CONNECTIVITY AND TRACKING PROBLEMS IN WIRELESS SENSOR NETWORKS

A Thesis Submitted

in Partial Fulfilment of the Requirements

for the Award of the Degree of

MASTER OF SCIENCE

In The Faculty of Science

by

D. YOGESHWARAN

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

March 2006

Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Prof. Srikanth K. Iyer at the Department of Mathematics, Indian Institute of Science, Bangalore 560012. I further declare that this work has not formed the basis for the award of any degree, diploma, fellowship, associateship or similar title of any other University or Institution.

D. Yogeshwaran

Certified

Prof. Srikanth K. Iyer

यथा शिखा मयूराणां नागानां मणयो यथा ।
तद्वेदांगशास्त्राणां गणितं मूर्धनि स्थितम् ।।

As are the crests on the heads of peacocks,
As are the gems on the heads of cobras,
So is Mathematics , at the top of all Sciences.

- Yajur Veda, circa 600 B.C.

As I prepare for the odyssey across the ocean of mathematics, after a short walk on its shores, I deem the moment fit to acknowledge the mosaic contributions to this enthralling walk of mine.

It is a pleasure to be eternally indebted to the pole stars of my journey –The Almighty and My Mother. Their presence has been a steadying influence on my life and research and will stay so perennially.

For a pleasant walk on the shores, I truly owe it to Prof. Srikanth K. Iyer. as my supervisor and a wonderful human. His affable, liberal and refreshing attitude to research has made my association with him all the more memorable. His enthusiasm and openness have greatly influenced me for the better.

For constraint of space, I refrain from enumerating the names of all my course teachers. Each of their courses has been useful to me in diverse ways. All are equal, but some are more equal. In this regard, I am very grateful to Prof. K.B. Athreya (ISU, USA) and Prof. M. K. Ghosh (IISc) for their courses in Probability and general advice. Also, it was hugely beneficial to attend the wonderful workshop on Stochastic processes at ISI, Bangalore.

Among other teachers, I greatly treasure the encouragement of Prof. B. R. Nagaraj (TIFR) during all times of my stay at IISc. It is not often students meet such encouraging teachers like him. I thank Prof. Prashanth S. (TIFR) for my summer project with him where I learnt a lot about ways of learning mathematics. Also, it has been worthwhile to work with Prof. D. Manjunath and his students Sundar and Nikhil of IITB.

Though of paramount importance are the people during my nascent years over here, even more invaluable are the people who have motivated me to tread this least likely path. For that, I was extremely fortunate to have had loving faculty at Sri Sathya Sai Institute of Higher Learning, A.P. I cannot but still drool over the inspiring course in Topology and beautiful insights into mathematics by Prof. C. J. M. Rao. I admire in the same breath, the fascinating introductions to many subjects of mathematics by Prof. M. Venkateswarlu, Prof. G. V. P. Rao and Madam K. Tiwari. My former *teachers and current friends – Dr. J. Balasubramaniam and Dr. D. J. Das - deserve a special mention for their support.*

To all my co-travelers over the years, earnest thanks for their presence and profuse apologies for not naming them all. But still, special thanks to Aatira and Anindya for their patient attendance to my lectures and Narayana for all the deep discussions about research. I do beholden the love and support of my brother, father and all my relatives.

The facilities and staff at IISc and TIFR have been of help in myriad ways. The picturesque campus with its buzzing activities remains unparalleled. The almost exhaustive collection of books at the libraries of IISc, the department and TIFR has been invaluable. One awe-inspiring memory of the place is the statue of J. N. Tata opposite to the main building. I salute the pioneering vision of that noble soul but for whom the nation would not have had this great milieu for research. For monetary interests not intruding into my academics, the credits go to MHRD, Government of India.

I owe a lot to all those giants who have unearthed beautiful pebbles from the shores and stunning pearls from the ocean. This work and walk would not have been possible without their enriching contributions. Books being the constant companion to a researcher, I have been excited by too many of them. But particular thanks to the books of

- *Peter Hall (ANU, Australia) on Coverage processes*
- *K. B. Athreya on Probability and Stochastic processes*
- *Peter Morters (Univ. of Bath U.K.) and Yuval Peres (UC, Berkeley) – lecture notes on Brownian Motion*
- *G. F. Simmons (Colorado Coll.) on Topology*
- *Sheldon Ross (UCLA) on Stochastic Models and Applications.*

As for dedications go, this work is lovingly dedicated to all my teachers from kinder-garden to post-graduation.

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Chapter 1

Introduction

Much of motivation for this thesis work comes from wireless sensor networks. In this chapter, we describe the models and the applications that have motivated the problems we have considered. The problems studied are connectivity in evolving one dimensional exponential random geometric graphs (RGGs) and target tracking in sensor networks. Though the problems are motivated by real-life models, some of the properties studied are more due to analytical tractability.

A one dimensional exponential random geometric graph can be described as follows: Given a set $\mathcal{X}_n = \{X_1, X_2, \dots X_n\}$ where X_i are i.i.d. d-dimensional random variables with common density f, a random geometric graph $G(\mathcal{X}_n, r)$ is an undirected graph with vertex set $\mathcal{X}_n$ and an undirected edge set $E = \{(X_i, X_j) : |X_i - X_j| \leq r\}$. $r > 0$ is called the cutoff range. Various asymptotic topological properties of RGGs have been studied in [1]. An exponential RGG is one where f is the exponential density. One may refer to [2] for importance and results on the exponential RGG. In most of the studies on RGGs, the nodes are assumed to be static. However [3] shows that networks can exploit mobility to achieve better throughput. This necessitates the study of RGGs with nodes evolving over time.

We consider two evolution models and study one of them in detail and briefly sketch the results for the other. In the first model, the inter-nodal gaps evolve according to an exponential AR(1) process that makes the stationary distribution of the node locations exponential. For this model we obtain the k-step transition probabilities of connectivity conditioned on the network being currently connected (or disconnected). We also derive the distribution of the first passage time for a connected network to become disconnected. We then describe a random birth-death model where at each instant, the node locations evolve according to an exponential AR(1) process. In addition, a node chosen at random is allowed to die while giving birth to a node at a random location. We derive properties similar to those above. While we restrict ourselves to the one-dimensional case but an important variation we have introduced to the basic theme is the non-uniform distribution of nodes.

For our second problem, we turn to target tracking in sensor networks. Consider the following model of a sensor network. The sensors are deployed according to a spatially homogenous Poisson process in Euclidean space $\mathcal{R}^d$. The sensing area of each sensor is a ball of random radius and a point is considered k-sensed if and only if it is in the sensing area of at least k sensors. It is easy to understand that when k-sensors spot a target, better estimate of the location of the target is obtained as opposed to the target being sensed by a single sensor. We analyze various measures of trackability - the ability of the network to track one dimensional linearly moving targets.

The coverage of an operational area by sensors can be described via a a two dimensional Boolean process. The Boolean process is a countable sequence of independent and identically distributed (i.i.d) sets centered at points of a stationary Poisson point process. This has been the focus of study in [4]. The tools of coverage processes facilitate a better study of the problem of trackability described above. The focus of the study in [4] has been the area coverage properties of a Boolean process. In order to study k-sensing, we need to extend the results of [4] to k-coverage. A point is said to be k-covered if only it lies in atleast k of the sets in the coverage process. We obtain a weak law of large numbers and a central limit theorem for area covered in Chapter 3.

In Chapter 4 we analyze the trackability of a sensor network. We first show that the sensing process induced on a straight line path by the area coverage process in $\mathcal{R}^2$ is a one dimensional Boolean process which in turn can be mapped to a M/G/∞ queue. This is then used to obtain two asymptotic results—a strong law and a central limit theorem for the fraction of a path that is covered by k or more sensors. The asymptotic results are obtained under the same conditions as that required for asymptotic coverage by a two dimensional Boolean process. Interestingly, the asymptotic fraction of the area k-covered by the sensors in $\mathcal{R}^2$ is the same as the fraction of a path k-sensed.

The strong law derived above helps us obtain the sensor density that is necessary to sense a given fraction of an arbitrary path with very high probability is derived. Expectation and variance of the fraction of a path covered for finite λ are also obtained. We then characterize the 'length to first sense', and sensing continuity measures like holes and clumps. Trackability measures that do not depend on the sensing radius like breach and support are also characterized. Also discussed are some generalizations of the results like characterization of the coverage process of $m-$dimensional 'straight line paths' by a $n(> m)$ dimensional sensor networks. Though we have delineated all the results in the context of sensor networks, it also can be applied to problems in atmospheric monitoring, intruder detection etc.

Next we study the backbone-client sensor network model. The model we consider is of base stations (backbone) and clients (sensors). The clients lie within a certain distance of the base station and communicate with that base station alone. These clients act as the sensor nodes which sense the target. We have considered the case when clients lie within a square of length $2R_0$ around the base station and each client senses within a square of length $2R_1$ around itself. This backbone-client network can be modeled as a Poisson Cluster Process. We derive the strong law and central limit theorem for fraction of the one dimensional linearly moving target sensed. We obtain other measures like breach and length to first sense.

The thesis is a compilation of our work done in [12], [5], [25], [24]. The thesis is organized as follows : In Chapter 2, we study the evolving one dimensional exponential RGG model described in the beginning. In Chapter 3 we extend results to the concept of k- coverage. Its applications to the sensor network model and other results pertaining to the model have been elaborated in Chapter 4. Similar kind of results have been obtained for the backbone sensor networks in Chapter 5. Finally we round off the thesis with a few remarks and possibilities for future work in Chapter 6.

Chapter 2

Evolving Random Geometric Graphs

2.1 Introduction and Preliminaries

Random geometric graphs (RGG) are being extensively studied in the context of wireless ad hoc networks, wireless sensor networks, interval graphs, etc. To obtain an RGG, N nodes are deployed according to a specified spatial probability distribution in an operational area. The operational area is defined by the support of the node distributions. Two nodes are connected by an edge if the distance between them, measured using a specified norm, is less than a critical distance r. In general, r can be a function of N. Topological properties of the resulting graph, are typically studied. [1] is an excellent introduction to this subject. The properties are usually obtained for uniform distribution of the nodes in the unit cube (of any dimension) and usually only asymptotic results as $N \to \infty$ are available. [13, 14] are notable examples where finite graphs have been considered. Using methods similar to that for the uniform distribution in a cube, [1] also obtains results for other distributions with finite support. An important assumption in all of the research is that the nodes are static, i.e., once they are deployed, their locations do not change. We can say that such stochastic characterizations of the RGGs are for ensembles. We introduce two variations to the usual analysis assumptions described above—(1) the node positions evolve over time according to an AR process, i.e., the nodes are mobile and (2) rather than being uniformly distributed in the unit cube, the nodes have a non uniform distribution in $(0, \infty)$. We restrict ourselves to the one-dimensional case. In an important study of mobility, [3] shows that ad hoc networks can exploit mobility to achieve $\mathcal{O}(1)$ throughput as opposed to the $\mathcal{O}(\sqrt{(}N/\log N))$ throughput obtained in [15]. While a fairly general mobility model is assumed, the analysis of [3] does not involve modeling of the topological properties of the network. A good introduction to node mobility models in the context of wireless networks is available in [16]. Extensive simulation results are also provided in [16] to obtain the performance of network protocols under different mobility patterns of the wireless network nodes. While there is significant simulation based research in obtaining network performance and properties when the nodes are mobile, to the best of our knowledge, there is no known study of the evolution of the topological properties with mobile nodes and we believe this is the first such study. We consider the following mobile, or evolving, network of N nodes in one dimension on $(0, \infty)$. We assume that the evolution is a discrete time process. Let X_l^t denote the position of the l-th ordered node from the origin at time t, for $t = 0, 1, \ldots,$ and $l = 1, 2, \ldots, N$. Let $Y_l^t := X_{l+1}^t - X_l^t$, for $l = 1, 2, \ldots, N-1$. Define $Y_0^t = X_1^t$. We consider a network where $\{Y_l^t\}$ evolves according to the autoregressive process

$$Y_l^{t+1} = aY_l^t + Z_l^t. \tag{2.1}$$

Here Z_l^t is a random variable independent of Y_l^t and is essentially the innovation of the AR(1) process. This corresponds to there being a constant drift of the nodes and a random perturbation. Further we assume that $Y_0^t, Y_1^t, \ldots, Y_{N-1}^t$ are independent for all t. The above model for Y_l^t implies an AR(1) model for X_l^t, $X_l^{t+1} = aX_l^t + W_l^{t+1}$, where $W_l^{t+1} = \sum_{k=0}^{l-1} Z_l^{t+1}$. Two special cases will have interesting properties and we will investigate them in detail in this paper.

Case 1: Let $0 < a < 1$ and define $Z_l^t = U_l^t \times V_l^t$, where $\{U_l^t\}_{t \geq 0}$ is a sequence of i.i.d. 0/1 Bernoulli random variables of mean $(1-a)$ and $\{V_l^t\}_{t \geq 0}$ is a sequence of i.i.d. exponential random variables of mean λ_l. In [17] it is shown that this corresponds to the $\{Y_l^t\}_{t>0}$ being a stationary exponential AR(1) sequence with autocorrelation function a^k, assuming that the inter-nodal gaps Y_l^0 are exponentially distributed with parameter λ_l. This means that the stationary distribution of Y_l^t is an exponential with mean $1/\lambda_l$. The density of Z_l^t, $f_{Z_l}(z)$ for this case has been derived in [17] to be $a\delta(z) + (1-a)\lambda_l e^{-\lambda_l z}$

Case 2: Here we extend Case 1 and choose $\lambda_l = (N - l)\lambda$, for $l = 0, \ldots, N-1$. In this case the distribution of the node locations corresponds to that of the ordered nodes when the node locations are i.i.d. exponential random variables with mean $1/\lambda$. The stationary

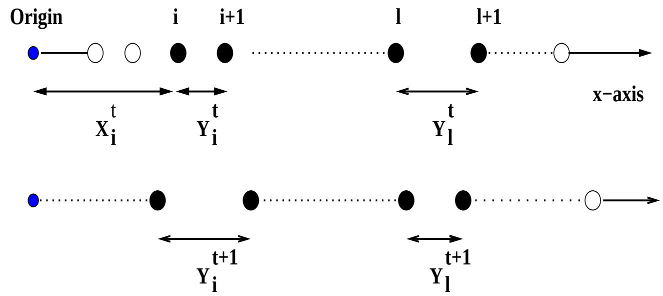

Figure 2.1: Inter-nodal spacings in the network evolving with time.

properties of the evolving RGGs from Case 2 correspond to the properties of the exponential RGGs. See [2, 18] for an extensive study of such RGGs where a number of topological properties like connectivity, span, existence of paths and components etc are obtained for finite N and also for the limiting case of $N \to \infty$. Here the time dependent behavior of the topological properties of the one-dimensional RGGs that evolve according to Eqn. 2.1 is characterized. Specifically we will obtain the finite node and asymptotic analysis for the k-step conditional probability of a connected network remaining connected and disconnected network being connected. This is described in Section 2.2. We then characterize the first passage time from connectivity to disconnectivity in Section 2.3 for finite N. In Section 2.4, we describe the random birth/death model and summarize a few results. Since we consider only a one-dimensional network, all the l_∞ and l_2 norms to measure the distance between two nodes are equivalent. Further, we will assume that r is fixed.

2.2 Conditional Connectivity

Consider an N-node network with Y_l^t evolving according to Case 1 above, as shown in Fig. 2.1 . Assume that at time t, the network is connected. Connectivity of the network implies that $Y_l^t < r$ for all l. The conditional distribution and density functions for Y_l^t in the connected network, denoted by $F_{Y_l|C}(t)$ and $f_{Y_l|C}(t)$ respectively, will be

$$F_{Y_l|C}(y_l) = \begin{cases} \frac{1-e^{-\lambda_l y_l}}{1-e^{-\lambda_l r}} & \text{for } y_l < r \\ 1 & y_l \geq r \end{cases}$$

$$f_{Y_l|C}(y_l) = \begin{cases} \frac{\lambda_l e^{-\lambda_l y_l}}{1-e^{-\lambda_l r}} & \text{for } y_l < r \\ 0 & \text{otherwise} \end{cases}$$

Let $\psi_k(N)$ be the probability that the network is connected at time $t+k$, conditioned on it being connected at time t. We begin by evaluating $\psi_1(N)$, the one-step conditional probability of staying connected in one step.

$$\begin{aligned}&\psi_1(N) := \\ &\mathsf{Pr}\left((Y_1^{t+1} < r, \dots Y_{N-1}^{t+1} < r) | (Y_1^t < r, \dots Y_{N-1}^t < r)\right) = \\ &\prod_{l=1}^{N-1} \mathsf{Pr}\left(Y_l^{t+1} < r | Y_l^t < r\right)\end{aligned} \tag{2.2}$$

We need to evaluate each term of the above product. For this, we have

$$\mathsf{Pr}\left(Y_l^{t+1} < r | Y_l^t < r\right) = \mathsf{Pr}\left(aY_l^t + Z_l^t < r | Y_l^t < r\right) \tag{2.3}$$

Let $Y_{l|C}^{t+1}$ denote Y_l^{t+1} conditioned on $Y_l^t < r$. Under the evolution model of Case 1, we can obtain the Laplace transform of the density of $Y_{l|C}^{t+1}$, denoted $\tilde{f}_{Y_{l|C}^{t+1}}(s)$, as follows. Since Z_l^t is independent of Y_l^t, $\tilde{f}_{Y_{l|C}^{t+1}}(s)$ is the product of $\tilde{f}_{Z_l^t}(s)$ and the Laplace transform of the conditional density of aY_l^t. Since Y_l^t is exponential with mean $1/\lambda_l$, we can write,

$$\tilde{f}_{Y_{l|C}^{t+1}}(s) = \frac{\lambda_l}{(1-e^{-\lambda_l r})(s+\lambda_l)}\left(1 - e^{-r(as+\lambda_l)}\right)$$

Inverting the above, we obtain the density of $Y_{l|C}^{t+1}$ to be

$$f_{Y_{l|C}^{t+1}}(y) = \frac{\lambda_l e^{-\lambda_l y}\left(U(y) - e^{-\lambda_l r(1-a)}U(y-ar)\right)}{(1-e^{-\lambda_l r})} \tag{2.4}$$

Here $U(y)$ is the Heaviside function. Integrating the above density from 0 to r, we get

$$\mathsf{Pr}\left(Y_l^{t+1} < r | Y_l^t < r\right) = \frac{1 - 2e^{-\lambda_l r} + e^{-\lambda_l r(2-a)}}{1 - e^{-\lambda_l r}} \tag{2.5}$$

Substituting the above in Eqn. 2.2, we have

$$\psi_1(N) = \prod_{l=1}^{N-1}\left(\frac{1 - 2e^{-\lambda_l r} + e^{-\lambda_l r(2-a)}}{1 - e^{-\lambda_l r}}\right) \tag{2.6}$$

LEMMA 2.2.1 *As $N \to \infty$, for Y_l^t evolving as in Case 2, $\psi_1(N)$, the one step conditional connectivity probability tends to a limit, i.e.,* $\lim_{N\to\infty}\psi_1(N) = \psi_1$.

PROOF: Using Eqn. 2.6, we have

$$\begin{aligned}\psi_1(N)-\psi_1(N+1) =& \prod_{l=1}^{N-1}\left(\frac{1-2e^{-(N-l)\lambda r}+e^{-(N-l)\lambda r(2-a)}}{1-e^{-(N-l)\lambda r}}\right)\\ &-\prod_{l=1}^{N}\left(\frac{1-2e^{-(N+1-l)\lambda r}+e^{-(N+1-l)\lambda r(2-a)}}{1-e^{-(N+1-l)\lambda r}}\right)\\ =&\left(\frac{e^{-N\lambda r}-e^{-N\lambda r(2-a)}}{1-e^{-N\lambda r}}\right)\psi_1(N)\end{aligned}$$

Clearly, the first term of the last equality above goes to zero as $N \to \infty$. Since $\psi_1(N)$ is a probability, $\psi_1(N) - \psi_1(N+1)$ goes to 0 as $N \to \infty$. The limit is clearly non zero because for the product in the expression for $\psi_1(N)$ in Eqn. 2.6, none of the individual terms goes to 0 as a $N \to \infty$. ∎

Now consider the case when the critical distance varies with time and this variation is homogeneous across the network, i.e., all nodes have the same critical distance. Let $\{r_n\}_{n\geq 0}$ be the sequence of critical distances. An example situation is when the power available at the sensor nodes decreases with time and it becomes important to limit the transmission range to conserve energy. Note though that with the development of smart batteries, the charge may be recovered and $\{r_n\}$ need not be a decreasing sequence. Clearly, the density of $Y_{l|C}^{t+1}$ depends on r_t. Then, along the lines of the derivation of Eqn. 2.4, except that the integration is from 0 to r_{t+1}, we obtain

$$\psi_1(N) = \prod_{l=1}^{N-1}\frac{1-e^{-\lambda_l r_{t+1}}-e^{-\lambda_l r_t}+e^{-\lambda_l(r_t(1-a)+r_{t+1})}}{(1-e^{-\lambda_l r_t})}$$

Thus, if the sequence $\{r_n\}$ is known, the conditional connectivity probabilities can be computed by substituting the values of r_t and r_{t+1} in the above equation.

k-Step Conditional Connectivity

Using arguments similar to those in Eqn. 2.2, we now derive the probability that the network will be connected after k steps at time $t+k$ conditioned on the network being connected at time t.

$$\psi_k(N) = \prod_{l=1}^{N-1}\mathsf{Pr}\left(Y_l^{t+k} < r | Y_l^t < r\right) \tag{2.7}$$

We note that

$$Y_l^{t+k} = aY_l^{t+k-1} + Z_l^{t+k-1} = a^k Y_l^t + \sum_{m=1}^{k} a^{m-1} Z_l^{t+k-m} \tag{2.8}$$

We begin by evaluating the probability density of $W_k^t := \sum_{m=1}^{k} a^{m-1} Z_l^{t+k-m}$. Since Z_l^t are all independent, characteristic function of W_k^t is given by

$$\tilde{f}_{W_k^t}(s) = \prod_{m=1}^{k} \frac{a^m s + \lambda_l}{a^{m-1} s + \lambda_l} = \frac{a^k s + \lambda_l}{s + \lambda_l}$$

From Eqn. 2.8, we can write

$$\mathsf{Pr}\left(Y_l^{t+k} < r | Y_l^t < r\right) = \mathsf{Pr}\left(a^k Y_l^t + W_k^t < r | Y_l^t < r\right)$$

For the evolution model of Case 1, we can obtain the Laplace transform of the density of $Y_{l|C}^{t+k}$, denoted by $\tilde{f}_{Y_{l|C}^{t+k}}(s)$, as follows. Since W_k^t is independent of Y_l^t, $\tilde{f}_{Y_{l|C}^{t+k}}(s)$ is the product of $\tilde{f}_{W_k^t}(s)$ and the Laplace transform of the conditional density of $a^k Y_l^t$.

$$\tilde{f}_{Y_{l|C}^{t+k}}(s) = \frac{\lambda_l}{(1 - e^{-\lambda_l r})(s + \lambda_l)} \left(1 - e^{-r(a^k s + \lambda_l)}\right)$$

Inverting the above, we get

$$f_{Y_{l|C}^{t+k}}(y) = \frac{\lambda_l e^{-\lambda_l y}\left(U(y) - e^{-\lambda_l r(1-a^k)} U(y - a^k r)\right)}{1 - e^{-\lambda_l}}$$

Integrating the above density from 0 to r, we get

$$\mathsf{Pr}\left(Y_l^{t+k} < r | Y_l^t < r\right) = \frac{1 - 2e^{-\lambda_l r} + e^{-\lambda_l r(2-a^k)}}{1 - e^{-\lambda_l r}}$$

Substituting the above in Eqn. 2.7, we have

$$\psi_k(N) = \prod_{l=1}^{N-1} \frac{1 - 2e^{-\lambda_l r} + e^{-\lambda_l r(2-a^k)}}{1 - e^{-\lambda_l r}} \tag{2.9}$$

As with the case of one step connectivity, we can obtain the asymptotics as $N \to \infty$.

LEMMA 2.2.2 *As $N \to \infty$, for Y_l^t evolving as in Case 2, $\psi_k(N)$, the one step conditional connectivity probability tends to a limit, i.e.,* $\lim_{N \to \infty} \psi_N = \psi_k$.

PROOF: Proceeding along the same lines as the proof of Lemma 2.2.1, we have

$$\psi_k(N) - \psi_k(N+1) = \left(\frac{e^{-N\lambda r} - e^{-N\lambda r(2-a^k)}}{1 - e^{-N\lambda r}}\right)\psi_k(N)$$

$\psi_k(N) - \psi_k(N+1)$ goes to 0 as $N \to \infty$ and the lemma follows. ∎

As $k \to \infty$, $\psi_k(N)$ should be go to the stationary probability of the network being connected. As we have mentioned before this is also the probability that the 'static' network is connected. From [2] and also taking limits of Eqn. 2.9 (for Case 2), we obtain

$$\psi_\infty(N) \quad := \quad \lim_{k\to\infty} \psi_k(N) = \prod_{l=1}^{N-1}\left(1 - e^{-(N-l)\lambda r}\right)$$

Connectivity Conditioned on Disconnectivity

Let S be the set of edges which are disconnected at instant t. For the network to get connected at $t+k$, all the edges in S need to get connected and the ones in $\bar{S}$ should stay connected. Then we have

$$\begin{aligned}
&\mathsf{Pr}\left(C(t+k)|D_S(t)\right)\\
&:= \mathsf{Pr}\left(\text{network conn. at } t+k|\{S\} \text{ disconn.,}\{\bar{S}\}\text{ conn. at } t\right)\\
&= \prod_{l\in\{S\}} \mathsf{Pr}\left((Y_l^{t+k} < r)|(Y_l^t > r)\right) \times\\
&\qquad \times \prod_{l\in\{\bar{S}\}} \mathsf{Pr}\left((Y_l^{t+k} < r)|(Y_l^t < r)\right)
\end{aligned} \tag{2.10}$$

Here $C(t)$ is the event that the network is connected at t and $D_S(t)$ is the event that the set of edges in S is disconnected at t. Including all the different compositions S can have, we can write

$$\mathsf{Pr}\left(\text{conn. at } t+k|\text{disconn. at } t\right) = \frac{\sum_{(\text{all } S)} \mathsf{Pr}\left(C|D_S\right)\mathsf{Pr}\left(S\right)}{\mathsf{Pr}\left(\text{disconn. at } t\right)}, \tag{2.11}$$

where $\mathsf{Pr}\left(S\right)$ is the probability of occurrence of that set of connected and disconnected edges and is given by

$$\begin{aligned}
\mathsf{Pr}\left(S\right) &= \mathsf{Pr}\left(\{S\} \text{ disconn., } \{\bar{S}\} \text{ conn. at } t\right)\\
&= \prod_{l\in\{S\}} \mathsf{Pr}\left(Y_l^t > r\right) \prod_{l\in\{\bar{S}\}} \mathsf{Pr}\left(Y_l^t < r\right)\\
&= \prod_{l\in\{S\}} \left(e^{-\lambda_l r}\right) \prod_{l\in\{\bar{S}\}} \left(1 - e^{-\lambda_l r}\right)
\end{aligned}$$

We note that

$$\begin{aligned}&\mathsf{Pr}\left(Y_l^{t+k}< r|Y_l^t> r\right)\\&=\frac{\mathsf{Pr}\left(Y_l^{t+k}< r\right)-\mathsf{Pr}\left(Y_l^t< r\right)\mathsf{Pr}\left(Y_l^{t+k}< r|Y_l^t< r\right)}{\mathsf{Pr}\left(Y_l^t> r\right)}\end{aligned}$$

Y_l^{t+k} and Y_l^t both have the same probability density function and hence $\mathsf{Pr}\left(Y_l^{t+k}< r\right) = \mathsf{Pr}\left(Y_l^t< r\right) = 1 - e^{-\lambda_l r}$. Using Eqn. 2.5, we have

$$\begin{aligned}&Pr(Y_l^{t+k}< r|Y_l^t> r) =\\&\frac{(1-e^{-\lambda_l r})\left(1-\mathsf{Pr}\left(Y_l^{t+k}< r|Y_l^t< r\right)\right)}{e^{-\lambda_l r}}=1-e^{-\lambda_l r(1-a^k)}\end{aligned}$$

Substituting the above in Eqn. 2.10, we have

$$\begin{aligned}&\mathsf{Pr}\left(C|D_S\right) =\\&\prod_{l\in\{S\}}\left(1-e^{-\lambda_l r(1-a^k)}\right)\prod_{l\in\{\bar{S}\}}\left(\frac{1-2e^{-\lambda_l r}+e^{-\lambda_l r(2-a^k)}}{1-e^{-\lambda_l r}}\right)\end{aligned}$$

Using this expression in Eqn. 2.11, we get

$$\begin{aligned}&\mathsf{Pr}\left(\text{conn. at } t+k|\text{ disconn. at } t\right)=\frac{1}{1-\prod_{j=1}^{N}(1-e^{-\lambda_j r})}\times\\&\sum_{(\text{all }S)}\prod_{l\in\{S\}}\left(1-e^{-\lambda_l r(1-a^k)}\right)e^{-\lambda_l r}\times\\&\qquad\prod_{l\in\{\bar{S}\}}\left((1-2e^{-\lambda_l r}+e^{-\lambda_l r(2-a^k)})\right)\end{aligned}$$

2.3 First Passage Time

We now evaluate the probability mass function of the first passage time, i.e., the probability that a connected network at time t becomes disconnected for the first time after t at $t+k$, $k>0$. Let T be the random variable corresponding to the first passage time. For $T>k$, the following k inequalities need to be satisfied simultaneously for all $l\in[1,N-1]$.

$$Y_l^{t+1}< r, Y_l^{t+2}< r\ldots Y_l^{t+k-1}< r, Y_l^{t+k}< r$$

Using Eqn. 2.8, we can translate the above equations into

$$\begin{aligned}
Y_l^{t+1} &= aY_l^t + Z_l^t < r \\
Y_l^{t+2} &= a^2Y_l^t + aZ_l^t + Z_l^{t+1} < r \\
&\dots \\
Y_l^{t+k} &= a^kY_l^t + a^{k-1}Z_l^t \dots aZ_l^{t+k-2} + Z_l^{t+k-1} < r
\end{aligned} \tag{2.12}$$

Define $W_l^j := a^jY_l^t + a^{j-1}Z_l^t \dots + aZ_l^{t+j-2}$ for $j = 2, 3, \dots k$, and $W_l^1 := aY_l^t$. From the independence of Z_l^t, the probability of the above inequalities being simultaneously satisfied conditioned on Y_l^t, $P_{k,l}(Y_l^t)$, is given by

$$\begin{aligned}
&P_{k,l}(Y_l^t) = \\
&\int_{z_l^t=0}^{r-W_l^1} f_{Z_l^t}(z_l^t)\, dz_l^t \dots \int_{z_l^{t+k-1}=0}^{r-W_l^k} f_{Z_l^{t+k-1}}(z_l^{t+k-1})\, dz_l^{t+k-1},
\end{aligned} \tag{2.13}$$

where $f_{Z_l^m}(\cdot)$ is the density of Z_l^m and is given by $a\delta(z_l^m) + (1-a)\lambda_l e^{-\lambda_l z_l^m}$ for $m = t, t+1, \dots, t+k-1$. Note that there are a total of k integrals. Denoting the last p integrals by $I_{l,k+1-p} = I_{l,k-i}$, we claim that this has a recursive form in i.

LEMMA 2.3.1 *$I_{l,k-i}$ for $i = 0, \dots, k-1$ has the following recursive form.*

$$\begin{aligned}
&I_{l,k-i} \\
&= 1 + (1-a)C_1^{k-i}(W_l^{k-i}) + (1-a^2)C_2^{k-i}(W_l^{k-i}) \dots \\
&\quad + (1-a^i)C_i^{k-i}(W_l^{k-i}) + (1-a^{i+1})C_{i+1}^{k-i}(W_l^{k-i}) \\
&= 1 - \\
&\quad (1-a)\left(\sum_{j=1}^{i}\left(C_j^{k-(i-1)}(aW_l^{k-i})e^{-\lambda_l(r-W_l^{k-i})(1-a^j)}\right)\right) \\
&\quad -(1-a)C_1^k(W_l^{k-i}) + \sum_{j=2}^{i+1}(1-a^j)C_{j-1}^{k-(i-1)}(aW_l^{k-i}),
\end{aligned} \tag{2.14}$$

where $C_i^j(W_l^j)$ is the coefficient of $(1-a^i)$ in the expression for $I_{l,j}$, as a function of W_l^j. Also $C_1^k(W_l^k) = e^{-\lambda_l(r-W_l^k)}$ can be easily verified.

PROOF: Proof is by induction. From Eqn. 2.13, we can write

$$I_{l,k-i-1} = \int_{z_l^{t+k-i-2}=0}^{r-W_l^{k-i-1}} I_{l,k-i} \times f_{Z_l^{t+k-i-2}}(z_l^{t+k-i-2})\, dz_l^{t+k-i-2}$$

We next use the induction hypothesis, substitute for $I_{l,k-i}$ from Eqn. 2.14, and integrate each term in the summation.

$$\int_{z_l^{t+k-i-2}=0}^{r-W_l^{k-i-1}} 1 \times f_{Z_l^{t+k-i-2}}(z_l^{t+k-i-2})\, dz_l^{t+k-i-2} = \\ 1-(1-a)e^{-\lambda_l(r-W_l^{k-i-1})} = 1 - C_1^k(W_l^{k-i-1}) \tag{2.15}$$

Next, $C_j^{k-i}(W_l^{k-i})$, the coefficient of $(1-a^j)$ in I_{k-i}, is a linear combination of terms of the form $T_j^{k-i}(W_l^{k-i}) = e^{-\lambda_l(g_{k-i}(r)-a^{j-1}W_l^{k-i})}$, where $g_*(r)$ is a linear function in r. We note that

$$\begin{aligned}
&\int_{z_l^{t+k-i-2}=0}^{r-W_l^{k-i-i}} T_j^{k-i}(W_l^{k-i})(1-a^j) f_{Z_l^{t+k-i-2}}(z_l^{t+k-i-2}) dz_l^{t+k-i-2} \\
&= \int_{z_l^{t+k-i-2}=0}^{r-W_l^{k-i-1}} e^{-\lambda_l(g_{k-i}(r)-a^{j-1}W_l^{k-i})}(1-a^j) \times \\
&\qquad\qquad f_{Z_l^{t+k-i-2}}(z_l^{t+k-i-2})\, dz_l^{t+k-i-2} \\
&= e^{-\lambda_l(g_{k-i}(r)-a^j W_l^{k-i-1})}(1-a^j) \times \\
&\int_{z_l^{t+k-i-2}=0}^{r-W_l^{k-i-1}} e^{\lambda_l a^j z^{t+k-i-2}} f_{Z_l^{t+k-i-2}}(z_l^{t+k-i-2})\, dz_l^{t+k-i-2} \\
&= T_j^{k-i}(aW_l^{k-i-1})(1-a^{j+1}) - \\
&\qquad T_j^{k-i}(aW_l^{k-i-1})e^{-\lambda_l(r-W_l^{k-i-1})(1-a^j)}(1-a)
\end{aligned} \tag{2.16}$$

Since $C_j^{k-i}(W_l^{k-i})$ is a linear combination of terms of the form $T_j^{k-i}(W_l^{k-i})$, we can conclude from the above equation that

$$\begin{aligned}
&\int_{z_l^{t+k-i-2}=0}^{r-W_l^{k-i-1}} C_j^{k-i}(W_l^{k-i}) f_{Z_l^{t+k-i-2}}(z_l^{t+k-i-2})\, dz_l^{t+k-i-2} = \\
&\quad C_j^{k-i}(aW_l^{k-i-1})(1-a^{j+1}) \\
&\qquad -C_j^{k-i}(aW_l^{k-i-1})e^{-\lambda(r-W_l^{k-i-1}(1-a^j))}(1-a)
\end{aligned}$$

Hence, for all $j \in [2, i+2]$, we have

$$C_j^{k-i-1}(W_l^{k-i-1}) = C_{j-1}^{k-i}(aW_l^{k-i-1}).$$

Also, using Eqn. 2.15 and Eqn. 2.16, we find that the coefficient of $(1-a)$ in $I_{l,k-i-1}$, $C_1^{k-i-1}(W_l^{k-i-1})$ is given by

$$\begin{aligned}&C_1^{k-i-1}(W_l^{k-i-1}) = \\ &-\sum_{j=1}^{i+1} C_j^{k-i}(aW_l^{k-i-1})e^{-\lambda_l(r-W_l^{k-i-1})(1-a^j)} - C_1^k(W_l^{k-i-1}).\end{aligned}$$

Thus, we have

$$\begin{aligned}&I_{l,k-(i+1)} = \\ &1-(1-a)\sum_{j=1}^{i+1} C_j^{k-i}(aW_l^{k-i-1})e^{-\lambda_l(r-W_l^{k-i-1})(1-a^j)} \\ &\quad -(1-a)C_1^k(W_l^{k-i-1}) + \sum_{j=2}^{i+2}(1-a^j)C_{j-1}^{k-i}(aW_l^{k-i-1})\end{aligned}$$

Hence the induction holds true and the expression for $I_{l,k-i}$ is indeed given by Eqn. 2.14. The recursive equations to compute the coefficients in the expression for $I_{l,k-i}$ are of the form

$$\begin{aligned}&C_j^{k-i-1}(W_l^{k-i-1}) = C_{j-1}^{k-i}(aW_l^{k-i-1}),\ j\in[2,i+2] \\ &C_1^{k-i-1}(W_l^{k-i-1}) = \\ &-\sum_{j=1}^{i+1} C_j^{k-i}(aW_l^{k-i-1})e^{-\lambda_l(r-\ W_l^{k-i-1})(1-a^j)} - C_1^k(W_l^{k-i-1}).\end{aligned}$$

To find the boundary condition, note that for $i=0$, we have from Eqn. 2.13

$$\begin{aligned}I_{l,k} &= \int_{z_l^{t+k-1}=0}^{r-W_l^k} f_{Z_l^{t+k-1}}(z_l^{t+k-1})\, dz_l^{t+k-1} \\ &= 1-(1-a)e^{-\lambda_l(r-W_l^k)} \\ &= 1+C_1^k(W_l^k)(1-a).\end{aligned}$$

Hence, $C_1^k(W_l^k) = e^{-\lambda_l(r-W_l^k)}$ provides the boundary condition for the set of recursive equations shown above. From Eqn. 2.13, we know that the required probability $P_{k,l}(Y_l^t) =$

$I_{l,k-(k-1)} = I_{l,1}$. Substituting $i = k-1$ in Eqn. 2.14 and noting that $W_l^1 = aY_l^t$, we get

$$
\begin{aligned}
&P_{k,l}(Y_l^t) = \\
&1 - \left(\sum_{j=1}^{k-1} \left(C_j^2(a^2 Y_l^t) e^{-\lambda_l (r - aY_l^t)(1-a^j)} \right) + C_1^k(aY_l^t) \right)(1-a) \\
&\quad + \sum_{j=2}^{k} C_{j-1}^2(a^2 Y_l^t)(1-a^j)
\end{aligned}
$$

The probability derived above is to be conditioned on the constraint that $Y_l^t < r$ and let it be denoted by $P_{k,l|C}$. We thus have,

$$
P_{k,l|C} = \frac{\int_{Y_l^t=0}^{r} P_{k,l}(Y_l^t) \lambda_l e^{-\lambda_l y_l^t}\, dY_l^t}{1 - e^{-\lambda_l r}}
$$

For the first passage time of the network, $T > k$, Eqn. 2.12 has to hold true for all $l = 1, 2, \ldots, N-1$. Since Y_l^t are all independent random variables, the corresponding probability, denoted by $P_{k|C}$ is given by

$$
\mathsf{Pr}\,(T > k) = P_{k|C} = \prod_{l=1}^{N-1} P_{k,l|C}
$$

■

2.4 Random Birth-Death Model

We now consider a model in which the N node network is distributed at time $t = 0$ according to ordered nodes from an exponential distribution with mean $1/\lambda$. In this model it is the evolution of nodes that will be governed by an exponential AR(1) process, instead of the spacings. Let K be a positive integer valued random variable with $P[K = k] = p_k$, $k \geq 0$. Since our model is time homogeneous, we drop the time subscripts in the equations below. At each time instant, given the current configuration of nodes $V = \{x_1, \ldots, x_N\}$, one of the following two events happen:
1) Let E_1 be the event that a node j, $1 \leq j \leq N$, chosen at random, moves K-steps according to the an exponential AR(1) model given by $X'_{n+1} = aX_n + Z_n$, where Z_n is a product of a

Bernoulli$(1-a)$ and an independent exponential random variable with mean $1/\lambda$. Thus, if the node j located at x_j moves, then its new location will be

$$x'_j = a^K x_j + \sum_{m=1}^{K} a^{m-1} Z_m.$$

2) Let E_2 denote the event that a node l chosen at random dies and another node $j \neq l$ chosen at random throws a new node whose location is given by the above equation, that is if node l dies and node j gives birth, then the new configuration will be given by the above equation with x'_j replaced by x'_l. Death in our models can be thought of as a node switching off (or going to sleep to save power) and birth as switching on. We can also decouple birth and death events in the above case, i.e., with probability p_1 event E_1 happens, with probability p_2 only birth happens and with probability $1-p_1-p_2$ only a death happens. Note that the ordered locations change at each time step, but it poses no problems in computing the one step conditional probabilities. Let $\Theta_1(V)$ denote the one step conditional probability that the new configuration after the above evolution is connected given the current configuration V of nodes. If we denote $\Theta_1(V)$ conditioned on E_1 as $\Psi_1(V)$ and $\Theta_1(V)$ conditioned on E_2 as $\Phi_1(V)$, then $\Theta(V) = p\Psi_1(V) + (1-p)\Phi_1(V)$, where $\mathsf{Pr}\,(E_1) = p$. Given a configuration V,

$$\begin{aligned} \Psi_1(V) &= \frac{1}{N}\sum_{i=1}^{N} \Psi_1(V,i) \qquad \text{and} \\ \Phi_1(V) &= \frac{1}{N(N-1)}\sum_{l=1}^{N}\sum_{i=1, i\neq l}^{N} \Phi_1(V,l,i), \end{aligned} \tag{2.17}$$

where $\Psi_1(V,i)$ is $\Psi_1(V)$ given that node i moves and $\Phi_1(V,l,i)$ is $\Phi_1(V)$ given that node l dies and node i gives birth. Recall that for any $k \in \mathbf{N}$, $W(k) = \sum_{m=1}^{k} a^{m-1} Z_m$ has the same distribution as a product of Bernoulli$(1-a^k)$ and an independent exponential random variable with mean $1/\lambda$. If $\Theta_1(V,k)$ is $\Theta_1(V)$ conditioned on $K=k$, then $\Theta_1(V,k)$, is obtained from $\Theta_1(V,1)$ by replacing a by a^k. $\Theta_1(V) = \sum_{k=1}^{\infty} \Theta_1(V,k)P(K=k)$. We make the observation that if in E_2, when node l dies and a new node is thrown from the same location, then this is nothing but the event E_1 given that node l moves. Thus, $\Psi_1(V,i) = \Phi_1(V,i,i)$. Thus, it suffices to compute $\Phi_1(V,l,i)$ only for choices of l, i for which it is positive and conditioned on $K=1$. Suppose the current configuration has exactly three components. We consider three sub cases.

(i) If V is such that $x_2 - x_1 > r$, $x_{j+1} - x_j > r$, for some $1 < j < N-1$, and $x_{i+1} - x_i \leq r$, for $i \neq 1, j$. In this case, only $\Phi_1(V, 1, i) \neq 0$.

$$\begin{aligned}\Phi_1(V,1,i) &= \big[(1-a)e^{\lambda a x_i}\big[e^{-\lambda(x_{j+1}-r)} - e^{-\lambda(x_j+r)}\big] \\ &\quad + aI_{[x_{j+1}-r \leq a x_i \leq x_j + r]}\big].\end{aligned}$$

(ii) If V is such that $x_N - x_{N-1} > r$, $x_{j+1} - x_j > r$, for some $1 < j < N-1$, and $x_{i+1} - x_i \leq r$, for $i \neq 1, j$. In this case

$$\begin{aligned}]\Phi_1(V,N,i) &= \big[(1-a)e^{\lambda a x_i}\big[e^{-\lambda(x_{j+1}-r)} - e^{-\lambda(x_j+r)}\big] \\ &\quad + aI_{[x_{j+1}-r \leq a x_i \leq x_j + r]}\big]\end{aligned}$$

(iii) If $x_2 - x_1 > r$, $x_N - x_{N-1} > r$ and $x_{j+1} - x_j \leq r$ for all $1 < j < N-1$.

$$\begin{aligned}\Phi_1(V,1,i) &= \big[(1-a)e^{\lambda a x_i}\big[e^{-\lambda(x_N-r)} - e^{-\lambda(x_{N-1}+r)}\big] \\ &\quad + aI_{[x_N-r \leq a x_i \leq x_{N-1} + r]}\big] \\ \Phi_1(V,N,i) &= \big[(1-a)e^{\lambda a x_i}\big[e^{-\lambda(x_2-r)} - e^{-\lambda(x_1+r)}\big] \\ &\quad + aI_{[x_2-r \leq a x_i \leq x_1 + r]}\big]\end{aligned}$$

Similarly explicit formulae can be derived when V has two and one component(s). The one step conditional probability of connectivity in all the three cases is then given by Eqn. 2.17.

Chapter 3

k-Coverage

3.1 Boolean Processes

In principle, a stochastic coverage process might be thought of as any random mechanism governing the positioning and configuration of random sets in the Euclidean space. And the applications of coverage processes range the gamut such as military applications, medical applications, image processing, industrial safety, stereology, packing problems etc.

By mostly working in the continuum, we are granted access to powerful tools from stochastic geometry and avoid the tedium of treating lattice types i.e, discrete models. We concentrate on higher dimensions as the theory there possesses a genuine spatial flavor. One dimensional results are essentially geometric interpretations of classical statistical theory about spacings of order statistics.

Now, we set up the terminology for one of the well known examples of coverage processes - the random (Poisson) distribution of sets in d-dimensional Euclidean space along the lines of [4]. The resulting coverage pattern is called the Boolean Model. Formally we define it as follows:

DEFINITION 3.1.1 *Let*

$$P \equiv \{\xi_i, i \geq 1\}$$

be a stationary Poisson process of intensity λ *in* $\mathcal{R}^d$*, the points* ξ_i *being indexed in any* *systematic order. Let* $S_1, S_2, \ldots$ *be i.i.d random sets, independent of* P*. Then*

$$C \equiv \{\xi_i + S_i, i \geq 1\}$$

is a Boolean model.

A spatial or stationary Poisson process P of intensity λ , is defined by the following two conditions :

- If $N(A)$ is defined as the number of points of P in A for a Borel set $A \subset \mathcal{R}^d$, then $N(A) \sim Po(\lambda\|A\|)$ i.e,

  $$\mathsf{Pr}\left(N(A) = n\right) = [\lambda\|A\|]^n e^{-\lambda\|A\|}/n!,$$

  for every $n \in \mathbb{N}$. And given there are m points in A, the m points are i.i.d. uniformly in A.

- And for $A_1, \ldots, A_l$ disjoint Borel sets of $\mathcal{R}^d$, $N(A_1), \ldots, N(A_l)$ are independent processes.

Both the conditions can be combined together to obtain

$$\mathsf{Pr}\left(N(A_1) = n_1, \ldots, N(A_l) = n_l\right) = \prod_{i=1}^{l} [\lambda\|A_i\|]^n e^{-\lambda\|A_i\|}/n_i!,$$

where $A_1, \ldots, A_n$ are any disjoint Borel sets in $\mathcal{R}^d$

Though we have considered P as spatial Poisson process, in general P can be taken to be any stochastic point process. In that case it is known as the germ-grain model. Also the discrete or lattice analogues are described in [4].

We call the S_i shapes to distinguish them from the sets $\xi_i + S_i$ and say that the Poisson process P drives the Boolean model and the shapes S_i generate the model. We always assume that the S_i are non-empty. Let the random shapes be distributed as S. Also we say a set or point is said to be $k-$ covered by the Boolean model if it is contained in at least k of the sets $\xi_i + S_i$. Formal definitions shall follow in the next section.

Bearing in mind that a stationary Poisson process is a “random distribution of points in space”, a Boolean model represents a sequence of random sets distributed in space.

Throughout the chapter we assume that for some $t > 0$, $|x| < t$ for all $x \in S$. Hence S is contained in a bounded domain, say A.

In [4] vacancy is defined as the content of a region not covered by a coverage process. We extend this vacancy to k-vacancy which is defined as the content of the region which is covered by atmost $k-1$ sets of the coverage processes. Under this notation, in [4] basic properties of 1-vacancy, and limit-theoretic approximations such as the central limit theorem and law of large numbers have been obtained.

We derive the expectation and variance of k-vacancy. Then we proceed to obtain the central limit theorem and law of large numbers.

3.2 k-Vacancy

Let R denote a Borel subset of $\mathcal{R}^d$ and C denote the d- dimensional Boolean process as in definition 3.1.1. For $k > 0$, the k-vacancy V_k within R is the d-dimensional content of the part covered by at most $k-1$ random sets of C.

$$V_k = V_k(R) \equiv \sum_{m=0}^{k-1} \int_R \chi_m(x)dx, \tag{3.1}$$

where,

$$\chi_m(x) = \begin{cases} 1 & \text{if for exactly m points in } P,\ x \in \xi_i + S_i \\ 0 & \text{otherwise.} \end{cases}$$

For the case $k = 1$, this is same as the definition in [4]. We denote $\chi(x) = \sum_{m=0}^{k-1} \chi_m(x)$. At times we write V_k as $V_k(\lambda, S)$ to denote its dependence on the two variables.

$$\begin{aligned} \mathsf{E}(\chi_m(x)) &= \mathsf{Pr}\,(\text{exactly m points in } P,\ x \in \xi_i + S_i) \\ &= \mathsf{Pr}\,(\text{exactly m points in } P,\ xi_i \in x - S_i) \\ &= \mathsf{Pr}\,(\text{exactly m points in } P,\ \xi_i \in S_i) \end{aligned}$$

If points $\xi^1, \ldots, \xi^N$ are placed independently and uniformly in A then (conditional on N) the probability that exactly m points lie in S is ${}^N C_m \{1 - \beta/\|A\|\}^{N-m}\{\beta/\|A\|\}^m$. If N is

Poisson distributed with mean $\lambda\|A\|$, then

$$\mathsf{E}\big({}^{N}C_m\{1-\beta/\|A\|\}^{N-m}\big) = \frac{e^{-\beta\lambda}(\lambda\|A\|)^m}{m!}.$$

Therefore,

$$\mathsf{Pr}\,(x \text{ covered by exactly m points of } P) = \frac{e^{-\beta\lambda}(\lambda\beta)^m}{m!}. \tag{3.2}$$

From (3.2) and Fubini's theorem,

$$\begin{aligned}
\mathsf{E}(V_k) &= \sum_{m=0}^{k-1}\int_R \mathsf{E}(\chi_m(x))\,dx \\
&= \sum_{m=0}^{k-1}\|R\|\mathsf{Pr}\,(\text{O covered by exactly m points}) \\
&= \sum_{m=0}^{k-1}\|R\|\frac{e^{-\beta\lambda}(\lambda\beta)^m}{m!}. \qquad (3.3)
\end{aligned}$$

The variance of V_k can be calculated similarly. Indeed,

$$\mathsf{E}(\chi(x_1)\chi(x_2)) = \sum_{m,n=0}^{k-1}\mathsf{E}(\chi_m(x_1)\chi_n(x_2))$$

$$= \sum_{m,n=0}^{k-1}\mathsf{Pr}\,(x_1 \text{ is covered exactly by m points and } x_2 \text{ is covered exactly by n points})$$

Note the following :

- If ξ_i covers x_1 and x_2, then $\xi_i \in B_1 = (x_1 - S_i)\cap(x_2 - S_i)$. Also $\|B_1\| \equiv \|(x_1 - x_2 + S)\cap S\|$ in distribution.
- If ξ_i covers x_1 and not x_2, then $\xi_i \in B_2 = (x_1 - S_i)\cap(x_2 - S_i^c)$. Also $\|B_2\| \equiv \|(x_1 - x_2 + S)\cap S^c\|$ in distribution.
- If ξ_i doesn't cover x_1 but covers x_2, then $\xi_i \in B_3 = (x_1 - S_i^c)\cap(x_2 - S_i)$. Also $\|B_3\| \equiv \|(x_1 - x_2 + S^c)\cap S\|$ in distribution.

Observe that B_i's defined above are mutually disjoint sets and B_2 and B_3 are identically distributed. These observations shall be used extensively. We calculate $\mathsf{E}(\chi_m(x_1)\chi_n(x_2))$ conditioned on l points falling in B_1. Then it is the probability that $m-l$ points lie in B_2 and $n-l$ points lie in B_3, which are disjoint events as the points under consideration form a Poisson spatial process. Since B_i,$i=1,2,3$ are random sets by the argument used in calculation of (2), we get

$$\begin{aligned}
\mathsf{E}(\chi_m(x_1)\chi_n(x_2)) &= \sum_{l=0}^{m\wedge n} \frac{[\lambda\mathsf{E}(\|B_1\|)]^l}{l!} e^{-\lambda\mathsf{E}(\|B_1\|)} \\
&\quad \times \frac{[\lambda\mathsf{E}(\|B_2\|)]^{m-l}}{(m-l)!} e^{-\lambda\mathsf{E}(\|B_2\|)} \\
&\quad \times \frac{[\lambda\mathsf{E}(\|B_3\|)]^{n-l}}{(n-l)!} e^{-\lambda\mathsf{E}(\|B_3\|)} \\
&= e^{-2\lambda\beta} e^{\lambda\|B_1\|} \sum_{l=0}^{m\wedge n} \frac{[\lambda\mathsf{E}(\|B_1\|)]^l}{l!} \times \frac{[\lambda\mathsf{E}(\|B_2\|)]^{m+n-2l}}{(m-l)!(n-l)!}.
\end{aligned} \tag{3.4}$$

Hence,

$$\begin{aligned}
Cov\{\chi(x_1)\chi(x_2)\} &= \sum_{m,n=0}^{k-1} [\mathsf{E}(\chi_m(x_1)\chi_n(x_2)) - \mathsf{E}(\chi_m(x_1))\,\mathsf{E}(\chi_n(x_2))] \\
&= e^{-2\lambda\beta} \sum_{m,n=0}^{k-1} \{e^{\lambda\mathsf{E}(\|B_1\|)} \sum_{l=0}^{m\wedge n} \frac{[\lambda\mathsf{E}(\|B_1\|)]^l}{l!} \times \frac{[\lambda E\|B_2\|]^{m+n-2l}}{(m-l)!(n-l)!} \\
&\quad - \frac{[\lambda\beta]^{m+n}}{m!n!}\}.
\end{aligned} \tag{3.5}$$

And hence,

$$\mathsf{VAR}(V_k) = \int\int_{R^2} Cov\{\chi(x_1)\chi(x_2)\} dx_1 dx_2 = \int\int_{\{|x_1-x_2|<t\}} Cov\{\chi(x_1)\chi(x_2)\} dx_1 dx_2. \tag{3.6}$$

3.3 Asymptotic Properties of k-Vacancy

Let $C(\delta,\lambda)$ be the Boolean model C in which shapes are distributed as $\delta S(\delta = \delta(\lambda))$.

THEOREM 3.3.1 *Let V_k denote the vacancy arising in R from the Boolean model $C(\delta,\lambda)$ defined above. If $\delta \to 0$ as $\lambda \to \infty$ such that $\delta^d\lambda \to \rho$ where $0 \le \rho < \infty$, then as $\lambda \to \infty$*

$$\mathsf{E}(V_k) \to \|R\|\mathsf{E}(V_k(\rho, S))\,. \tag{3.7}$$

$$\mathsf{E}(|V_k - \mathsf{E}(V_k)\,|^p) \to 0 \tag{3.8}$$

for $1 \le p \le \infty$.

$$\lambda\mathsf{VAR}(V_k) \to \sigma^2(S), \tag{3.9}$$

Where

$$\begin{aligned}\sigma^2(S) \;&=\; \rho\|R\|e^{-2\rho\beta}\left[\sum_{m,n=0}^{k-1}\int_{\mathcal{R}^d}\left(e^{\rho\mathsf{E}(\|D_1\|)}\sum_{l=0}^{m\wedge n}\frac{[\rho\mathsf{E}(\|D_1\|)]^l}{l!}\times\frac{[\rho\mathsf{E}(\|D_2\|)]^{m+n-l}}{(m-l)!(n-l)!}\right.\right.\\ &\quad\left.\left. - \frac{[\rho\beta]^{m+n}}{m!n!}dy\right)\right],\end{aligned} \tag{3.10}$$

where $D_1 = (y+S)\cap S,\ D_2 = (y+S)\cap S^c$.

Proof : (3.7) follows from (3.3) directly. The proof of remaining statements rests on (3.5) and (3.6). First we shall prove $\mathsf{VAR}(V_k) \to 0$, in fact more than that. Since V_k is bounded, $\mathsf{VAR}(V_k)$ is also bounded. Therefore, by dominated convergence theorem we are required to show that for almost all x_1, x_2, $Cov\chi_m(x_1)\chi_n(x_2) \to 0$. For a fixed x_1 and $x_2 (\neq x_1)$,

$$\lambda\mathsf{E}(\|(x_1 - x_2 + \delta S)\cap\delta S\|) = \delta^d\lambda\mathsf{E}\big(\|[\delta^{-1}(x_1-x_2)+S]\cap S\|\big)\,.$$

And from the boundedness of S, $[\delta^{-1}(x_1 - x_2) + S] \cap S = \emptyset$ for $\delta < |x_1 - x_2|/t$. Since $\delta^d\lambda$ converges, for large λ we have, $\lambda\mathsf{E}(\|(x_1 - x_2 + \delta S)\cap\delta S\|) = 0$. And for the same δ, $B_2 = S$. And hence for that δ we have $Cov\chi_m(x_1)\chi_n(x_2) = 0$. Therefore $\mathsf{VAR}(V_k) \to 0$. Hence $V_k - \mathsf{E}(V_k)$ converges in probability and also in L_1. Since $0 \le \|V_k\| \le R$,by dominated convergence theorem we have (3.8).

Finally we establish (3.9). In view of (3.4), (3.5) and (3.6), it suffices to prove that $\lambda\int\int_{R^2} Cov(\chi_m(x_1)\chi_n(x_2))$ converges to the integral on the r.h.s of (3.10).

$$e^{2\delta^d\lambda\beta}\int_{R^2} Cov(\chi_m(x_1)\chi_n(x_2))dx_1dx_2 = \delta^d\int_R f_\delta(x)dx,$$

where

$$f_\delta(x) = \int_{\delta^{-1}(x-R)} \left(e^{\delta^d \lambda \|D_1\|} \sum_{l=0}^{m\wedge n} \frac{[\delta^d \lambda \mathsf{E}(\|D_1\|)]^l}{l!} \times \frac{[\delta^d \lambda \mathsf{E}(\|D_2\|)]^{m+n-2l}}{(m-l)!(n-l)!} - \frac{[\delta^d \lambda \beta]^{m+n}}{m!n!} \right) dy.$$

For $|y| > t$, we have the integrand of $f_\delta(x)$ to be 0 and hence $Sup_\delta\ f_\delta(x) < \infty$.

Now R is Riemann measurable and $\|R\| > 0$, then for almost all $x \in R$, $x - R$ contains a sphere centered on the origin, in which case $\delta^{-1}(x - R) \to \mathcal{R}^d$ as $\delta \to 0$. Hence

$$f_\delta(x) \to c_0 \equiv \int_{\mathcal{R}^d} \left(e^{\rho \mathsf{E}(\|D_1\|)} \sum_{l=0}^{m\wedge n} \frac{[\rho \mathsf{E}(\|D_1\|)]^l}{l!} \times \frac{[\rho \mathsf{E}(\|D_2\|)]^{m+n-2l}}{(m-l)!(n-l)!} - \frac{[\rho\beta]^{m+n}}{m!n!} \right) dy,$$

whence by dominated convergence,

$$e^{2\delta^d \lambda \beta} \delta^d \lambda \int \int_{R^2} Cov(\chi_m(x_1)\chi_n(x_2)) \to \rho \int_R c_0 dx = \rho c_0 \|R\|,$$

as $\lambda \to \infty$. This proves (3.9). ∎

REMARK 1: If one considers the case $k = 1$ as in [4], then it is enough to assume S with finite second moment to prove the above theorems. This is so because the covariance is of the form $e^x - 1$ and the inequality $e^x - 1 \leq xe^x$ is used in proofs of both (3.8) and (3.9) .

The above theorem is essentially a "weak law of large numbers" for vacancy within large regions. We complement it with a central limit theorem.

THEOREM 3.3.2 *We make the same assumptions as in Theorem 1. Then,*

$$\sqrt{\lambda}\{V_k - \mathsf{E}(V_k)\} \to N(0, \sigma^2),$$

in distribution where σ^2 is as defined in (3.10).

Proof : To get the pictorial representation of following construction, look at Fig. 3.1 (page 28). Let r be a large positive constant. Divide all of $\mathcal{R}^d$ into a regular lattice of d-dimensional cubes of side length $cr\delta$, and each cube separated from its nearest neighbor by a spacing strip of width $2c\delta$. Let A_1 denote the union of those cubes which are wholly within R. A_2

The union of the rectangular boxes that form the spacings and contained wholly within R. And A_3, is the intersection of R with all those cubes are spacings which are not completely within or without R. Then vacancy V_k within R may be written as,

$$V_k = V^1 + V^2 + V^3.$$

where V^i is the vacancy within the region A_i. Since R is Riemann measurable then the content of R evaluated over increasingly fine dissections, is approximable arbitrarily closely by both inner and outer sums. In the case of dissection the difference between inner and outer sums is greater than $\|A_3\|$ and so

$$\|A_3\| \to 0. \tag{3.11}$$

Furthermore the total content of all those between cubes that lie entirely within R is dominated by a constant multiple of $(1/cr\delta)c\delta = r^{-1}$. Therefore,

$$\|A_2\| \leq a.r^{-1}, \tag{3.12}$$

where a does not depend on r. We may deduce from (3.4) and also the fact that variance is less than the second moment, that vacancy V^i within region A_i satisfies

$$\begin{aligned} \mathsf{VAR}(V^i) &\leq \sum_{m,n=0}^{k-1} \int_{A_i} dx_1 \int_{\mathcal{R}^d} \mathsf{E}(\chi_m(x_1)\chi_n(x_2))\, dx_2 \\ &\leq (k-1)^2 M \|A_i\|. \end{aligned}$$

In the last inequality we have used the fact that the second moment is bounded. In view of (3.11) and (3.12), this implies

$$\lim_{\lambda\to\infty} \lambda \mathsf{VAR}(V^3) = 0, \tag{3.13}$$

and

$$\lim_{r\to\infty} \limsup_{\lambda\to\infty} \lambda \mathsf{VAR}(V^2) = 0. \tag{3.14}$$

Consequently our goal of central limit theorem will be achieved if we prove that

$$\{V^1 - \mathsf{E}(V^1)\}/(\mathsf{VAR}(V^1))^{1/2} \xrightarrow{d} N(0,1) \qquad \text{and} \qquad \lim_{r\to\infty} \limsup_{\lambda\to\infty} |\lambda \mathsf{VAR}(V^1) - \sigma^2| = 0. \tag{3.15}$$

Let $n = n(\lambda)$ denote the number of small cubes of side-length $cr\delta$ that make up region A_1, and let $\mathcal{D}_i$ denote the ith of these cubes for $1 \leq i \leq n$. Write U_i for the contribution to V^1

from $\mathcal{D}_i$, then $V^1 = \sum_i U_i$. Since each random shape is contained within a sphere of radius $c\delta$ and the cubes $\mathcal{D}_i$ are distant at least $2c\delta$ apart, then no random shape can intersect more than one cube. Therefore the variables U_i are independently distributed, given λ. Hence

$$\begin{aligned}
\mathsf{VAR}\big(V^1\big) &= \sum_i \mathsf{VAR}(U_i) = n\mathsf{VAR}(U_1) \\
&= ne^{-2\delta^d\lambda\beta} \sum_{m,n=0}^{k-1} \int\int_{\mathcal{D}_1^2} \Big[e^{\lambda\mathsf{E}(\|(x_1-x_2+\delta S)\cap\delta S\|)} \sum_{l=0}^{m\wedge n} \frac{[\lambda\mathsf{E}(\|(x_1-x_2+\delta S)\cap\delta S\|)]^l}{l!} \times \\
&\quad \frac{[\lambda\mathsf{E}(\|(x_1-x_2+\delta S)\cap\delta S^c\|)]^{m+n-2l}}{(m-l)!(n-l)!} - \frac{[\delta^d\lambda\beta]^{m+n}}{m!n!}\Big] dx_1dx_2 \\
&\sim n\delta^{2d}e^{-2\rho\beta} \sum_{m,n=0}^{k-1} \int\int_{\mathcal{D}^2} [e^{\rho\mathsf{E}(\|B_1\|)} \sum_{l=0}^{m\wedge n} \frac{[\rho\mathsf{E}(\|B_1\|)]^l}{l!} \times \frac{[\rho\mathsf{E}(\|B_2\|)]^{m+n-2l}}{(m-l)!(n-l)!} \\
&\quad - \frac{[\rho\beta]^{m+n}}{m!n!}]dx_1dx_2,
\end{aligned}$$

where $\mathcal{D}$ is any d-dimensional cube of side length cr with the same orientation as $\mathcal{D}_1$. Also,

$$\mathsf{E}(|U_i - \mathsf{E}(U_i)\,|)^3 \leq \|\mathcal{D}_i\|\mathsf{VAR}(U_i) = (cr\delta)^d\mathsf{VAR}(U_i)\,,$$

and so, since $n = O(\lambda)$ as $\lambda \to \infty$,

$$\begin{aligned}
\{\sum_i \mathsf{E}(|U_i - \mathsf{E}(U_i)\,|)^3\}/\{\sum_i \mathsf{VAR}(U_i)\}^{3/2} &\leq (cr\delta)^d/\{\sum_i \mathsf{VAR}(U_i)\}^{1/2} \\
&= O(\lambda^{-1}\lambda^{1/2}) \to 0.
\end{aligned}$$

And first part of (3.15) follows from this estimate and Lyapunov's Central Limit theorem. To prove Second part of (3.15) it suffices to prove

$$\lim_{r\to\infty} \limsup_{\lambda\to\infty} |\frac{\mathsf{VAR}(V) - \mathsf{VAR}(V^1)}{\mathsf{VAR}(V)}| = 0. \tag{3.16}$$

Now,

$$\begin{aligned}
|\mathsf{VAR}(V) - \mathsf{VAR}\big(V^1\big)\,| &= |\mathsf{E}\big(\{V - V^1 - E(V - V^1)\}\{V + V^1 - E(V + V^1)\}\big)\,| \\
&= |\mathsf{E}\,\big(\{V^2 + V^3 - E(V^2 + V^3)\} \\
&\quad \{2V - V^2 - V^3 - E(2V - V^2 - V^3)\}\big)\,| \\
&\leq 4\{\mathsf{VAR}\big(V^2\big) + \mathsf{VAR}\big(V^3\big)\}^{1/2} \\
&\quad \{\mathsf{VAR}(V) + \mathsf{VAR}\big(V^2\big) + \mathsf{VAR}\big(V^3\big)\}^{1/2}
\end{aligned} \tag{3.17}$$

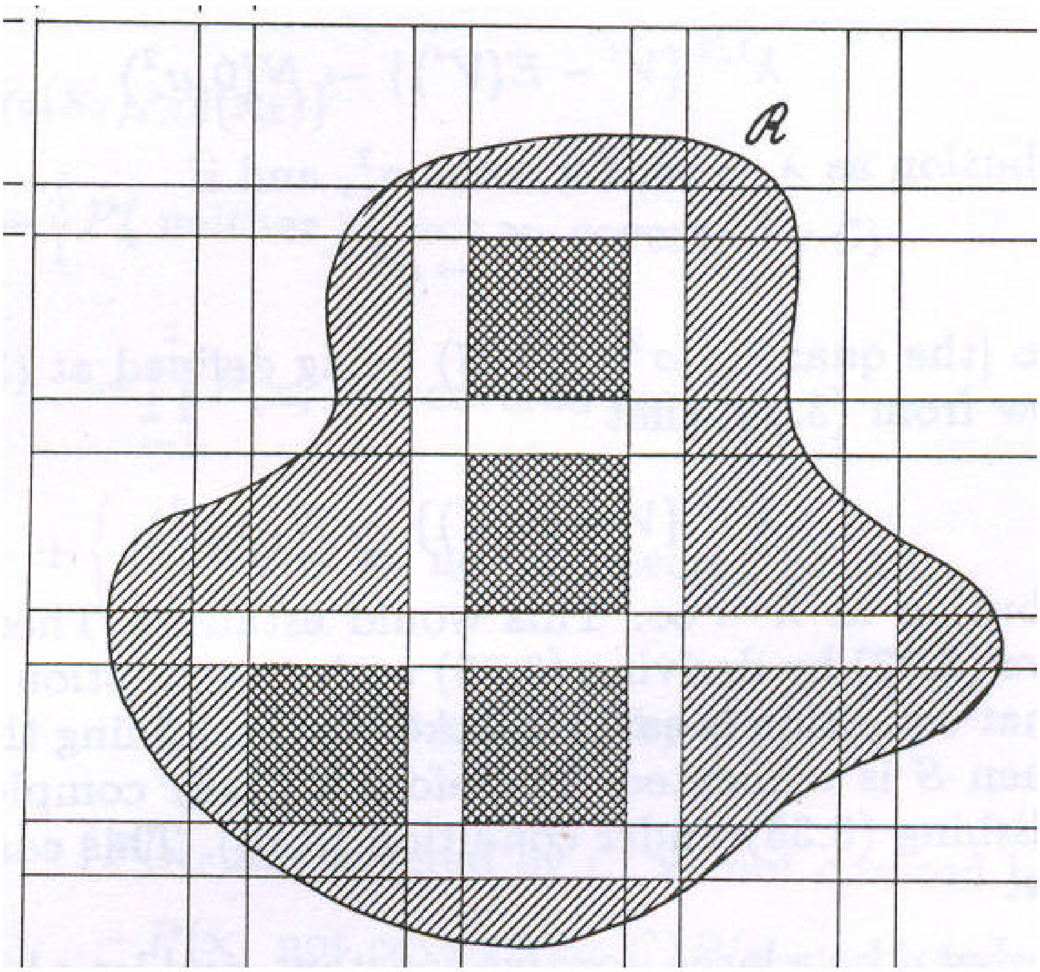

Fig 3.1: Definition of A_1, A_2 and A_3. The "cubes" D_i are cross-hatched; their union is A_1. The shaded region is A_3, and the blank region within R is A_2. (A definition of D_i is given following (3.15))

using the Cauchy-Schwarz inequality. (3.16) follows from (3.10), (3.13), (3.14) and (3.17). ■

Chapter 4

Target Tracking in Sensor Networks

4.1 Introduction

Sensor networks are formed from a large number of randomly deployed sensor nodes. These sensor nodes sense a phenomenon, possibly process the collected sensing data in a collaborative manner and route the results to an end user. The phenomenon that is being sensed could be a localized event, e.g., an acoustic point source, or it could be a spatial phenomenon spread throughout the operational area of the sensor network, e.g., target tracking and atmospheric monitoring. Each sensor node will have a footprint over which it can perform the measurements and a random sensor network may not sense the entire operational area. Accuracy of processing depends on the sensing granularity of the network. This is best illustrated in a tracking application by the concept of *trackability* which is the focus of this chapter.

Consider a sensor network for target tracking. A typical trajectory estimation algorithm for tracking of a moving target would work as follows. Whenever the target can be sensed by a sufficient number of sensors, point estimates of the location are obtained. These estimates are then appropriately filtered to estimate the trajectory for the times when the target is not sufficiently sensed. The quality of the trajectory estimates will depend on the fraction of the trajectory that is being sensed by a specified minimum number of sensors, which therefore is a measure of the tracking ability of the sensor network. If the complete trajectory is not being sufficiently sensed, then an immediate measure of trackability is the 'length to first sense,' i.e., the distance traveled by the target in the operational area before it is sensed. This can also be interpreted as the time to detect an intruder in an intrusion detection network. Another measure of trackability would be the length of a continuous segment that is tracked by a given number (or a given minimum number) of sensors, a measure of the 'sensing continuity'. Clearly, the above properties are indicators of the accuracy with which the network can track the target i.e., the *trackability* of the network. In this chapter we analyze a random sensor network for trackability measures. In addition to obtaining the above measures, after formally defining them, we also obtain trackability measures that have been defined in the literature, like 'breach' and 'support' [6].

We use the following sensing model. The sensors are deployed according to a spatial Poisson process. The sensing area of each sensor is a circle of random radius and a point is considered sensed if and only if it is in the sensing area of at least k sensors. Thus the coverage of the operational area by the sensors is a two dimensional Boolean process. We analyze the properties of the coverage process on an arbitrary straight line path. Thus our interest is in the statistical properties of the coverage of a one-dimensional path induced by a two-dimensional coverage process of the sensors. The trackability measures are essentially the coverage statistics of this one-dimensional process.

The area coverage properties have been extensively studied in the literature, most notably in [4]. The properties of the induced one-dimensional process seems to have not received the same attention and we develop a method to analyze such a process. To the best of our knowledge this is the first such analysis. The two are clearly intimately related because non-trivial coverage of the two dimensional region will be required to obtain non-trivial coverage of one dimensional paths. However the nature of the relationship is not clear and we explore it in this chapter. We will obtain asymptotic results for the one dimensional path process under the same limiting regime as that required for obtaining non-trivial coverage results for a two dimensional area process.

4.2 Performance Measures and System Model

Let Ω be the operational area of the sensor network and let $\{s_i\}$ be the set of sensors with sensor s_i located at $X_i \in \Omega$. The following two measures of the 'goodness' of deployment with respect to sensing a path are defined in [6]. For a given deployment and a path $L \subset \Omega$, the *breach* of L, $Br(L)$, is defined as

$$\min_{i} \min_{x \in L} \quad ||X_i - x||.$$

and the *support* for the path L, $Su(L)$, is defined as

$$Su(L) = \max_{x \in L} \quad ||x - X_{i*}||.$$

Here X_{i*} is the distance of the closest sensor to path L and the norms above are the Euclidean norms. Observe that these measures are independent of the sensing radius. We will obtain the mean and variance of breach and support in Section 4.6.

To develop other trackability measures we first define the sensing process. For every (s_i, x), $x \in \Omega$, a *sensing* function, $\phi(s_i, x)$, that captures the ability of sensor s_i to sense a target at point x. Note that $\phi(s_i, x)$ could be a random variable. This leads us to define a *sensor intensity* function

$$\psi(x, \theta, \phi) = \begin{cases} 1 & \text{if } \mathcal{V}\left(\phi(s_1, x), \phi(s_2, x), \ldots\right) \geq \theta \\ 0 & \text{otherwise.} \end{cases}$$

Here $\mathcal{V}$ is an operator and θ is some constant. $\psi(x, \theta, \phi)$ captures the summary effect of all the sensors at point x and we consider a point to be sensed only when $\psi(x, \theta, \phi)$ is 1.

$\phi(\cdot)$ and $\psi(\cdot)$ defined above lead us to the next measure of trackability that we consider in this chapter—*exposure* of a path L.

Definition 4.2.1 *Exposure,* $\mathsf{X}_\psi(L)$*, of a path* L *in* Ω *is the*

$$\mathsf{X}_\psi(L) = \frac{\int_{x \in L} \psi(x, \theta, \phi) \ dx}{|L|},$$

where $|L|$ *denotes the length of the path* L*. This is essentially the same as that defined in [7] except that we also normalize it to the length of the path.*

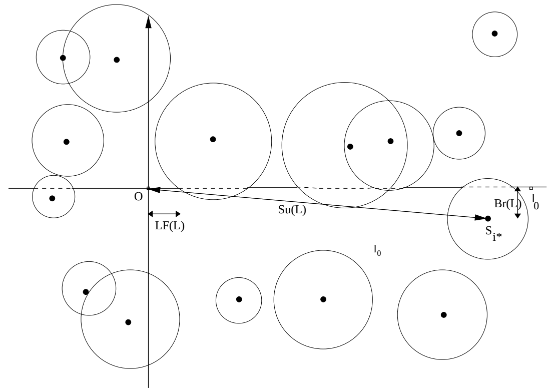

Figure 4.1: Figure shows an instance of a sensor network that performs thresholded sensing. The dots represents the sensors and the circle is its sensing area. Path L is the segment $[0, l_0]$. The dotted parts of L are the clumps on it and the thick parts are its holes. $LF(L)$ is the length to first $1-$sense, S_{i*} the closest sensor to L, $Br(L)$ its breach and $Su(L)$ its support.

In this chapter we will primarily discuss thresholded sensing where we assume that sensor s_i has a random *sensing radius* R_i within which it can sense perfectly and beyond which it cannot sense, i.e., if the location of the s_i is X_i, then

$$\phi_T(s_i, x) \quad = \quad \begin{cases} 1 & \text{if } \|x - X_i\| \leq R_i \\ 0 & \text{otherwise} \end{cases}$$

The subscript T refers to thresholded sensing. Further we can specialize thresholded sensing into *k-thresholded sensing* where we define

$$\psi_T(x, k) = \begin{cases} 1 & \text{if } \sum_i \phi_T(s_i, x) \geq k \\ 0 & \text{otherwise} \end{cases}$$

Here, point x is sensed only if it is in the sensing range of at least k sensors. An example of the use of such a sensor intensity function is in position localization that requires range estimates from at least three sensors.

Observe that $\psi_T(x, k)$, $x \in L$, is a random point process on L. The measures of trackability which we define next are essentially statistics of this random process. Thus, exposure is the fraction of the path that will be sensed. A measure of sensing continuity is clump that is defined as follows for this model.

DEFINITION 4.2.2 *For $k-$thresholded sensing, a* clump *on a path L is a contiguous segment of L for which $\psi_T(x,k) = 1$. Any segment of L between two consecutive clumps is a* hole.

The length to first sense can be defined as follows.

DEFINITION 4.2.3 *For $k-$thresholded sensing, we define the length-to-first-sense for a path L, $LF(L)$, as the distance to the first point on L where $\psi_T(x,k) = 1$. We will say that $LF(L) = l_0$ if the L is not sensed, l_0 being the length of L.*

When $\phi(s_i, x)$ is a continuous function, the sensing process is said to be *non-thresholded*. The trackability measures for the non-thresholded sensing can be lower bounded by the measures for an appropriately defined thresholded sensing case. We discuss this connection in detail in Section 4.7.

We let Ω be $\Re^2$ and $\{X_i\}$, the set of sensor locations, form a spatial Poisson process in $\Re^2$ of density λ. An excellent discussion on the physical interpretations of this model is available in [4]. Further, the Poisson process has been extensively used to model the sensor locations, e.g., [8, 19]. $\{R_i\}_{\{i>0\}}$ is assumed to be a sequence of positive i.i.d. random variables whose density has finite support. Without loss of generality, we assume the support to be $[0, 1]$. Let $f_{R_i}(r)$ denote the density of R_i and let $\beta := \mathsf{E}(R_i)$. The coverage process of these sensors deployed as above is a special case of the two dimensional Boolean process or the 'germ-grain' process defined in 3.1

Clearly, $\psi_T(x, k)$, $x \in \Re^2$, is a two dimensional Boolean process, the statistical properties of which are well studied as coverage processes, e.g., [4, 19, 20]. However, as we have mentioned earlier, our interest in this chapter is to study the properties of $\psi_T(x, k)$, $x \in L$ where L is an arbitrary straight line path in Ω. These properties depend on the statistics of one-dimensional sets embedded in a two-dimensional space.

We mention here that although much of the chapter is on sensor networks in $\Re^2$, extensions of some of the results to finite Ω are discussed in Section 4.7.

Prior work on trackability is primarily on intruder detection and on algorithmic studies. A notable exception is [8] where the notion of detectability, the probability that an object on a path L is detected, is discussed and some asymptotic results are given. Algorithmic results have been described in [6,7,21] to identify the best and worst sensed paths in a network when the sensor locations and the sensing radii are known. Statistical results via simulation are also presented in these papers. [9] studies the same properties for the case of a network with a single sensor. [10] also obtains the probability of detecting a target and uses this to develop a sequential deployment strategy to meet a QoS defined by the false alarm probability. Note that detectability of [8] is the same as exposure of [10]. There have also been papers that study connectivity and coverage together. [22] proves that if the radio-range is at least twice the sensing range, complete coverage implies connectivity. [22, 23] propose algorithms to schedule sleep intervals in large scale networks while meeting the required degree of coverage and connectivity requirements.

4.3 Sensing Process on a Straight Line Path

Let $L \in \Omega$ be an arbitrary straight line path of finite length. Let $\mathcal{L}$ be the line obtained by extending L in both directions. Since the Boolean process is shift invariant, without loss of generality we can take $\mathcal{L}$ to be the $X-$axis of the co-ordinate axes. Since R_i has a support of $[0, 1]$, only sensors within a perpendicular distance of 1 from $\mathcal{L}$ may sense any part of L and are of interest to us.

Construct a point process on $\mathcal{L}$ as follows. Mark all sensors that track some part of $\mathcal{L}$, i.e., mark a sensor if and only if its perpendicular distance to $\mathcal{L}$ is less than its sensing radius R_i. Project all the marked sensors onto $\mathcal{L}$ along the perpendicular to it. Denote the resulting point process on $\mathcal{L}$ by $\bar{F}$.

Lemma 4.3.1 *$\bar{F}$ is a Poisson arrival process on $\mathcal{L}$ with rate $\bar{\lambda} = 2\lambda\beta$.*

Proof:

We prove the lemma by showing that the probability of an arrival in any differential length dl of $\mathcal{L}$ is $\bar{\lambda}dl$ and that the arrivals have the independent increment property.

There is an arrival of $\bar{F}$ in $[l - dl/2, l + dl/2]$ if there is a marked sensor in the differential strip, dP, of thickness dl. Since the sensing radius has support in $[0, 1]$, the length of the strip over which a marked sensor could be present is within 1 unit on either side of $\mathcal{L}$. We restrict dP to this range. This strip is centered at $l \in \mathcal{L}$. This is shown in Fig. 4.2.

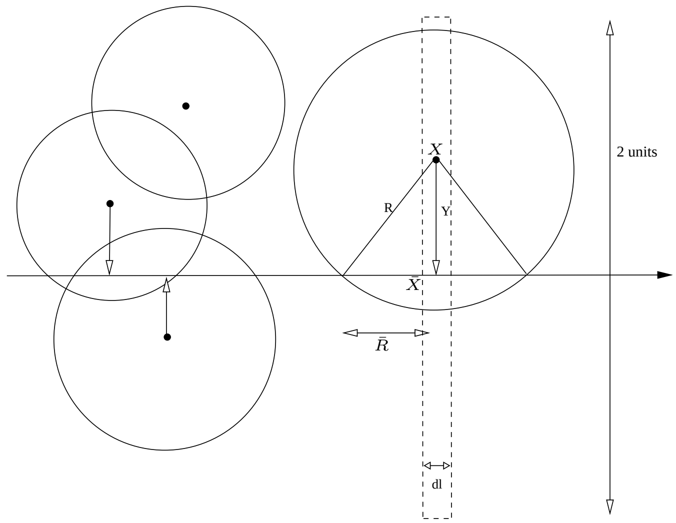

Figure 4.2: Figure shows the projection of a sensor onto $\mathcal{L}$. The dots represent the sensors and the circles their sensing area. The region enclosed by vertical dotted lines is the differential region dl.

A sensor being present in dP and it being marked are independent events. Thus the probability that there is a marked sensor in dP is the product of the probability of there being a sensor in dP, approximately $(\lambda 2dl)$, and the probability that this is marked. We obtain this latter probability next.

If there is a sensor (say s with sensing radius R) in dP, then from the Poisson distribution of the sensors its location is uniformly distributed in dP. This implies that the perpendicular distance of the sensor in dP to $\mathcal{L}$, say Y, will be uniformly distributed in $[0,1]$. For s to be marked, its R must be greater than Y. Therefore

$$\begin{aligned} \mathsf{Pr}\left(s \text{ is marked}\right) &= \int_0^1 \Pr(Y \leq r) f_R(r)\ dr \\ &= \int_0^1 r f_R(r)\ dr \ = \ \beta \end{aligned}$$

and

$$\mathsf{Pr}\left(\text{Arrival in}[l, l+dl]\right) \simeq 2\lambda\beta\ dl.$$

To prove the independent increment property, consider two non overlapping segments on $\mathcal{L}$, L_1 and L_2. Arrivals of $\bar{F}$ in L_1 and L_2 are decided by the presence of marked sensors in the rectangular regions B_1 and B_2 of height 2 and width equal to the width of L_1 and L_2

and centered at L_1 and L_2. Clearly B_1 and B_2 are non overlapping and by the independent increment property of the Poisson process of the sensor deployment, the point process $\bar{F}$ has the independent increment property. ∎

Consider an arbitrary marked sensor s located at X and having sensing radius R. Let Y be the perpendicular distance of s to $\mathcal{L}$ and $\bar{X}$ be the projection of X on $\mathcal{L}$. Note that both R, Y are random variables and $\bar{X} \in \bar{F}$. Recall that for thresholded sensing s will sense all points that are within a distance of R from it. This means that on $\mathcal{L}$, s will sense the segment $[\bar{X} - \bar{R}, \bar{X} + \bar{R}]$ where $\bar{R} = \sqrt{R^2 - Y^2}$. See Figure 4.2

LEMMA 4.3.2 *$\bar{R}$ is independent of $\bar{X}$ and its density, $f_{\bar{R}}(\bar{r})$, is*

$$f_{\bar{R}}(\bar{r}) = \begin{cases} \frac{\bar{r}}{\beta} \int_{\bar{r}}^{1} \frac{f_R(r)}{\sqrt{r^2 - \bar{r}^2}} \, dr & \text{for } 0 \le \bar{r} \le 1 \\ 0 & \text{otherwise} \end{cases}$$

PROOF:

Since sensor s is marked, $R \geq Y$. As discussed above, the segment on $\mathcal{L}$ sensed by s is $[\bar{X} - \bar{R}, \bar{X} + \bar{R}]$ where $\bar{R} = \sqrt{R^2 - Y^2}$. Since the sensor nodes are distributed as a homogeneous Poisson process, Y is independent of $\bar{X}$. Further, R is independent of X and it follows that $\bar{R}$ is independent of $\bar{X}$. By a simple transformation of random variables the distribution function of $\bar{R}$, $F_{\bar{R}}(\bar{r})$, can be written as

$$F_{\bar{R}}(\bar{r}) = \int_{r,y:\sqrt{r^2 - y^2} \leq \bar{r}} f_{\{R,Y|R \geq Y\}}(r, y) \, dr \, dy \tag{4.1}$$

Here $f_{R,Y}(\cdot, \cdot)$ is the joint density of R and Y. The joint density conditioned on the event that the sensor is marked can be written as

$$f_{\{R,Y|R \geq Y\}}(r, y) = \frac{f_{R,Y}(r, y)}{\Pr(R \geq Y)}$$

Recall from the proof of Lemma 4.3.1 that the probability that a sensor within a distance of 1 from $\mathcal{L}$ is marked is β. Further, as discussed before, Y is uniformly distributed in $[0, 1]$ and is independent of R. Therefore

$$f_{(R,Y|R \geq Y)}(r, y) = \frac{f_R(r)}{\beta}$$

Substituting for $f_{(R,Y|R\geq Y)}$ in Eqn. 4.1 and then differentiating with respect to $\bar{r}$ we get the density function of $\bar{R}$ as

$$f_{\bar{R}}(\bar{r}) = \begin{cases} \frac{\bar{r}}{\beta}\int_{\bar{r}}^{1} \frac{f_R(r)}{\sqrt{r^2-\bar{r}^2}}\,dr & \text{for } 0 \leq \bar{r} \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

■

The 'regions' on $\mathcal{L}$ that are sensed are the collection of segments $\{\bar{X}_i + \bar{I}_i\}$ where $\{\bar{X}_i\}$ is a Poisson process, I_i is the random interval $[-\bar{R}_i, \bar{R}_i]$ and $\bar{R}_i$s are i.i.d random variables. Therefore, the sensing process on $\mathcal{L}$ is a one-dimensional Boolean process. This means that the trackability of any straight line path can be studied as the coverage of a straight line of equal length by an appropriately defined one-dimensional Boolean process. It is easy to see that the latter is just an M/G/∞ queue where the projected sensors are akin to the customer arrivals and the sensed segment of the path is the corresponding service time. There is one difference though. For the one dimensional Boolean process described above, the centers of the sensing intervals are derived from a Poisson process whereas in the M/G/∞ queue, the left endpoints of the service period form a Poisson process. Fortunately, as seen from the following lemma of [4], there is a statistical equivalence between the two processes.

LEMMA 4.3.3 *Consider a one-dimensional Boolean process* $\{X_i + C_i\}$ *where* $\{X_i\}$ *is a Poisson process,* C_i *is the random interval* $[-T_i, T_i]$ *and the* T_i*s are i.i.d positive random variables. Then* $\{X_i + C_i\}$ *has the same laws as the one-dimensional Boolean process* $\{X_i + C_i'\}$ *where* C_i' *is the random interval* $[0, 2T_i]$.

The above discussion now leads us to state the key theorem of this chapter.

THEOREM 4.3.4 *For thresholded sensing, the projected point process and the collection of sensed segments form a one-dimensional Boolean process with laws identical to the one-dimensional Boolean process* $\{\bar{X}_i + \bar{C}_i\}$ *where* $\{\bar{X}_i\}$ *is a Poisson point process of density* $\bar{\lambda} = 2\lambda\beta$, $\bar{C}_i$ *is the random interval* $[0, 2\bar{R}_i]$ *and the* $\bar{R}_i$*s are i.i.d random variables with density as in Eqn. 4.2.*

From the M/G/∞ analogy, the theorem also says that the sensing process on $\mathcal{L}$ is statistically equivalent to an M/G/∞ queue with arrival rate $\bar{\lambda} = 2\lambda\beta$, and service time density given by

$$g(x) = \begin{cases} \frac{x}{4\beta} \int_{\frac{x}{2}}^{1} \frac{f_R(r)}{\sqrt{r^2 - \frac{x^2}{4}}} \, dr & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

It can be shown that $\mathsf{E}(\bar{R}) = \frac{\pi \mathsf{E}(R^2)}{4\mathsf{E}(R)}$. For the special case when the sensing radii are fixed at 1, $f_{\bar{R}}(r) = \frac{\bar{r}}{\sqrt{1^2 - \bar{r}^2}}$ for $0 \le \bar{r} \le 1$ and 0 outside. Further, $\beta = 1$ and $\mathsf{E}(\bar{R}) = \frac{\pi}{4}$

4.4 Exposure—Fraction k-Sensed

From the previous section, the event that a fraction α of the line segment is sensed by exactly k sensors corresponds to the event that in an M/G/∞ queue, for a fraction α of an observation period of duration l_0, there are exactly k customers in the system. From the ergodicity of the M/G/∞ queue, as $l_0 \to \infty$, this probability has a Poisson distribution with mean $\mu := \bar{\lambda} 2\mathsf{E}(\bar{R}) = \pi \lambda \mathsf{E}(R^2)$. Therefore, for a straight line path L of length l_0, as $l_0 \to \infty$, the limiting fraction of a path that will be $k-$sensed is $\sum_{i=k}^{\infty} \frac{\mu^i e^{-\mu}}{i!}$.

We now obtain asymptotic results for finite length paths with increasing density of sensor nodes and decreasing sensing radii. The sensing radii are scaled by δ, i.e., the sensing radii are distributed as δR, such that $\delta^2 \lambda \to \rho$, $0 \le \rho < \infty$, as $\lambda \to \infty$. This scaling of the sensing radii allows us to derive sensing properties in the large density limit. With this scaling, the sensing statistics of the finite length segment L will be the same as the limiting statistics of L as $l_0 \to \infty$ in a network where $\delta = 1$ and the sensor density is fixed at ρ. This follows from the discussion on the scaling properties of Boolean processes in [4]. From this we can state the following strong law.

THEOREM 4.4.1 *Let $\bar{\rho} = \pi \rho \mathsf{E}(R^2)$. Let $\delta \to 0$ and $\lambda \to \infty$ such that $\delta^2 \lambda \to \rho$, $0 \le \rho < \infty$. Then, with probability* 1,

$$\alpha_k := \mathsf{X}_{\psi_T(x,k)(L)} \to \sum_{i=k}^{\infty} \frac{\bar{\rho}^i e^{-\bar{\rho}}}{i!}. \tag{4.3}$$

■

From Theorem 3.3.1 the limiting value of α_k for any path is same as the fraction of any finite area that is almost surely k-covered in the same limiting regime! While this seems reasonable it is not obvious that this should be the case.

We now consider the critical ρ that is required to achieve a specified α_k. Eqn. 4.3 can be solved to obtain ρ_c, the minimum ρ required to asymptotically sense at least a fraction α $(0 < \alpha \leq 1)$ of L ρ_c will be given by

$$\rho_c = \frac{\bar{\rho}_c}{\pi \mathsf{E}(R^2)}$$

where $\bar{\rho}_c$ is the solution to Eqn. 4.3. Eqn. 4.3 can be explicitly solved to obtain a closed form for ρ_c only for $k = 1$, for which

$$\rho_c = \frac{-\ln(1-\alpha)}{\pi \mathsf{E}(R^2)}.$$

Although the critical ρ derived above is an asymptotic result, the minimum sensor density required to sense at least a fraction α of L in a finite network can be approximated by $\frac{\rho_c}{\delta^2}$. We now characterize this approximation for $k = 1$. From Theorem 4.3.4, $\bar{\lambda} = 2\beta\lambda$. Therefore, scaling R by δ such that $\lambda^2\delta \to \rho$ as $\lambda \to \infty$ and $\delta \to 0$ implies that $\delta\bar{\lambda} \to 2\beta\rho$. This means that Theorem 5.4.4 from Chapter 3 that gives a central limit theorem for the length of the path that is not 1-sensed, $\mathsf{V}_{\psi_T(x,1k}(L)$, is applicable and we can adapt it to obtain a central limit theorem for $\mathsf{X}_{\psi_T(x,k)}(L)$ by observing that $\mathsf{X}_{\psi_T(x,k)}(L) = 1 - \frac{\mathsf{V}_{\psi_T(x,k)}(L)}{l_0}$. We thus have

THEOREM 4.4.2 *If $\delta \to 0$ as $\lambda \to \infty$ such that $\delta^2\lambda \to \frac{-\ln(1-\alpha_k)}{\pi \mathsf{E}(R^2)}$ then*

$$\sqrt{2\beta\delta\lambda}\left(\mathsf{X}_{\psi_T(x,k)}(L) - \alpha_k\right) \quad \to \quad N(0, \sigma^2),$$

where

$$\begin{aligned}\sigma^2(S) \quad &= \quad \frac{2\bar{\rho}}{(1-\alpha_k)^2\ell_0}\left[\sum_{m,n=0}^{k-1}\int_0^1\left(e^{\bar{\rho}\int_x^1(1-F_{\bar{R}}(y)dy)}\sum_{l=0}^{m\wedge n}\frac{[\bar{\rho}(\int_x^\infty(1-F_{\bar{R}}(y)dy))]^l}{l!}\right.\right.\\ &\quad \left.\left.\times\frac{[\bar{\rho}(\int_0^x(1-F_{\bar{R}}(y)dy))]^{m+n-l}}{(m-l)!(n-l)!} - \frac{[\bar{\rho}\alpha]^{m+n}}{m!n!}dy\right)\right]. \qquad (4.4)\end{aligned}$$

where $\bar{\rho} = \frac{-2\beta ln(1-\alpha)}{\pi \mathsf{E}(R^2)}$. and $F_{\bar{R}}(\cdot)$ is the distribution function of $\bar{R}$ from Theorem 4.3.4. ■

For the proof note that $\delta\bar{\lambda} \to \bar{\rho}$. Also the terms in 3.10 of Chapter 3 simplify as

$$\mathsf{E}(\|D_1\|) = \int_x^\infty (1 - F_{\bar{R}}(y)dy),$$

and

$$\mathsf{E}(\|D_2\|) = \int_0^x (1 - F_{\bar{R}}(y)dy).$$

From Theorem 3.5 of [4] note that the asymptotic variance of the fraction of a path covered and the fraction of the area covered are not the same.

The above are asymptotic results and provide useful insights into the behavior of high density networks and/or large operational areas. For finite λ, we can obtain the expectation and variance of $\mathsf{X}_{psi_T(x,k)}(L))$ by a simple application of the results of Section 3.2(equations 3.3 and 3.6) as

$$\begin{aligned}
\mathsf{E}\left(\mathsf{X}_{\psi_T(x,k)}\right)(L) &= 1 - \sum_{i=k}^{\infty} [\lambda\pi\mathsf{E}\left(R^2\right)]^i e^{\lambda\pi\mathsf{E}\left(R^2\right)}/i! \\
\mathsf{VAR}\left(\mathsf{X}_{\psi_T(x,k)}\right)(L) &= e^{-2\pi\mathsf{E}\left(R^2\right)} \left[\sum_{m,n=0}^{k-1} \int_0^1 \left(e^{2\beta\lambda\int_x^\infty (1-F_{\bar{R}}(y)dy)} \sum_{l=0}^{m\wedge n} \frac{[2\beta\lambda(\int_x^\infty (1-F_{\bar{R}}(y)dy))]^l}{l!} \right.\right. \\
&\quad \left.\left. \times \frac{[2\beta\lambda(\int_0^x (1-F_{\bar{R}}(y)dy))]^{m+n-l}}{(m-l)!(n-l)!} - \frac{\pi\mathsf{E}(R^2)}{m!n!} dy \right)\right].
\end{aligned}$$

4.5 1−Sensing: Length to First k-Sense and Sensing Continuity

Since the Boolean process is shift invariant, without loss of generality, L can be taken to be the segment $[0, l_0]$. Also, in the following we consider networks with finite λ.

Length to First k-Sense

Let the regions $x < 0$, $0 \le x < l_0$ and $l_0 \le x$ in $\Re^2$ be denoted by W_1, W_2 and W_3 respectively. Define the regions $W_i(d)$ $i = 1, 2, 3$ as follows. $W_1(d)$ is the rectangle with $(0, d)$ and $(l_0, -d)$ as the opposite corners. $W_2(d)$, $W_3(d)$ are the semicircular regions of radius d with centers at $(0, 0)$ and $(0, l_0)$ respectively. See Fig. 4.3.

Let $E_u := E_1 \cap E_2 \cap E_3$, where E_i, $i = 1, 2, 3$, is the event that no part of L is 1−sensed by any sensor in W_i. Let E_d be the complement of E_u, i.e., the event that L is 1-sensed. Since the W_is are non-overlapping, the E_is are independent and $\mathsf{Pr}(E_u)$ is the product of the probabilities of the E_i. We calculate these probabilities next.

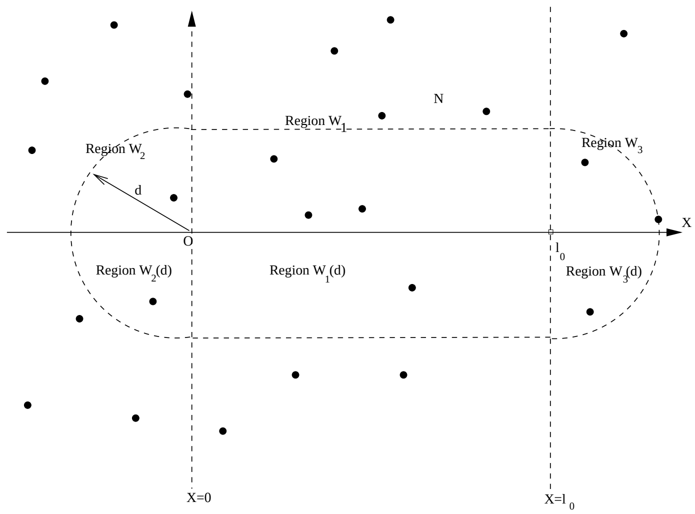

Figure 4.3: The dotted lines mark the boundary of the regions $W_1(d)$, $W_2(d)$ and $W_3(d)$. The two line $x = 0$ and $x = l_0$ also mark the boundary of W_1, W_2 and W_3.

Since the sensing radius has support in $[0, 1]$, only sensors in $W_i(1)$ can sense any part of L. Further, since the sensor locations form a spatial Poisson process of density λ, the number of sensors in $W_1(1)$, N_1, will be a Poisson random variable with mean $2l_0\lambda$ and from the proof of Lemma 4.3.1, the probability that any of these sensors will sense L is β. Therefore

$$\mathsf{Pr}\,(E_1) = \mathsf{E}\big((1-\beta)^{N_1}\big) = e^{-2l_0\lambda\beta}$$

From symmetry, the probabilities of E_2 and E_3 are equal and we evaluate the probability of E_2. Given that a sensor is in $W_2(1)$, its location is uniformly distributed in that region. Hence the probability that this sensor does not sense $(0, 0)$ is

$$\frac{1}{2}\int_0^1 f_R(r)(1-r^2)dr = \frac{1-\mathsf{E}(R^2)}{2}$$

N_2, the number of sensors in $W_2(1)$ is a Poisson random variable with mean $\lambda\frac{\pi}{2}$. Therefore

$$\mathsf{Pr}\,(E_2) = \mathsf{E}\big((1-\mathsf{E}\big(R^2\big))^{N_2}\big) = e^{-\frac{\pi}{2}\lambda\mathsf{E}\left(R^2\right)}$$

Therefore, it follows that

$$\mathsf{Pr}\,(E_d) = 1 - e^{-\lambda(\pi\mathsf{E}\left(R^2\right)+l_0\mathsf{E}(R))} \tag{4.5}$$

We use this to obtain the distribution of $LF(L)$, $F_{LF(L)}(x)$. Clearly, $\mathsf{Pr}\,(LF(L) < x)$, is 0 for $x < 0$ and 1 for $x > l_0$. For $x \in [0, l_0)$, $F_{LF(L)}(x)$ is the probability that $[0, x)$ is sensed which can be obtained from Eqn. 4.5 by replacing l_0 by x. Therefore

$$F_{LF(L)}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda\pi \mathsf{E}\left(R^2\right) - \lambda \mathsf{E}(R)x} & \text{if } 0 \le x < l_0 \\ 1 & \text{if } x > l_0 \end{cases}$$

From above, notice that the probability density of $LF(L)$ has point masses at 0 and l_0 corresponding to the probabilities of the beginning of the path being sensed and the path not being sensed at all. Observe that this is just the truncated exponential distribution. Easily one can extend the above formula to the case of k-tracking. The corresponding formula is

$$F_{LF(L)}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda\pi \mathsf{E}\left(R^2\right) - \lambda \mathsf{E}(R)x} \sum_{m=0}^{k-1} [\lambda^2 \pi x \mathsf{E}(R^2)\, \mathsf{E}(R)]^m / m! & \text{if } 0 \le x < l_0 \\ 1 & \text{if } x > l_0 \end{cases}$$

The remaining results are for the case of 1-sensing.

Sensing Continuity: Clumps and Holes Since the sensing process on the path is one-dimensional Boolean process, for 1-sensing, the hole lengths are clearly exponentially distributed with rate $\bar{\lambda}$, i.e., the hole length density is $f_H(x) = \bar{\lambda} e^{-\bar{\lambda} x}$. From Theorem 2.2 of [4] the characteristic function, $\gamma_Z(s)$, of length of a clump, Z, is

$$\gamma_Z(s) = \frac{\bar{\lambda} + s}{\bar{\lambda}} - \left(\bar{\lambda} \int_0^\infty \exp\left(-st - \bar{\lambda} \int_0^t (1 - F_{\bar{R}}(x)) dx \right) dt \right)^{-1}.$$

The expectation of the clump length is $\mathsf{E}(Z) = \bar{\lambda}^{-1}(e^{2\bar{\lambda}\mathsf{E}\left(\bar{R}\right)} - 1)$ and its variance is

$$\begin{aligned} \mathsf{VAR}(Z) &= -\left(\bar{\lambda}^{-2} (e^{2\bar{\lambda}\mathsf{E}\left(\bar{R}\right)} - 1)^2 \right) + \\ &+ \frac{2e^{2\bar{\lambda}\mathsf{E}\left(\bar{R}\right)}}{\bar{\lambda}} \int_0^\infty \left(\exp\left(\bar{\lambda} \int_y^\infty \{1 - F_{\bar{R}}(x)\} dx \right) - 1 \right) dy \end{aligned}$$

Theorems 2.3 and 2.4 of [4] can be used to obtain limiting distributions for the clump lengths,

THEOREM 4.5.1 *As $\lambda \to \infty$, the distribution of $\frac{Z}{\mathbf{E}(Z)}$ goes to an exponential with mean* 1. ∎

Further observe that,

$$\log x \int_x^\infty \{1 - F_{\bar{R}}(r)\} \to 0$$

as $x \to 0$. Here $F_{\bar{R}}(r)$ is the distribution function of $\bar{R}$. Therefore from Theorem 2.4 in [4] we have

THEOREM 4.5.2 *If the sensing radii R are scaled by δ, $\delta \to 0$, and if $\lambda \to \infty$ such that*

$$4\mathsf{E}\left(\bar{R}\right)\beta\lambda\delta^2 = \ln\left(\frac{2\beta\delta\lambda}{u}\right) + o(1)$$

then, in the limit, the distribution of Z goes to an exponential with mean u, a constant. ■

We next obtain asymptotic results for the number of holes and clumps in L. A one-dimensional Boolean process is essentially a renewal process on $\mathcal{L}$, with renewal cycle length, D, equal to $H + Z$. Since H and Z are independent

$$\mathsf{E}(D) = \mathsf{E}(H) + \mathsf{E}(Z) = \frac{e^{\pi\mathsf{E}\left(R^2\right)\lambda}}{2\lambda\beta}$$

and $\mathsf{VAR}(D) = \frac{1}{\lambda^2} + \mathsf{VAR}(Z)$. The number of holes, $N_H(L)$, and clumps, $N_Z(L)$, in L will be equal to, or 1 less than the number of renewals in L. Therefore, from the renewal theorem as $l_0 \to \infty$, $\frac{N_H(L)}{l_0}$ and $\frac{N_Z(L)}{l_0}$ will converge to $2\lambda\beta e^{-\pi\lambda\mathsf{E}\left(R^2\right)}$. This result, as in Section 4.4, can be extended to obtain asymptotics for $N_H(L)$ and $N_Z(L)$ when l_0 is finite by scaling R to δR such that $\delta \to 0$ and $\delta^2\lambda \to \rho$.

THEOREM 4.5.3 *If $\delta \to 0$ and $\lambda \to \infty$ such that $\delta^2\lambda \to \rho$, $0 \le \rho < \infty$, then with probability 1*

$$\frac{N_H(L)}{l_0} = \frac{N_Z(L)}{l_0} \to 2\rho\beta e^{-\pi\rho\mathsf{E}\left(R^2\right)}$$

Further, it is possible to derive a central limit theorem for N_H and N_Z from the central limit theorem for number of renewals. Additional statistics may be obtained by suitably using the results from Chapter 4 of [4].

4.6 Breach and Support

Recall that breach is the distance of the closest sensor to L. Therefore, $F_{Br(L)}(d)$ is the probability that there is at least one sensor within a distance of d from L. The latter is the probability that there is at least one sensor in the region $W_i(d) \bigcup W_2(d) \bigcup W_3(d)$, which is of area $2l_0d + \pi d^2$ (see Fig. 4.3). Since the sensor locations form a spatial Poisson process, we get

$$F_{Br(L)}(d) = 1 - e^{-(\lambda\pi d^2 + \lambda l_0 d)}$$

To calculate the support , we first derive the distribution of B_i, the distance to L of the closest sensor in W_i. Let $N_i(d)$ denote the number of sensors in $W_i(d)$ and $E_i(d)$ denote the event that $N_i(d) \neq 0$.

Observe that, in the limit as $d \to \infty$, the region $W_i(d)$ approaches W_i and the probability of $E_i(d)$ approaches 1. Thus the density of B_i can be obtained by first conditioning on $E_i(d)$ and then taking the limit $d \to \infty$. We use this strategy below to calculate the densities .

Given that there are sensors in $W_1(d)$, they will be uniformly distributed in $W_1(d)$. This means that the perpendicular distance of the sensors to L, which is also their shortest distance to L, has a uniform density in $[0, d]$. Therefore conditioned on $E_1(d)$ and $N_1(d)$, B_1 will be the minimum of $N_1(d)$ random variables that are independent and uniformly distributed in $[0, d]$.

$$f_{B_1|E_1(d)}(x) = \begin{cases} \mathsf{E}\left(\frac{N_1(d)}{d} \ \left(1 - \frac{x}{d}\right)^{N_1(d)-1}\right) & \text{if } x < d; \\ 0 & \text{otherwise} \end{cases}$$

Further, conditioned on $E_1(d)$, the density of $N_1(d)$ is

$$p_{(N_1(d)|E_1(d))}(n) = \frac{(\lambda l_0 d)^n e^{-\lambda l_0 d}}{n!(1 - e^{-\lambda l_0 d})}$$

From the above

$$f_{B_1|E_1(d)}(x) = \begin{cases} \frac{\lambda l_0 e^{-l_0\lambda x}}{1-e^{-\lambda l_0 d}} & \text{if } x < d; \\ 0 & \text{otherwise} \end{cases}$$

As before, we obtain the marginal density of B_i by taking the limit as $d \to \infty$.

$$f_{B_1}(x) = \lambda l_0 e^{-l_0\lambda x} \tag{4.6}$$

Clearly B_2 and B_3 will be identically distributed. Further, given that there are $N_2(d)$ sensors in $W_2(d)$, they will be independently and uniformly distributed in the region. This means that their distance from the origin which is also their shortest distance to L, will have a density $\frac{2x}{d^2}$ in $[0, d]$ and 0 outside. As in the case of B_1, we obtain $f_{B_2}(x)$ by first conditioning on $N_2(d)$, $E_2(d)$ and evaluating the marginal. This turns out to be

$$f_{B_3}(x) = f_{B_2}(x) = \lambda \pi x \ e^{-\frac{\lambda \pi x^2}{2}} \tag{4.7}$$

Recall, that support, $Su(L)$, is the maximum Euclidean distance of the closest sensor from the path L. In the case of straight line paths this will essentially be the distance of the closest sensor to the furthest end point of the straight line.

Let E_c denote the event that the closest sensor is in W_1 and E_c^c the complement event. The density of the support, $f_{Su(L)}(x)$, can therefore be obtained as

$$f_{Su(L)}(x) = \mathsf{Pr}\left(E_c\right) \ f_{Su(L)|E_c}(x) + \mathsf{Pr}\left(E_c^c\right) \ f_{Su(L)|E_c^c}(x)$$

The event E_c is the event that B_1 is less than B_2 and B_3. Therefore

$$\mathsf{Pr}\left(E_c\right) = \int_{x=0}^{\infty} f_{B_1}(x)(1 - F_{B_2}(x))(1 - F_{B_3}(x)) \ dx.$$

Also, given E_c, $Su(L) = \sqrt{B_1^2 + \bar{T}^2}$. Here $\bar{T}$ is the distance of the point $\bar{X}$, the projection of the closest sensor onto $\mathcal{L}$, to furthest end point of L. (See Fig. 4.3.) Since the sensors are deployed as a homogeneous Poisson process, $\bar{X}$ is uniformly distributed in $[0, l_0]$ which means $\bar{T}$ is uniformly distributed in $[\frac{l_0}{2}, l_0]$. Therefore

$$f_{Su(L)|E_c}(x) = \int_{l_0/2}^{x} \frac{2x f_{B_1}(\sqrt{x^2 - t^2})}{l_0 \sqrt{x^2 - t^2}} dt.$$

In the case when the closest is in W_2 or W_3 the support can be written as $Su(L) = \sqrt{(B_2 \sin(\theta))^2 + (l_0 + B_2 \cos(\theta))^2}$. Here θ is the angle made by the line joining the closest sensor to the closest end of L. Since the sensor will be uniformly distributed in the semicircular regions, θ will be uniform in $[0, \pi]$. We have not been able to obtain a closed form expression for either the distribution or the moments of the support.

4.7 Generalizations

So far we have analyzed the thresholded sensing of straight line paths. In this section, we extend these results to more general settings.

Paths and Networks in Higher Dimensions

Our first generalization is to extend the results for networks and paths of higher dimensions. As an example, consider a network deployed in three-dimensions that is used to track the movement of a fleet of airborne objects, say a flock of birds, and also to track a specific object or bird. Since the 'cross section' of the fleet or flock will be significantly higher than a single object, we may treat the latter as a point object and the former as a planar set. To track the fleet, the three-dimensional sensor network must sense a 'two dimensional path' while in tracking the individual element, a one-dimensional path must be sensed.

Consider an $m-$dimensional straight line path L. When $m = 1$, L is a straight line and when $m = 2$ is a rectangle. For a general m, L will be a $m-$dimensional hypercuboid. Let $\{X_i, C_i\}$ form a $n-$dimensional Boolean process. Here $\{X_i\}$ is an n-dimensional Poisson process of intensity λ, C_i is a hypersphere of radius R_i and R_i are i.i.d random variables with density $f_R(\cdot)$ with support in $[0, 1]$. We are interested in the coverage/sensing properties of L by $\{X_i, C_i\}$. Let $\mathcal{L}$ be the plane obtained by extending L along the m directions in which it is has a non-zero measure. As before, we project marked points i.e., a point at X_i onto $\mathcal{L}$ if the set $X_i + C_i$ intersects $\mathcal{L}$. This results in a point process $\{\bar{X}_i\}$. Let $\bar{C}_i$ be the intersection of $X_i + C_i$ and $\mathcal{L}$.

The following results can be obtained as above.

Lemma 4.7.1 *The point process, $\{\bar{X}_i\}$, on $\mathcal{L}$ is a Poisson process of intensity $2\beta_{n-m}\lambda$ where $\beta_{n-m} = \mathsf{E}\big(R_i^{n-m}\big)$.*

Proof Consider a differential element dS on $\mathcal{L}$ of $m-$dimensional volume dV. The process $\{\bar{X}_i\}$ is a Poisson point process on $\mathcal{L}$ if the probability there is a projected point i.e., $\bar{X}_i$ for some i, in dS, is $\bar{\lambda}dV$ and that the arrivals also have the independent increments property.

There is a projected point in dS if there is a corresponding X in the $n-$dimensional differential element, dP, centered at dS and of a 2 units length along the $n - m$ directions that are perpendicular to $\mathcal{L}$. The probability of this event is the product of the probability that there is a sensor in dP (which is approximately λdV) and the probability that this sensor is marked. We calculate this next.

Given that a sensor (say s of sensing radius R) is in dP, it will be uniformly distributed there and the perpendicular distance of s from $\mathcal{L}$, say Y, will have a density

$$f_Y(y) = \begin{cases} (n-m)y^{n-m-1} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

Therefore the probability that s is marked is

$$\begin{aligned} \mathsf{Pr}\,(s \text{ is marked}) &= \int_0^1 Pr(Y \leq R) f_R(r) dr \\ &= \int_0^1 r^{n-m} f_R(r) dr = \beta_{n-m} \end{aligned}$$

The argument for independent increment property is identical to the argument in proof of Lemma 4.3.1. ∎

LEMMA 4.7.2 *The set $\bar{C}_i$ is an $m-$dimensional hypersphere, centered at $\bar{X}_i$ and has a radius $\bar{R}_i$ where the density of R_i is*

$$f_{\bar{R}_i}(\bar{r}) = \begin{cases} \frac{(n-m)\bar{r}}{\beta_{n-m}} \times \\ \quad \int_{\bar{r}}^1 f_R(r)(r^2 - \bar{r}^2)^{\frac{n-m-2}{2}} dr & \text{if } 0 \leq \bar{r} \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof Consider a marked sensor S at X with a sensing radius R. Let Y be its perpendicular distance from $\mathcal{L}$. Note that as the sensor is marked $R > Y$.

The region on $\mathcal{L}$ sensed by S will be the hypersphere with radius $\bar{R} = \sqrt{R^2 - Y^2}$. As before, by a transformation of random variables we have the distribution function, $F_{\bar{R}}(\cdot)$, can be written as

$$F_{\bar{R}}(\bar{r}) = \int_{r,y:\sqrt{r^2-y^2}=\bar{r}} f_{(R,Y|R \geq Y)}(r,y)\; dr\; dy \tag{4.9}$$

The probability that a sensor, which is within a unit distance from $\mathcal{L}$, is marked is β_{n-m} (proof of Lemma 4.7.1). Further, Y is independent of R and its is density is given by Eqn. 4.8. Therefore

$$f_{(R,Y|R>Y)}(r,y) = \frac{(n-m)f_R(r)y^{n-m-1}}{\beta_{m-n}} \tag{4.10}$$

Substituting for $f_{(R,Y|R>Y)}(r,y)$ in Eqn. 4.9 and differentiating with respect to $\bar{r}$ we get

$$f_{\bar{R}_i}(\bar{r}) = \begin{cases} \frac{(n-m)\bar{r}}{\beta_{n-m}} \times \\ \int_{\bar{r}}^1 f_R(r)(r^2 - \bar{r}^2)^{\frac{n-m-2}{2}} dr & \text{if } 0 \leq \bar{r} \leq 1 \\ 0 & \text{otherwise. } \blacksquare \end{cases}$$

THEOREM 4.7.3 *The projected point process $\{\bar{X}_i\}$ and the sets $\{\bar{C}_i\}$ constitute an $m-$dimensional Boolean process.*

Non-Thresholded Sensing

Recall that, in non-thresholded sensing $\phi(X_i, x)$ is a continuous function that decreases as the distance of the point from the sensor increases. In general $\phi(X_i, x)$ is taken to be $\frac{\eta}{d(X_i,x)^\gamma}$ and a point is considered sensed only if $\sum_i \frac{\eta}{d(X_i,x)^\gamma} \geq \theta$. Here $d(X_i, x)$ is the Euclidean distance between X_i and x.

Recall, that breach and support are not dependent on the sensing model and will remain the same. Further, it is shown in [8] that the region sensed by a network under the 1-thresholded model with the sensing radii is fixed at $\left(\frac{\eta}{\theta}\right)^{1/\gamma}$ will be a subset of the region sensed by the network when the sensing is non-thresholded. Therefore, the exposure and the clump lengths for the non-thresholded model are lower bounded while the length to first sense is upper bounded by their values obtained for $1-$thresholded sensing.

Finite Operational Area

Consider a square A in which sensors are deployed in a Poisson manner i.e., the number of sensors in $R \subset A$ is a Poisson random variable and is independent of the number of sensors in any non-overlapping region. Let P be a straight line path with the two end points on opposite edges of A. Since the sensors are distributed only inside A, the results we have derived above, upper bound the sensing properties of P. Further, the results derived in the limit $\delta \to 0$ hold, as in that limit, even if sensors are deployed outside A the length sensed by these sensors will be negligible.

Chapter 5

Tracking in Backbone Networks

5.1 Introduction

From the previous chapter, one would have got an overview of the myriad applications of sensor networks. In practice, varied models of sensor networks are employed. In great many cases, the sensor nodes do not directly relay the information about the phenomenon sensed by them. The information is routed through a server or a base station. A set of nodes (sensors) are affiliated to a base station to which they send the information and the base station routes it to the end user.

Consider such a kind of sensor network for target tracking. Whenever a target lies in the vicinity of a node, it is considered to be sensed by that node. The node relays the information to the base station which obtains estimates of the trajectory of the target based on the information received from other nodes affiliated to it and other base stations by the data available. The measures of the trackability we are interested remain the same as in the previous chapter.

We use the following sensing model. The base stations are deployed according to a Poisson spatial process with parameter $\lambda > 0$. In a box of radius R_0 around each base station the sensor nodes are scattered according to independent Poisson spatial processes with parameter $\mu > 0$. The sensing area of each node is a box of radius R_1 around them. The ℓ_∞ norm is used here to make analysis easier. A point is considered sensed if it lies within the sensing area of atleast one of the sensor nodes. Thus the coverage of the operational area is two dimensional Poisson cluster process. This can be viewed as two dimensional Boolean process as well. We shall analyze the properties of the induced coverage process on an one dimensional linear path. The measures we consider are essentially coverage statistics of this process.

In the case of a simple Boolean model, the area coverage as well as trackability has been studied in [4], [5]. But using methods analogous to the latter work, we analyze the properties of the induced one dimensional process. The two processes are intimately related as will be seen in the work. We shall obtain mainly asymptotic results for the proportion of the target tracked under this coverage process.

5.2 System Model

A (stationary) Poisson Cluster Process(PCP) on $\mathbb{R}^2$ consists of three components.

1. A stationary Poisson Point Process $\mathbf{N}$ on $\mathcal{R}^2$ with intensity $\lambda > 0$.

2. A bivariate probability density $f(x, y)$ on $\mathbb{R}^2$.

3. A distribution $\mathbf{G}$ concentrated on positive integers with a moment generating function $\phi_{\mathbf{G}}(\mathbf{t})$.

   We shall assume f(x,y) to be uniformly distributed on ℓ_∞ ball of radius R_0, $R_0 > 0$ centered at origin. i.e,

$$f(x, y) = \begin{cases} \frac{1}{4R^2} & \text{if } max(|x|, |y|) \leq R \\ 0 & \text{otherwise} \end{cases}$$

   We shall assume $\mathbf{G}$ to be a Poisson Spatial process with intensity μ. Hence $\phi_{\mathbf{G}}(\mathbf{t}) = e^{\mu(e^t - 1)}$.

The structure of the PCP is as follows : first the points of $\mathbf{N}$ act as cluster centers (called as the process of cluster centers) and then at each cluster center are formed clusters independently of other cluster centers, with the number of points in a cluster distributed as $\mathbf{G}$.

Conditional on this number, say k, and the location of the cluster center say (x, y), the k points (called as the subsidiary process) of the cluster are i.i.d. about the center with density $f(v - x, w - y)$. Thus the subsidiary process is a Poisson spatial process with intensity μ on the ℓ_∞ ball of radius R_0 centered at (x, y).

Now we shall describe our coverage process $\mathbf{C}_0$. Let $\{Z_1, Z_2, \ldots\}$ be the points of the Poisson process N. For each $i = 1, 2, \ldots,$ let G_i be i.i.d. random variables with distribution G. Let $\{Z_{ij} : 1 \leq j \leq G_i\}$ be i.i.d. points distributed according to the p.d.f. f around Z_i. Then the stationary point process $\mathbf{P}_0$ is the collection of points $\mathbf{P}_0 = \{Z_i + Z_{ij} \in \mathcal{R}^2 : 1 \leq j \leq G_i, i \geq 1\}$.

Let S_0 be the ℓ_∞ ball of radius R_1 centered at the origin in $\mathbb{R}^2$. Define $\mathbf{C}_{ij} = Z_i + Z_{ij} + S_0$, $1 \leq j \leq G_i, i \geq 1$. The process $\mathbf{C}_0 := \cup_{\substack{1 \leq j \leq G_i \\ i \geq 1}} C_{ij}$ is called the coverage process generated by $\mathbf{P}_0$ and driven by S_0 (in the terminology of Section 3.1).

We assume the operational area as $\mathbb{R}^2$. The points Z_i of the process N are the locations of the backbone nodes and the sensors are located at points $Z_i + Z_{ij}$ with corresponding coverage area C_{ij}. Such a model of wireless network has been considered in [11] and its percolation properties and covering algorithms has been studied in the cited reference. In general, R_1 is much smaller than R_0.

Prior work on trackability of target has been done in the case of Boolean models as described in the previous chapter. Note that $\mathbf{C}_0$ is a Boolean model driven by $\xi_i = Z_i$ and generated by $C_i = \cup_{1 \leq j \leq G_i} Z_{ij} + S_0$. And we consider the statistical properties of the measures defined for a path $L \subset \Omega$. Recall that $\psi_T(x)$ was defined in 4.2 as the characteristic function of coverage of x. We are interested in the random process $\psi_T(x)$, where $x \in L$. Analogously, We also show that when L is a straight line path, the sensing process on L i.e. $\psi_T(x)$, $x \in L$, has the same laws as a one-dimensional Boolean process. However in this case the induced one-dimensional Boolean process is more complicated. Neverthless, this enables us to use the available literature on one dimensional Boolean processes and characterize our measures of trackability.

5.3 Sensing Process on a straight line path

Let $L \in \Omega$ be an arbitrary straight line path of finite length. Let l be the line obtained by extending L in both directions. Since the Boolean process is shift invariant, without loss of generality we can take l to be the $X-$axis of the co-ordinate axes (by this we also have made the assumption that l is perpendicular to one of the axes).

We shall now construct a 1-D Boolean Model, $\mathbf{C}$. Let l be the line which is obtained by extending the line segment MN at both ends. Without loss of generality we can assume l is x-axis. We say a cluster center is marked iff there is a positive probability that at least one of the subsidiary points in its cluster will be at a perpendicular distance less than the coverage radius R_1 from the X-axis. In effect, we mark all cluster centers whose subsidiary process will observe some part of the target. Since with positive probability a subsidiary point has to lie within a distance R_1, the cluster center has to lie within a distance of $R_0 + R_1$. Hence all cluster centers that lie within a distance of $R_0 + R_1$ are marked. Now construct the point process $\mathbf{P}$ on X-axis by projecting all the marked sensors onto it. We need to show $\mathbf{P}$ is a spatial Poisson process with intensity $\bar{\lambda} = 2\lambda(R_0 + R_1)$. To prove that the $\mathbf{P}$ is a Poisson point process it is sufficient to prove that the probability of a arrival in a differential length dl of the line is $\bar{\lambda}dl$ and that this probability is independent of arrivals in any other interval.

The event that there is an arrival in the interval $(l, l + dl)$ is equivalent to the event that there is a marked cluster center in a corresponding differential strip, dP, of thickness dl with center on the line l and length $2(R_0 + R_1)$. Since when a cluster center lies in dP , there is a positive probability of one of its subsidiary points covering x-axis. Hence the probability of an arrival in $(l, l+dl)$ is the probability there is a cluster center in dP. Hence the probability is $\lambda\|dP\| = 2\lambda(R_0 + R_1)$. The property of independent arrivals in non-overlapping interval follows from the fact that the original birth process is a spatial Poisson process.

Let $\bar{X}$ be the location of a marked cluster center. Then, $\bar{X} = (X, Y)$ where $X \in \mathbf{P}$ and Y is uniformly distributed on $[-R_0 - R_1, R_0 + R_1]$. Let $X + S = X + S(\mu, R_0, R_1)$ be the portion of the X-axis covered by the sensors within the cluster around a marked sensor. Let S_Y denote S conditioned on this marked sensor being at a distance Y from the x-axis. To this end if $(X + U, V)$ is the location of a sensor belonging to the cluster around $\bar{X}$, that is $-R_0 \leq U \leq R_0, Y - R_0 \leq V \leq Y + R_0$, then this sensor covers the region

$[X+U-R_1, X+U+R_1]$, provided $|V| \leq R_1$. Hence the sensor at (U,V) is relevant to coverage only if $\max(-R_1, Y-R_0) \leq V \leq \min(Y+R_0, R_1)$. Projecting sensors that satisfy this condition onto the $x-$axis will yield a spatial Poisson process of intensity $\mu g(Y, R_0, R_1)$, on $[X-R_0, X+R_0]$, where $g(Y,R_0,R_1) = 2R_0(\min(Y+R_0,R_1) - \max(-R_1, Y-R_0))$. If $X+U_1, X+U_2, \ldots, X+U_{N(Y)}$ are the projected points then $N(Y)$ is Poisson with intensity $\mu g(Y,R_0,R_1)$, and $S_Y = \cup_{j=1}^{N_Y}[U_j - R_1, U_j + R_1]$. Thus the tracking of a linear target in a Poisson cluster field is equivalent to the coverage properties of the one dimensional Boolean model $\mathbf{C}$ generated by $\mathbf{P}$ and driven by S, that is,

$$\mathbf{C} = \cup_{X_i \in \mathbf{P}} X_i + S_i,$$

where the S_i are distributed as S. We will call $\mathbf{C}$ as the projection of the Boolean model $\mathbf{C}_0$.

We now state an important theorem. Let S be the random set described above.

THEOREM 5.3.1 *For thresholded sensing, the projected point process and the collection of sensed segments form a one-dimensional Boolean process with laws identical to the one-dimensional Boolean process $\{\bar{X}_i + \bar{C}_i\}$ where $\{\bar{X}_i\}$ is a Poisson point process of density $\bar{\lambda} = 2\lambda(R_0 + R_1)$, and the $\bar{C}_i$ are i.i.d as S.*

5.4 Exposure - Fraction Sensed

The trackability of the line L of length ℓ_0 lying in the region $B_0 \in \mathcal{R}^2$ of interest by the coverage process $\mathbf{C}_0$, is the same as the coverage of L by the Boolean model $\mathbf{C}$. In this section we obtain asymptotic results for exposure $X_\psi(L)$ (defined in Section 4.2) for finite length paths with increasing densities of sensor nodes and decreasing sensor radii.

First we calculate the exposure for a fixed λ, μ, R_0, R_1 and then pass onto the asymptotic results. From [4] (Sec 3.1), we know $\mathsf{E}(X_\psi(L)) = 1 - \exp\{-\overline{\lambda}\mathsf{E}(\|S\|)\}$. In inference of properties of exposure we shall use Theorem 3.3 of [4] which guarantees that complete coverage of L is equivalent to $X_\psi(L) = 0$. From symmetry of S_y about the x-axis,

$$\begin{aligned} \mathsf{E}(\|S\|) &= \frac{1}{2(R_0+R_1)} \int_{-(R_0+R_1)}^{(R_0+R_1)} \mathsf{E}(\|S_y\|)\, dy \\ &= \frac{1}{(R_0+R_1)} \int_0^{(R_0+R_1)} \mathsf{E}(\|S_y\|)\, dy, \end{aligned} \tag{5.1}$$

where S_y is the region covered by a Boolean model generated by a spatial Poisson process in $[0, 2R_0]$ with intensity $\mu g(y, R_0, R_1)(= 2R_0(\min(Y + R_0, R_1) - \max(-R_1, Y - R_0))$ and driven by the set $[0, 2R_1]$. Then $\mathsf{E}(S_y)$ is the exposure in $[0, 2R_0]$ (see Remark 2) under this Boolean model. Hence $\mathsf{E}(\|S_y\|) = 2R_0 - 2R_0 \exp\{-2\mu g(y)R_1\}$. Since

$$\int_{-R_1-R_0}^{R_1+R_0} \exp\{-2\mu R_1 g(y)\} = \frac{1}{\mu R_1}(1 - e^{-4\mu R_1^2}) + 2(R_0 - R_1)e^{-4\mu R_1^2},$$

we get from (5.1),

$$\overline{\lambda}\mathsf{E}(\|S\|) = \lambda 4(R_0 + R_1)R_0\{1 - \frac{1}{\mu 2R_1(R_0+R_1)}(1 - e^{-4\mu R_1^2}) + \frac{R_0 - R_1}{R_0 + R_1} e^{-4\mu R_1^2}\} \tag{5.2}$$

Let us denote the r.h.s of (5.2) as $f(\lambda, \mu, R_0, R_1)$.

Now we consider the Boolean model $C_0(\delta, \lambda, \mu)$. This is the Boolean model $\mathbf{C}_0$ in which R_0 is scaled to δR_0 and R_1 is scaled to δR_1. Now we state an asymptotic result for exposure under suitable scaling of δ, λ and μ.

THEOREM 5.4.1 *Let $X_\psi(L)$ be the exposure as defined in Chapter 4 and $C_0(\delta, \lambda, \mu)$ the Boolean model as defined above. If $\delta \to 0$ as $\lambda \to \infty$ and $\mu \to \infty$ such that $\delta^2\lambda \to \rho_1$ and $\delta^2\mu \to \rho_2$ where $0 \leq \rho_i \leq \infty$ for $i = 1, 2$, then*

$$\mathsf{E}(X_\psi(L)) \to 1 - e^{-f(\rho_1, \rho_2, R_0, R_1)} \tag{5.3}$$

and for each $1 \leq a < \infty$

$$\mathsf{E}(|X_\psi(L) - \mathsf{E}(X_\psi(L))\,|)^a \to 0$$

(5.3) follows from direct substitution in (5.2) and the L_a convergence follows as in the proof of Theorem 3.3.1 in Chapter 3.

REMARK 1: The difference with Hall's result is that though both are 1-D Boolean models the model we consider arises by projecting a 2-D coverage processes onto a line and hence the scaling has to be as required by the 2-D model. This is intuitively clear since a non-trivial sensing can be obtained only under a non-trivial coverage of the 2-D space where the target moves. Hence we have δ^2 as compared to δ in Hall. This is expected too. Also we can note that the cases of either $\rho_i, i = 1, 2$ being ∞ or 0. If $\rho_1 = \infty$ then $E(X_\psi(L)) = 1$ which by our observation at the beginning of the section implies that L is completely tracked.

REMARK 2: In the calculation of $E(\|S_Y\|)$ we have included the effect of points outside of the Boolean model (with Poisson $\mu g(Y)$) which lie outside $[0, 2R_0]$. and also neglected the coverage of points lying near the border covering region outside $[0, 2R_0]$. This is justified, since as $\delta \to 0$ they don't really affect the vacancy within $[0, 2R_0]$.

We look at the coverage process $\mathbf{C}_0$. Now we shall look at the variance of the exposure. Again from [4] (Sec 3.2) we get,

$$\mathsf{VAR}(X_\psi(L)) = \frac{1}{\ell_0} \exp[-2\lambda \mathsf{E}(\|S\|)] \int\int_{L^2} (\exp[\lambda \mathsf{E}(\|(x_1 - x_2 + S) \cap S\|)] - 1) dx_1 dx_2$$

and

$$\lambda \mathsf{E}\big(\|S\|^2\big) \exp[-2\lambda \mathsf{E}(\|S\|)] \leq \ell_0 \mathsf{VAR}(X_\psi(L)) \leq \lambda \mathsf{E}\big(\|S\|^2\big) \exp[-\lambda \mathsf{E}(\|S\|)]$$

Though we cannot compute variance explicitly the above inequality will help us to get good bounds on it. And also for a fixed λ, variance decreases when the random set S^0 undergoes a 'random orthogonal transformation'. Since $\mathsf{E}(\|S\|^2) \leq 4R_0^2$ as $S \subseteq [0, 2R_0]$, from Hall we get the following lemma.

Lemma 5.4.2 *We make the same assumptions as in Theorem 5.4.1. Then,*

$$\lambda \mathsf{VAR}(X_\psi(L)) = \mathcal{O}(1)$$

and

$$\begin{aligned} \lambda \mathsf{VAR}(X_\psi(L)) \to \sigma^2(S) &= \frac{1}{\ell_0} \rho_1 \exp[-4\rho_1(R_0 + R_1)\mathsf{E}(\|S(\rho_2, R_0, R_1)\|)] \\ &\quad \int_{\mathbb{R}} (\exp[2\rho_1(R_1 + R_0)\mathsf{E}(\|(x + S) \cap S\|)] - 1) dx \end{aligned} \tag{5.4}$$

We note that though μ doesn't occur explicitly in (5.4), it is involved in $\mathsf{E}(\|S\|)$ and the term in the exponential of the integrand. Intuitively we can observe that λ and μ need to tend to infinity at the same rate else it will result in trivial cases as no coverage or full coverage. And $\mu \to \infty$ according to Theorem 5.4.1 is necessary to ensure proper convergence of $\mathsf{E}(\|S\|)$ and term in the integrand.

Now we give a strong law for exposure as the area of the region of interest increases while the sensor densities and sensor radii remain fixed. Put $B_b = bB$. And L_b denote the line ℓ within B_b of length ℓ_b. From theorem 3.6 of Hall,

LEMMA 5.4.3 *As $b \to \infty$,*

$$X_\psi(L_b) \to 1 - \exp[-2\lambda(R + r)\mathsf{E}(\|S\|)] \tag{5.5}$$

a.s.

In Lemmas (5.4.2),(5.4.3) expression for $E(\|S\|)$ using (5.1) and (5.2) can be substituted to yield explicit formulae. Theorem 5.4.1 is basically a "weak law of large numbers" for exposure within a large region. In general for weak laws we divide the sum of independent random variables, but here it is replaced with scaling by δ. And also we have not independent random variables but 'almost' independent random variables which are exposure within smaller parts of the region. Now we complement the weak law with a central limit theorem derived using theorem 3.5 of [4].

LEMMA 5.4.4 *We make assumptions as in Theorem 5.4.1. Then*

$$\sqrt{2\delta\lambda(R_0 + R_1)}\{X_\psi(L) - \mathsf{E}(X_\psi(L))\} \to N(0, \sigma^2) \tag{5.6}$$

where σ^2 is as defined in lemma 5.4.2.

5.5 Statistical Analysis of thresholded Sensing

Since the Boolean process is shift invariant, without loss of generality, L can be taken to be the segment $[0, l_0]$. In this section we statistically characterize the trackability of L. In the following two subsections we denote by v and X a typical point of the subsidiary process and the parent process respectively.

Breach

Let B_0 and B_1 denote balls of radii x and $x + R_0$ around $[0, l_0]$. Let $Br(L)$ be the breach of L.

$$\begin{aligned}
\mathsf{Pr}\left(Br(L) \leq x\right) &= \mathsf{Pr}\left(\exists\ v \in B_0\right) \\
&= \mathsf{E}(\mathsf{Pr}\left(\exists\ v \in B_0 | \exists\ X \in B_1\right)) \\
&= \frac{1 - e^{-\lambda\|B_1\|}}{\|B_1\|} \int_{X \in B_1} (1 - e^{-\|B_2(X,R_0) \cap B_0\|}) dX
\end{aligned}$$

Length to First Sense

Let E denote the event $[0, x]$,$x < l_0$ is not sensed. B_0 and B_1 are rectangles centered at $(x/2, 0)$ of height $2R_1$ and $2(R_0 + R_1)$ respectively and width $2R_1 + x$ and $2(R_0 + R_1) + x$ respectively. Also $\|B_p((X,Y), R_0) \cap B_0\| = (R_0 + R_1 - |Y|)(R_0 + R_1 - |X|)$.

Let the length-to-first sense be $LF(L)$. Set $LF(L) = l_0$ when L is not sensed. Note that

$$\mathsf{Pr}\left(LF(L) > x\right) = \begin{cases} 1 & \text{if } x < 0 \\ \mathsf{Pr}\left(E\right) & \text{if } 0 \leq x < l_0 \\ 0 & \text{if } x \geq l_0 \end{cases}$$

$$\begin{aligned}
\mathsf{Pr}\left(E\right) &= \mathsf{Pr}\left(\exists \text{ no } v \in B_0\right) \\
&= \mathsf{E}(\mathsf{Pr}\left(\exists \text{ no } v \in B_0 | \exists (X,Y) \in B_1\right)) \\
&= \frac{1 - e^{-\lambda\|B_1\|}}{\|B_1\|} \int_{(X,Y) \in B_1} e^{-\|B_p((X,Y),R_0) \cap B_0\|} dX dY
\end{aligned}$$

Suppose we denote by E_d the event that $[0, l_0]$ is tracked. Let $\bar{B}_0 = B_q[[0, l_0], R_1]$ and $\bar{B}_1 = B_p[\bar{B}_0, R_0]$. Then

$$\mathsf{Pr}\left(E_d\right) = \frac{1 - e^{-\lambda\|\bar{B}_1\|}}{\|\bar{B}_1\|} \int_{(X,Y) \in \bar{B}_1} e^{-\|B_p((X,Y),R_0) \cap \bar{B}_0\|} dX dY$$

and

$$\mathsf{Pr}\left(LF(L) > x | E_d\right) = \begin{cases} 1 & \text{if } x < 0 \\ \mathsf{Pr}\left(E\right) / \mathsf{Pr}\left(E_d\right) & \text{if } 0 \leq x < l_0 \\ 0 & \text{if } x \geq l_0 \end{cases}$$

Chapter 6

Conclusion

In this thesis, we have studied varied problems motivated by applications to wireless sensor networks. In Chapter 2, we had studied evolving random geometric graphs in one dimension. In Chapter 3, we extended the work of [4] in coverage processes with a more general definition of coverage. This we used effectively in Chapter 4 to characterize the ability of a sensor network to track a target moving linearly in its field. Also though not to a generic model, but in a particular model of a backbone network, we deduced certain results regarding the tracking of a linearly moving target in the field. Chapter 5 describes the results obtained on the same.

Now we would like to point out further possibilities of investigation along similar lines.

- The study of evolving RGGs can be considered in higher dimensions. Also more sophisticated models where all the nodes evolve can be considered. But this will complicate the solutions to a great extent.

- We have extended only the weak law and central limit theorem of [4] to the case of k-coverage. It might be interesting to extend results related to counting and clumping to the case of k-coverage.

- A much interesting study of coverage processes as [4] points out is to extend the results of coverage processes to the case of Poisson cluster processes and Cox processes (i.e, random non-homogeneous Poisson processes). With certain approximations we have derived elementary results for the former case. But it might be a worthwhile study to get better results.

- We have considered only linearly moving targets. It makes sense to consider random walks. It might be good to start with particular cases of Self-avoiding random walks and then study more general cases. Another interesting object might be the study of trackability of Brownian paths which we are currently investigating. The work of Erwin Bolthausen and Alan-Sol Sznitman around 1990 deals with some of these questions. A good reference would be [27].

- One might even think of examining tracking in a more general backbone network. But it might require different kind of tools and techniques. Essentially the equivalence to a sensing process (like the Boolean process) on a line might not be possible.

- Multifarious coverage processes also can be studied as investigated in [26].

Bibliography

[1] M. D. Penrose - *Random Geomtric Graphs* Oxford University Press, 2003.

[2] B. Gupta, Srikanth. K. Iyer, D. Manjunath - "On the Topological Properties of the One Dimensional Exponential Random Geometric Graph," *Submitted* 2005.

[3] D. Tse, M. Grossgaluser - "Mobility increases the capacity of wireless ad-hoc networks, " *Proceedings of IEEE INFOCOM-2001*, 2001.

[4] P. Hall. - *Introduction to the Theory of Coverage Process* John Wiley and Sons, 1988.

[5] S. Sundhar Ram, D. Manjunath, Srikanth. K. Iyer and D. Yogeshwaran - "On Path Tracking Properties of Random Sensor Networks", *Submitted* 2006.

[6] S. Megerian, F. Koushanfar, M Potkonjak, and M. B. Srivastava, - "Worst- Case and Best-Case Coverage in Sensor Networks," in *IEEE Transactions on Mobile Commputing*

[7] S. Megerian, F. Koushanfar, G. Qu, G. Veltri, and G. Potkonjak, - "Exposure in Wireless Sensor Networks: Theory and Practical Solutions," *Journal of Wireless Networks*, Vol.8 2002.

[8] B. Liu and D. Towsley, - "A Study on the Coverage of Large Scale Network," in *Proc.* ACM *MobiHoc*, 2004.

[9] Q. Huang, - "Solving an Open Sensor Exposure Problem using Variational Calculus",`www.cse.seas.wustl.edu/techreportfiles/getreport.asp?237.`

[10] T. Clouqueur, V. Phipatanasuphorn, P. Ramanathan, and K. Saluja, - "Sensor Deployment Strategy for Source Tracking," in *First ACM International Workshop on Wireless Sensor Networks and Application*, 2002.

[11] L.Booth, J.Bruck, M.Franceschetti, R.Meester, - "Covering algorithms, continuum percolation and the geometry of wireless networks", *Ann. Appl. Probab.* 13 (2003), no. 2, 722–741

[12] Nikhil Karamchandani, D. Manjunath, Srikanth. K. Iyer, D. Yogeshwaran - "Evolving Random Geometric Graph Models for Mobile Wireless Networks", accepted at *4th International Symposium on Modeling and Optimization in Mobile,Ad-hoc and Wireless Networks (Wiopt '06)*, April 2006.

[13] M. P. Desai and D. Manjunath, "On the connectivity in finite ad-hoc networks," *IEEE Communication Letters,* vol. 6, no. 10, pp. 237239, 2002.

[14] M. P. Desai and D. Manjunath, "On range matrices and wirelesss networks in d-dimensions," in *Proceedings of WiOpt-05,* Trentino, Italy, April 2005.

[15] P. Gupta and P. R. Kumar, "The capacity of wireless networks," *IEEE Transactions on Information Theory,* vol. 46, no. 2, pp. 388404, March 2000.

[16] T. Camp, J. Boleng, and V. Davies, "A survey of mobility models for ad hoc network research," *Wireless Communications and Mobile Computing: Special issue on Mobile AdHoc Networking: Research, Trends and Applications,* vol. 2, no. 5, pp. 483502, 2002.

[17] D. P. Gaver and P. A. W. Lewis, "First order autoregressive gamma sequences and point processes," *Advances in Applied Probability,* vol. 12, pp. 727745, 1980.

[18] N. Karamchandani, D. Manjunath, and S. K. Iyer, "On the clustering properties of exponential random networks," in *Proceedings of IEEE Intnl. Symp. World of Wireless, Mobile and Multimedia Networks (WoWMoM),* Taormina, Italy, 2005.

[19] H. Zhang and J. Hou, "On Deriving the Upper Bound of α-Lifetime for Large Sensor Networks", in *Proceedings of ACM MobiHoc,* 2004.

[20] S. Athreya, R. Roy, and A. Sarkar, "On the coverage of space by random sets", *Advances in Applied Probability,* vol. 36, no. 1, pp. 1-18, March 2004.

[21] X. Y. Li, P. J. Wan, and O. Frieder, "Coverage in Wireless Ad-Hoc Sensor Networks," *IEEE Transactions on Computers,* vol. 52, no. 6, pp. 753-763, 2003.

[22] X. Wang et. al., "Integrated Coverage and Connectivity Configuration in Wireless Sensor Networks," in *Proceedings of the First ACM Conference on Embedded Networked Sensor Systems,* 2003.

[23] H. Zhang and J. .C. Hou, "Maintaining sensing coverage and connectivity in large sensor networks," *Ad Hoc and Sensor Wireless Networks: An International Journal,* vol. 1, no. 1, pp. 89123, January 2005.

[24] D. Yogeshwaran, Srikanth. K. Iyer, D. Manjunath and S. Sundher Ram - "On Path Tracking Properties of Backbone Sensor Networks", *Pre-print* 2006.

[25] D. Yogeshwaran, Srikanth. K. Iyer, D. Manjunath and S. Sundher Ram - "K-coverage and Applications to Trackability in Sensor Networks", *Pre-print* 2006.

[26] F. L. Baccelli and B. Blaszczyszyn - "On a coverage process ranging from the Boolean model to the Poisson-Voronoi tessellation with applications to wireless communications", *Advances in Applied Probability* 33 (2001), no. 2.

[27] Sznitman, Alain-Sol - *Brownian motion, obstacles and random media.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.