ZEROES OF POLYNOMIALS

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The Permanent

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why permanent?

Let us consider the probléme des ménages: At a round table n couples are to be seated. The n wives have already occupied the seats $1, 3, \ldots, 2n - 1$. No husband is allowed to seat next to his wife. In how many ways, the men can be seated?

Count the number of permutations σ of n symbols such that neither $\sigma(i) = i$ nor $\sigma(i + i) = i$ modulo n.

the permanent again!

This is the permanent of the matrix $J - I - I'$, where

J is the all 1 matrix,

I is the identity matrix and

I' is the matrix with 1 at $(i, i+1)$ position and $(n, 1)$

This matrix is of the form

$$\begin{pmatrix} 0 & 0 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 1 & \cdots & 0 \end{pmatrix}$$

Doubly Stochastic Matrices

All entries are non-negative, and further, every row sums to one, and every column sums to one.

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$$\partial^{(1,\ldots,n)} f_A = \text{per}(A).$$

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Real Stability

A polynomial f in $\mathbb{R}[z_1, \dots, z_n]$ is said to be stable with respect to a region $\Omega \subseteq \mathbb{C}^n$ if no root of f lies in Ω. Polynomials with no roots in the region

$$\mathcal{H}_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \Im(z_i) > 0, i = 1, 2, \dots, n\}$$

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When f is a univariate polynomial, real stability amounts to saying that all the roots of f are real, or f is real-rooted.

A Lower Bound for the Derivative At Zero.

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$$f'(0) \geqslant \left(\frac{d-1}{d}\right)^{d-1} \inf_{t>0} \frac{f(t)}{t},$$

where d is the degree of f.

Restriction

It is easy to see that if $f(z_1, \ldots, z_n)$ is a stable polynomial, then $f(a, z_2, \ldots, z_n)$ is also stable if $\Im(a) > 0$.

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Hurwitz's Theorem

Let Ω be a connected, open set and $\{f_n : n \geqslant 0\}$ be a sequence of holomorphic functions which converge uniformly on compact subsets of Ω to a holomorphic function f.

If the f_n's are not zero at any point in Ω,
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If $f(z_1, \dots, z_n)$ is real stable, then for all a in the closure of $\mathcal{H}_n$, the polynomial $f(a, z_2, \dots, z_n)$ is also real stable.

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Differentiation.

If f is real stable, then $\partial_1 f$ is also real stable.

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Permanent: Continued

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Moreover, for any $0 \leqslant i \leqslant n$, the polynomial

$$g_i(z_1, \ldots, z_i) = \frac{\partial^{n-i} f_A}{\partial z_{i+1} \ldots \partial z_n}(z_1, \ldots, z_i, 0, \ldots, 0)$$

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Let $b_1, \ldots, b_{i-1}$ be fixed positive reals. Notice that:

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$$g_{i-1}(b_1, \ldots, b_{i-1}) = \partial_i g_i(b_1, \ldots, b_{i-1}, 0).$$

Now, since all entries of A are non-negative, then it follows, from the lower bound for $f'(0)$ and closure under restriction of stability, that

$$\partial_i g_i(b_1, \ldots, b_{i-1}, 0) \geqslant \left(\frac{d_i - 1}{d_i}\right)^{d_i - 1} \inf_{t>0} \frac{g_i(b_1, \ldots, b_{i-1}, b_i)}{b_i},$$

where d_i is the degree of the polynomial $g_i(b_1, \ldots, b_{i-1}, z_i)$.

Fixing $s_1, \ldots, s_{i-1}$, let s_i be defined to be

$$\arg \inf_{t>0} \frac{g_i(s_1, \ldots, s_{i-1}, t)}{t}.$$

Set $d = \max_{i=1}^{n} d_i$.

Applying the inequality for $g_i, i = 0, \dots, n-1$, we obtain that $\mathrm{per}(A) = g_0$, which is at least

$$\left(\frac{d-1}{d}\right)^{d-1} \frac{g_1(s_1)}{s_1} \geqslant \cdots \geqslant \left(\frac{d-1}{d}\right)^{(d-1)n} \frac{g_n(s_1, \dots, s_n)}{\prod_{i=1}^{n} s_i}.$$

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On the other hand, we have

$$\frac{f_A(s_1, \dots, s_n)}{\prod_{i=1}^n s_i} \geqslant \inf_{b_1 > 0, \dots, b_n > 0} \frac{f_A(b_1, \dots, b_n)}{\prod_{i=1}^n b_i}.$$

AM-GM Inequality

If $\lambda_1, \ldots, \lambda_n$ and $x_1, \ldots, x_n$ are positive real numbers with $\sum_{i=1}^{n} \lambda_i = 1$,

$$\sum_{i=1}^{n} \lambda_i x_i \geqslant \prod_{i=1}^{n} x_i^{\lambda_i}.$$

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$$\frac{f_A(b_1, \ldots, b_n)}{\prod_{j=1}^{n} b_j} \geqslant 1.$$

Putting everything together

$$\text{Per}(A) \geqslant \left(\frac{d-1}{d}\right)^{(d-1)n} \cdot \frac{f_A(s_1, \ldots, s_n)}{\prod_{i=1}^{n} s_i}$$

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Noting that $\left(\frac{d-1}{d}\right)^{d-1} \geqslant \frac{1}{e}$, we have proved

$$\text{per}(A) \geqslant \left(\frac{1}{e}\right)^{n}.$$

ZEROES OF POLYNOMIALS

Applications

A matching in a graph is a collection of edges such that no pair of edges have any common end points.

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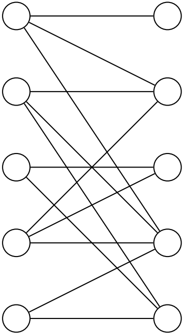
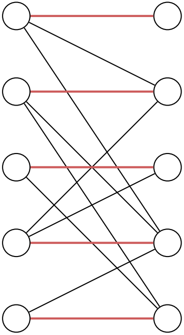
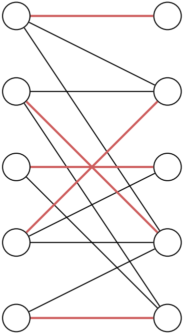
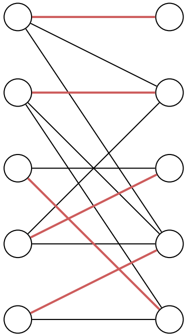
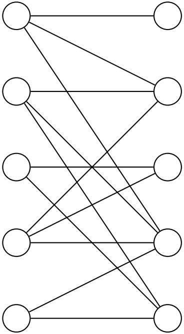
A matching M is perfect if every vertex of the graph is incident to some edge of M.



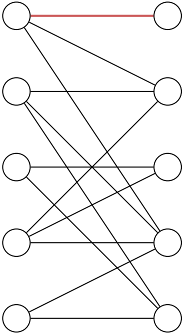








$$M = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$



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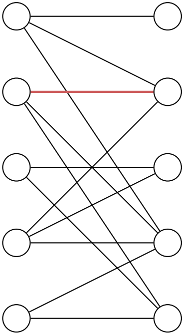

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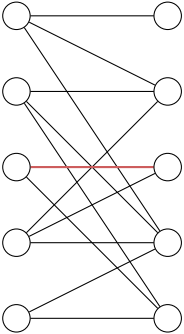

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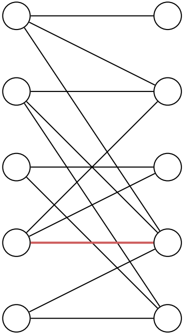

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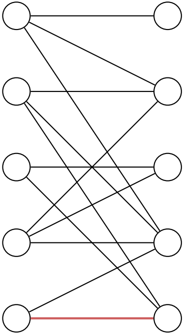
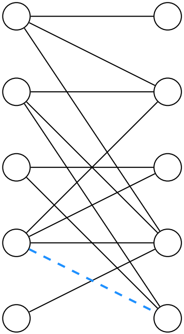


$$M = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & \color{red}{1} \end{pmatrix}$$



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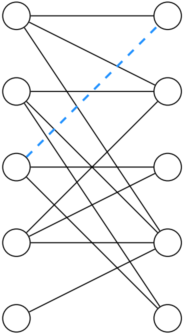

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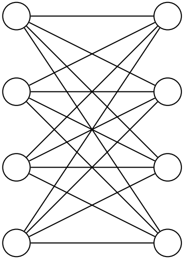

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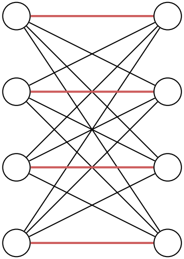

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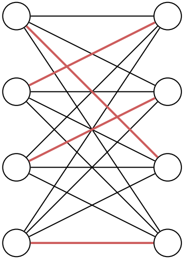

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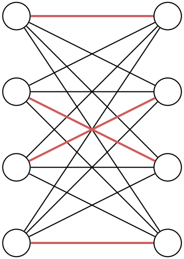

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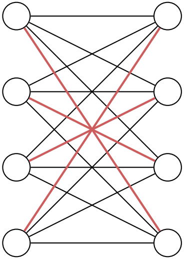

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$$\mathrm{perm}(A) = \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} a_{i,\sigma(i)} \right).$$

Permanent of $A(G) =$ # of perfect matchings in G.

Note that every perfect matching M in G corresponds to a unique permutation $\sigma_M \in S_n$ such that $\sigma_M(i)$ is equal to k such that v_i is matched to u_j in M.

Conversely, every permutation $\sigma \in S_n$ which is a perfect matching corresponds to a 1-term in perm$(A(G))$, and all other terms are 0.

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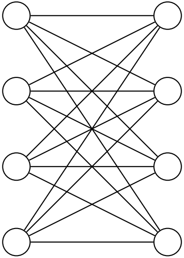
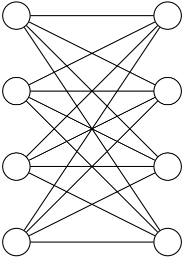


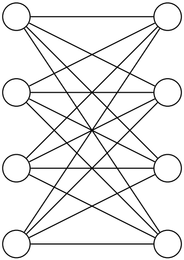
The graph G.

$$A(G) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$



The graph G.

$$\frac{1}{n} \cdot A(G) = \begin{pmatrix} 1/n & 1/n & 1/n & 1/n \\ 1/n & 1/n & 1/n & 1/n \\ 1/n & 1/n & 1/n & 1/n \\ 1/n & 1/n & 1/n & 1/n \end{pmatrix}$$



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The matrix $(1/n)A(G)$ is doubly stochastic.

A k-regular graph is a graph where every vertex has exactly k neighbors.

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In particular, for a k-regular bipartite graph G, the matrix $A(G)$ has k ones in every row and every column.

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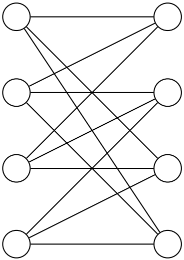

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$$\frac{1}{k} \cdot M = \begin{pmatrix} 1/k & 0 & \cdots & 1/k & 1/k \\ 1/k & 1/k & \cdots & 0 & 1/k \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1/k & 1/k & \cdots & 1/k & 0 \\ 0 & 1/k & \cdots & 1/k & 1/k \end{pmatrix}$$

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The matrix $(1/k)M$ is doubly stochastic.

$$\mathrm{Per}(M) \geqslant \left(\frac{1}{e}\right)^n \cdot k^n = \left(\frac{k}{e}\right)^n$$

$$\frac{1}{k} \cdot M = \begin{pmatrix} 1/k & 0 & \cdots & 1/k & 1/k \\ 1/k & 1/k & \cdots & 0 & 1/k \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1/k & 1/k & \cdots & 1/k & 0 \\ 0 & 1/k & \cdots & 1/k & 1/k \end{pmatrix}$$

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Number of matchings in a k-regular bipartite graph $\geqslant \left(\frac{k}{e}\right)^n$

Thank You!