Towards connectivities of random geometric complexes

D. Yogeshwaran

Indian Statistical Institute Bangalore.

Joint work with:

Srikanth K. Iyer, Indian Institute of Science, Bangalore

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Random geometric graphs

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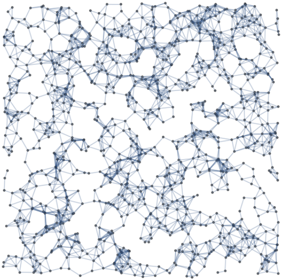

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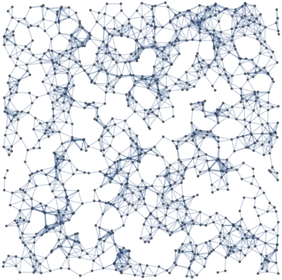

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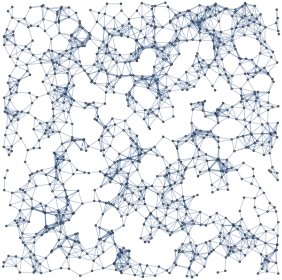

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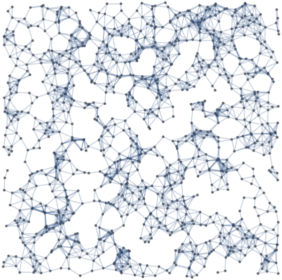

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- $\mathcal{P}_n = \{X_1, \ldots, X_{N_n}\}$ - X_i i.i.d. uniform and $N_n \sim Poi(n)$.
- Random geometric graph : $G(\mathcal{P}_n, r)$: Vertices, $V = \mathcal{P}_n$, Edges : $x_i \sim x_j$ if $0 < |x_i - x_j| \leq 2r$, $r > 0$.

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- Sharp connectivity threshold: (Penrose '97) :

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- Threshold for isolated nodes : $J_{n,0}$. For $w(n) \to \infty$,

$$P(J_{n,0} = 0) \to \begin{cases} 0 & \text{if } n\theta_d 2^d r_n^d = \log n - w(n) \\ 1 & \text{if } n\theta_d 2^d r_n^d = \log n + w(n) \end{cases}$$

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- G_0^U : Vertices - $F^0 \subset V$. Edges $\subset F^1$ i.e., the usual graph.

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$$C(\mathcal{X}, r) \subset R(\mathcal{X}, r) \subset C(\mathcal{X}, \sqrt{2}r).$$

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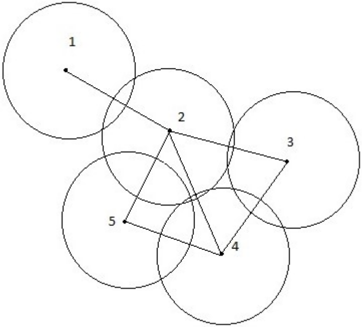

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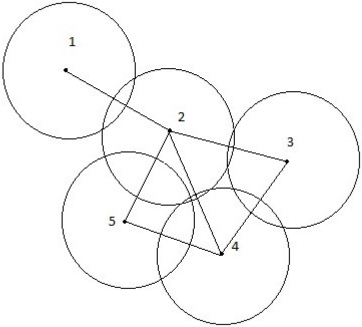

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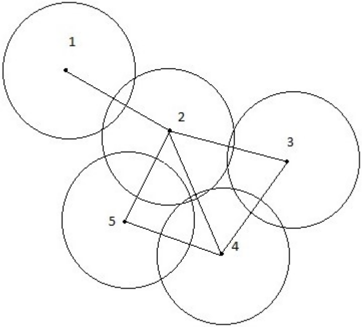

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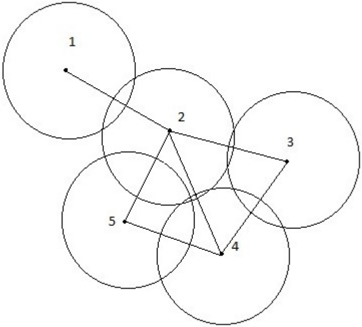

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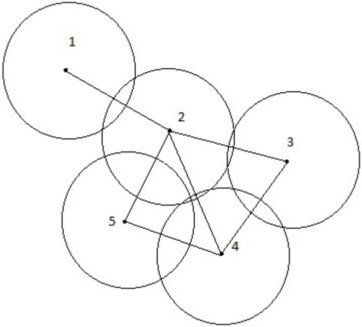

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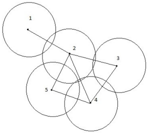

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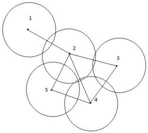

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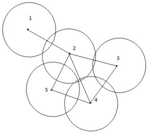

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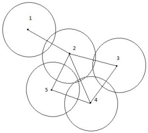

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- Combinatorial structure matters for up/down connectivity !

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- Topological data analysis : Carlsson, 2014 ; Adler, 2015 ; Bobrowski-Kahle, 2017.

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- $G_k^*(X_n(r))$: $* = U/D$ i.e., up / down.
  $X_n(r) := X(\mathcal{P}_n, r), X = C/R$ i.e., Čech / Vietoris-Rips.

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- Known only for Linial-Meshulam random complex ([Kahle-Pittel, 2014]).

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$$P(\beta_k(C_n(r_n)) = 0) \to \begin{cases} 0 & \text{if } n\theta_d r_n^d = (1-\epsilon)\log n \\ 1 & \text{if } n\theta_d r_n^d = (1+\epsilon)\log n \end{cases}$$

- "DY-SKI (2018+)" : For $\epsilon \in (0,1)$

$$P\big(G_k^U(C_n(r_n)) \text{ is connected}\big) \to \begin{cases} 0 & \text{if } n\theta_d r_n^d = (1-\epsilon)\log n \\ 1 & \text{if } n\theta_d r_n^d = (1+\epsilon)\log n \end{cases}$$

- Known only for Linial-Meshulam random complex ([Kahle-Pittel, 2014]).
- Also threshold for coverage - ([P. Hall, 1988])

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- No finite components exist above vanishing threshold !

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- Why not Connectivity thresholds ? Can there be two infinite components ?

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- Same for $J_1^{*U}(R_n(r_n))$.
- $nm_1 r_n^d = \log n - w_n \to \infty \Rightarrow P(\beta_1(R_n(r_n)) = 0) \to 0$.

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- For fixed "$r \in (\log n - \log\log n, \log n)$", no isolated edge but infinitely many appear and disappear very quickly !

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- Rate of $c_n \to m_k \Rightarrow$ 2nd order term $= k \log\log n$ or $(k-2)\log\log n$ or $(k-1)\log\log n$.

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- Lower bounds involve second-moments and some more Palm calculus.

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- Key geometric Lemma: If $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{(k+1)}$ are isolated k-faces ($(k+1)$-cliques) in $R(\mathbf{x} \cup \mathbf{y} \cup \mathcal{P}, 1)$ then for some $\epsilon, \beta > 0$, either $|Q(\mathbf{x}) \vee Q(\mathbf{y})| \geq m_k + \epsilon$ or $|Q(\mathbf{x}) \setminus Q(\mathbf{y})| \geq \beta$.

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- "Bobrowski-D.Y." $\lambda_2(G(\mathcal{P}_n \cap B_1(O), 1)) \to 1/2$ a.s. !!!

References

- Srikanth K. Iyer and D.Y. (2018). Thresholds for vanishing of 'Isolated' faces in random Čech and Vietoris-Rips complexes. arXiv:1802.08224
- O. Bobrowski and S. Weinberger (2017), On the vanishing of homology in random Čech complexes. *Rand. Struct & Alg.*
- O. Bobrowski and M. Kahle (2017) Topology of random geometric complexes: A survey. *J. Appl. & Comp. Topology*
- M. D. Penrose. (2016) Inhomogeneous random graphs, isolated vertices, and Poisson approximation, *arXiv:1507.07132*
- C. Hoffmann, M. Kahle and E. Paquette (2016), Spectral gaps of random graphs and applications to random topology. *arXiv:1201.0425*