Differentiation (?)!

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The ingredients

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$. This is just the inside of the unit circle $\mathbb{T}$. We have functions defined on it – $\varphi_{\alpha,a}(z) = \alpha\frac{z-a}{1-\bar{a}z}$, where a is in $\mathbb{D}$ and α is in $\mathbb{T}$.
Each of these is the ratio of two polynomials of degree one. One can't ask for anything simpler!
If we go forward from $\mathbb{D}$ with one these functions, then we land in $\mathbb{D}$ again. Moreover, we always have a function of the form $\varphi_{\beta,b}$ with which we can return back to the first copy of the $\mathbb{D}$ undoing the effect of $\varphi_{\alpha,a}$.
We say that each of the $\varphi_{\alpha,a}$ admits an inverse, namely the function $\varphi_{\beta,b}$. The set of these functions, namely, $\{\varphi_{\alpha,a} : \alpha \in \mathbb{T}, a \in \mathbb{D}\}$ forms a group G under composition of functions and is called the Möbius group.

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The Hilbert space of square integrable functions

Consider the space $L^2_a(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$ which are square integrable with respect to the area measure. This consists of the functions (these are polynomials that refuse to stop):

$$\{f : f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots\}$$

with $|a_0|^2 + |a_1|^2 + 2|a_2|^2 + \cdots + (n+1)|a_n|^2 + \cdots < \infty.$
The space $L^2_a(\mathbb{D})$ of functions defined on $\mathbb{D}$ is a Hilbert space.
Let $\mathcal{A}(\mathbb{D})$ be the set of holomorphic functions (again, polynomials that refuse to stop) which are continuous on the union of the two sets $\mathbb{D}$ and $\mathbb{T}$, it is an algebra. We have described three mathematical objects, namely
the Möbius group G,
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imprmitivity

Both the group G and the algebra $\mathcal{A}(\mathbb{D})$ "live" on the Hilbert space $L_a^2(\mathbb{D})$. Here is how this works! The action U of the group G on the Hilbert space $L_a^2(\mathbb{D})$ is given by the formula:

$$\big(U(\varphi)h\big)(z) = \varphi'(z)(h \circ \varphi)(z),\, h \in L_a^2(\mathbb{D})$$

while the action ϱ of the algebra $\mathcal{A}(\mathbb{D})$ is obtained by a mere multiplication –

$$\big(\varrho(f)h\big)(z) = f(z)h(z),\, h \in L_a^2(\mathbb{D})$$

What is more, U is a (actually, in general, projective) group homomorphism and ϱ is an algebra homomorphism. These satisfy the imprimitivity relation (a form of Weyl commutation relation):

$$\varrho(\varphi \cdot f) = U(\varphi)^* \varrho(f) U(\varphi),\, f \in \mathcal{A}(\mathbb{D}),\, \varphi \in G,$$

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Multiplier representations

Let $\mathcal{H}$ be a space of functions, say, on the unit disc or the unit circle. Suppose that the homomorphism $\varrho : \mathcal{A}(\mathbb{D}) \to \mathcal{L}(\mathcal{H})$ defined by the rule $\varrho(f) = M_f,\ f \in \mathcal{A}(\mathbb{D})$ is bounded.

Let $U : G \to \mathcal{L}(\mathcal{H})$ be of the form $U(\varphi) = M_{J_\varphi} R_\varphi,$ where M_{J_φ} is the multiplication by J_φ and R_φ is the composition by φ. The map U is a homomorphism if and only if the multiplier identity

$$J_{\varphi\psi}(z) = J_\varphi(\psi(z)) J_\psi(z),\ \varphi, \psi \in G$$

is valid for the function $J : G \times \mathbb{D} \to \mathbb{C}$. In this case U is said to be a multiplier representation.

If there is a multiplier representation, say U, of the group G on the Hilbert space $\mathcal{H}$, then the imprimitivity relationship

$$(M_{J_\varphi} R_\varphi)^* \varrho(\varphi \cdot f)(M_{J_\varphi} R_\varphi) = \varrho(f), \varphi \in G\ f \in \mathcal{A}(\mathbb{D}).$$

is forced.

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A question

Recall that the multiplier identity for $J : G \times \mathbb{D} \to \mathbb{C}$, where J_φ is the derivative φ', is the familiar chain rule.

Let us also emphasize that the map $U : G \to \mathcal{O}(\mathbb{D})$ defined by the rule $\big(U(\varphi)f\big)(z) = J_\varphi(z)f(\varphi(z))$ is a homomorphism only if J satisfies the multiplier identity.

Clearly, any power of the derivative $J_\varphi^{(\lambda)} := (\varphi')^\lambda, \lambda > 0$ will continue to obey the multiplier identity.

Surprisingly, these are all the possible complex valued multipliers for the Möbius group.

Given the multiplier $J^{(\lambda)}$, it is easy to find a Hilbert space $\mathcal{H}^{(\lambda)}$ such that U and ϱ, defined as before, acts on it satisfying the imprimitivity condition.

How do we construct multipliers taking values, say, in $n \times n$ matrices?

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A derivation

Assume that we have found a derivation $\mathfrak{d} : G \times \mathbb{D} \to \mathbb{C}^{n\times n}$ satisfying the multiplier identity, that is,

$$\mathfrak{d}(gh, z) = \mathfrak{d}(g, h(z))\mathfrak{d}(h, z),\ g, h \in G,\ z \in \mathbb{D}.$$

For $0 < \lambda \in \mathbb{R}$, define the map $\Gamma : G \to \mathcal{E}(\mathcal{O}(\mathbb{D}, \mathbb{C}^n))$ by the rule

$$\big(\Gamma(g^{-1})f\big)(z) = J^{(\lambda)}(g, z)\mathfrak{d}(g, z)f(g(z)),\ f \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n),\ g \in G.$$

Not only Γ is a homomorphism but any homomorphism must be of this form. It would be therefore desirable to find all the possible derivations $\mathfrak{d}$.

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the construction

Let $D(g,z)$ be the diagonal matrix whose (ℓ,ℓ) entry is $J^{(-m+j)}(g,z)I_{d_j}, \quad d_0+\cdots+d_j < \ell \leq d_{j+1}, \quad d_0+\cdots d_m = n.$
Also, the ratio

$$-\frac{1}{2}\frac{g''(z)}{(g'(z))^{\frac{3}{2}}},\ g\in G,\ z\in\mathbb{D}$$

is independent of z, which we denote by b_g.
Let $Y_i : \mathbb{C}^{d_j} \to \mathbb{C}^{d_{j+1}}$ be a set of m linear transformations and Y be the corresponding shift operator on $\mathbb{C}^n$.

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The proof

Theorem

The multiplier identity holds for the derivation $\mathfrak{d}$ defined by the rule:

$$(g, z) \mapsto J^{(\lambda)}(g, z) D(g, z)^{\frac{1}{2}} \exp(-b_g Y) D(g, z)^{\frac{1}{2}}, \ g \in G, \ z \in \mathbb{D}.$$

Proof: It is easy to verify that $D(g_1 g_2, z) = D(g_1, g_2(z)) D(g_2, z)$ using the chain rule. Now,

$$\mathfrak{d}(g_1 g_2, z) = D(g_1 g_2, z)^{\frac{1}{2}} \exp(-b_{g_1 g_2} Y) D(g_1 g_2, z)^{\frac{1}{2}}.$$

proof contd.

However, we have

$$
\begin{aligned}
-b_{g_1 g_2} &= \frac{1}{2}\frac{(g_1 g_2)''(z)}{\big((g_1 g_2)'(z)\big)^{3/2}} \\
&= \frac{1}{2}\frac{\big(g_1'(g_2(z))g_2'(z)\big)'}{\big(g_1'(g_2(z))g_2'(z)\big)^{3/2}} \\
&= \frac{1}{2}\frac{g_1''(g_2(z))(g_2'(z))^2 + g_1'(g_2(z))g_2''(z)}{\big(g_1'(g_2(z))g_2'(z)\big)^{3/2}} \\
&= \frac{1}{2}\Big\{\frac{g_1''(g_2(z))}{(g_1'(g_2(z)))^{3/2}} g_2'(z)^{1/2} + \frac{g_2''(z)}{(g_2'(z))^{3/2}}(g_1'(g_2(z)))^{-1/2}\Big\} \\
&= -b_{g_1}(g_2'(z))^{1/2} - b_{g_2}(g_1'(g_2(z)))^{-1/2}.
\end{aligned}
$$

proof contd.

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$e^{(-b_{g_1})}$$

$$e^{(-b_{g_2})}$$

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$$\mathfrak{d}(g_1 g_2, z)$$

These commute!

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$$\|$$

$$D(g_1, g_2(z))^{\frac{1}{2}} e^{(-b_{g_1} Y)} D(g_2, z)^{\frac{1}{2}}$$

$$D(g_1, g_2(z))^{\frac{1}{2}} e^{(-b_{g_2} Y)} D(g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$A^{-1} e^X A = e^{(A^{-1} X A)}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1, g_2(z))^{\frac{1}{2}} e^{(-b_{g_1} Y)} D(g_2, z)^{\frac{1}{2}}$$

$$D(g_1, g_2(z))^{\frac{1}{2}} e^{(-b_{g_2} Y)} D(g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$A^{-1} e^X A = e^{(A^{-1} X A)}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} D(g_2, z)^{-\frac{1}{2}} e^{(-b_{g_1} Y)} D(g_2, z)^{\frac{1}{2}}$$

$$D(g_1, g_2(z))^{\frac{1}{2}} e^{(-b_{g_2} Y)} D(g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

Inserting an identity...

$$A^{-1} e^X A = e^{(A^{-1} X A)}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} D(g_2, z)^{-\frac{1}{2}} e^{(-b_{g_1} Y)} D(g_2, z)^{\frac{1}{2}}$$

$$D(g_1, g_2(z))^{\frac{1}{2}} e^{(-b_{g_2} Y)} D(g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$A^{-1} e^X A = e^{(A^{-1} X A)}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$D(g_1, g_2(z))^{\frac{1}{2}} e^{(-b_{g_2} Y)} D(g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$A^{-1} e^X A = e^{(A^{-1} X A)}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$D(g_1, g_2(z))^{\frac{1}{2}} e^{(-b_{g_2} Y)} D(g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$A^{-1} e^X A = e^{(A^{-1} X A)}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$D(g_1, g_2(z))^{\frac{1}{2}} e^{(-b_{g_2} Y)} D(g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$A^{-1} e^{X} A = e^{(A^{-1} X A)}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$D(g_1, g_2(z))^{\frac{1}{2}} e^{(-b_{g_2} Y)} D(g_1, g_2(z))^{-\frac{1}{2}} D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

Innserting an identity, again.

$$A^{-1} e^X A = e^{(A^{-1} X A)}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$D(g_1, g_2(z))^{\frac{1}{2}} e^{(-b_{g_2} Y)} D(g_1, g_2(z))^{-\frac{1}{2}} D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$A^{-1} e^X A = e^{(A^{-1} X A)}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$A^{-1} e^X A = e^{(A^{-1} X A)}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1, g_2(z))^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$D(g_2, z)^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} = D(g_1 g_2, z)^{\frac{1}{2}}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1 g_2, z)^{\frac{1}{2}} e^{\left(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}}\right)}$$

$$e^{\left(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}}\right)} D(g_1 g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$D(g_2, z)^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} = D(g_1 g_2, z)^{\frac{1}{2}}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1 g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1 g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$D(g_2, z)^{\frac{1}{2}} D(g_2, z)^{\frac{1}{2}} = D(g_1 g_2, z)^{\frac{1}{2}}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1 g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1 g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$(D(g_2, z)^{-1/2} Y D(g_2, z)^{1/2})_{i(i+1)} = d_i^{-\frac{1}{2}} Y_i d_{(i+1)}^{\frac{1}{2}}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1 g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1 g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

All other entries are zero.

$$(D(g_2, z)^{-1/2} Y D(g_2, z)^{1/2})_{i(i+1)} = d_i^{-\frac{1}{2}} Y_i d_{(i+1)}^{\frac{1}{2}}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1 g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1 g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$(D(g_2, z)^{-1/2} Y D(g_2, z)^{1/2})_{i(i+1)} = d_i^{-\frac{1}{2}} Y_i d_{(i+1)}^{\frac{1}{2}}$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1g_2, z)$$

$$(D(g_2, z)^{-1/2} Y D(g_2, z)^{1/2})_{i(i+1)} = d_i^{-\frac{1}{2}} d_{(i+1)}^{\frac{1}{2}} Y_i$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1 g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1 g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$(D(g_2, z)^{-1/2} Y D(g_2, z)^{1/2})_{i(i+1)} = (d_{i+1}/d_i)^{\frac{1}{2}} Y_i$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1 g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1 g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$(D(g_2, z)^{-1/2} Y D(g_2, z)^{1/2})_{i(i+1)} = (g_2'(z))^{\frac{1}{2}} Y_i$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1 g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1 g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$(D(g_1, g_2(z))^{-1/2} Y D(g_1, g_2(z))^{1/2})_{i(i+1)} = (g_1'(g_2(z)))^{\frac{1}{2}} Y_i$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1 g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1 g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1 g_2, z)$$

$$(D(g_1, g_2(z))^{-1/2} Y D(g_1, g_2(z))^{1/2})_{i(i+1)} = (g_1'(g_2(z)))^{\frac{1}{2}} Y_i$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1g_2, z)$$

$$(D(g_1, g_2(z))^{-1/2} Y D(g_1, g_2(z))^{1/2})_{i(i+1)} = (g_1'(g_2(z)))^{\frac{1}{2}} Y_i$$

$$\mathfrak{d}(g_1, g_2(z))\mathfrak{d}(g_2, z)$$

$$\|$$

$$D(g_1g_2, z)^{\frac{1}{2}} e^{(-b_{g_1} D(g_2,z)^{-\frac{1}{2}} Y D(g_2,z)^{\frac{1}{2}})}$$

$$e^{(-b_{g_2} D(g_1,g_2(z))^{-\frac{1}{2}} Y D(g_1,g_2(z))^{-\frac{1}{2}})} D(g_1g_2, z)^{\frac{1}{2}}$$

$$\|$$

$$\mathfrak{d}(g_1g_2, z)$$

$$(D(g_1, g_2(z))^{-1/2} Y D(g_1, g_2(z))^{1/2})_{i(i+1)} = (g_1'(g_2(z)))^{\frac{1}{2}} Y_i$$

Thank you!