ZEROES OF POLYNOMIALS

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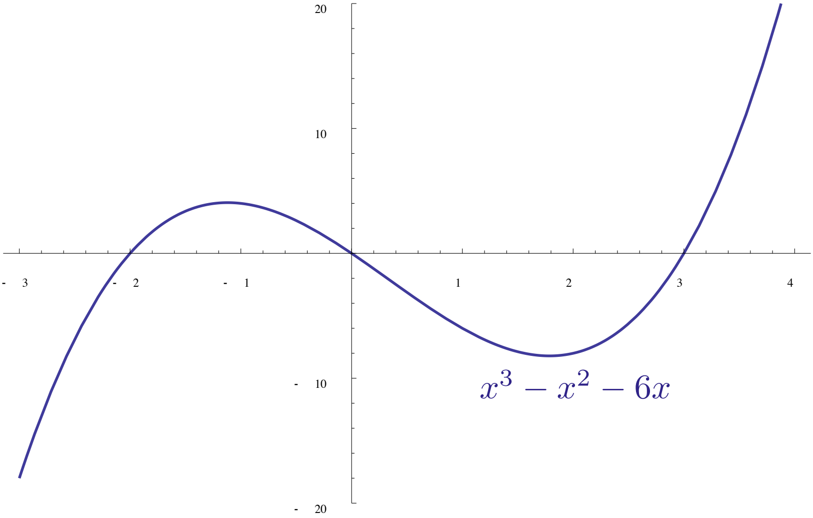
The roots of the derivative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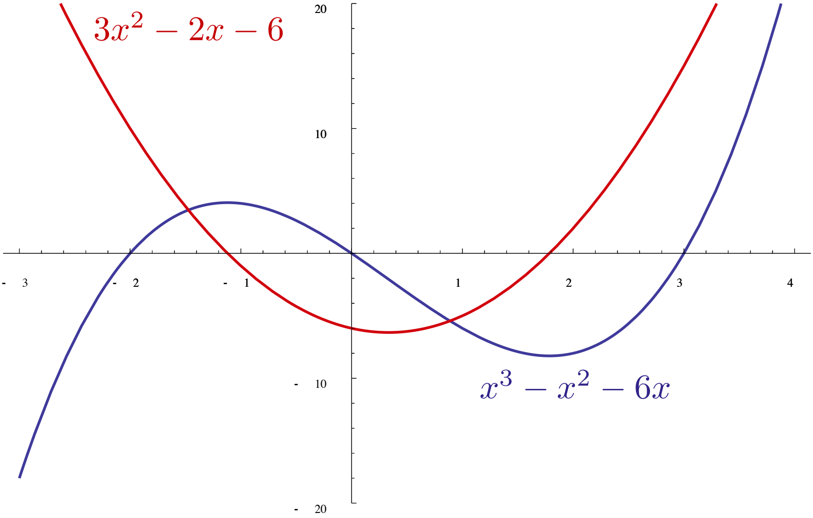
$x^3 - x^2 - 6x$



$3x^2 - 2x - 6$
$x^3 - x^2 - 6x$
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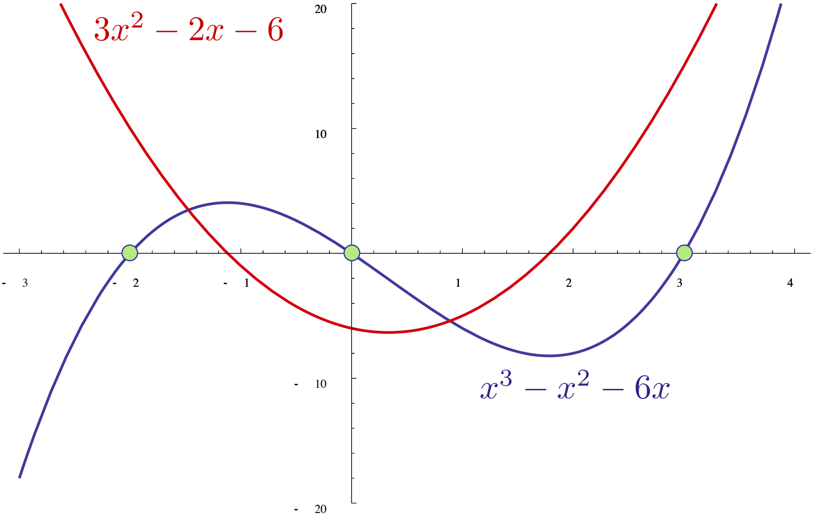

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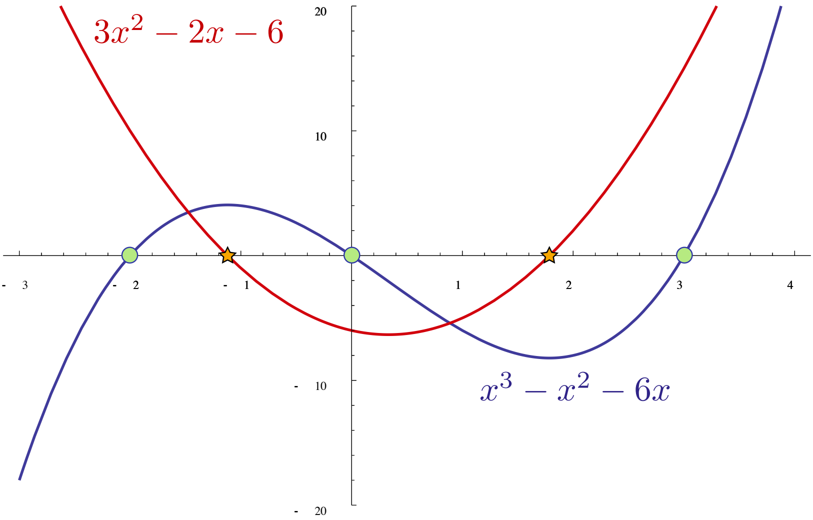

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Between consecutive real roots of the polynomial $p = x^3 - x^2 - 6x$,
we find one root of its derivative!

Let x_1, x_2 be two "consecutive" real roots of p.
Is it true that we always have one root of p' between x_1 and x_2?

Recall: Rolle's Theorem

If a real-valued function f is:

- continuous on a closed interval $[a, b]$,
- differentiable on the open interval (a, b),
- and $f(a) = f(b)$,

---

then there exists a c in the open interval (a, b) such that $f'(c) = 0$.

Let f be the polynomial p, and let the interval $[a, b]$ be given by $[x_1, x_2]$, where x_1 and x_2 are real roots of p.

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Note that $p(x_1) = p(x_2) = 0$.

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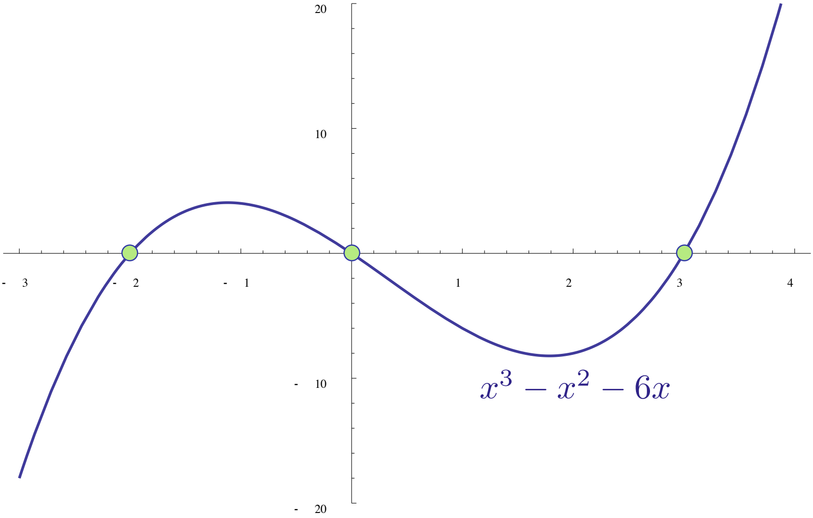
Observe that c is a root of the derivative!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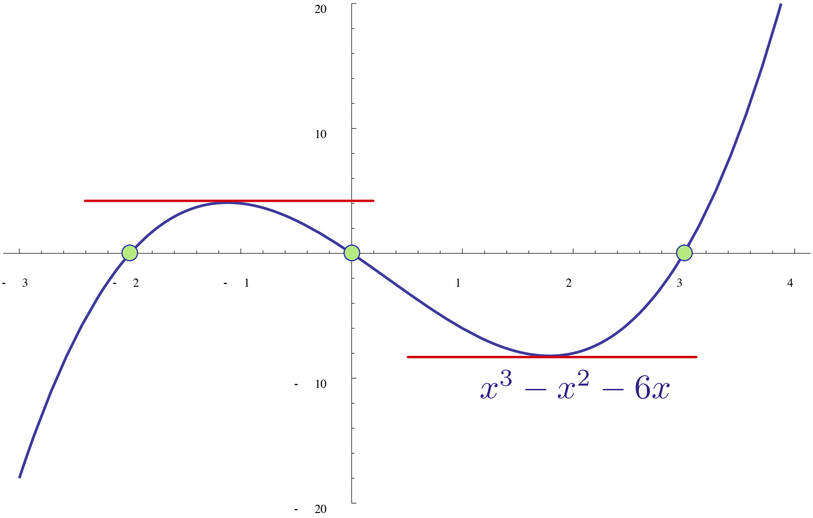
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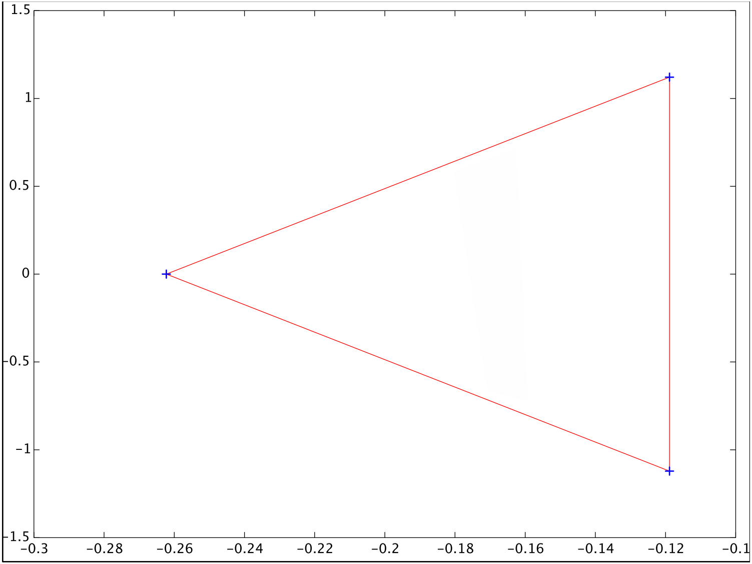
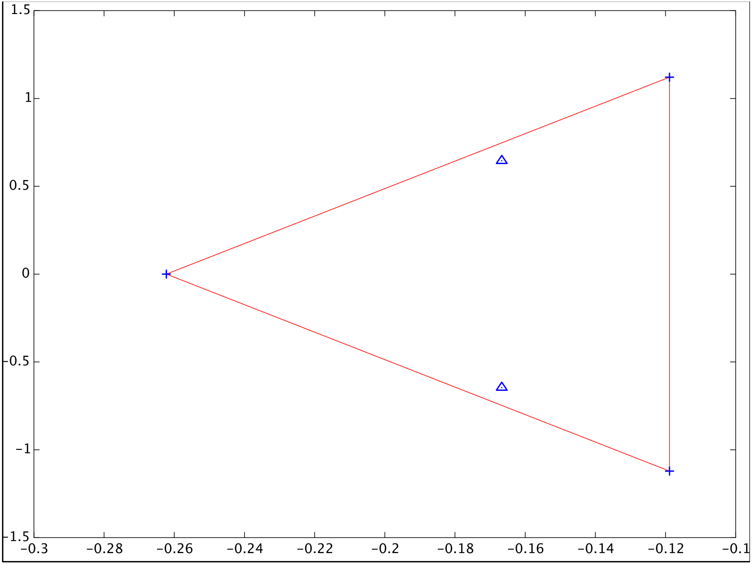
What can we say about complex roots?

Let us look at some examples.

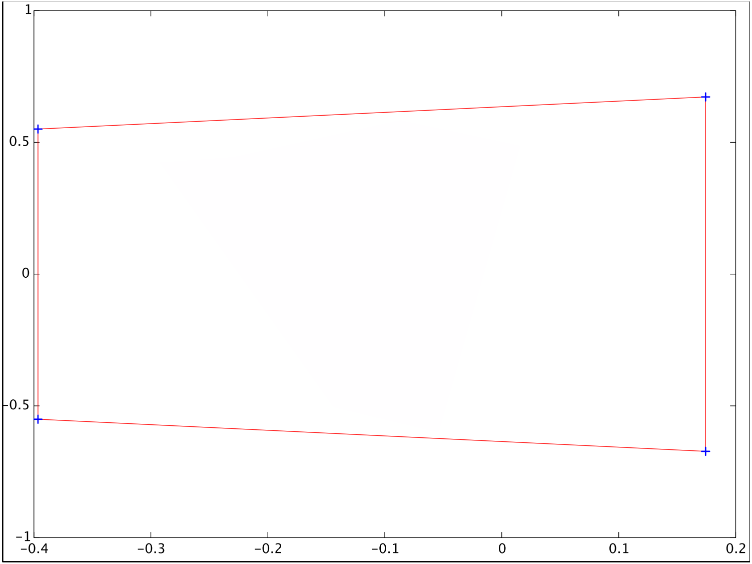


$$6x^3 + 3x^2 + 8x + 2$$

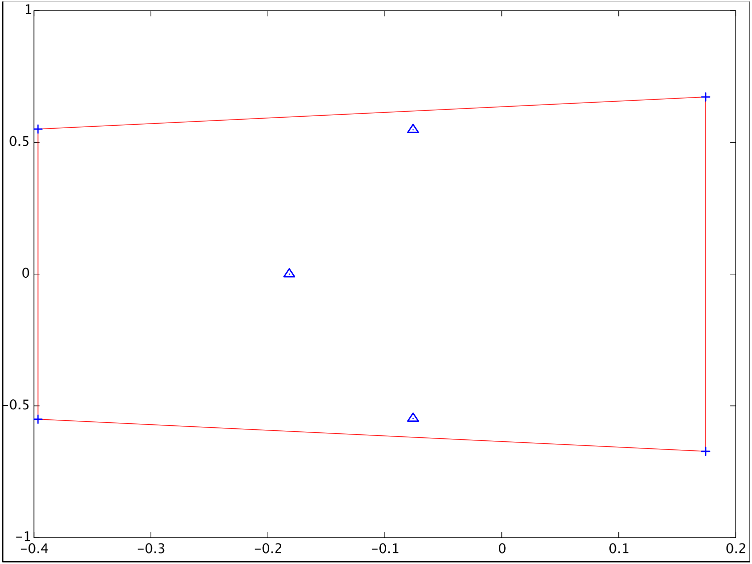


$$6x^3 + 3x^2 + 8x + 2$$

$$18x^2 + 6x + 8$$



$$9x^4 + 4x^3 + 6x^2 + 2x + 2$$



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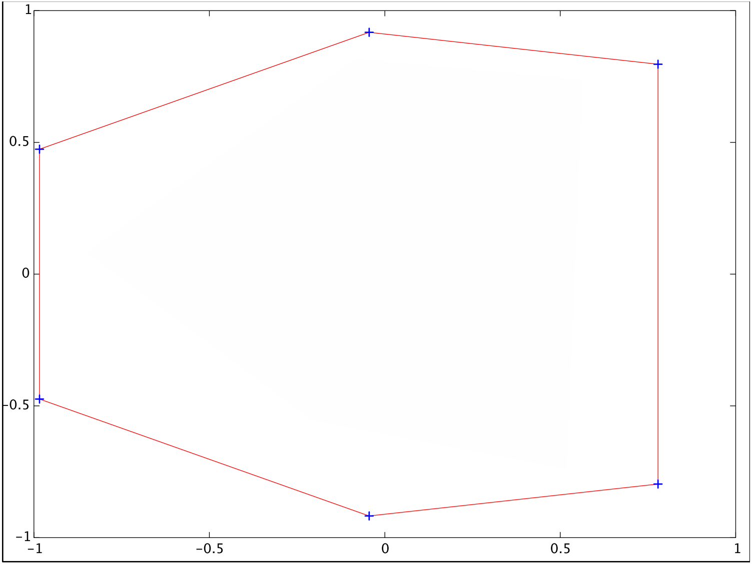
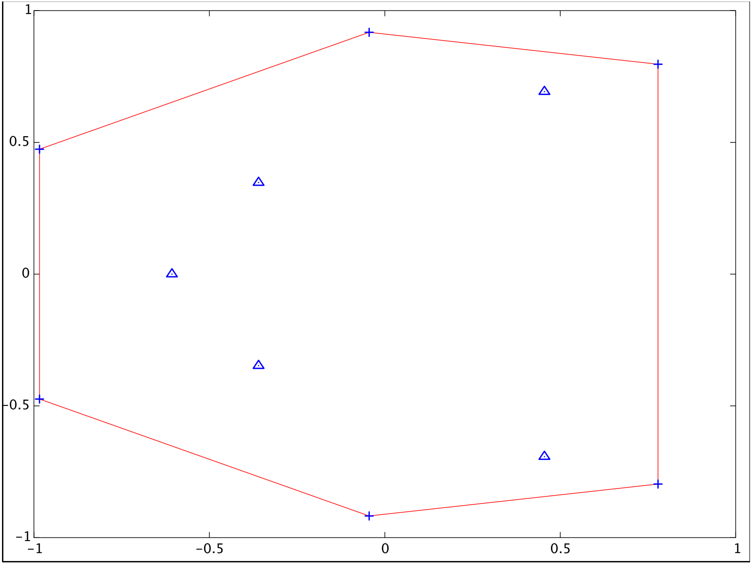
$$36x^3 + 12x^2 + 12x + 2$$



$$8x^6 + 4x^5 + 2x^4 + 7x^3 + 8x^2 + 5x + 10$$



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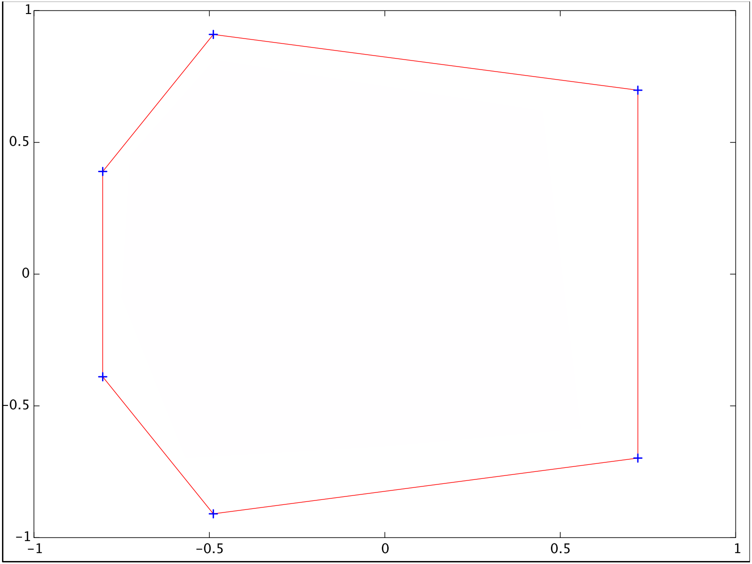
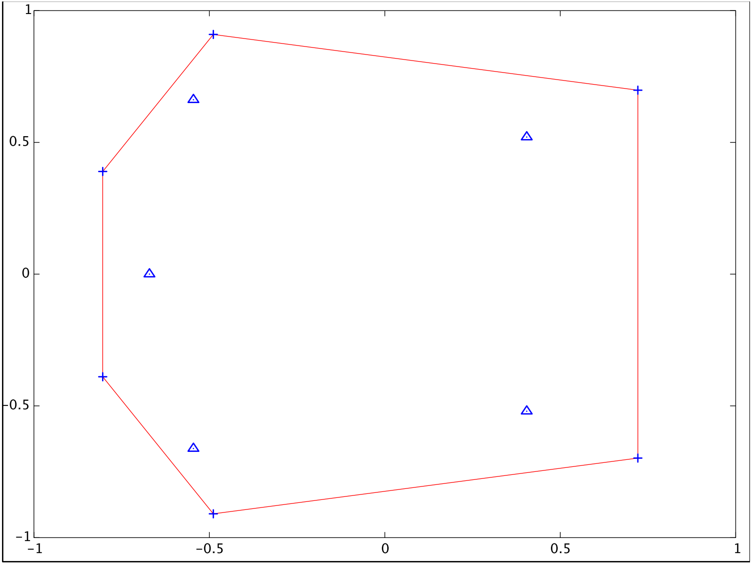
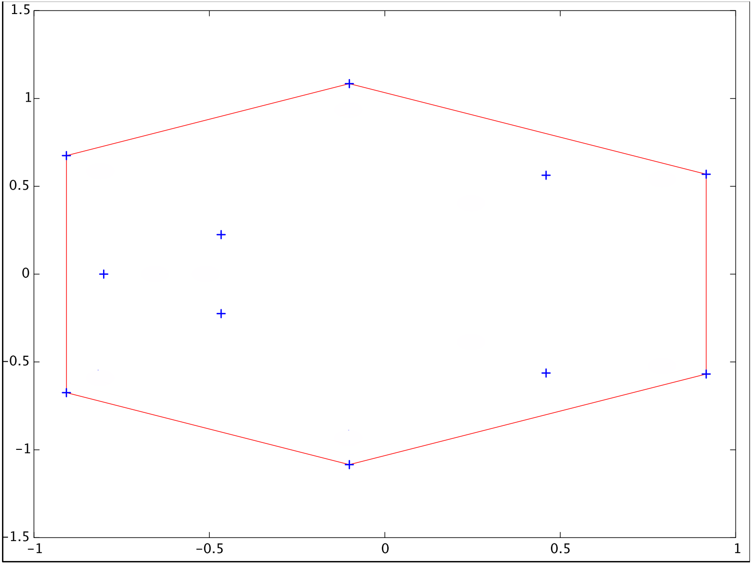


$$7x^6 + 8x^5 + 5x^4 + 1x^3 + 5x^2 + 9x + 6$$



$$7x^6 + 8x^5 + 5x^4 + 1x^3 + 5x^2 + 9x + 6$$

$$42x^5 + 40x^4 + 20x^3 + 3x^2 + 10x + 9$$



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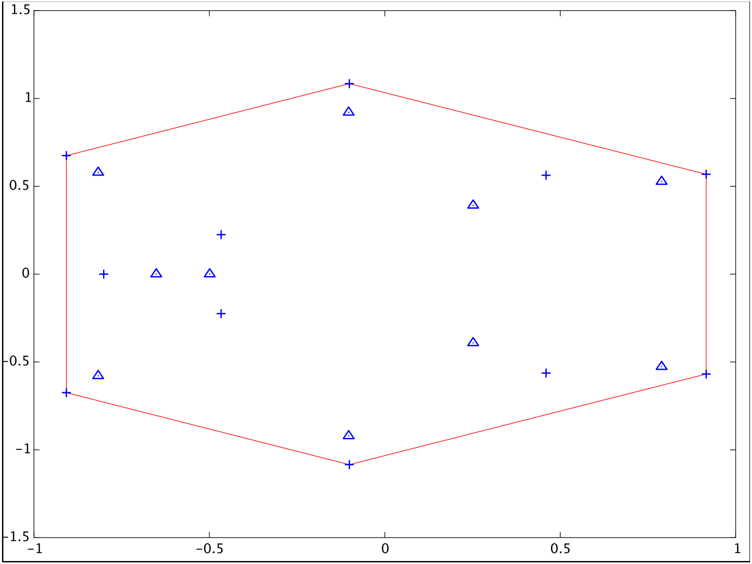

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$$55x^{10} + 50x^{9} + 18x^{8} + 14x^{6} + 18x^{5} + 45x^{4} + 28x^{3} + 4x + 3$$

Goal: To show that the roots of P' can be written as a convex combination of the roots of P.

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Note that the $\beta_i's$ are positive and sum to one.

ZEROES OF POLYNOMIALS

Trigonometric Polynomials

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The complex numbers c_k are called the (Fourier) coefficients of f.

A trig polynomial f is the zero function if and only if all its coefficients vanish.

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We can recover the coefficients c_k of a trig polynomial by integration:

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We show: A trig polynomial is real-valued if and only if $c_k = \overline{c_{-k}}$ for all k.

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The difference of the two far sides is the zero function; hence the conclusion follows.

(Fejer-Riesz) Let f be a trig polynomial with $f(\theta) \geqslant 0$ for all θ.
Then there is a complex polynomial p such that $f(\theta) = |p(e^{\imath\theta})|^2$.

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Assume f is of degree d and write $f(\theta) = \sum_{k=-n}^{n} c_k e^{\imath k\theta}$, where $c_{-k} = \overline{c_k}$ since f is real-valued. Note also that $c_d \neq 0$.

Define a polynomial q in one complex variable by:

$$q(z) = z^{d} \sum_{k=-d}^{d} c_k z^k.$$

Let $\xi_1, \ldots, \xi_{2d}$ be the roots of the polynomial q.

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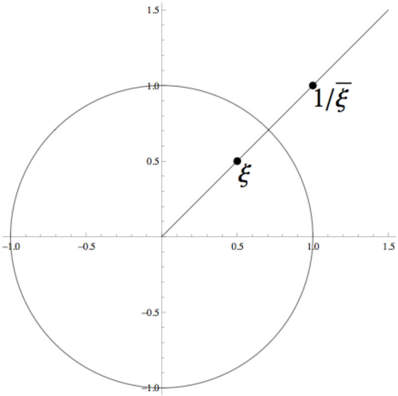
Claim that the reality of f implies that if ξ is a root of q, then $(\overline{\xi})^{-1}$ is also a root of q.

This point is called the reflection of ξ in the circle.



1.5
1.0
0.5
−0.5
−1.0
−1.0
−0.5
0.5
1.0
1.5
ξ
$1/\overline{\xi}$

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By the Fundamental Theorem of Algebra, we may factor the polynomial q into linear factors.

For z on the circle we can replace the factor $z - \left(\overline{\xi}\right)^{-1}$ with $\frac{1}{\overline{z}} - \frac{1}{\overline{\xi}} = \frac{\overline{\xi} - \overline{z}}{\overline{\xi}\,\overline{z}}$.

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Let $p(z) = C \prod_{k=1}^{d} (z - \xi_k)$.

Here we use all the roots in the unit disc and half the roots on the circle. Note that $q = |p|^2$ on the circle.

Since $f \geqslant 0$ on the circle, we have:

$$f(\theta) = |f(\theta)| = |q(e^{\imath\theta})| = |p(e^{\imath\theta})|^2.$$

ZEROES OF POLYNOMIALS

The Fundamental Theorem of Algebra

A continuous real valued function defined on a closed rectangle (or, a disc) in the plane is bounded and takes on an absolute minimum and an absolute maximum value.

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We now show: If f is a continuous real valued function on the plane such that $f(x, y)$ goes to infinity as (x, y) go to infinity, then f takes an absolute minimum value at some point of the plane.

Set $f_0 = |f(0, 0)|$. We may choose $r > 0$ such that:

$$f(x, y) > f_0 \text{ for all } (x, y) \text{ with } x^2 + y^2 \geqslant r.$$

Choose a rectangle R containing the disc of radius r centred at the origin.

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Pick m in the rectangle R such that the minimum of f occurs at m.

Since $(0, 0)$ is in the rectangle R, it follows that $f(m)$ is at most f_0. Since outside the rectangle R, the value of f is at least f_0, the value of f at m is the minimum of f on the whole plane, not just on R!

Let $f(z) = a_n z^n + \cdots + a_0$, where, for $i = 0, 1, \ldots n$, the a_i are in $\mathbb{C}$, and $a_n \neq 0$.

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Setting $z = x + \imath y$, we may write

$$f(z) = p(x, y) + \imath q(x, y),$$

where p, q are polynomials in two real variables x and y.

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where p, q are polynomials in two real variables x and y. Thus,

$$|f(z)| = (p(x, y)^2 + q(x, y)^2)^{\frac{1}{2}}$$

may be thought of as a continuous function the two real variables.

We have

$$|f(z)| = |a_n|\,|z^n|\,|1 + \frac{b_{n-1}}{z} + \frac{b_{n-2}}{z^2} + \cdots \frac{b_0}{z^n}|,$$

where $b_i = \frac{a_i}{a_n}$ for $0 \leqslant i \leqslant n-1$.

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where $b_i = \frac{a_i}{a_n}$ for $0 \leqslant i \leqslant n-1$. Now,

$$|1 + \frac{b_{n-1}}{z} + \frac{b_{n-2}}{z^2} + \cdots \frac{b_0}{z^n}| \geqslant |1| - |\frac{b_{n-1}}{z} + \frac{b_{n-2}}{z^2} + \cdots \frac{b_0}{z^n}|.$$

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The term we are subtracting on the right is at most

$$\frac{|b_{n-1}|}{|z|} + \frac{|b_{n-2}|}{|z|^2} + \cdots \frac{|b_0|}{|z|^n},$$

and this approaches zero as $|z|$ approaches infinity.

Thus the quantity on the left of this inequality, for large enough $|z|$, is at least $\frac{1}{2}$.

Hence $|f(z)|$ is at least $\frac{|a_n|}{2}|z|^n$ for large $|z|$.

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Hence $|f(z)|$ is at least $\frac{|a_n|}{2}\,|z|^n$ for large $|z|$.

Therefore, there must exist a point $m = a + \imath b$ at which $|f|$ attains its absolute minimum.

We will show that $f(m)$ must be zero.

SIMPLIFICATION 1

Let $g(z) = f(z + m)$. The polynomial g is again of degree n, it takes on the same set of values as f.

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But $|g|$ is minimum at $z = 0$, where the value is $|f(0 + m)| = |f(m)|$.

We now assume that $g(0) = \alpha_0$ is not zero.

SIMPLIFICATION 2

Replace g by $h := \frac{g}{\alpha_0}$. This new function h has its absolute minimum at $z = 0$, and the minimum value of h, which is taken at 0, is 1.

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Clearly, $h(0) = 1$, and h is of the form:

$$h(z) = \beta_n z^n + \cdots + 1, \text{ where } \beta_i = \frac{\alpha_i}{\alpha_0} (1 \leqslant i \leqslant n),$$

where α_i's are the coefficients of the polynomial g.

SIMPLIFICATION 3

We know that $\beta_n \neq 0$. Pick the smallest $k \leqslant n$ such that $\beta_k \neq 0$. We don't rule out the possibility of k being equal to either 1 or n, yet.

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If we replace the polynomial $h(z)$ by $h(cz)$, where c is some fixed complex number, then none of the properties of h change.

Choose c to be the kth root $(\frac{-1}{\beta_k})^{\frac{1}{k}}$. The new polynomial $h(cz)$ with this choice of c has the representation

$$1 - z^k + \beta_{k+1} z^{k+1} + \cdots \beta_n z^n.$$

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So, we may assume, without loss of generality that $\beta_k = -1$. If $k = n$, then $h(z) = 1 - z^n$ and we are done. So, we may assume, again without loss of generality, that $k < n$.

The main point: We need to show that the minimum absolute value of

$$1 - z^k + \beta_{k+1} z^{k+1} + \cdots \beta_n z^n$$

is less than 1 arriving at a contradiction arising out of our assumption that $f(m)$ is not 0.

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We shall indeed show that $|h(z)| < 1$ for small positive real z. to see this, choose z to be real and with $0 < z < 1$.

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For a small positive number z, we have $0 < 1 - z^k(1 - w(z)) < 1$. Since $|h(z)| < 1 - z^k(1 - w(z))$, it follows that $|h|$ takes values smaller than 1 and therefore $|h(z)|$ cannot have its minimum at 0.

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This concludes the proof of the Fundamental Theorem of Algebra.

CONSEUQENCE 1

Let f be a polynomial of degree n. Then given any complex number $w \in \mathbb{C}$, we have that $f(w) = 0$ if and only if there exists a polynomial g of degree $n-1$ such that $f(z) = (z-w)g(z), z \in \mathbb{C}$.

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The proof in the forward direction follows since

$$f(w) := a_n w^n + \cdots + a_1 w + a_0 = 0,$$

implying

$$f(z) = a_n z^n + \cdots + a_1 z + a_0 - \big(a_n w^n + \cdots + a_1 w + a_0\big),$$

which is easily seen to be of the required form.

CONSEQUENCE 2

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If f is any polynomial of degree n, then it must vanish at some w, and hence is of the form $(z - w)g(z)$ for some polynomial g of degree $n - 1$. The polynomial g has $n - 1$ zeros by the induction hypothesis.