Topology of some random complexes.

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Technion, Haifa.

SGSIA, Torun, June 2013.

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OUR GOAL :

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OUR GOAL : Topology of the Boolean model on different point processes

Geometric graphs

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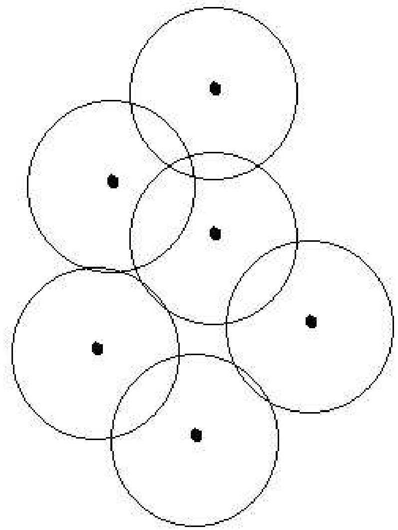

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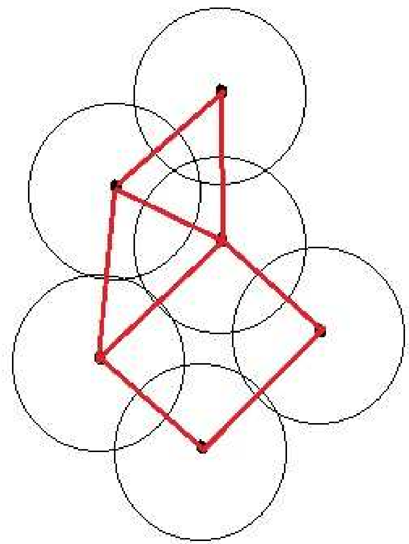

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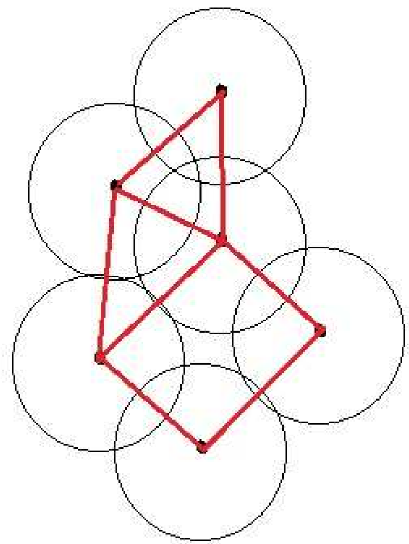

Vertices = $\Phi = \{X_i\}$, finite set of points.

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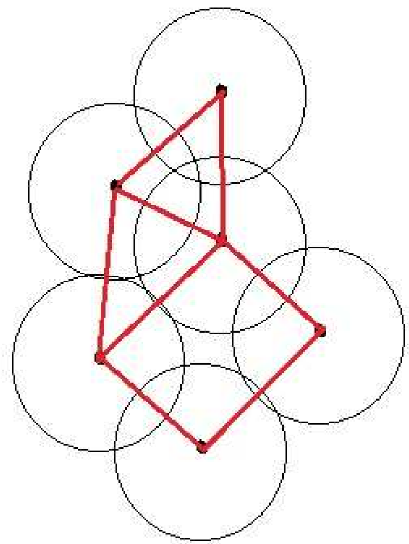

Vertices = $\Phi = \{X_i\}$, finite set of points.

Edges = $\{(X, Y) : B_X(r) \cap B_Y(r) \neq \emptyset\}$.

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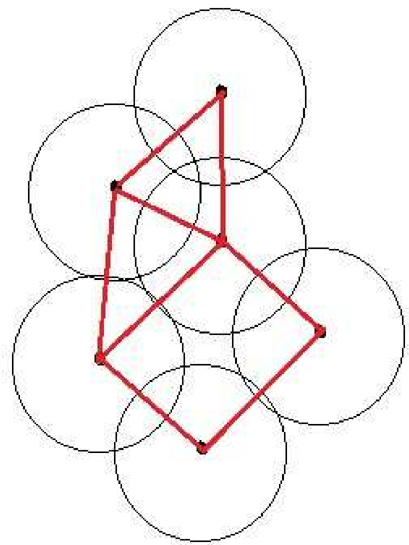

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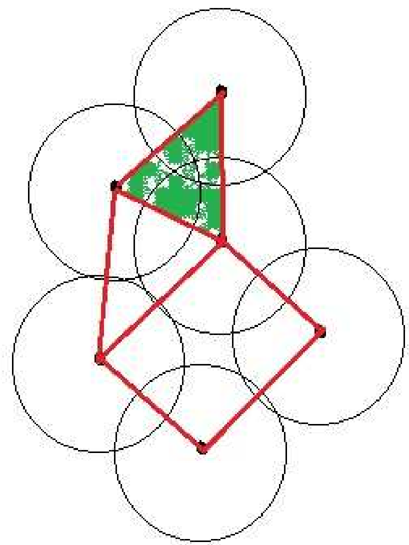

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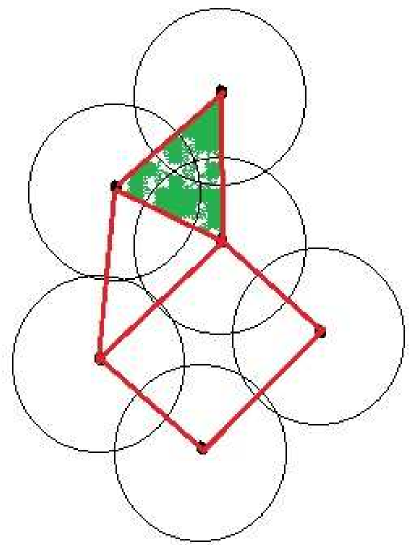

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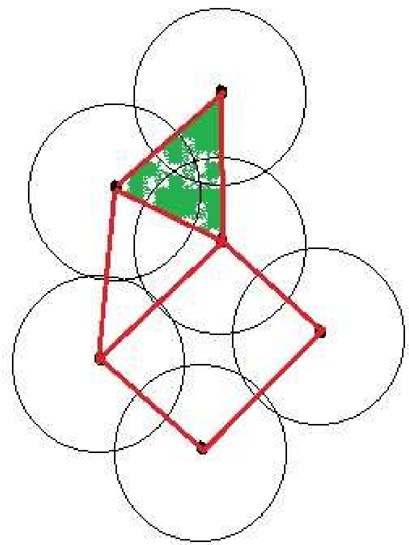

Vertices = Φ ; k**-simplices/face =** $\{X_0, \ldots, X_k\}$ **if**

$$\cap_{i=0}^{k} B_{X_i}(r) \neq \emptyset.$$

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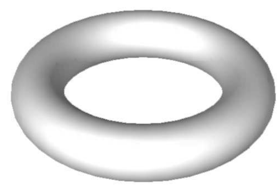

Figure 9: $\beta_0(T) = 1, \beta_1(T) = 2, \beta_2(T) = 1.$

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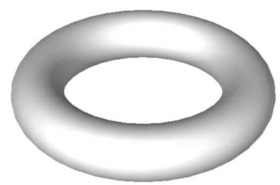

Figure 11: $\beta_0(T) = 1, \beta_1(T) = 2, \beta_2(T) = 1.$

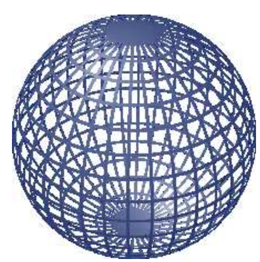

Figure 12: $\beta_0(T) = 1, \beta_1(T) = 0, \beta_2(T) = 1.$

Betti number heuristics

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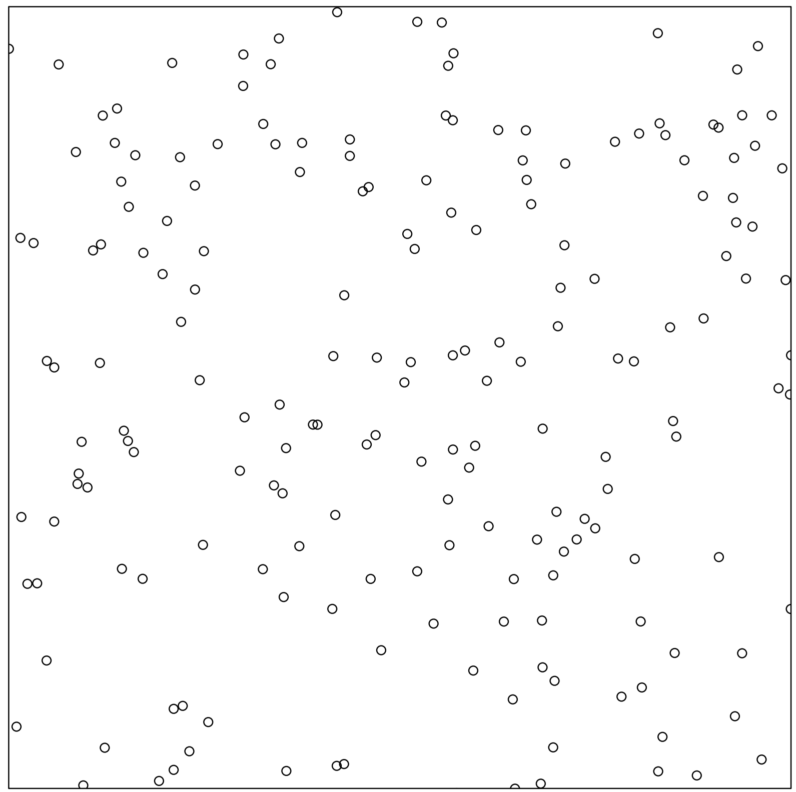

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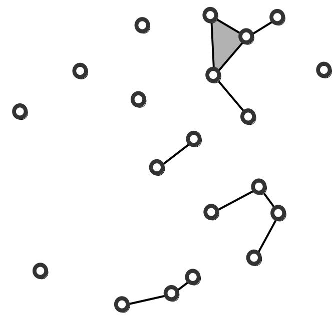

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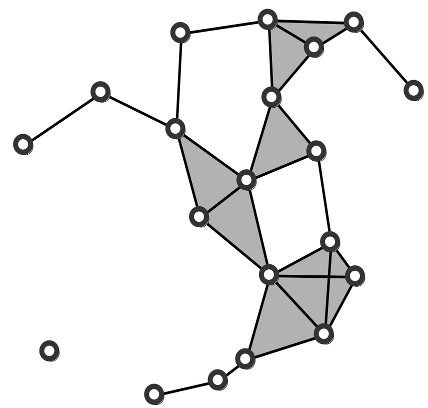

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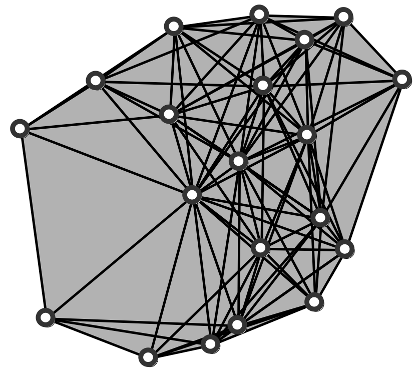

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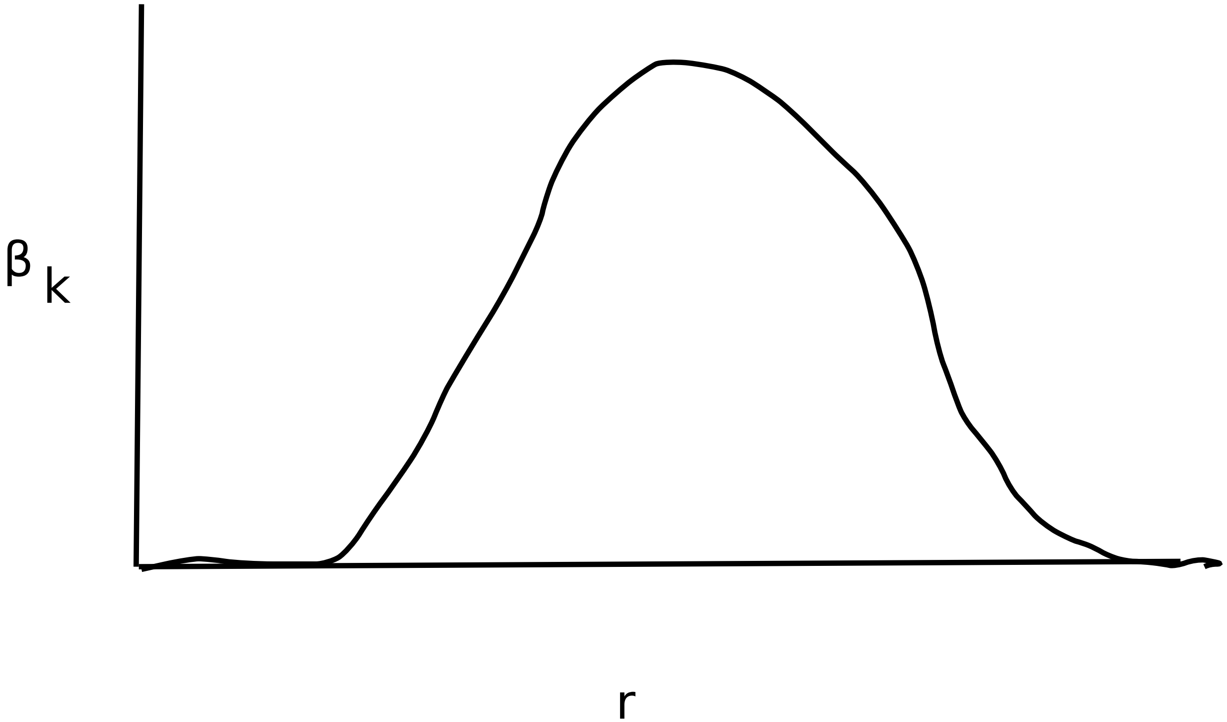

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Ginibre point process - Eigenvalues of $n \times n$ matrix with i.i.d. $N_{\mathbb{C}}(0,1)$ entries as $n \to \infty$.

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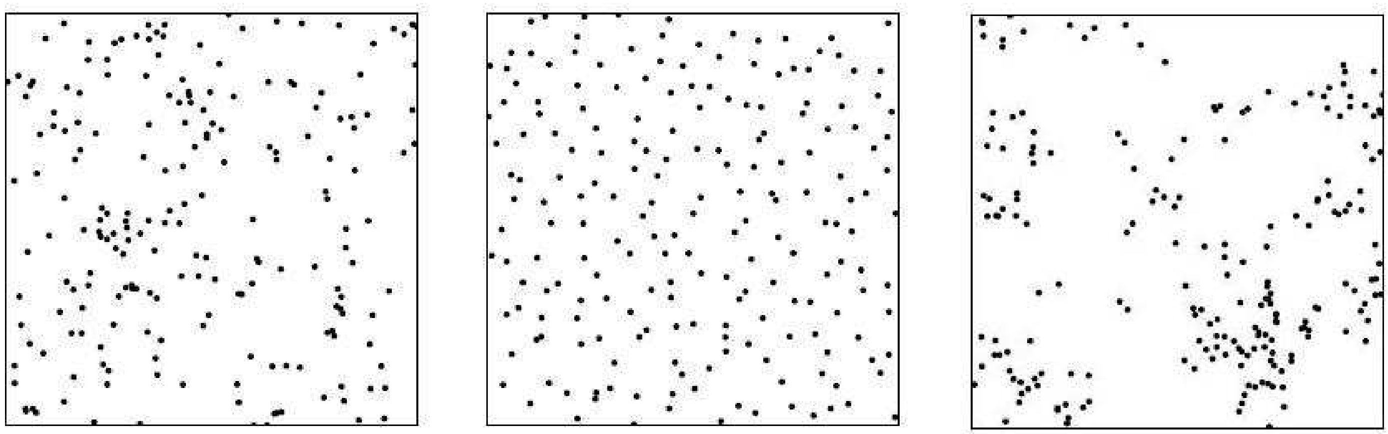

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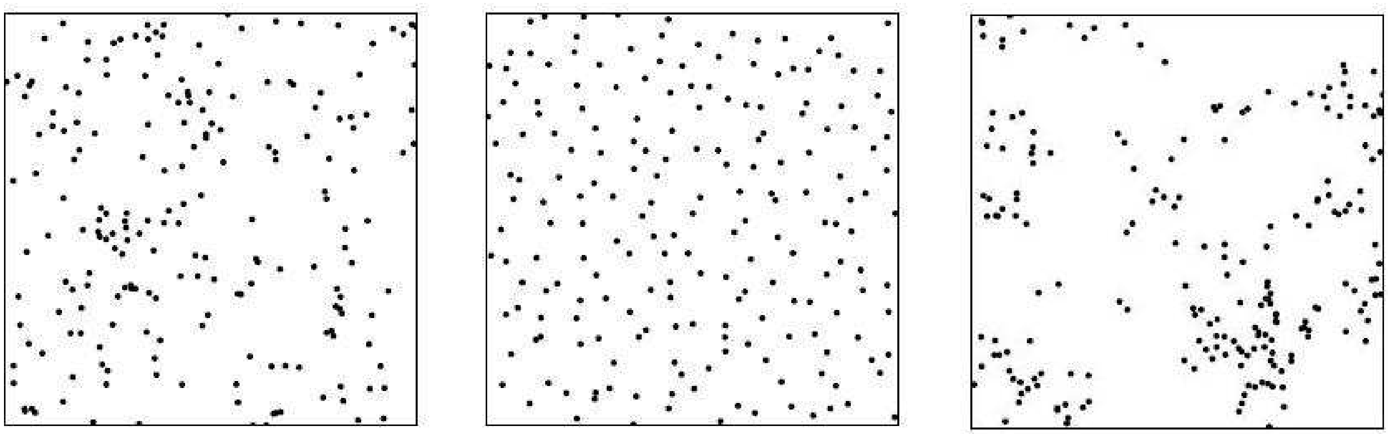

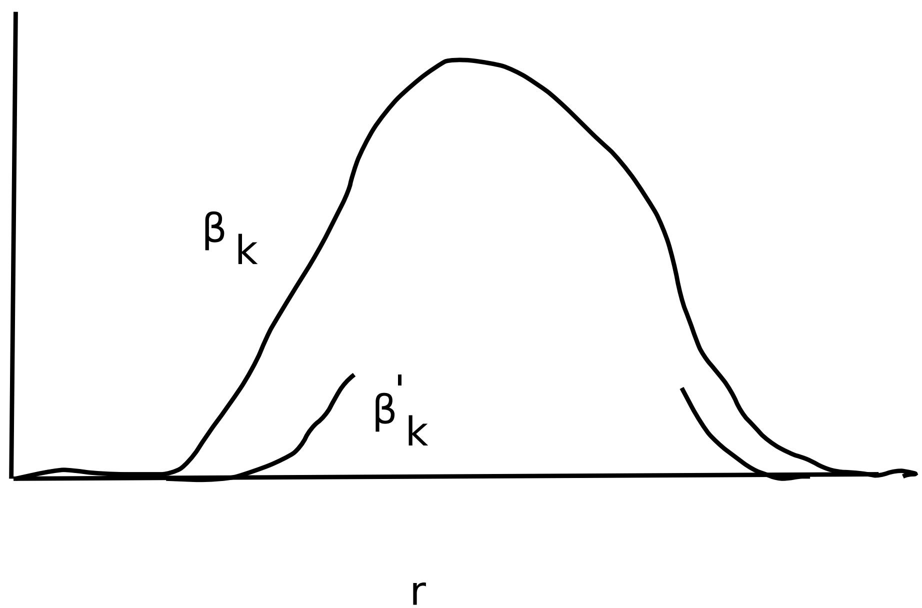

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Poi	$\sim n r_n^{d(k+1)}$	$\Theta(n)$	0

Mathew Kahle - Elizabeth Meckes.

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$$\rho^{(k)}(0, \ldots, 0) = 0 \; ; \; \mathsf{E}\left(\prod_{i=1}^{k} \Phi(B_i)\right) := \int_{\prod B_i} \rho^{(k)}(x) \mathsf{d}x.$$

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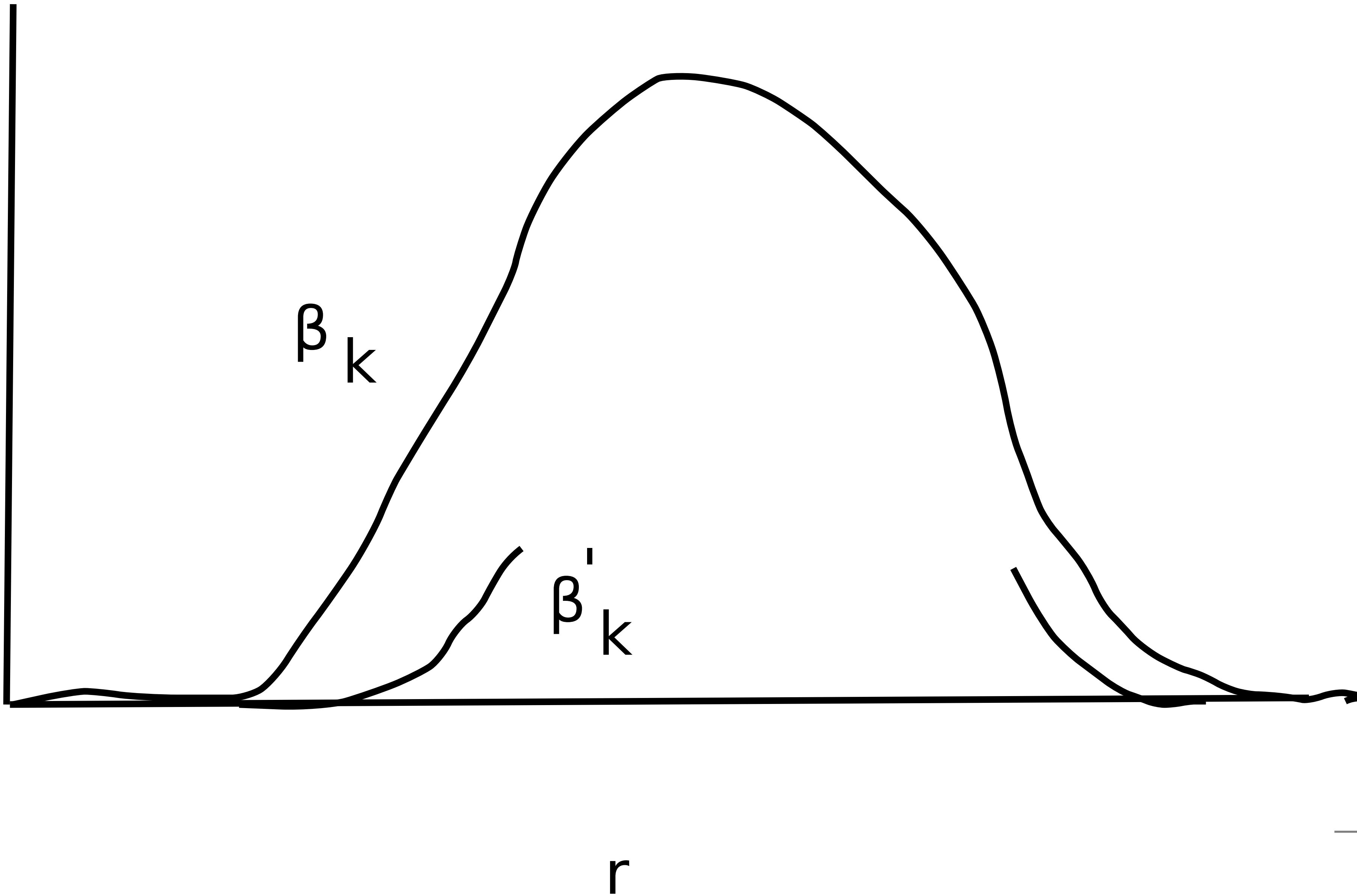
weak sub-P $+$	$o(n r_n^{d(k+1)})$	$O(n)$	0
Gin	$\sim n r_n^{(k+1)(k+4)}$	$\sim n$	$\geq C(\log n)^{\frac{1}{4}}$

In Picture



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Morse critical points : Link between differential and algebraic topology.

References :

- D. Yogeshwaran and R. J. Adler : On topology of some random complexes over stationary point processes. arXiv:1211.0061

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- Mathew Kahle : Random geometric complexes.
- Omer Bobrowski and R. J. Adler : Distance function, critical points and topology of some random complexes


Thank You
Mahalo
Kiitos
Tack
Thanks
Toda
Grazie
Obrigado
Takk
Merci
Gracias
நன்றி
Dziękuję