Phase-type distributed Power in SINR Coverage Process

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1 Introduction

In [2], the authors have defined a new coverage process motivated by many SINR (signal to interference noise ratio) models in wireless communications. In [3], it was observed that assuming Rayleigh fading of powers (i.e, exponentially distributed powers) makes the analysis more tractable. In particular, quantities such as the probability of coverage could be given neater expressions.

In this report we seek to explore
(i) Further results assuming Rayleigh fading.
(ii) And to extend these results to the case when the powers have a Phase-type distribution.

The importance of extending results to Phase-type distribution arises out of the following fact: Phase-type distributions are dense in the space of all distributions(Refer [4], Ch.2). Hence, for any distribution of the powers one can obtain the probability of communication as approximately as required.

For the sake of independent reading, we shall again describe the model here as in [3].

We consider an infinite planar network. Let

$$\Phi = \{(X_i, (S_i, T_i))\}$$

be a marked Poisson point process with intensity λ on the plane $\mathcal{R}^2$, where we have the following.

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- $\Phi = \{X_i\}$ denotes the locations of stations;
- $\{S_i\}$ denotes the powers emitted by the stations; the random variables $\{S_i\}$ will always be assumed independent and identically distributed (i.i.d) with mean $1/\mu$.
- $\{T_i\}$ are the SINR thresholds corresponding to some channel bit rates or bit-error rates; here for simplicity, we will take $T_i \equiv T$, a constant.

In addition to this marked point process, the model is based on a function $l(x, y)$ that gives the attenuation(path loss) from y to x in $\mathcal{R}^2$. We will assume that the path loss depends only on the distance, i.e, with a slight abuse of the notation $l(x, y) = l(|x - y|)$.

As one can see our main aim in the paper is to calculate the various quantities for as general a S_i as possible. The remaining of the report is organized as follows : Section 2, we obtain some results when the powers are exponentially distributed (i.e, Rayleigh fading). And we extend the results to the case when the powers have a Phase-type distribution in Section 3.

2 Exponentially Distributed Powers

We assume in this section that S_i are i.i.d exponential with mean $1/\mu_i$.

Let us suppose there is a station located at x that transmits with power S and requires SINR T.The station can communicate to the user at distance R iff

$$\frac{Sl(R)}{W + I_\Phi(y)} \geq T,$$

where $I_\Phi(y) = \sum_{X_i \in \Phi} S_i l(|y - X_i|)$ is the shot-noise process of Φ. Let the event be $\delta(R, \Phi)$. We denote $\mathsf{E}(\delta(R, \Phi))$ by $p_R(\lambda)$. We know that

$$p_R(\lambda) = \exp\{-2\pi\lambda \int_0^\infty \frac{u}{1 + l(R)/Tl(u)} du\} \Psi_W(\mu T / l(R)).$$

For the above definitions and results, one can refer [3].

We want to then calculate the chance that stations located at $x_1, \ldots, x_n$ that transmit with powers $S_1, \ldots, S_n$ respectively can communicate to a point y which is located at a distance $R_1, \ldots, R_n$ respectively from them. We denote this probability by $p_{(R_1,\ldots,R_n)}(\lambda)$.

$$\mathsf{Pr}\left(\frac{S_i l(R_i)}{W + I_\Phi + \sum_{j \neq i} S_j l(R_j)} \geq T \forall i\right) = \mathsf{Pr}\left(S_i l(R_i) \geq \frac{T}{1+T}(W + I_\Phi + \sum_j S_j l(R_j)) \forall i\right).$$

Since $l(R_i)$ is constant $S_i l(R_i)$ is again exponential with parameter $\bar{\mu}_i = \frac{\mu_i}{l(R_i)}$. We shall denote it by X_i and denote $min(X_1, \ldots, X_n)$ by Z. Then Z is again exponential with $\sum \mu / l(R_i)$. Hence the previous equation is now

$$\begin{aligned}
\mathsf{Pr}\left(X_i \geq \frac{T}{1+T}(W + I_\Phi + \sum_{j \leq n} X_j) \forall i\right) &= \mathsf{Pr}\left(min(X_1, \ldots, X_n) \geq \frac{T}{1+T}(W + I_\Phi + \sum_j X_j)\right) \\
&= \mathsf{Pr}\left(Z \geq \frac{T}{1+T}(W + I_\Phi + nZ + \Delta)\right)
\end{aligned}$$

where Z and Δ are defined by $\sum_{j=1}^n X_j = nZ + \Delta$. By the memorylessness of exponential distribution, Z and Δ are independent. The distribution of Δ is given as follows,

$$\mathsf{Pr}\left(\Delta > x\right) = \sum_{i=1}^n \frac{\bar{\mu}_i}{\sum_j \bar{\mu}_j} \mathsf{Pr}\left(\sum_{k=1, k \neq i}^n X_k > x\right)$$

$$\begin{aligned}
p_{(R_1, \ldots, R_n)}(\lambda) &= \mathsf{Pr}\left(Z \geq \frac{T}{1-(n-1)T}(W + I_\Phi + \Delta)\right) \\
&= \int_0^\infty e^{-\mu s T/(1-(n-1)T)} d\mathsf{Pr}\left((W + I_\Phi + \Delta \leq s\right) \\
&= \Psi_{I_\Phi}(\mu s T/(1-(n-1)T) \Psi_\Delta(\mu s T/(1-(n-1)T) \\
&\quad \Psi_W(\mu s T/(1-(n-1)T)
\end{aligned}$$

The notation used is $\Psi_X(.)$ for the Laplace Transform of the variable X. Since X_i's are independent Ψ_Δ is nothing but the product of the laplace transform of n exponential random variables.

3 Phase-Type Distributions

Suppose as in the above section, the station located at x transmits with power S which has phase-type distribution $PH(\alpha, B, d)$ where α is $d-$ dimensional initial distribution vector, B is the $(d+1)X(d+1)$ intensity matrix and d is the cardinality of the state space. Then the probability of successful communication,

$$\begin{aligned}
p_R(\lambda) &= \mathsf{Pr}\left(S \geq \frac{T}{l(R)}(W + I_\Phi)\right) \\
&= \int_0^\infty \alpha \exp sTB/l(R) e^T d\mathsf{Pr}\,(W + I_\Phi \leq s) \\
&= \sum_n \int_0^\infty \alpha \frac{(sTB/l(R))^n}{n!} e^T d\mathsf{Pr}\,(W + I_\Phi \leq s) \\
&= \sum_n \alpha \frac{(TB/l(R))^n}{n!} e^T \int_0^\infty s^n d\mathsf{Pr}\,(W + I_\Phi \leq s) \\
&= \sum_n \alpha \frac{(TB/l(R))^n}{n!} e^T \mathsf{E}((W + I_\Phi)^n) \qquad (1)
\end{aligned}$$

Since the laplace transforms of the shot-noise is known, the nth moments of the shot-noise can be obtained from it. Hence it is possible to numerically evaluate the Eqn. (1).

Now we would like to calculate the probability that n stations can communicate to a point. Let the n stations transmit with powers $S_1, \ldots, S_n$ which have phase-type distributions $PH(\alpha_1, B_1, d_1)\ , \ldots, PH(\alpha_n, B_n, d_n)$. Let the stations be at a distance $R_1, \ldots R_n$. Then $p_{R_1,\ldots,R_n}(\lambda)$ is

$$\mathsf{Pr}\left(\frac{S_i l(R_i)}{W + I_\Phi + \sum_{j \neq i} S_j l(R_j)} \geq T \forall i\right) = \mathsf{Pr}\left(P_i l(R_i) \geq \frac{T}{1+T}(W + I_\Phi + \sum_j S_j l(R_j)) \forall i\right).$$

Since $l(R_i)$ is constant $S_i l(R_i)$ is again phase-type distributed as $PH(\alpha_i, B_i/l(R_i), d_i)$. We shall denote it by τ_i and denote $min(\tau_1, \ldots, \tau_n)$ by τ. Then τ is again phase-type distributed as $PH(\bigotimes_{i=1}^n \alpha_i, \bigoplus i = 1^n B_i, \prod_{i=1}^n d_i)$ where $\bigotimes$ and $\bigoplus$ are the Kronecker product and Kronecker sum respetively.(Refer Sec 5, [**?**]).

If we denote the the markov chain corresponding to τ_i as X^i, then τ is a stopping time for the markov chain $Y = (X^1, \ldots, X^n)$. Hence,

$$\begin{aligned}
p_{R_1,\ldots,R_n}(\lambda) &= \mathsf{Pr}\left(\tau_i \geq \frac{T}{1+T}(W + I_\Phi + \sum_j \tau_j) \forall i\right) \\
&= \mathsf{Pr}\left(\tau \geq \frac{T}{1+T}(W + I_\Phi + \sum_j \tau_j)\right) \\
&= \mathsf{Pr}\left(\tau \geq \frac{T}{1+T}(W + I_\Phi + n\tau + \Delta)\right)
\end{aligned}$$

Now $\Delta = \sum_i (\tau_i - \tau) \sim \sum_i PH(X^i_\tau, B_i/l(R_i), d_i)$. Unlike in the previous section, we have Δ dependent on τ. Hence to overcome it, we condition on (τ, Y_τ). Hence

$$\begin{aligned} p_{R_1,\dots,R_n}(\lambda) &= \mathsf{E}\left(\mathsf{Pr}\left(\tau \geq \frac{T}{1+T}(W + I_\Phi + n\tau + \Delta)|(\tau, Y_\tau)\right)\right) \\ &= \mathsf{E}\left(F_W * F_{I_\Phi} * F_\Delta(\tau\frac{1-(1-n)T}{T})\right) \end{aligned} \tag{2}$$

The above formula can be treated as an algorithm for evaluation of the desired probability. Since it is possible to numerically evaluate the distributions of Δ, I_Φ and W, one can also numerically evaluate the equation.

Though the case of a general phase-type distribution is complicated, if one were to assume a simple phase-type distribution P such that $\mathsf{Pr}\,(P > t) = \sum_i^k \alpha_i e^{-\lambda_i t}$, then one can obtain expressions similar to the previous section. This particular distribution corresponds to a markov chain which chooses a state $i \in \{1, \dots, k\}$ with probability α_i and stays for exponential time. It is also possible to do similar calculations for Erlang distribution.

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