A sheaf theoretic model for analytic Hilbert modules

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(joint work with Shibananda Biswas)

Let Ω be a bounded open connected set in $\mathbb{C}^m$ and $\mathcal{M}$ be a Hilbert module over the function algebra $\mathcal{A}(\Omega)$ (see [6]). The study of the natural class $\mathrm{B}_n(\Omega)$, discussed below, was initiated in [1, 2]. A different approach was given in [3]. Let $D_{\mathbf{T}} : \mathcal{M} \to \mathcal{M} \oplus \cdots \oplus \mathcal{M}$ be the operator $f \mapsto \big(T_1 f, \ldots, T_m f\big)$, where T_i is the operator determined by the adjoint of the module action $(z_i, f) \mapsto z_i \cdot f$, $1 \leq i \leq m$, $f \in \mathcal{M}$. Let $\mathrm{B}_n(\Omega)$ be the set of those Hilbert modules $\mathcal{M}$ for which $\operatorname{ran} D_{\mathbf{T}-w}$ is closed, $\operatorname{span}_{w\in\Omega} \ker D_{\mathbf{T}-w}$ is dense and $\dim \ker D_{\mathbf{T}-w} = n$ for all $w \in \Omega$. A Hilbert module $\mathcal{M}$ in $\mathrm{B}_n(\Omega)$ determines a holomorphic Hermitian vector bundle on Ω. It is then proved that isomorphic Hilbert modules correspond to equivalent vector bundles and vice-versa (see [1, 2]). Also, these papers provide a model for the Hilbert modules in $\mathrm{B}_n(\Omega)$ by showing that they can be realized as a Hilbert space consisting of holomorphic functions on Ω possessing a reproducing kernel. The module action is then simply the pointwise multiplication. Examples are Hardy and the Bergman modules over the ball and the poly-disc in $\mathbb{C}^m$. However, many natural examples of Hilbert modules fail to be in the class $\mathrm{B}_n(\Omega)$. For instance, $H^2_0(\mathbb{D}^2) := \{f \in H^2(\mathbb{D}^2) : f(0) = 0\}$ is not in $\mathrm{B}_n(\mathbb{D}^2)$. The problem is that the dimension of the joint kernel $\mathbb{K}(w) := \ker D_{\mathbf{T}-w}$ is no longer a constant (cf. [4]):

$$\dim H^2_0(\mathbb{D}^2) \otimes_{\mathcal{A}(\mathbb{D}^2)} \mathbb{C}_w = \begin{cases} 1 & \text{if } w \neq (0,0) \\ 2 & \text{if } w = (0,0). \end{cases}$$

Here $\mathbb{C}_w$ is the one dimensional module over the algebra $\mathcal{A}(\mathbb{D}^2)$, where the module action is given by the map $(f, w) \mapsto f(w)$ for $f \in \mathcal{A}(\mathbb{D}^2)$ and $w \in \mathbb{C}$. We outline an attempt to systematically study examples like the one given above using methods of complex analytic geometry.

For a Hilbert module $\mathcal{M}$ over a function algebra $\mathcal{A}(\Omega)$, not necessarily in the class $\mathrm{B}_1(\Omega)$, motivated by the correspondence of vector bundles with locally free sheaf, we construct a sheaf of modules $\mathcal{S}^{\mathcal{M}}(\Omega)$ over $\mathcal{O}(\Omega)$ corresponding to $\mathcal{M}$. We assume that $\mathcal{M}$ possesses all the properties for it to be in the class $\mathrm{B}_1(\Omega)$ except that the dimension of the joint kernel $\mathbb{K}(w)$ need not be constant. We note that sheaf models have occured, as a very useful tool, in the study of analytic Hilbert modules (cf. [7]). Although, the model we describe below is somewhat different.

Let $\mathcal{S}^{\mathcal{M}}(\Omega)$ be the subsheaf of the sheaf of holomorphic functions $\mathcal{O}(\Omega)$ whose stalk at $w \in \Omega$ is $\big\{(f_1)_w \mathcal{O}_w + \cdots + (f_n)_w \mathcal{O}_w : f_1, \ldots, f_n \in \mathcal{M}\big\}$, or equivalently, $\mathcal{S}^{\mathcal{M}}(U) = \Big\{ \sum_{i=1}^n \big(f_{i|U}\big) g_i : f_i \in \mathcal{M}, g_i \in \mathcal{O}(U) \Big\}$ for U open in Ω.

Proposition 1. *The sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ is coherent.*

Proof. The sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ is generated by the set of functions $\{f : f \in \mathcal{M}\}$. Let $\mathcal{S}^{\mathcal{M}}_J(\Omega)$ be the subsheaf generated by the set of functions $J = \{f_1, \ldots, f_\ell\} \subseteq \mathcal{M} \subseteq \mathcal{O}(\Omega)$. Thus $\mathcal{S}^{\mathcal{M}}_J(\Omega)$ is coherent. An application of Noether's Lemma [8] then guarantees that $\mathcal{S}^{\mathcal{M}}(\Omega) = \cup_{J\ \text{finite}} \mathcal{S}^{\mathcal{M}}_J(\Omega)$ is coherent. ☐

We note that the coherence of the sheaf implies, in particular, that the stalk $(\mathcal{S}^{\mathcal{M}})_w$ at $w \in \Omega$ is generated by a finite number of elements $g_1, \ldots, g_n$ from $\mathcal{O}(\Omega)$.

If K is the reproducing kernel for $\mathcal{M}$ and $w_0 \in \Omega$ is a fixed but arbitrary point, then for w in a small neighborhood U of w_0, we obtain the following decomposition theorem.

Theorem 1. *Suppose g_i^0, $1 \leq i \leq n$, be a minimal set of generators for the stalk $(\mathcal{S}^{\mathcal{M}})_0 := (\mathcal{S}^{\mathcal{M}})_{w_0}$. Then we have*

$$K(\cdot, w) := K_w = g_1^0(w) K_w^{(1)} + \cdots + g_n^0(w) K_w^{(n)},$$

where $K^{(p)} : U \to \mathcal{M}$, $1 \leq k \leq n$, is anti-holomorphic. Moreover, the elements $K_{w_0}^{(p)}$, $1 \leq p \leq n$ are linearly independent in $\mathcal{M}$, they are eigenvectors for the adjoint of the action of $\mathcal{A}(\Omega)$ on the Hilbert module $\mathcal{M}$ at w_0 and are uniquely determined by these generators.

We also point out that the Grammian $G(w) = (\!(\langle K_w^{(p)}, K_w^{(q)} \rangle)\!)_{p,q=1}^n$ is invertible in a small neighborhood of w_0 and is independent of the generators $g_1, \ldots, g_n$. Thus $t : w \mapsto (K_w^{(1)}, \ldots, K_w^{(n)})$ defines a holomorphic map into the Grassmannian $\mathrm{G}(\mathcal{H}, n)$ on the open set U. The pull-back E_0 of the canonical bundle on $\mathrm{G}(\mathcal{H}, n)$ under this map then define a holomorphic Hermitian bundle on U. Clearly, the decomposition of K given in our Theorem is not canonical in anyway. So, we can't expect the corresponding vector bundle E_0 to reflect the properties of the Hilbert module $\mathcal{M}$. However, it is possible to obtain a canonical decomposition following the construction in [3]. It then turns out that the equivalence class of the corresponding vector bundle E_0 obtained from this canonical decomposition is an invariant for the isomorphism class of the Hilbert module $\mathcal{M}$. These invariants are by no means easy to compute. At the end of this note, we indicate, how to construct invariants which are more easily computable.

For now, the following Corollary to the decomposition theorem is immediate.

Corollary 1. *The dimension of the joint kernel $\mathbb{K}(w)$ is greater or equal to the number of minimal generators of the stalk $(\mathcal{S}^{\mathcal{M}})_w$ at $w \in \Omega$.*

Now is the appropriate time to raise a basic question. Let $\mathfrak{m}_w \subseteq \mathcal{A}(\Omega)$ be the maximal ideal of functions vanishing at w. Since we have assumed $\mathfrak{m}_w\mathcal{M}$ is closed, it follows that the dimension of the joint kernel $\mathbb{K}(w)$ equals the dimension of the quotient module $\mathcal{M}/(\mathfrak{m}_w\mathcal{M})$. However it is not clear if one may impose natural hypothesis on $\mathcal{M}$ to ensure

$$\dim \mathcal{M}/(\mathfrak{m}_w\mathcal{M}) = \dim \mathbb{K}(w) = \dim (\mathcal{S}^{\mathcal{M}})_w / (\mathfrak{m}(\mathcal{O}_w)(\mathcal{S}^{\mathcal{M}})_w),$$

where $\mathfrak{m}(\mathcal{O}_w)$ is the maximal ideal in $\mathcal{O}_w$, as well.

More generally, suppose $p_1, \ldots, p_n$ generate $\mathcal{M}$. Then $\dim \mathbb{K}(w) \leq n$ for all $w \in \Omega$. If the common zero set V of these is $\{0\}$ then $(p_1)_0, \ldots, (p_n)_0$ need not be a minimal set of generators for $(\mathcal{S}^{\mathcal{M}})_0$. However, we show that they do if we assume $p_1, \ldots, p_n$ are homogeneous of degree k, say. Further more, basis for $\mathbb{K}(0)$ is the set of vectors:

$$\big\{p_1(\bar{\boldsymbol{\partial}})\} K(\cdot, w)_{|\, w=0}, \ldots, p_n(\bar{\boldsymbol{\partial}})\} K(\cdot, w)_{|\, w=0}\big\},$$

where $\bar{\boldsymbol{\partial}} = (\bar{\partial}_1, \dots, \bar{\partial}_m)$.

Going back to the example of $H^2_0(\mathbb{D}^2)$, we see that it has two generators, namely z_1 and z_2. Clearly, the joint kernel $\mathbb{K}(w) := \ker D_{(M_1^* - \bar{w}_1, M_2^* - \bar{w}_2)}$ at $w = (w_1, w_2)$ is spanned by $\{z_1 \otimes_{\mathcal{A}(\mathbb{D}^2)} 1_w, z_2 \otimes_{\mathcal{A}(\mathbb{D}^2)} 1_w\} = \{w_1 K_{H^2_0(\mathbb{D}^2)}(z, w), w_2 K_{H^2_0(\mathbb{D}^2)}(z, w)\}$ which consists of two vectors that are linearly dependent except when $w = (0, 0)$. We also easily verify that

$$\left(\mathcal{S}^{H^2_0(\mathbb{D}^2)}\right)_w \cong \begin{cases} \mathcal{O}_w & w \neq (0,0) \\ \mathfrak{m}\left(\mathcal{O}_0\right) & w = (0,0). \end{cases}$$

Since the reproducing kernel

$$K_{H^2_0(\mathbb{D}^2)}(z, w) \quad = \quad K_{H^2(\mathbb{D}^2)}(z, w) - 1 = \frac{z_1 \bar{w}_1 + z_2 \bar{w}_2 - z_1 z_2 \bar{w}_1 \bar{w}_2}{(1 - z_1 \bar{w}_1)(1 - z_2 \bar{w}_2)},$$

we find there are several choices for $K_w^{(1)}$ and $K_w^{(2)}$, $w \in U$. However, all of these choices disappear if we set $\bar{w}_1 \theta_1 = \bar{w}_2$ for $w_1 \neq 0$, and take the limit:

$$\lim_{(w_1, w_2) \to 0} \frac{K_{H^2_0(\mathbb{D}^2)}(z, w)}{\bar{w}_1} = K_0^{(1)}(z) + \theta_1 K_0^{(2)}(z) = z_1 + \theta_1 z_2$$

because $K_0^{(1)}$ and $K_0^{(2)}$ are uniquely deteremined by Theorem 1. Similarly, for $\bar{w}_2 \theta_2 = \bar{w}_1$ for $w_2 \neq 0$, we have

$$\lim_{(w_1, w_2) \to 0} \frac{K_{H^2_0(\mathbb{D}^2)}(z, w)}{\bar{w}_2} = K_0^{(2)}(z) + \theta_2 K_0^{(1)}(z) = z_2 + \theta_2 z_1.$$

Thus we have a Hermitian line bundle on the complex projective space $\mathbb{P}^1$ given by the frame $\theta_1 \mapsto z_1 + \theta_1 z_2$ and $\theta_2 \mapsto z_2 + \theta_2 z_1$. The curvature of this line bundle is then an invariant for the Hilbert module $H^2_0(\mathbb{D}^2)$ as shown in [5]. This curvature is easily calculated and is given by the formula $\mathcal{K}(\theta) = (1 + |\theta|^2)^{-2}$.

The decomposition theorem yields similar results in many other examples.

References

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