Poisson-Poisson Cluster SINR Coverage Process

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1 Introduction

In [1], a new coverage process was defined. It was motivated by the various SINR(Signal to Interference Noise Ratio) models in wireless communications. The underlying point process was assumed to be a Poisson process. In this report, we extend the theory to the case when the underlying point process is a Poisson-Poisson cluster process.

Firstly, we define the stochastic geometric model. We give sufficient conditions similar to [1] for the model to be well-defined. Finally, certain results analogous to the cited paper are derived. The reader can also refer to [2] for some more results on such a model and [3] for percolation in such a model.

The necessity of this extension arises mainly because in certain models of wireless communications the antennae are clustered or bunched together. And cluster processes model such cases better than Poisson process. Our main goal in this direction is comparison of the two models. In SINR models one would expect the clustering to have a negative impact on the connectivity of the network. The report is organized as follows: Section 2 describes the model in generality and various assumptions for the model to be well-defined. The concluding remark describes the specific model we shall analyse in lot more detail. The sufficient conditions for the assumptions on the model to be satisfied are given in Section 3. Finally, in Section 4 we derive formulae for the coverage probability.

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2 Description of Model

2.1 Generic Stochastic Geometric Model

Our underlying marked point process is $\Phi = \{(X_i + X_{ij} \in \mathcal{R}^d, Z_{ij})\}$, where $\mathcal{C} = \{X_i\}$ is a Poisson process with intensity $\lambda(.)$ and $(\{X_{ij}\}|X_i)_i$ are i.i.d Poisson process with intensity $\mu(.)$. The marks $Z_{ij} = (V_i, S_{ij}, A_{ij})$ are such that $(S_{ij}, V_i) \in \mathbb{D}X\mathbb{D}_1$, product of two metric spaces and $A_{ij} = (a_{ij}, b_{ij}, c_{ij}) \in \mathcal{R}^3$. We shall call the point process $\mathcal{C}$ as the center process and the point process $\{X_{ij}\}$ as the cluster process associated with X_i. This is viewed as follows : there is a main point process with each point of whose is associated a subsidiary point process. By choosing the parameters $\lambda(.)$ and $\mu(.)$ one can get varied cluster processes. The mark V_i is common to all the points of the cluster process associated with X_i. In applications, this can be viewed as a common parameter for all the points in a cluster. And S_{ij} is a more individual parameter associated with the point.

Apart from these, the model is based on the 'response' function $L : \mathbb{D}X\mathbb{D}_1X\mathcal{R}^d \to \mathcal{R}^+$, which is continuous in its final argument and such that $L(s, v, x) \to 0$ when $|x| \to \infty$.
For some results, we shall omit the v argument from the function L. In such cases, we shall not write the v variable in the function.

Individual Cells We shall now define the cell attached to the point $X_0 + X_{00}$ as the following set

$$C_{00} = C_{00}(\Phi) = \{y : a_{00}L(S_{00}, V_0, y - X_0 - X_{00}) \geq b_{00}I_\Phi(y) + c_{00}\}, \quad (1)$$

where

$$I_\Phi(y) = \sum_i \sum_j L(S_{ij}, V_i, y - X_i - X_{ij}) = \int_{\mathcal{R}^d X\mathcal{R}^d X\mathbb{D}} L(s, y - z)\Phi(d(z, s)). \quad (2)$$

The second formula is obtained by viewing the point process as the random point measure $\Phi = \sum_i \sum_j \delta_{(X_i + X_{ij}, Z_{ij})}$. The function $I_\Phi(y)$ denotes the value of a *shot-noise* process (see e.g. [5, 6, 11, 13]) of $(X_i + X_{ij}, S_{ij})$ at the point y for the response function L.

Coverage Process Naturally, the coverage process is defined as the union of all cells i.e,

$$\Xi = \Xi(\Phi) = \cup_{i,j} C_{ij}(\Phi). \quad (3)$$

2.2 Assumptions on $\lambda(.)$ and $\mu(.)$

Point process For the point process Φ to be a random point measure, we need the number of points in any bounded set to be finite. Formally, define $N(A) = \#\{\Phi \cap A\}$, where A is any bounded subset of $\mathcal{R}^d$. And we require $\mathsf{E}(N(A)) < \infty$. We assume throughout the paper that our point process is a random point measure. We shall give some formulas to check the assumption as well as demonstrate it with a few simple examples.

Individual Cells and Coverage Process As stated in Secn 2.3 of [1], we will require the random function to have finite expectation and $I_\Phi(y)$ to be a.s continuous in y(even lower semi-continuity is sufficient).

Also we want the expected number of cells that intersect a bounded set to be finite. This shall imply that the coverage process Ξ is closed.

And to avoid degenerate cases we assume that $a, b, c \in \mathcal{R}^+$ and $\mathsf{P}\,(a_{00} = c_{00} = 0) = 0$. Also, we shall mention at the required places stationarity assumptions on the centre or cluster processes. Note that the stationarity of the center process guarantees the stationarity of the entire process Φ.

Remark 1 *For more detailed exposition of coverage process and other stochastic geometric tools, the reader is referred to [4, 7, 8]. The mentioned texts, in particular, shall explain the necessity of the above assumptions.*

Remark 2 *We refrain from explaining some examples and certain special cases of the model defined above. The reader is referred to Secn 2.4 and Secn 2.5 of [1] where a very good account of motivating examples are described.*

Remark 3 *Many times in the following sections, we shall calculate some quantities explicitly under the assumptions $a = 1+T, b = T, c = TW$, where $T > 0$ is called the 'threshold' and W is the external noise independent of the other random variables. Also, we shall take $L(s, z) = sl(|z|)$. Under these assumptions,*

$$C = C(\Phi) = \{y : Sl(|y - X_0 - X|) \geq T(I_\Phi(y) + W)\}.$$

This can be viewed as follows: The signal of power S emitted by station at $X_0 - X$ with 'path-loss' or 'attenuation' function l is received at level $Sl(|y - X_0 - X|)$ and this has to be at least T times bigger than the interference from other antennae and external noise for the point y to receive it correctly. We shall call these assumptions as SINR assumptions.

3 Sufficient Conditions for the model to be well defined

3.1 Individual Cells

In this section, we give some sufficient conditions analogous to [1] for the model to be well-defined. Remember that in the previous section, we had said of the point process as well as the individual cells to satisfy certain conditions. We shall give sufficient conditions for them to meet the required conditions. We omit some of the proofs as they are similar to the cited paper. As we shall use the Campbell's formula (see e.g. [7], eq. (4.4.3), p. 119) often , we refrain from mentioning it everywhere. We use it to represent various expectations as integrals.

PROPOSITION 4 *Let H denote the law of $S_{00} \in \mathbb{D}$ and H_1 denote the law of $V_0 \in \mathbb{D}_1$ (see for eg. [12]). If for each $y \in \mathcal{R}^d$, there exists a ball $B(y, \epsilon_y)$ such that,*

$$\int_{\mathcal{R}^d x \mathbb{D} x \mathcal{R}^d x \mathbb{D}_1} sup_{z \in B(y,\epsilon_y)} L(s, v, z - x_0 - x) \lambda(dx_0) H(ds) \mu(dx) H_1(dv) < \infty \tag{4}$$

then w.p.1, the function $I_\Phi(y)$ is continuous w.r.t y.

The proof follows from noting that same proposition in [1] holds for any general point process.

3.2 Coverage Process

In the previous section, we had mentioned of the expected number of points in a bounded Borel set being finite. Let,

$$N(A) := \sum_{i,j} \mathbf{1}[X_i + X_{ij} \in A],$$

where A is a bounded Borel set in $\mathcal{R}^d$. We require $\mathsf{E}(N(A)) < \infty$. Let us denote Φ^0 as the random measure corresponding to the center process.

$$\begin{aligned} \mathsf{E}(N(A)) &= \mathsf{E}(\mathsf{E}(N(A)|\mathcal{C})) \\ &= \mathsf{E}\Big(\int_{\mathcal{R}^d} \int_{A-x} \mu(dx_0) \Phi^0(dx) \Big) \\ &= \int_{\mathcal{R}^d} \int_{A-x} \mu(dx_0) \lambda(dx). \qquad (5) \end{aligned}$$

(6)

The above equation gives us a formula to verify whether the given point process satisfies the required condition. Now, we shall look at the some of the standard cases where the given condition is satisfied.

3.2.1 $\lambda(dx) = \lambda dx$ and $\mu(\mathcal{R}^d) < \infty$

This is when the center process is a homogeneous Poisson point process and the cluster process is a finite process.

$$\begin{aligned} \mathsf{E}(N(A)) &= \lambda \int_{\mathcal{R}^d} \int_{A-x} \mu(dx_0)dx \\ &= \lambda \int_{\mathcal{R}^d} \int_{A-x_0} dx\mu(dx_0) \\ &= \lambda\mu(\mathcal{R}^d)\|A\| < \infty \end{aligned}$$

Here $\lambda\mu(\mathcal{R}^d)$ is the intensity of the point process. Similarly, one can show the same holds for $\lambda(\mathcal{R}^d) < \infty$ and $\mu(dx_0) = \mu dx_0$.

3.2.2 $\lambda(dx) = \lambda dx$ and $\mu(dx_0) = \mu_0(x)\mathbf{1}_B(x)for\|B\| < \infty$

Under these assumptions also, one can show that the required condition is satisfied. The explicit formulas can be derived for the special case when $\mu_0(x) = \mu$ and $B = B_p[0, R]$, i.e, a a ball of radius R in L_p norm. The intensity for this point process shall be $\lambda\mu\|B_p[0, R]\| = C_p\lambda\mu R^d$, where $C_p = \|B_p[0, 1]\|$.

As said in the previous section, the number of cells intersecting a bounded set needs to be finite for the coverage process to be well-defined.

4 Cell Characteristics

One of the important tools we need for some of the calculations is of Palm probability of the point process. For this we use the results concerning the same from [10]. The reader can refer to the above paper and [9] for further results on the same.

In particular, we shall use the following result which can be deduced from Proposition 2 of [10].

PROPOSITION 5 *Let N_{lf} be the set of locally finite sets of $\mathcal{R}^d$ equipped with the usual σ-algebra $\mathfrak{N}_{lf}$ by the sets $F_{B,n} = \{x \in N_{lf} : \sharp(x \cap B) = n\}$ for $n = 0, 1, \ldots$ and bounded Borel subsets $B \subset \mathcal{R}^d$.*

Let $\xi \in \mathcal{R}^d$. Now, define the marked point process Φ_ξ independent of Φ as follows :

$$\Phi_\xi = \{(X_\xi + \tilde{X}_i, \tilde{S}_i)\},$$

where

$$\mathsf{P}\left(X_\xi \in D\right) = \frac{\mu(\xi + dx)}{\mu(\mathcal{R}^d)},$$

for Lebesgue sets $D \subset \mathcal{R}^d$. And $(\tilde{X}_i)$ are Poisson point process with intensity $\mu(.)$.

The reduced palm distribution for ξ denoted by $P^!_\xi$ is given as

$$P^!_\xi(F) = \mathsf{P}\left(\Phi \cup \Phi_\xi \in F\right).$$

4.1 Coverage Probability

Now we want to analyze the cell $C(\tilde{x}; \Phi)$ attached to a point located at $\tilde{x}$ of the marked point process Φ under the reduced palm distribution $P^!_\xi$. Due to Proposition 5, we have the law of this set is same as that of the random closed set,

$$C(\tilde{x}; \Phi \cup \Phi_{\tilde{x}}) = \{y : aL(S, V, y - \tilde{x}) \geq b(I_\Phi(y) + I_{\Phi_{\tilde{x}}}(y) + L(S, V, y - \tilde{x})) + c\}$$

under P, where Φ is the original Poisson point process and $\Phi_{\tilde{x}}$ is as defined in the previous proposition. And $(S, V, (a, b, c))$ is an 'additional mark' distributed independently of other marks.

Denote by probability $p_{\tilde{x}}(y)$ that the point $y \in \mathcal{R}^d$ is covered by $C(\tilde{x}; \Phi \cup \Phi_{\tilde{x}})$. We have,

$$p_{\tilde{x}}(y) = \mathsf{P}\left(aL(S, V, y - \tilde{x}) \geq b(I_\Phi(y) + I_{\Phi_{\tilde{x}}}(y) + L(S, V, y - \tilde{x})) + c\right).$$

We assume $\lambda(dx) = \lambda dx$ and $\mu(dx_0) = \mu \mathbf{1}_{B_p[0,R]}(x)dx$. SINR assumptions as in Remark 3 are also assumed. Even though some of the calculations can be worked in more generality, but this helps us to analyse the model better. The assumption on $\lambda(dx)$ ensures that the process is stationary, hence the probability of signal reception at y of the antenna at $\tilde{x}$ depends only on $|y - \tilde{x}|$. Hence without loss of generality, we can take $y = 0$ and $|\tilde{x}| = r > 0$. So, we can denote the probability of coverage as p_r. The dependence on λ, μ, R is not explicitly mentioned here. Let S have an exponential distribution with parameter η. Also, we omit 0 in the shot-noise functions and denote $I_{\Phi_{\tilde{x}}}$ by I_{Φ_0}. Thus,

$$\begin{aligned} p_r &= \mathsf{P}\left(S \geq \frac{T}{l(r)}(I_\Phi + I_{\Phi_0} + W)\right) \\ &= \int_0^\infty e^{-\eta T s/l(r)} d\mathsf{P}\left(I_\Phi + I_{\Phi_0} + W \leq s\right) \\ &= \Psi_\Phi(\eta T/l(r))\Psi_{\Phi_0}(\eta T/l(r))\Psi_W(\eta T/l(r)), \end{aligned} \tag{7}$$

Ψ_X denotes the Laplace transform of the random variable X defined as $\Psi_X(\xi) = \mathsf{E}\big(e^{-\xi X}\big)$. In the above expressions we have slightly abused the notation by denoting Ψ_Φ the Laplace transform of the shot-noise of the corresponding point process. Also, note that Ψ_W doesn't depend on λ. The above calculation is similar to the one in Lemma 3.1, [2]. As in that proof, we use additive shot noise theory.

For a general Poisson shot noise in $\mathcal{R}^2$, we know that,

$$\Psi(\xi) = \exp\{-\lambda \int_{\mathcal{R}^2} [1 - \mathsf{E}\Big(e^{-\xi Q(|x|)}\Big)]dx\},$$

where $Q(|x|)$ is the level of signal power received from the antenna at x at 0. In our case, we can view $Q(|x|) = \sum_j S_j l(|x + X_j|)$ with x as the point of center process and X_j as the cluster points associated with x. S_j is the power of signal emitted by X_j. And by another application of the Laplace transform formula of a Poisson shot noise, we get

$$\mathsf{E}\Big(e^{-\xi \sum_j S_j l(|x+X_j|)}\Big) = \exp\{-\frac{\mu}{C_p R^2} \int_{B_p[0,R]} (1 - \mathsf{E}\Big(e^{-\xi S l(|x+x_0|)}\Big))dx_0\}.$$

Hence,

$$\begin{aligned} \Psi_\Phi(\xi) &= exp\{-\lambda \int_{\mathcal{R}^2} [1 - exp\{-\frac{\mu}{C_p R^2} \int_{B_p[0,R]} (1 - \mathsf{E}\Big(e^{-\xi S l(|x+x_0|)}\Big))dx_0\}]dx\} \\ &= exp\{-\lambda \int_{\mathcal{R}^2} [1 - exp\{-\frac{\mu}{C_p R^2} \int_{B_p[0,R]} \frac{dx_0}{1 + \frac{\eta}{\xi l(|x+x_0|)}}\}]dx\} \end{aligned} \tag{8}$$

The last equality is using the fact $\mathsf{E}\big(e^{-\xi S}\big) = \frac{\eta}{\eta+\xi}$ for $S \sim EXP(\eta)$.

$$\Psi_\Phi(\eta T/l(r)) = \exp\{-\lambda \int_{\mathcal{R}^2} [1 - \exp\{-\mu \int_{B_p[0,R]} \frac{dx_0}{1 + \frac{l(r)}{T l(|x+x_0|)}}\}]dx.$$

$$= \exp\{-\lambda \int_0^{2\pi} \int_0^\infty [1 - \exp\{-\mu \int_0^{2\pi} \int_0^R \frac{r_2 dr_2 d\theta_2}{1 + \frac{l(r)}{T l(\sqrt{r_1^2 + r_2^2 + 2 r_1 r_2 cos(\theta_2 - \theta_1)})}}\}]\, r_1 dr_1\, d\theta_1\}\}$$

Similarly, we can calculate Ψ_{Φ_0}.

$$\begin{aligned}
\Psi_{\Phi_0}(\xi) &= \mathsf{E}\Big(e^{-\xi I_{\Phi_0}}\Big) \\
&= \mathsf{E}\Big(\mathsf{E}\Big(e^{-\xi I_{\Phi_0}}|X_r\Big)\Big) \\
&= \mathsf{E}\Bigg(\exp\{-\mu\int_{B_p[0,R]}(1-\mathsf{E}\Big(e^{-\xi Sl(|X_r+x_0|)}\Big))dx_0\}\Bigg) \\
&= \int_{B_p[\tilde{x},R]}\exp\{-\mu\int_{B_p[0,R]}(1-\mathsf{E}\Big(e^{-\xi Sl(|x+x_0|)}\Big))dx_0\}\frac{1}{C_pR^2}dx \\
&= \int_{B_p[\tilde{x},R]}\exp\{-\mu\int_{B_p[0,R]}\frac{dx_0}{1+\frac{\eta}{\xi l(|x+x_0|)}}\}\frac{1}{C_pR^2}dx
\end{aligned}$$

Thus,

$$\Psi_{\Phi_0}(\eta T/l(r)) = \int_{B_p[\tilde{x},R]}\exp\{-\mu\int_{B_p[0,R]}\frac{dx_0}{1+\frac{l(r)}{Tl(|x+x_0|)}}\}\frac{1}{C_pR^2}dx$$

$$= \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R\exp\{-\mu\int_0^{2\pi}\int_0^R\frac{r_2dr_2d\theta_2}{1+\frac{l(r)}{Tl(\sqrt{r_1^2+r_2^2+2r_1r_2cos(\theta_2-\theta_1)})}}\}r_1dr_1d\theta_1$$

And so the probability of coverage can be obtained by substituting (9) and (9) in (7).

5 Conclusion

To conclude, in this article we have calculated some of the basic quantities for a SINR coverage process when the underlying point process is a Poisson-Poisson cluster process. Further scope exists in comparing the coverage properties of the model with the one introduced in [1](where the underlying point process is a Poisson point process).

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