (1), the operator M splits into the sum of two operators. The first one is the direct sum of $(m+1)$ copies of the standard isometric shift operator and the second one belongs to the Hilbert-Schmidt class. This implies that it is bounded and is in the Cowen - Douglas class.
Using Theorem 1 one can verify, in a standard way, that $g(M) = U_g^{-1} M U_g$ proving that M is homogeneous.

Theorem 3. *For every $m \geq 1$, the operators $M^{(\lambda,\boldsymbol{\mu})}$, $\lambda > \frac{m}{2}; \mu_1, \ldots, \mu_m > 0$ are mutually unitarily inequivalent.*

One essential ingredient of the proof is the following.

Theorem 4. *Suppose E is a Hermitian holomorphic vector bundle over $\mathbb{D}$ and for every $g \in G$ there exists an automorphism $\hat{g}$ of E whose action on $\mathbb{D}$ coincides with g. Then $\tilde{G}$ acts on E by automorphisms in a unique way.*

In the proof one considers the connected component $\hat{G}$ of the full automorphism group of E. There is a natural homomorphism $\hat{G} \to G$ which has a compact kernel N. One shows that $\hat{G}$ is reductive so it contains a normal subgroup which is a covering of G.

Proof of Theorem 3

For the proof of Theorem 3 we consider the representation of the Lie algebra $\mathfrak{g}$ induced by U and extend it to the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$, still denoting it by U.

Restricting U to the triangular subalgebra $\mathfrak{t}$ of $\mathfrak{g}^{\mathbb{C}}$ (the Lie algebra of the stabilizer of 0 in $SL(2, \mathbb{C})$ acting on the extended complex plane), direct computation using Theorem 1 shows that $(U(X)f)(0) = \rho_{\lambda,\boldsymbol{\mu}}(X)f(0)$ for all $X \in \mathfrak{t}$ and $f \in \mathbf{A}^{(\lambda,\boldsymbol{\mu})}$, with $\rho_{\lambda,\boldsymbol{\mu}}(X)$ independent of f, and therefore, giving a representation of $\mathfrak{t}$ on the finite dimensional Hilbert space $\mathbb{C}^{m+1}$. The unitary equivalence class of $\rho_{\lambda,\boldsymbol{\mu}}$ is uniquely determined by U, and hence by the operator $M^{(\lambda,\boldsymbol{\mu})}$.

The Reproducing kernel at (0,0)

Explicit computation of $\rho_{\lambda,\boldsymbol{\mu}}$ shows that different pairs $(\lambda, \boldsymbol{\mu})$ give inequivalent representations, proving the theorem.
In the following theorem S stands for the matrix with entries $S_{\ell p} = \delta_{p+1,\ell}\,\ell$, T its transpose. $D = D(z\bar{w})$ is diagonal with entries $D_{\ell\ell} = (1 - z\bar{w})^{m-\ell}$. Let $K^{\lambda,\boldsymbol{\mu}}$ be the diagonal matrix with

$$(K^{\lambda,\boldsymbol{\mu}})_{\ell\ell} = \sum_{j=0}^{m} \mu_j^2 \sum_{j=0}^{\ell} \binom{\ell}{j}^2 \frac{(\ell - j)!}{(2\lambda_j)_{\ell-j}}.$$

The reproducing kerenel

Theorem 5. *The space* $\mathbf{A}^{(\lambda,\boldsymbol{\mu})}$ *has a reproducing kernel given by*

$$\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(z,w) = (1 - z\bar{w})^{-2\lambda-m} D(z\bar{w}) e^{\bar{w}S} K^{\lambda,\boldsymbol{\mu}} e^{zT} D(z\bar{w}).$$

The normalized reproducing kernel in the sense of Curto and Salinas is

$$\mathbf{B}_0^{(\lambda,\boldsymbol{\mu})}(z,w) = e^{-zT}(K^{\lambda,\boldsymbol{\mu}})^{-1}\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(z,w)(K^{\lambda,\boldsymbol{\mu}})^{-1}e^{-\bar{w}S}.$$

This is proved by computing the effect of the map Γ on the well known reproducing kernels of the spaces $A^{(\lambda_j)}$ and using the identity

$$\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(z,w) = J(g,z)\mathbf{B}^{(\lambda,\boldsymbol{\mu})}(gz,gw)J(g,w)^*,\ g \in \tilde{G}. \quad \mathbf{(2)}$$

Irreducibility

Theorem 6. *The operators $M^{(\lambda,\boldsymbol{\mu})}$, $\lambda > \frac{m}{2}; \mu_1, \ldots, \mu_m > 0$ are irreducible.*

One proves this by first refining some arguments of Curto and Salinas to show that any orthogonal projection commuting with M corresponds to an orthogonal projection P in $\mathbb{C}^{m+1}$ such that $P\mathbf{B}_0^{(\lambda,\boldsymbol{\mu})}(z,w) = \mathbf{B}_0^{(\lambda,\boldsymbol{\mu})}(z,w)P$ for all $z, w \in \mathbb{D}$. Such a P then commutes with all coefficients of the power series development $\mathbf{B}_0^{(\lambda,\boldsymbol{\mu})}$ at $z = w = 0$. Using the explicit expressions in Theorem 5 one can show that no non-trivial P can exist.

More general results

Every Hermitian holomorphic vector bundle homogeneous under $\tilde{G}$ can be obtained by the process of holomorphic induction from representations of $\mathfrak{t}$ on $\mathbb{C}^{m+1}$ that are Hermitian on the real reductive part of $\mathfrak{t}$. Unitary equivalence classes of such representations are in one-to-one correspondence with isomorphism classes of bundles. The $\rho_{\lambda,\boldsymbol{\mu}}$ occurring earlier are certain conjugates of the restriction of the standard $(m+1)$-dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{t}$ tensored with a one-dimensional representation.

More general results

Any finite dimensional representation of $\mathfrak{t}$ can be written as a tensor product of a one-dimensional representation ρ_λ characterized by a real parameter λ and another one ρ_0 normalized in some way.
Even though the results are less explicit in the general case than for the operators constructed here, we can prove the following.

Theorem 7. *To any ρ_0 there corresponds a number λ_0 such that the homogeneous bundle induced by $\rho_\lambda \otimes \rho_0$ for $\lambda > \lambda_0$ is the Cowen - Douglas bundle of a homogeneous operator.*

A number of results in this Note extend to the case of operator tuples of Cowen - Douglas class on bounded symmetric domains in several complex variables.

References

[1] B. Bagchi and G. Misra, *Homogeneous tuples of multiplication operators on twisted Bergman space*, J. Funct. Anal. **136** (1996), 171 – 213.

[2] B. Bagchi and G. Misra, *Homogeneous operators and projective representations of the Möbius group: a survey*, Proc. Ind. Acad. Sc.(Math. Sci.) **111** (2001), 415–437.

[3] ______, *The homogeneous shifts*, J. Funct. Anal. **204** (2003), 293 –319.

[4] M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Acta Math. **141** (1978), 187–261.

[5] R. E. Curto and N. Salinas, *Generalized Bergman kernels and the Cowen - Douglas theory*, Amer J. Math. **106** (1984), 447–488.

[6] A. A. Kirillov, *Elements of the theory of representations*, Springer-Verlag, 1976.

[7] D. R. Wilkins, *Homogeneous vector bundles and Cowen - Douglas operators*, International J. of Math. **4** (1993), 503 – 520.