MATHEMATICAL MORPHOLOGY IN GEOSCIENCES AND GISci

B.S. DAYA SAGAR
http://www.isibang.ac.in/~bsdsagar
Systems Science and Informatics Unit (SSIU)
Indian Statistical Institute-Bangalore Centre

Systems Science and Informatics Unit (SSIU)

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Centre for Earth & Space Sciences, UoH, Hyderabad
II.II. Terrestrial Analysis

- Scale invariance and Power-laws in networks
- Shape-dependant power-laws
- Granulometric analysis
First step in drainage basin analyses is the classification of stream orders by Horton-Strahler’s ordering system (Horton, 1945; Strahler, 1957). The order of the whole tree is defined to be the order of the root. This ordering system has been found to correlate well with important basin properties in a wide range of environments.

This figure shows a sample network classified based on Horton-Strahler’s ordering system. The network is divided into orders, with the outlet at the bottom. The network is labeled as a Cameron Highland channel network.
Two topological quantities bifurcation ratio ($R_b$) and length ratio ($R_l$)

$$R_b = \frac{N_i}{N_{i+1}} \quad R_l = \frac{L_i}{L_{i-1}}$$

Networks extraction and their properties: Morphometry

Besides these two ratios, the universal similarity of stream network can be shown through Hack’s law and Hurst’s law as follows:

- **Hack’s Law**: $L_{mc} \propto A^h$
  
  where $A$ is the area of basin with main channel length $L_{mc}$.

- **Hurst’s law**: $L_\perp \propto L_{||}^H$
  
  where $L_{||}$ is the longitudinal length and $L$ transverse length respectively.
Allometric power-laws are derived between the basic measures such as basin area, basin perimeter, channel length, longitudinal length and transverse length.

Observed that these power-laws are of universal type as they exhibit similar scaling relationships at all scales.

Existing allometric power-laws: Decomposed basins & networks.
The number of decomposed sub-basins of respective orders from the simulated 6th order F-DEM include:

- two 5\textsuperscript{th}
- five 4\textsuperscript{th}
- ten 3\textsuperscript{rd}
- thirty six 2\textsuperscript{nd}, and
- eighty six 1\textsuperscript{st} order basins.
Existing allometric power-laws:

**Decomposed basins and networks**

Decomposed sub-basins are
- two 4\textsuperscript{th}
- eight 3\textsuperscript{rd}
- twenty-eight 2\textsuperscript{nd}, and
- one hundred twenty-four 1\textsuperscript{st} order basins.
Basic measures for a basin, (a) basin area, (b) total channel length, (c) main channel length, (d) basin perimeter, (e) longitudinal length and (f) transverse length.
Scale Invariant allometric power-laws

Allometric relationships among various areal and length parameters for all sub-basins of F-DEM and TOPSAR DEM.
Our results shown for basins derived from F-DEM and TOPSAR DEM are in good accord with power-laws derived from Optimal Channel Networks (Maritan et. al., 2002) and Random Self-Similar Networks (Veitzer and Gupta 2000) and certain natural river basins.
Network topology and watershed geometry are important features in terrain characterization.

Travel-time networks are sequence of networks generated by removing the extremities of the network iteratively. Hit-or-Miss transformation and Thinning transformations is used in generating travel-time network. Half-plane closing-based algorithm (Soille, 2005) is employed to generate convex hulls for these travel-time networks.

Length of the travel-time network and area of the corresponding convex hull are used to derive new scaling exponents.
Proposed scaling relationships:
Travel-time networks

- The process of deleting the end points from the networks is named as pruning.
- To decompose the stream network subsets from \( n = 1 \) to \( N \), structuring template of \( B_1 \) and \( B_2 \) are decomposed into various subsets, \( B_n^i \) where \( i = 1, 2, \ldots, 8 \) and \( n = 1, 2 \)

\[
\begin{align*}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_1^1 = & 0 & 1 & 0 & B_1^2 = & 0 & 1 & 0 & B_1^3 = & 0 & 1 & 0 & B_1^4 = & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
X & 1 & X & 0 & 0 & X & 0 & 0 & 1 & X & 0 & 0
\end{align*}
\]

- Both structuring templates are disjointed into eight directions. The intersecting portion of eroded \( S \) and eroded \( S_c \) by disjointed templates \( \{ B_1^k \} \) and \( \{ B_2^k \} \) \( k = 1, 2, \ldots, 8 \) respectively are computed to derive pruned version of \( S \).

\[
\begin{align*}
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
B_2^1 = & 1 & 0 & 1 & B_2^2 = & 1 & 0 & 1 & B_2^3 = & 1 & 0 & 1 & B_2^4 = & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & X & 1 & X & X & 1 & X & X & 0 & 0
\end{align*}
\]

- The X’s in the structuring templates signifies the ‘don’t care’ condition – it doesn’t matter whether the pixel in that location has a value of 0 or 1.

\[
\begin{align*}
X & 0 & X & 1 & 1 & X & 1 & 1 & 1 & X & 1 & 1 \\
B_2^5 = & 1 & 0 & 1 & B_2^6 = & 1 & 0 & 0 & B_2^7 = & 1 & 0 & 1 & B_2^8 = & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & X & 0 & X & X & 1 & 1
\end{align*}
\]
Mathematically, 

$$ S \ast B = (S \ominus B_1^k) \cap (S^c \ominus B_2^k) \text{ where } B = B_1^k \cup B_2^k $$

By subtracting \((S \ast B)\) from \(S\), a pruned version of \(S\) is obtained and expressed as 

$$ S_1 = S \otimes \{B\} \text{ where, } S \otimes \{B\} = S - (S \ast B) $$

\(\{B\}\) is the sequence of 

$$ \{(B_1, B_1^2, \ldots, B_1^8), (B_2, B_2^2, \ldots, B_2^8)\} $$

After pruning of \(S\) in first pass with \(B_1\), the process continue with pruning with \(B_2\) and so on until \(S\) is pruned in the last pass with \(B_8\).

$$ S \otimes \{B\} = ((\cdots ((S \otimes B_1^1) \otimes B_2^2) \cdots) \otimes B_8^8) $$

The whole process removes the first-encountered open pixels of \(S\) and produces \(S_1\).

Repeating the same process on \(S_1\) will produce \(S_2\). The process is repeated until no further changes occur, where the closed outlet is reached.
Proposed scaling relationships: Convex hull

Convex hull is the smallest convex set that contains all the points of the network.

Since convex hull represents the basin of network, convex hulls of the travel-time networks are generated.
Properties of the pruned network:

1. \[ S = \bigcup_{n=0}^{N-1} (S_n - S_{n+1}) \]

2. \[ S_N \subset S_{N-1} \subset \cdots \subset S_2 \subset S_1 \subset S \]

3. \( S, S_1, S_2, \ldots, S_N \) obtained by iterative pruning.

The final convex polygon containing all the points of \( S \) yields \( C(S) \).
Proposed scaling relationships

- Network – pruning – network length = $S_n$
- Convex hull computed – convex hull area = $C(S_n)$
- Convexity measures, $CM = \frac{1}{L(S_n)^\beta}$
  $$L(S_n) \sim A[C(S_n)]^\alpha$$
  $$CM(S_n) \sim \frac{1}{A[C(S_n)]^\lambda}$$

Graph of lengths of the sequential pruned networks versus the corresponding areas of convex hulls.

Relationship between channel lengths and convexity measures.

Relationship between areas of convex hulls and convexity measures.
Proposed scaling relationships

- Sample basin
- Simulated F-DEM basins
- Cameron basins
- Petaling basins
Proposed scaling relationships

<table>
<thead>
<tr>
<th>Network</th>
<th>α, (R²)</th>
<th>σ, R²</th>
<th>λ, R²</th>
<th>Rₚ</th>
<th>Rᵢ</th>
<th>h</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample</td>
<td>0.5693, (0.9671)</td>
<td>0.6988, (0.8325)</td>
<td>0.4307, (0.9439)</td>
<td>3.84</td>
<td>1.66</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Basin 1 (Cameron)</td>
<td>0.5777, (0.9883)</td>
<td>0.7109, (0.9358)</td>
<td>0.4223, (0.9783)</td>
<td>3.60</td>
<td>2.21</td>
<td>0.5414</td>
<td>0.9714</td>
</tr>
<tr>
<td>Basin 2 (Cameron)</td>
<td>0.5774, (0.9925)</td>
<td>0.7189, (0.9586)</td>
<td>0.4226, (0.9861)</td>
<td>4.35</td>
<td>2.25</td>
<td>0.5561</td>
<td>1</td>
</tr>
<tr>
<td>Basin 3 (Cameron)</td>
<td>0.5799, (0.9934)</td>
<td>0.7131, (0.963)</td>
<td>0.4201, (0.9875)</td>
<td>3.31</td>
<td>2.39</td>
<td>0.5612</td>
<td>0.9256</td>
</tr>
<tr>
<td>Basin 4 (Cameron)</td>
<td>0.5521, (0.9835)</td>
<td>0.7814, (0.92)</td>
<td>0.4479, (0.9752)</td>
<td>4.47</td>
<td>3.18</td>
<td>0.5671</td>
<td>0.9506</td>
</tr>
<tr>
<td>Basin 5 (Cameron)</td>
<td>0.5798, (0.9905)</td>
<td>0.7083, (0.9469)</td>
<td>0.4202, (0.982)</td>
<td>3.31</td>
<td>2.16</td>
<td>0.5766</td>
<td>0.9162</td>
</tr>
<tr>
<td>Basin 6 (Cameron)</td>
<td>0.5819, (0.9865)</td>
<td>0.6955, (0.925)</td>
<td>0.4181, (0.9743)</td>
<td>4.00</td>
<td>2.64</td>
<td>0.5746</td>
<td>0.8597</td>
</tr>
<tr>
<td>Basin 7 (Cameron)</td>
<td>0.5885, (0.9887)</td>
<td>0.68, (0.9348)</td>
<td>0.4115, (0.9772)</td>
<td>2.82</td>
<td>2.39</td>
<td>0.5548</td>
<td>0.895</td>
</tr>
<tr>
<td>Basin 1 (Petaling)</td>
<td>0.5462, (0.969)</td>
<td>0.7741, (0.8561)</td>
<td>0.4538, (0.9557)</td>
<td>5.00</td>
<td>2.57</td>
<td>0.5568</td>
<td>0.9319</td>
</tr>
<tr>
<td>Basin 2 (Petaling)</td>
<td>0.5393, (0.9899)</td>
<td>0.8357, (0.9532)</td>
<td>0.4607, (0.9863)</td>
<td>4.00</td>
<td>3.51</td>
<td>0.5828</td>
<td>0.8623</td>
</tr>
<tr>
<td>Basin 3 (Petaling)</td>
<td>0.5198, (0.9852)</td>
<td>0.8953, (0.9367)</td>
<td>0.4802, (0.9827)</td>
<td>4.24</td>
<td>3.30</td>
<td>0.597</td>
<td>0.9019</td>
</tr>
<tr>
<td>Basin 4 (Petaling)</td>
<td>0.5592, (0.9938)</td>
<td>0.7771, (0.9684)</td>
<td>0.4408, (0.99)</td>
<td>4.24</td>
<td>2.96</td>
<td>0.5807</td>
<td>0.8902</td>
</tr>
<tr>
<td>Basin 5 (Petaling)</td>
<td>0.5729, (0.9906)</td>
<td>0.729, (0.9492)</td>
<td>0.4271, (0.9832)</td>
<td>4.79</td>
<td>3.96</td>
<td>0.5844</td>
<td>0.8704</td>
</tr>
<tr>
<td>Basin 6 (Petaling)</td>
<td>0.5547, (0.9872)</td>
<td>0.7798, (0.937)</td>
<td>0.4453, (0.9804)</td>
<td>4.89</td>
<td>3.42</td>
<td>0.5713</td>
<td>0.9116</td>
</tr>
<tr>
<td>Basin 7 (Petaling)</td>
<td>0.6059, (0.9929)</td>
<td>0.6387, (0.9551)</td>
<td>0.3941, (0.9834)</td>
<td>3.60</td>
<td>3.39</td>
<td>0.5865</td>
<td>0.8312</td>
</tr>
</tbody>
</table>

Allometric power-laws between travel-time channel networks, convex hulls, and convexity measures for model network, networks of Hortonian fractal DEM, and networks of fourteen basins of Cameron Highlands and Petaling region.

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B. S. Daya Sagar
Proposed scaling relationships

These proposed scaling exponents are shown for basins derived from simulated F-DEM and TOPSAR DEMs.

These exponents are scale-independent.

At macroscopic level, these exponents complement with other existing scaling coefficients can be used to identify commonly sharing generic mechanisms in different river basins.
II.II.II. Scale Invariant But Shape Dependent Power-laws
Objectives

To propose morphology based method via fragmentation rules to compute scale invariant but shape-dependent measures of non-network space of a basin.

To make comparisons between morphometry based parameters / dimensions and dimensions derived for non-network space.

Topologically Invariant networks with variant geometric organization
Proposed Technique

Step 1: Channel network is traced from topographic map.

Step 2: Channel network is dilated and eroded iteratively until the entire basin is filled up with white space. This step is to generate catchment boundary automatically. Dilation followed by erosion is called structural closing, which will smoothen the image.

Step 3: Generate the basin with channel network and non-network space with boundary by subtracting the channel network from the catchment boundary achieved in Step 2.

Step 4: Structural opening (erosion followed by dilation) is performed recursively in basin achieved in Step 3 to fill the entire basin of non-network space with varying size of octagons.

Step 5: Assign unique color for each size of octagons.

Step 6: Compute morphometry for the basin.

Step 7: Compute shape dependent dimension.
Power law relationship

- As per the previous fig. the slopes of the best-fit lines ($\alpha_N$ and $\alpha_A$) for number-radius and area-radius relationships yield 2.37 and 1.34.
- These slope values of the best-fit lines provide shape dependent dimensions as $D_N = \alpha_N - 1$ and $D_A = \alpha_A$.
- As in previous Fig., $D_N$ and $D_A$ for non-network space yield 1.37 and 1.34.
- A Power-law relationship is shown in earlier Fig. with an exponent value 1.79 between the area and number of NODs observed with increasing radius of structuring template.

(a) Appollonian Space, and (b) after decomposition by means of octagon.
Algorithm Implementation:

Step 1: **Channel network of sub basin 1**

Step 2: **Close-Hull Generation**

Iterative dilation of channel network of basin 1

Iterative erosion applied to previous Fig
Step 3: Non-network space of basin 1

Iterative erosion applied to step-3 Fig.

Iterative erosion applied to previous Fig.

Iterative dilation applied to previous Fig.
Step 4: Non-Network Space Decomposition

Iterative erosion applied to previous Fig.  

Iterative dilation applied to previous Fig.
Decomposition of Non-network space into non-overlapping disks of octagon shape of several sizes for basin 1

Non-Network Spaces Packed with Non-Overlapping Disks of basins 2 to 8
Dimensions derived from morphometry of network and non network space

<table>
<thead>
<tr>
<th>Basin #</th>
<th>Network FD</th>
<th>Log Rs/Log Rn</th>
<th>R vs A</th>
<th>R vs N</th>
<th>A vs N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.83</td>
<td>1.93</td>
<td>1.34</td>
<td>2.06</td>
<td>1.50</td>
</tr>
<tr>
<td>2</td>
<td>0.86</td>
<td>1.63</td>
<td>1.33</td>
<td>1.23</td>
<td>1.59</td>
</tr>
<tr>
<td>3</td>
<td>0.98</td>
<td>1.41</td>
<td>1.02</td>
<td>1.87</td>
<td>1.80</td>
</tr>
<tr>
<td>4</td>
<td>2.07</td>
<td>2.01</td>
<td>1.43</td>
<td>2.17</td>
<td>1.52</td>
</tr>
<tr>
<td>5</td>
<td>1.73</td>
<td>1.90</td>
<td>1.34</td>
<td>1.94</td>
<td>1.43</td>
</tr>
<tr>
<td>6</td>
<td>1.84</td>
<td>2.04</td>
<td>1.13</td>
<td>1.87</td>
<td>1.63</td>
</tr>
<tr>
<td>7</td>
<td>1.33</td>
<td>1.61</td>
<td>1.23</td>
<td>2.08</td>
<td>1.70</td>
</tr>
<tr>
<td>8</td>
<td>1.65</td>
<td>2.06</td>
<td>1.61</td>
<td>2.38</td>
<td>1.49</td>
</tr>
</tbody>
</table>

Morphometric parameter computations achieved through decomposition of non-network space
Basin number versus varied dimensions derived from morphometry of networks and non-network spaces