Random polytopes in $\mathbb{C}$-Convex domains

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OUTLINE

1. Random Real Polyhedra - Optimal vs Random approximations.
2. Polyhedra in complex variables - Optimal approximation.
3. Random Complex Polyhedra - A theorem and a conjecture.

rRp

RANDOM REAL POLYHEDRA

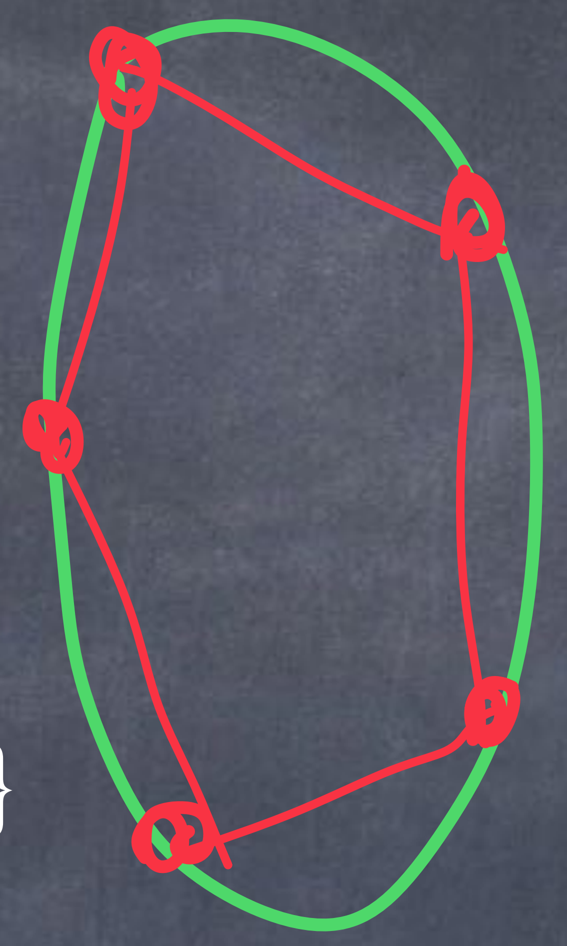
EUCLIDEAN POLYHEDRAL APPROXIMATION

$K \subset \mathbb{R}^d, d \geq 2-$ Smooth, Compact, convex set. Eg. Unit ball, Ellipsoids....

INSCRIBED POLYHEDRA

$$P_n := Conv - Hull\left(\{x_1, \ldots, x_n\}\right)$$

$$x_1, \ldots, x_n \in bK.$$



Optimal approximation:

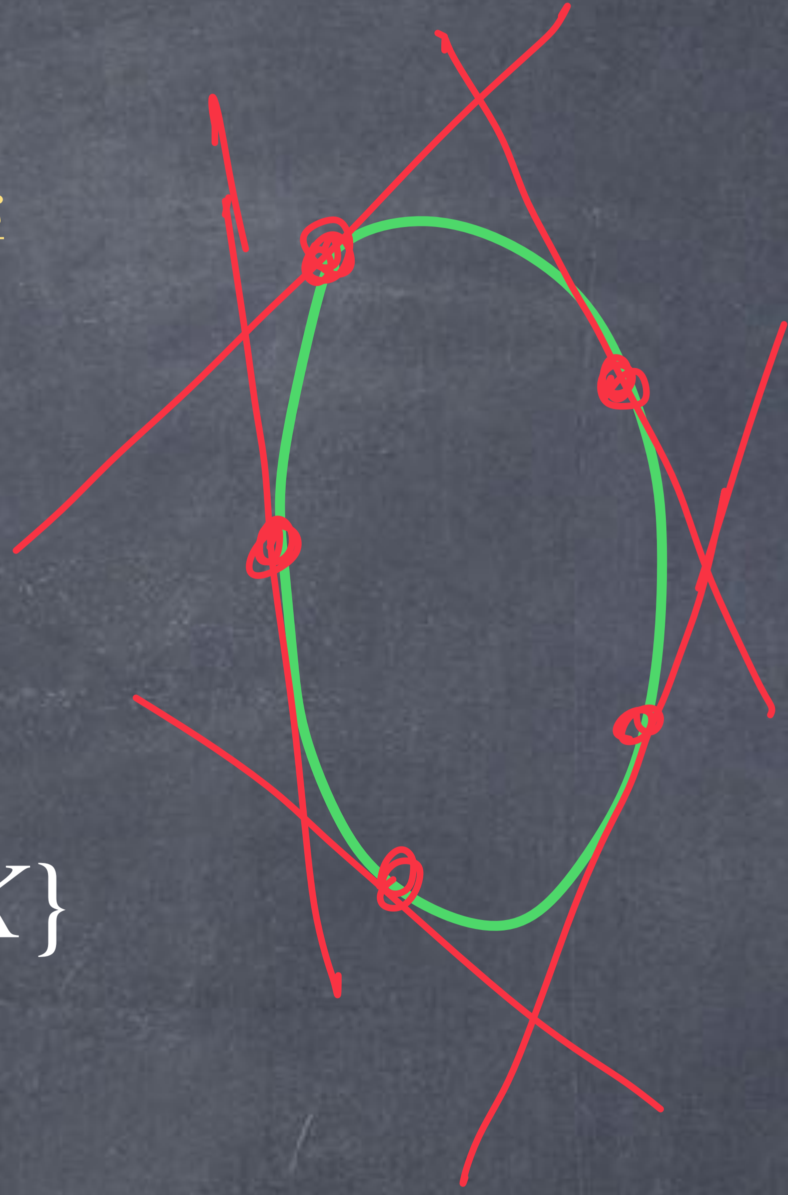
$$v_n(K) := \inf\{Vol(K \backslash P_n) : x_1, \ldots, x_n \in bK\}$$

McClure & Vitale 1975; Gruber 1993.

$$v_n(K) \sim \frac{1}{2} del_{d-1}\, \sigma_{bla}(bK)^{1+2/(d-1)}\, n^{-2/(d-1)}$$

CIRCUMSCRIBED POLYHEDRA

P_n - Intersection of tangent half-planes at $x_1, \ldots, x_n \in bK$ and containing K.



Optimal approximation:

$$v_n(K) := \inf\{Vol(P_n \backslash K) : x_1, \ldots, x_n \in bK\}$$

Gruber 1993.

$$v_n(K) \sim \frac{1}{2} div_{d-1}\, \sigma_{bla}(bK)^{1+2/(d-1)}\, n^{-2/(d-1)}$$

Domain constant -$\sigma_{bla}(x) = \kappa_K^{1/(d+1)}(x)\, \sigma(x),\ x \in bK,\ \kappa_K-$Gaussian curvature, $\sigma = \sigma_{euc}$

Blaschke affine surface area measure.

Exponent - 2/(d-1)

Dimension Constants - Delone tilings and Dirichlet-Voronoi tilings.

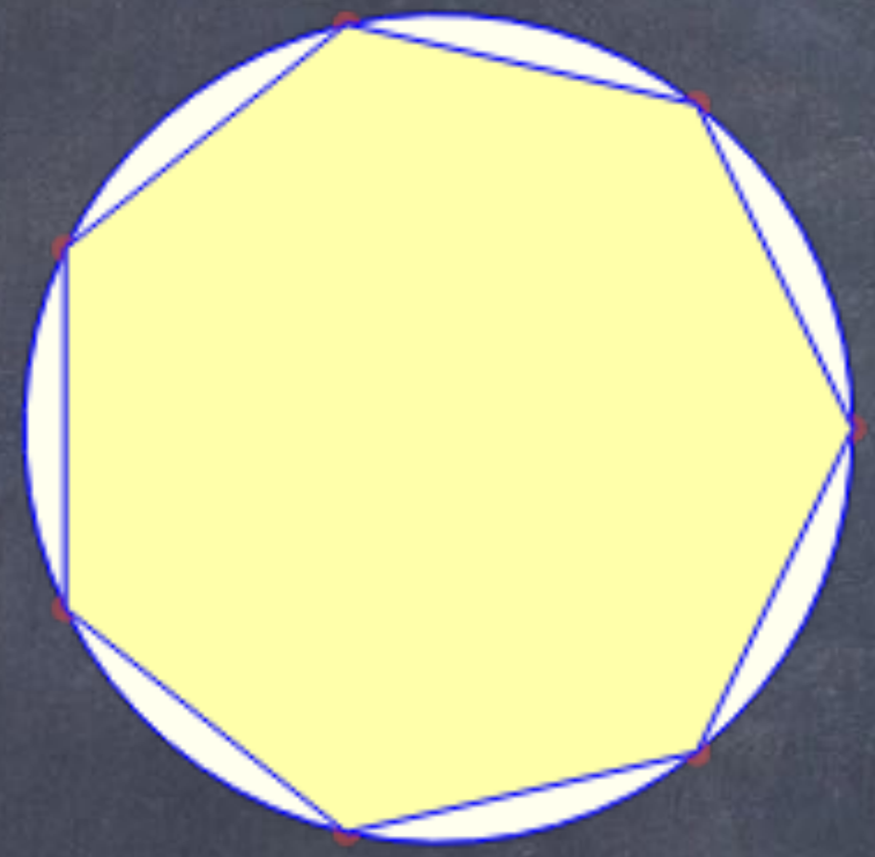
Equi-Affine transformations preserve convex sets, polyhedra and all that !

A SIMPLE ILLUSTRATION

$d = 2,\ K = \mathbb{B},$ Unit ball.

INSCRIBED POLYHEDRA

P_n – inscribed regular polygon i.e., n equi-spaced points.



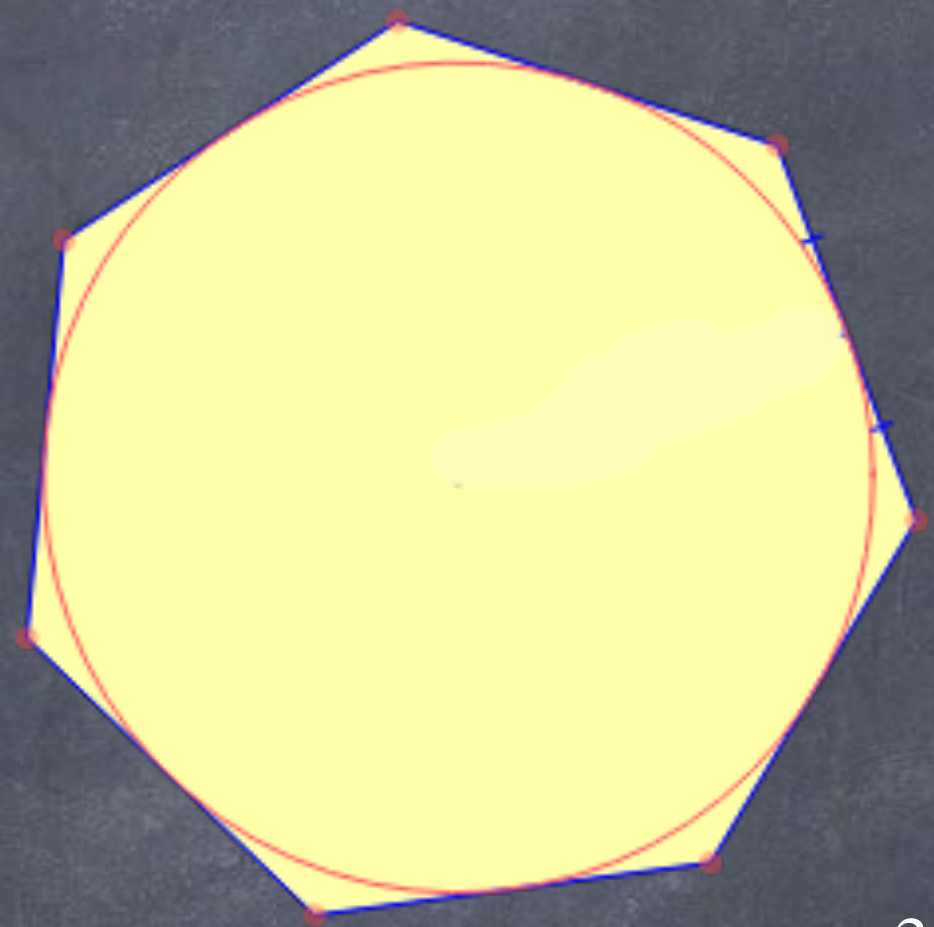
$$vol(K \backslash P_n) = \pi - \frac{n}{2} sin(\frac{2\pi}{n}) \sim \frac{2\pi^3}{3} n^{-2}$$

General Optimal approximation result

$$v_n(K) \sim \frac{1}{2} del_{d-1}\, \sigma_{bla}(bK)^{1+2/(d-1)}\, n^{-2/(d-1)}$$

CIRCUMSCRIBED POLYHEDRA

P_n - circumscribed regular polygon.



$$vol(P_n \backslash K) = n\, tan(\frac{\pi}{n}) - \pi \sim \frac{\pi^3}{3} n^{-2}$$

General Optimal approximation result

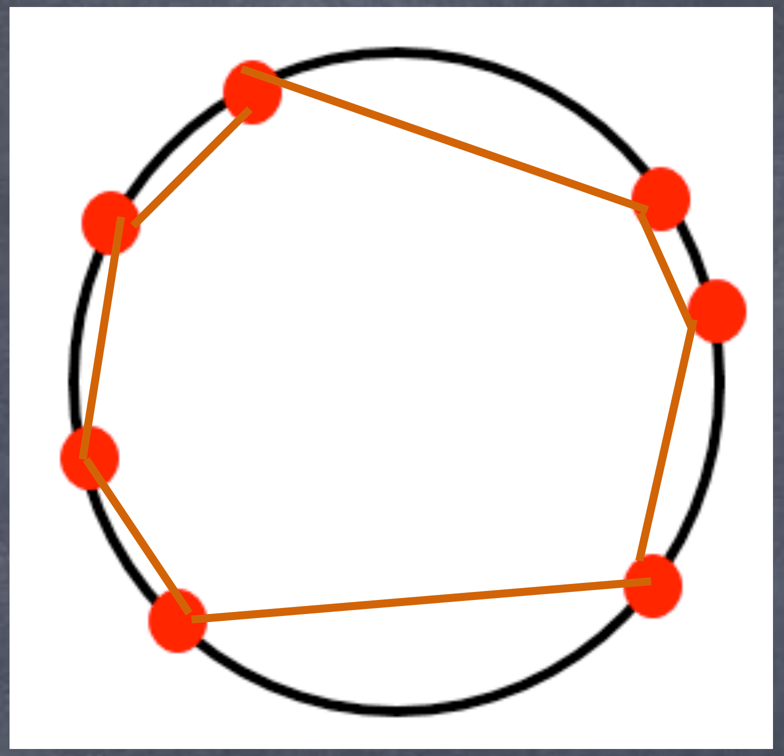
$$v_n(K) \sim \frac{1}{2} div_{d-1}\, \sigma_{bla}(bK)^{1+2/(d-1)}\, n^{-2/(d-1)}$$

RANDOM POLYHEDRAL APPROXIMATION

$K \subset \mathbb{R}^d, d \geq 2.$ Compact, convex set. $X_1, \ldots, X_n$ i.i.d. with cts density $f : bK \to (0,\infty)$

INSCRIBED POLYHEDRA

$P_n := Conv - Hull(\{X_1, \ldots, X_n\})$

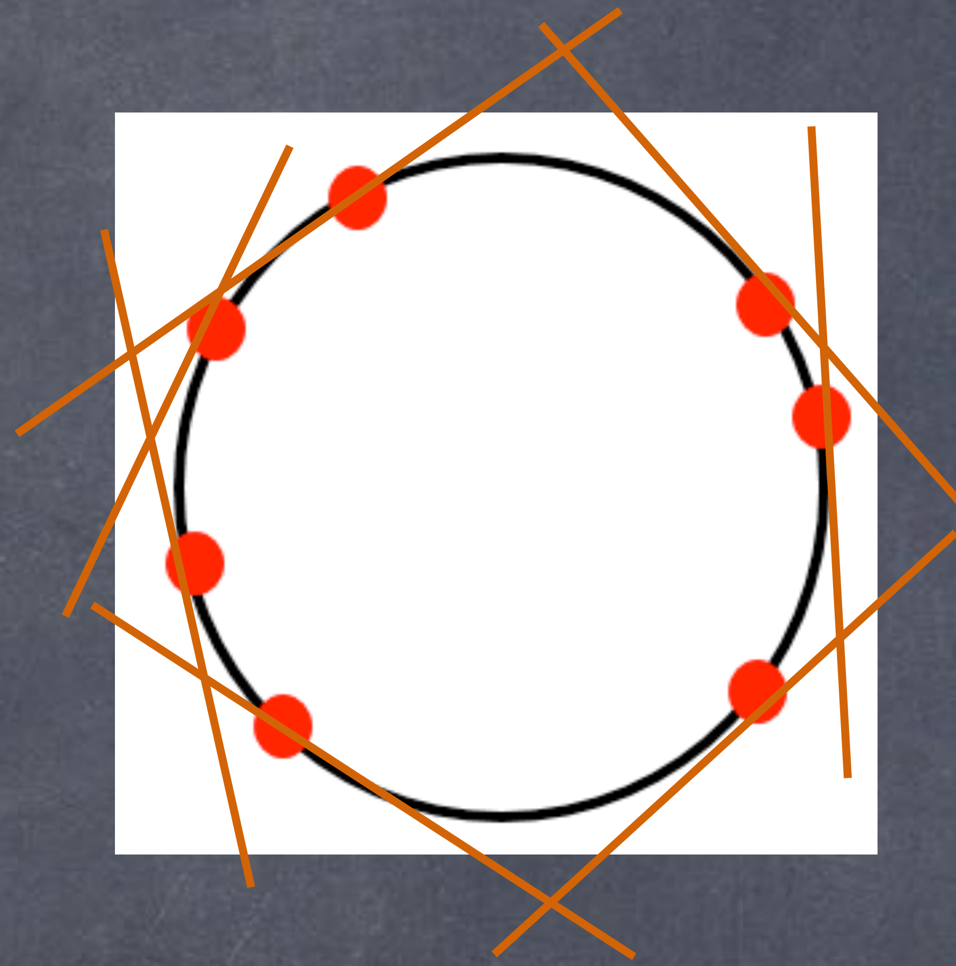


Schneider 1986, Müller 1990, Schütt & Werner 2000, Reitzner 2002.

$$Vol(K \backslash P_n) \overset{p}{\sim} \alpha_i(d)\ v^*(K, f)\ n^{-2/d-1}$$

CIRCUMSCRIBED POLYHEDRA

P_n - Intersection of tangent half-planes at $X_1, \ldots, X_n$ and containing K.



Böröczky & Reitzner 2004.

$$Vol(P_n \backslash K) \overset{p}{\sim} \alpha_c(d)\ v^*(K, f)\ n^{-2/d-1}$$

$$v^*(K) := \inf\{v^*(K, f) : f > 0, cts\} = \sigma_{bla}(bK)^{1+2/(d-1)}$$

Optimal and Random approximation.

Same rate of decay. Same domain dependent constants - $\sigma_{bla}(bK)$. Same Exponent - $2/(d-1)$.

Dimension constants differ

WHY ?

Mathematics: Interesting mix of geometric analysis and probability.

"The probabilistic description of random figures, e.g. distributions, moments, asymptotic coefficients, etc., should yield new integral geometric entities containing information about the geometry of the random figures. This general point of view has been the motivation of all my papers about geometric probabilities."

R. Sulanke in a 2004 letter to I. Bárány about Rényi & Sulanke (1963)

Statistical Estimation: Infer geometry of domain (for eg., Volume, mean-width) using random polyhedra.

Baldin & Reiß (2018), Last & Molchanov (2022+)

Optimization: Useful to approximate arbitrary sets by Polyhedra.

Optimal Polyhedra are very specific constructions. Random polyhedra are easier to construct.

See Hug (2013) survey introduction for more applications.

POLYHEDRA IN SEVERAL COMPLEX VARIABLES

“ If I knew what it is I am doing, it wouldn’t be called research. Would it ?”

For an SCV perspective, watch Purvi Gupta’s Talk in Virtual East-West SCV Seminar.

CONVEXITY : REAL AND IMAGINARY

REAL CONVEXITY

$K \subset \mathbb{R}^d$, smooth $bK \Leftrightarrow \; K := \{\rho < 0\}$, "nice" ρ

Convex if $K \cap L$ is connected for all real affine lines L.
(& simply connected)

Convex if Hessian (ρ) is semi-positive definite on the tangent space everywhere on bK.

Strongly Convex if Hessian (ρ) is positive definite.

K is Convex $\Rightarrow K \cong \mathbb{B}$

Preserved under affine transformations.

$$x \mapsto \left(b_{10} + b_{11}x_1 + \ldots + b_{1d}x_d, \ldots, b_{d0} + b_{d1}x_1 + \ldots + b_{dd}x_d\right)$$

$\mathbb{C}-$CONVEXITY

$D \subset \mathbb{C}^d$, smooth $bD \Leftrightarrow \; D := \{\rho < 0\}$, "nice" ρ

$\mathbb{C}-$Convex if $D \cap L$ is connected and simply connected for all complex affine lines L.

$\mathbb{C}-$Convex if Hessian (ρ) is semi-positive definite on the complex tangent space (i.e., maximal complex subspace of the tangent space) everywhere on bD.

Strongly $\mathbb{C}$-Convex if Hessian (ρ) is pos. definite.

D is $\mathbb{C}-$Convex $\Rightarrow D \cong \mathbb{B}$

Preserved under linear fractional (Möbius) transformations

$$(z_1, \ldots, z_d) \mapsto \left(\frac{b_{10} + b_{11}z_1 + \ldots + b_{1d}z_d}{b_{00} + b_{01}z_1 + \ldots + b_{0d}z_d}, \ldots, \frac{b_{d0} + b_{d1}z_1 + \ldots + b_{dd}z_d}{b_{00} + b_{01}z_1 + \ldots + b_{0d}z_d}\right)$$

Locally strongly Convexiable under Möbius transformations.

(Strong) Convexity in $\mathbb{R}^{2d}$ implies (Strong) $\mathbb{C}-$Convexity in $\mathbb{C}^d$!

$\mathbb{C}-$CONVEX DOMAINS : SOME ILLUSTRATIONS

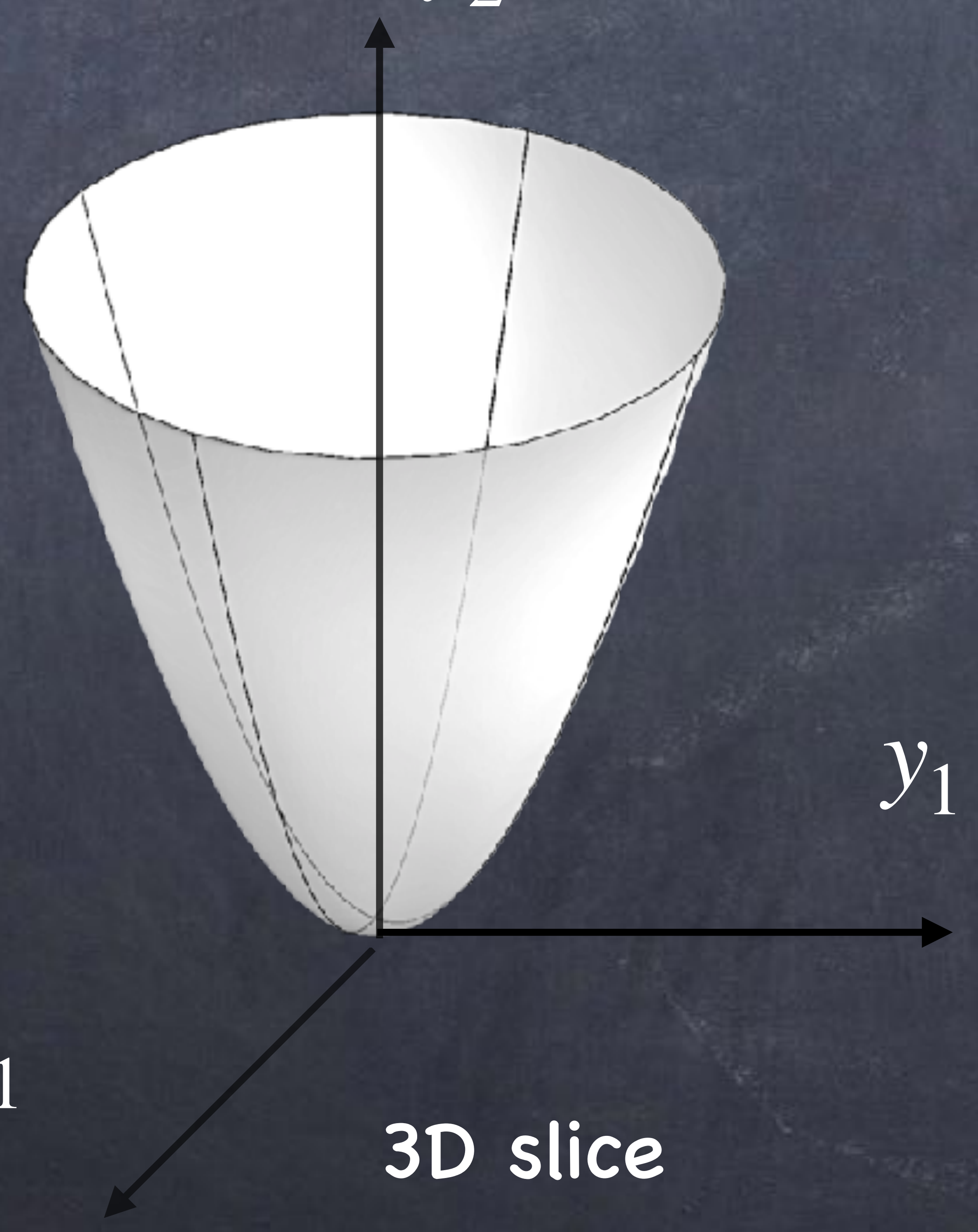
$\mathbb{C}-$Convex if $D \cap L$ is connected and simply connected for all complex affine lines L.

$\Rightarrow D \subset \mathbb{C}$ is $\mathbb{C}-$convex if D is a simply connected domain.

$\mathbb{C}-$Convex if Hessian (ρ) is semi-positive definite on the complex tangent space (i.e., maximal complex subspace of the tangent space). Strongly $\mathbb{C}-$Convex if Hessian (ρ) is positive definite.

Eg 1: $D = \{y_2 > |z_1|^2\}$; $\partial\rho(z) = (\bar{z}_1, -\frac{i}{2})$

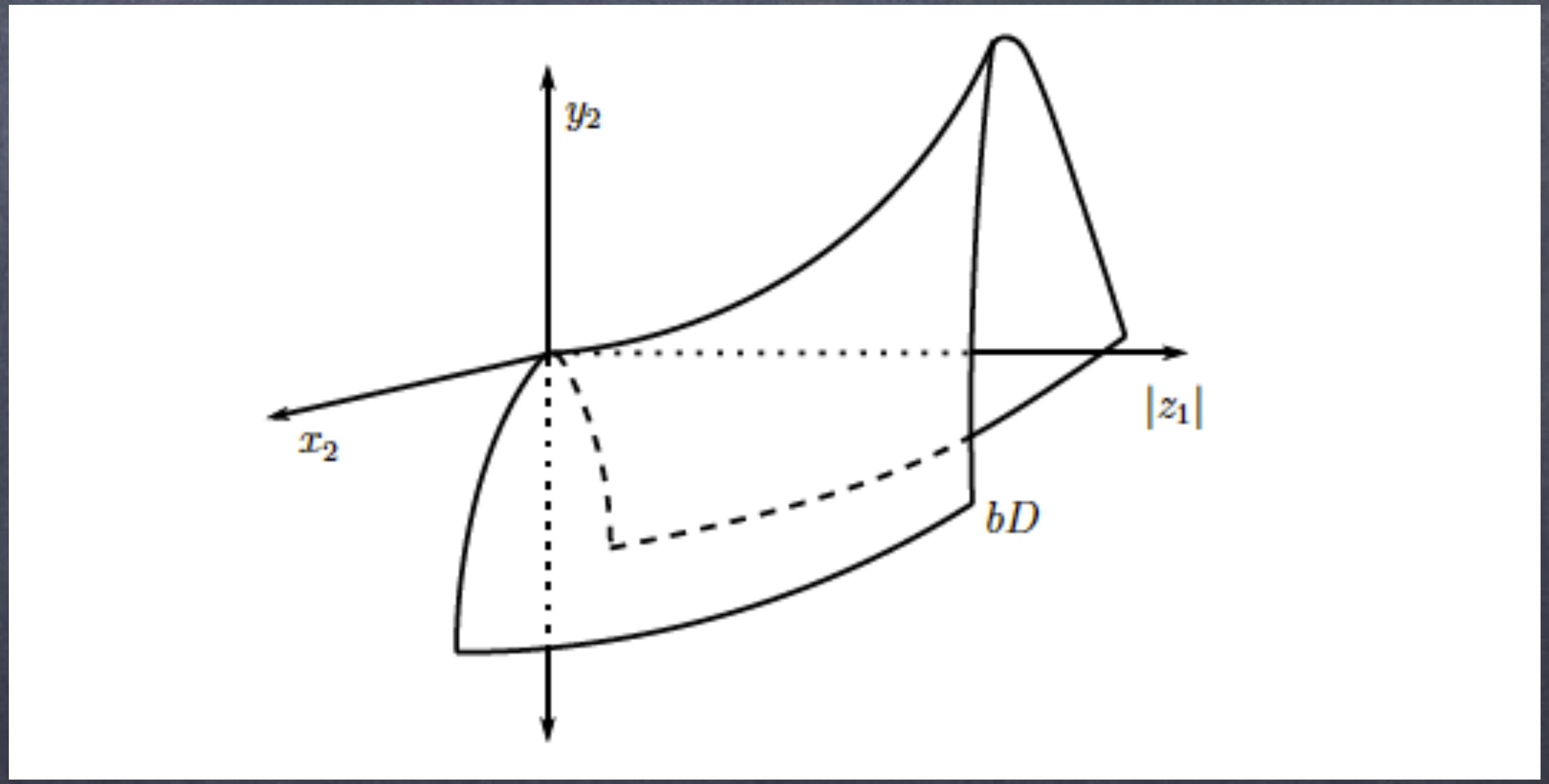
$z_k = (x_k, y_k) = x_k + iy_k$



3D slice

in $x_2 = 0$

Eg 2: $D = \{y_2 > |z_1|^2 - x_2^2\}$; $\partial\rho(z) = (\bar{z}_1, -2Re(z_2) - \frac{i}{2})$



Eg 3: Möbius transformations of Strongly convex domains.

POLYHEDRAL CONSTRUCTION IN ONE LINE

GIVEN

DOMAIN - $D \subset \mathbb{C}^d$, a smooth strongly $\mathbb{C}-$convex domain.

SOURCE - $\varphi = \{w_1, \ldots, w_n\} \subset bD$ DEPTH - $\delta : bD \to (0,\infty)$

Obstacle to Polyhedral Construction - No natural convex hull construction, tangent planes are lower dimensional,...

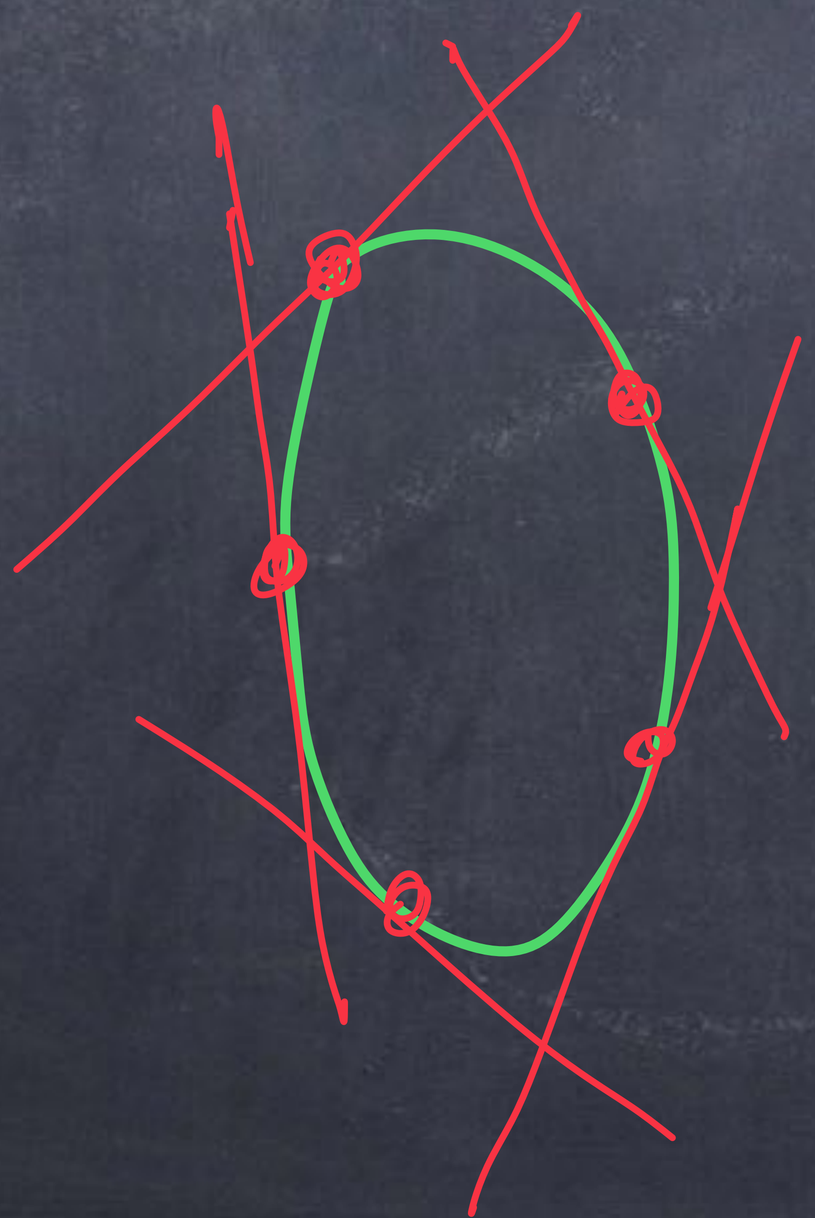
Affine Complex tangent space: $H_w bD = w$ + complex tangent space at w $= \{v + w \in \mathbb{C}^d : \langle \partial\rho(w), v\rangle = 0\}$

A construction Cut-out tube of depth $\delta(w_i)$ around $H_{w_i}(bD)$.

REAL ANALOGUES.

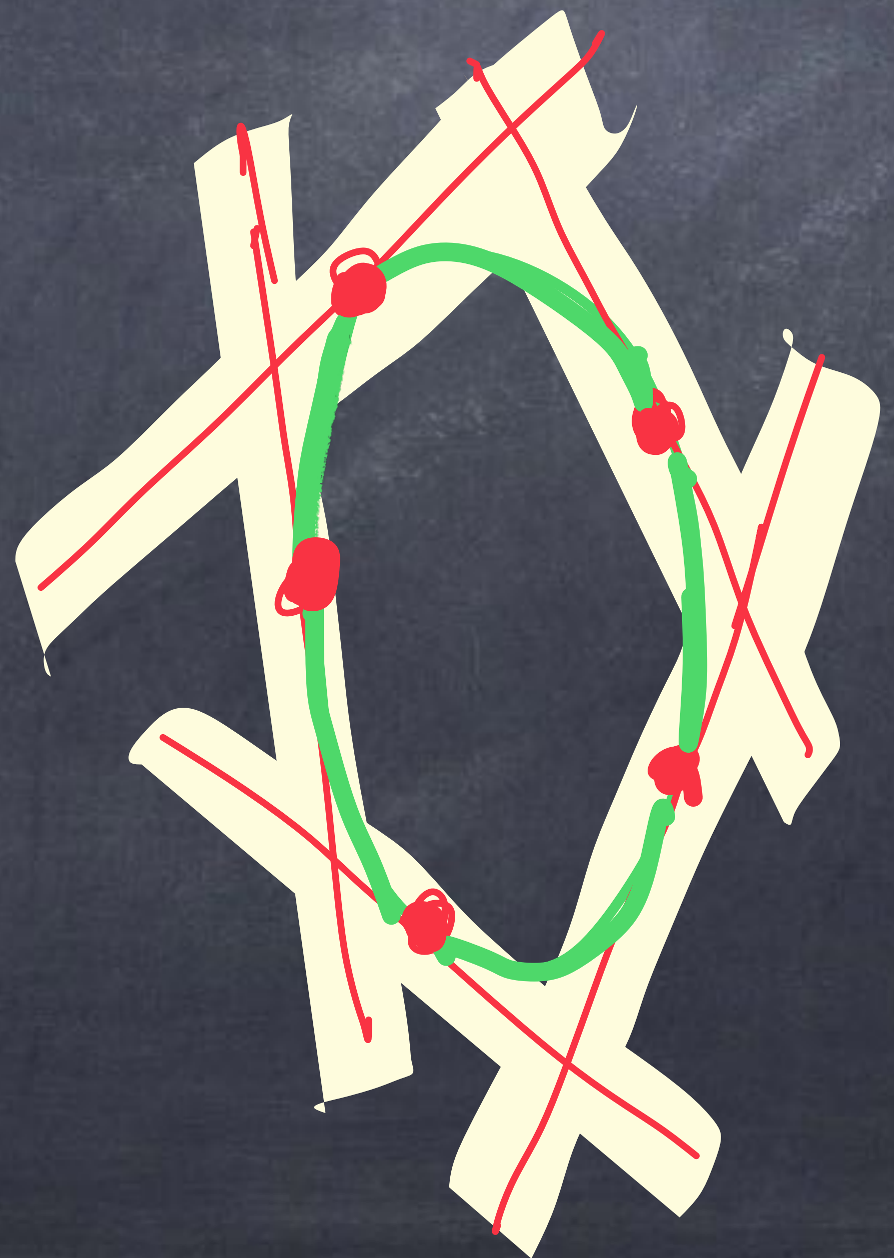
Circumscribed.

$\delta \equiv 0$



Cutting out

$\delta-$tubes



Glasauer-Schneider 1996,

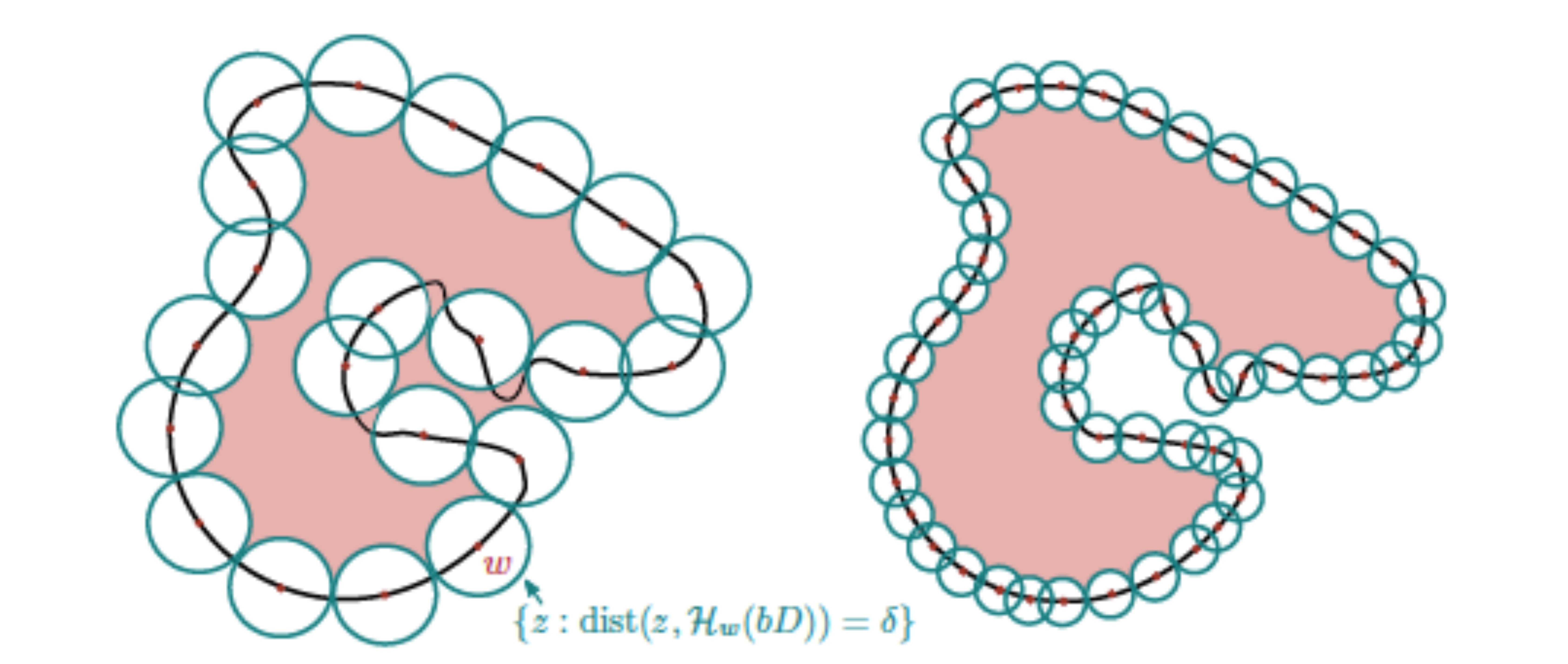
Ludwig 1999.

1-d CAUCHY-LERAY POLYHEDRA

d=1. A connected domain $D \subset \mathbb{C}$ is Strongly $\mathbb{C}-$convex iff D is simply connected.

$H_w(bD) = \{w\},\ w \in bD\ ;\ \{z : d(z, H_w(bD)) \leq \delta\} = B_w(\delta)$ i.e., a regular Euclidean ball.

Cauchy-Leray polyhedra are obtained by "cutting out" balls at $w_1, \ldots, w_n$ from D and union of the remaining connected components that intersect D.



TUBULAR NEIGHBOURHOODS IN HIGHER DIMENSIONS

d > 1. $D \subset \mathbb{C}^d$, a smooth strongly $\mathbb{C}-$convex domain. Source - $\varphi = \{w_1, \ldots, w_n\} \subset bD$, $\delta : bD \to (0,\infty)$ - Depth.

$H_w bD-$ Affine Complex tangent space to bD at w = w + complex tangent space at w $= \{v + w \in \mathbb{C}^d : \langle \partial\rho(w), v\rangle = 0\}.$

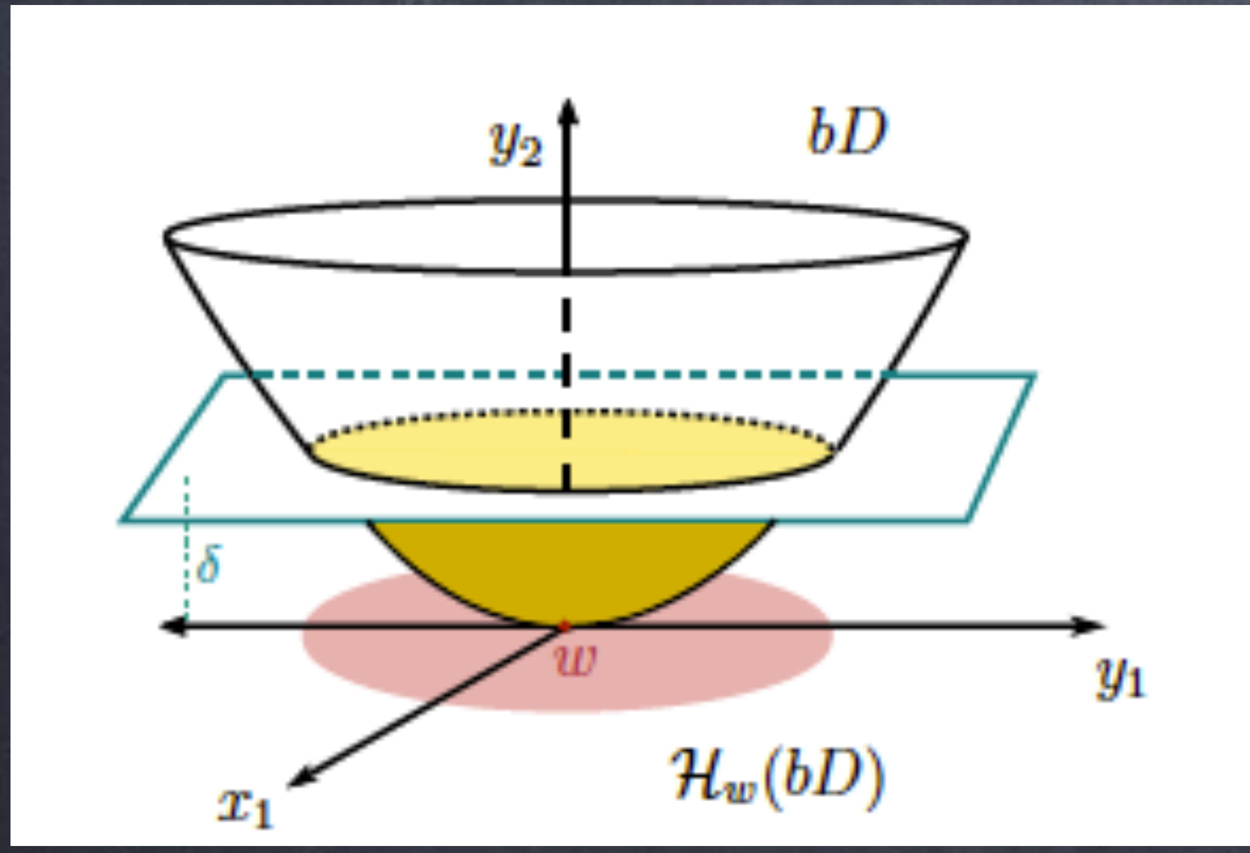
Tubular neighbourhood of $H_w bD$ - $\{ z \in \mathbb{C}^d \ : \ d(z, H_w bD) \leq \delta(w) \}$

Eg 1: $D = \{y_2 > |z_1|^2\}$; $\partial\rho(z) = (\bar{z}_1, -\frac{i}{2})$

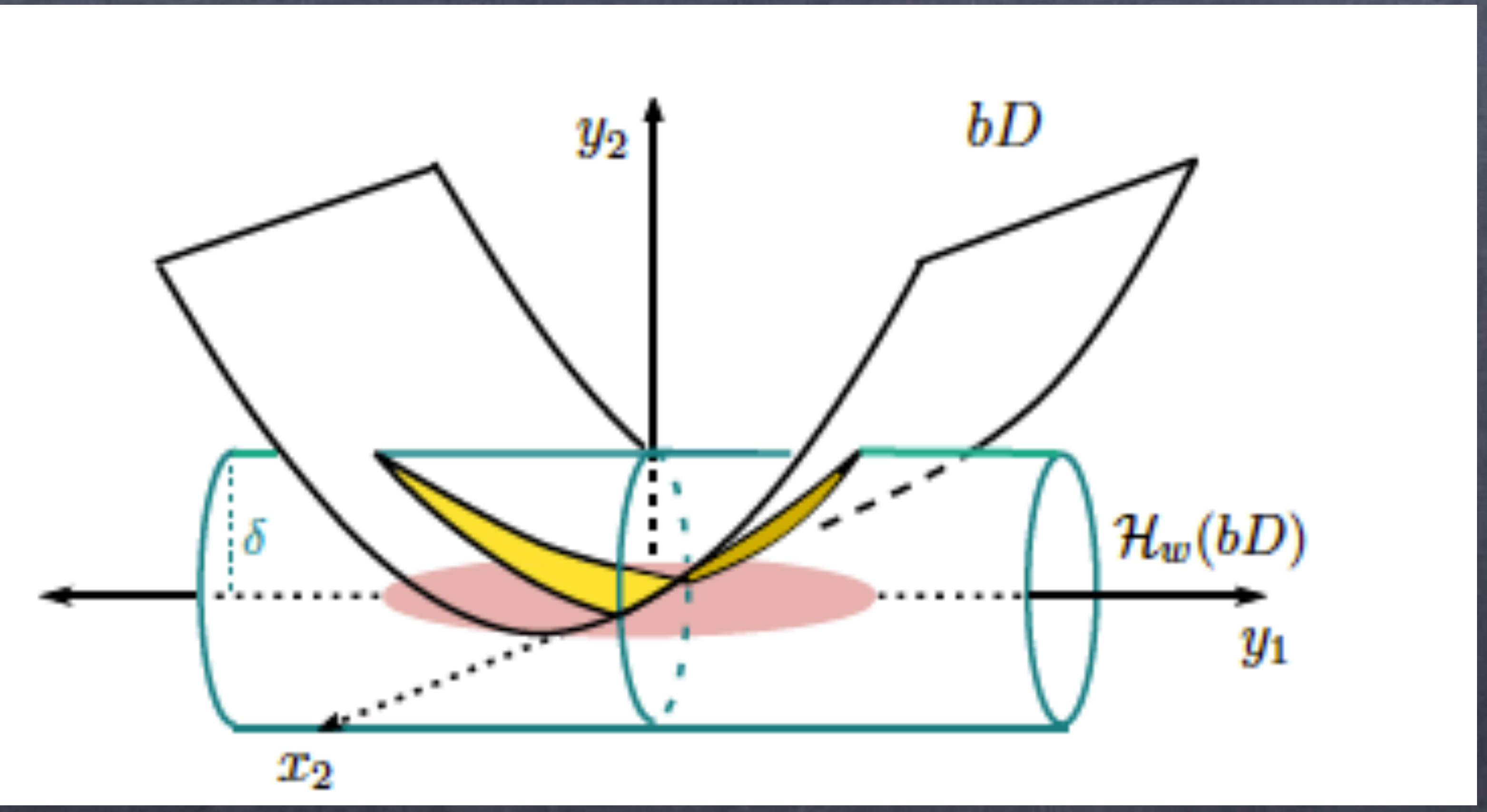
$w = 0;\ H_w bD = \{z_2 = 0\} = \{x_2 = y_2 = 0\}$

Eg 2: $D = \{y_2 > |z_1|^2 - x_2^2\}$; $\partial\rho(z) = (\bar{z}_1, -2Re(z_2) - \frac{i}{2})$

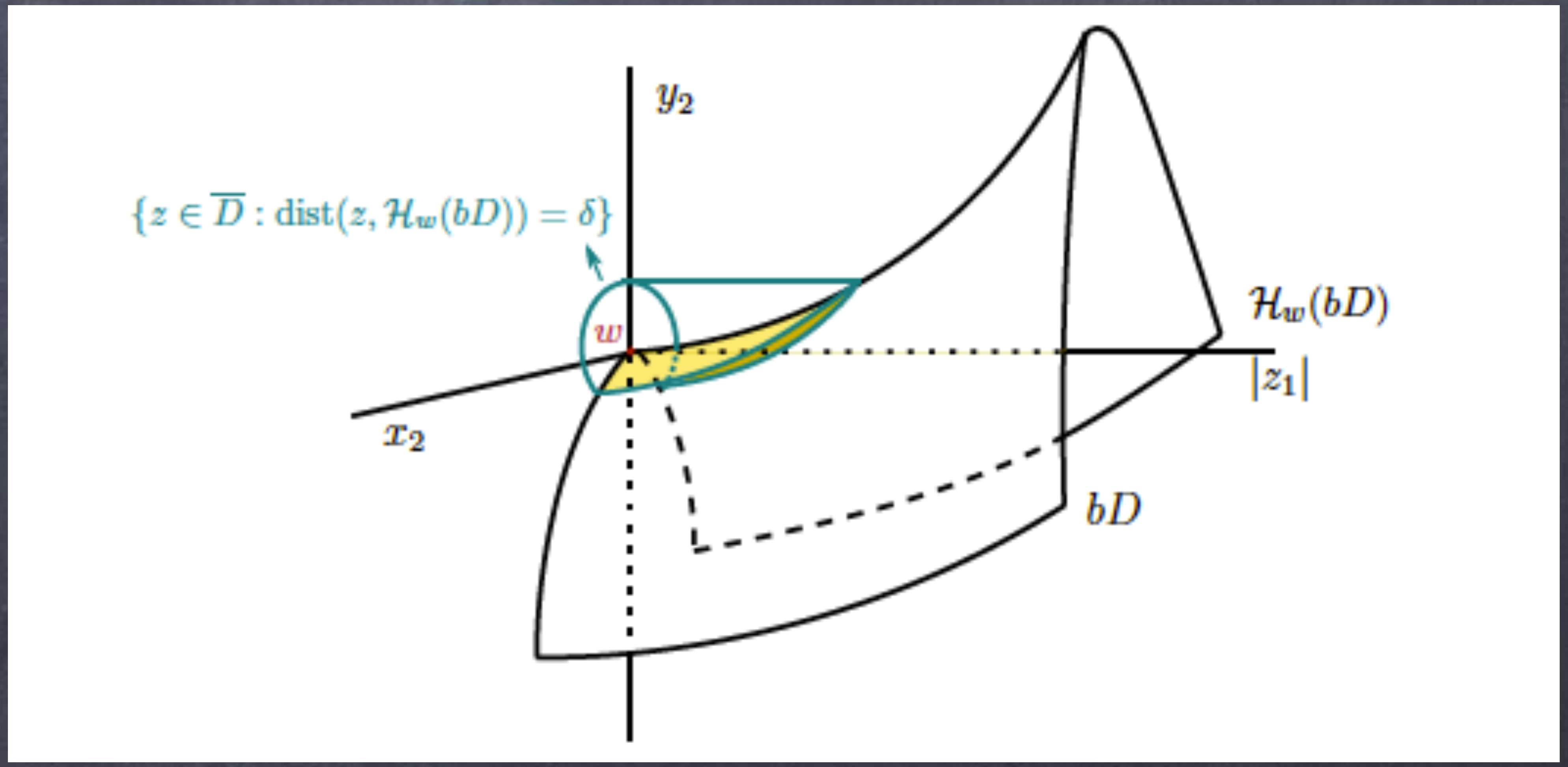
$w = 0;\ H_w bD = \{z_2 = 0\} = \{x_2 = y_2 = 0\}$



$x_2 = 0$ plane.



$x_1 = 0$ plane.



CAUCHY-LERAY POLYHEDRA IN HIGHER DIMENSIONS

d > 1. $D \subset \mathbb{C}^d$, a smooth strongly $\mathbb{C}-$convex domain.

Source - $\varphi = \{w_1, \ldots, w_n\} \subset bD, \ \delta : bD \to (0,\infty)$ - Depth.

$H_w bD-$ Affine Complex tangent space at w

(= w + complex tangent space at w $= \{v + w \in \mathbb{C}^d : \langle \partial\rho(w), v\rangle = 0\}$).

Tubular neighbourhood of $H_w bD$ - $\{ z \in \mathbb{C}^d \ : \ d(z, H_w bD) \leq \delta(w) \}$

Polyhedra - $P(\varphi; \delta)$ Union of connected components (intersecting D) of

$$\cap_{i=1}^{n} \{ z \in \mathbb{C}^d \ : \ d(z, H_{w_i} bD) > \delta(w_i) \} \ ;$$

Möbius transformations preserve strongly $\mathbb{C}-$convex domain, polyhedra and all that !

OPTIMAL APPROXIMATION

d > 1. $D \subset \mathbb{C}^d$, a smooth strongly $\mathbb{C}-$convex domain. Source - $\varphi = \{w_1, \ldots, w_n\} \subset bD, \ \delta : bD \to (0,\infty)$ - Depth.

Polyhedra - $P(\varphi; \delta)$ Union of connected components (intersecting D) of

$\cap_{i=1}^n \{ z \in \mathbb{C}^d \ : \ d(z, H_{w_i} bD) > \delta(w_i) \}$; $H_w bD-$ Affine Complex tangent space to bD at w.

Optimal approximation: $v_n(D) := \inf\{ Vol(D \backslash P) : P = P(\varphi, \delta) \subset D, \varphi = \{w_1, \ldots, w_n\} \subset bD, \delta : bD \to (0,\infty)\}$

Optimal approximation among contained polyhedra with at most n 'facets'.

THEOREM (Gupta, 2022+)

$$v_n(D) \sim \tilde{\sigma}_{mf}(bD)^{1+1/d} \ n^{-1/d}$$

Exponents - 1/d = 2/2d ; Real case - 2/(d-1)

Domain constant - $\tilde{\sigma}_{mf}$ - related to Möbius-Fefferman hyper surface measure

Real case - σ_{bla} - Blaschke affine surface measure.

Conjecture - $\tilde{\sigma}_{mf} = kor_d^{-d/(d+1)} \ \sigma_{mf}$; σ_{mf} - Möbius-Fefferman hyper surface measure

Precursor: Optimal approximation for pseudo-convex domains with Fefferman measure; see Gupta (2017).

RCp

RANDOM COMPLEX POLYHEDRA

RANDOM COMPLEX POLYHEDRA - SETUP

$D \subset \mathbb{C}^d, d \geq 2.$ Smooth, Strongly $\mathbb{C}-$Convex, compact set.

$\beta_n := \{X_1, \ldots, X_n\} \subset bD\ ; X_1, \ldots, X_n -$ i.i.d. with continuous density $f : bD \to (0,\infty)$.

Depth function $g : bD \to (0,\infty)$ continuous function. $\delta_n(x) := \left(\frac{\log n}{n}\right)^{1/d} g(x)\,,\ x \in bD.$

Polyhedra - $P_n := P(\beta_n; \delta_n)$ Union of connected components (intersecting D) of

$$\cap_{i=1}^{n} \{\ z \in \mathbb{C}^d\ :\ d(z, H_{X_i} bD) > \delta_n(X_i)\ \}\ ;\ H_w bD -$$ Affine Complex tangent space at w.

Metric of approximation. Containment not guaranteed and so we need a different metric.

$$\delta_V(n) = \delta_V(D, P_n) := Vol(D \backslash P_n)\mathbf{1}[P_n \subset D] + Vol(D)\mathbf{1}[P_n \not\subset D]$$

Asymptotics for $\delta_V(n)$?

Best depth function g and density f ?

RANDOM APPROXIMATION : A THEOREM.

$D \subset \mathbb{C}^d, d \geq 2.$ Strongly $\mathbb{C}-$Convex, compact set. $\beta_n := \{X_1, \ldots, X_n\} \subset bD$; $\delta_n(x) := \left(\frac{\log n}{n}\right)^{1/d} g(x)$, $x \in bD$

Random Polyhedra. $P_n := P(\beta_n; \delta_n)$

Approximation Metric. $\delta_V(n) := \delta_V(D, P_n) := Vol(D \backslash P_n)\mathbf{1}[P_n \subset D] + Vol(D)\mathbf{1}[P_n \not\subset D]$

Containment Condition. Assume $\mathbb{P}\big(P_n \subset D\big) \to 1$ as $n \to \infty$.

Log factor is necessary as containment $\equiv$ coverage on bD i.e., $\{P_n \subset D\} = \{bD \subset \cup_{i=1}^n B(X_i; \delta_n(X_i))\}$

THEOREM (Siva Athreya, Purvi Gupta, D. Y., 2022)

$$\left(\frac{n}{\log n}\right)^{\frac{1}{d}} \delta_V(n) \xrightarrow{P} \int_{bD} g(z)\, d\sigma(z)$$

i.e., $\forall \epsilon > 0$, $\mathbb{P}\left[\left| \left(\frac{n}{\log n}\right)^{\frac{1}{d}} \delta_V(n) - \int_{bD} g(z)\, d\sigma(z) \right| > \epsilon\right] \to 0$, as $n \to \infty$.

Exponent - 1/d = 2/2d ; Real case - 2/(d-1)

Theorem reduces finding good volume approximation to finding good coverage on the boundary !!!

BEST RANDOM APPROXIMATION : A CONJECTURE

THEOREM (Siva Athreya, Purvi Gupta, D. Y., 2022) $\left(\frac{n}{\log n}\right)^{\frac{1}{d}} \delta_V(n) \xrightarrow{P} \int_{bD} g(z)\, d\sigma(z)$

Best approximation for a f. $\nu_D(f) := \inf \{ \int_{bD} g(z)\, d\sigma(z) : g$ cts, $\mathbb{P}\big(P_n \subset D\big) \to 1\},\ f > 0,$ a cts density on bD.

Best overall approximation. $\nu_D^* := \inf \{\nu_D(f) : f > 0,$ a continuous density on $bD\}$

CONJECTURE (Siva Athreya, Purvi Gupta, D. Y., 2022) $\nu_d^* = (h_d\, \omega_{2d-2})^{\frac{-1}{d}}\, \sigma_{mf}(bD)^{1+\frac{1}{d}}$

Möbius-Fefferman hyper surface measure. $\sigma_{mf}(z) := (16\, \gamma_D(z))^{\frac{1}{2d+2}}\, \sigma(z),\quad \gamma_D-$ complex-restricted curvature of bD

Coverage Heuristics: $bD \subset \cup_{i=1}^{n} B(X_i; \delta_n(X_i))$ iff $(4\sqrt{\gamma_D(z)})^{-1}\, h_d\, \omega_{2d-2}\, f(z)\, g(z)^d > 1\ \forall\ z \in bD.$

Janson, 1986; Hall 1985.

RANDOM APPROXIMATION : PROOF IDEA

THEOREM (Siva Athreya, Purvi Gupta, D. Y., 2022) $\left(\frac{n}{\log n}\right)^{\frac{1}{d}} \delta_V(n) \xrightarrow{P} \int_{bD} g(z)\, d\sigma(z)$

i.e., $\forall \epsilon > 0,\ \mathbb{P}\left[\left| \left(\frac{n}{\log n}\right)^{\frac{1}{d}} \delta_V(n) - \int_{bD} g(z)\, d\sigma(z) \right| > \epsilon\right] \to 0$, as $n \to \infty$.

Idea :

1. Approximate $D \backslash P_n$ via a random tubular neighbourhood of bD - depth depends on $\{X_1, \ldots, X_n\}$.
2. Use Weyl-type tube formula due to Roccaforte, 2013.
3. Random depth at z $\approx \left(\frac{\log n}{n}\right)^{1/d} g(z)$

SUMMARY OF REAL vs COMPLEX

COMPLEX CONTAINED POLYHEDRA

$D \subset \mathbb{C}^d, d \geq 2.$ Smooth, Compact, Strongly $\mathbb{C}-$convex set.

P_n- Cut out tubular neighbourhoods around complex tangent planes at $x_1, \ldots, x_n$

Optimal approximation: (Gupta, 2022+)

$$v_n(D) \sim \tilde{\sigma}_{MF}(bD)^{1+1/d} \; n^{-1/d}$$

$$\frac{1}{d} = \frac{2}{2d}$$

Best Random approximation: (CONJECTURE)

$$v_n^*(K) \overset{p}{\sim} (h_d\, \omega_{2d-2})^{-1/d}\, \sigma_{MF}(bD)^{1+1/d} \left(\frac{\log n}{n}\right)^{1/d}$$

Other Results:

Variance upper bounds for $Vol(D \backslash P_n)$ (via Poincaré inequality)

Normal approximation for $Vol(D \backslash P_n)$ assuming var. low. bd. (via second-order Poincaré inequality)

All the above results for Poisson process too and variance lower bound.

REAL INSCRIBED POLYHEDRA

$K \subset \mathbb{R}^d, d \geq 2.$ Smooth, Compact, convex set.

$$P_n := Conv - Hull(\{x_1, \ldots, x_n\})$$

Optimal approximation:

$$v_n(K) \sim \frac{1}{2}\, del_{d-1}\, \sigma_{bla}(bK)^{1+2/(d-1)}\, n^{-2/(d-1)}$$

Best Random approximation:

$$v_n^*(K) \overset{p}{\sim} \alpha_i(d)\, \sigma_{bla}(bK)^{1+2/(d-1)}\, n^{-2/(d-1)}$$

Other Results:

Variance asymptotics for $Vol(D \backslash P_n)$

Normal approximation for $Vol(D \backslash P_n)$

All the above results for Poisson process too.

And more.....

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