Random Minimal Spanning Acycles.

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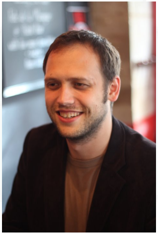

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- Geometric Model : Penrose 1997, Hsing-Rootzen, 2005.

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  (ii) Columns $\{e\}_{e \in S}$ are linearly independent (acyle)

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Betti numbers (formally in one slide) !

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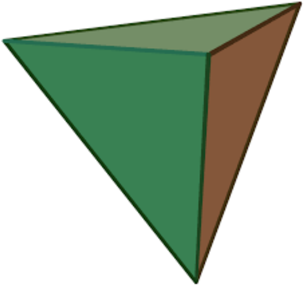

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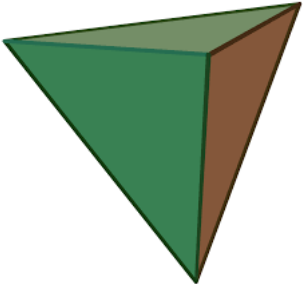

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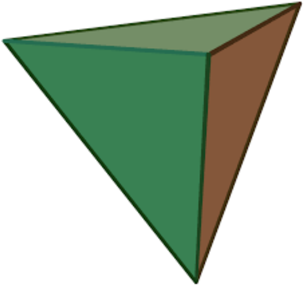

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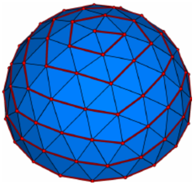
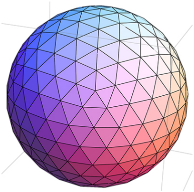
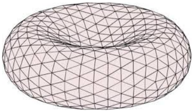
More Examples



2-Hemisphere
$\beta_k = 0, k \geq 1$



2-Sphere
$\beta_2 = 1$, else 0



2-Torus
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- Recall that ST is edges E such that
  $\beta_0(V \cup E) = \beta_0(G), \beta_1(V \cup E) = 0$.

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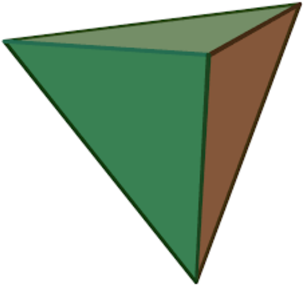

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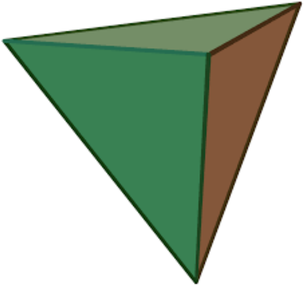

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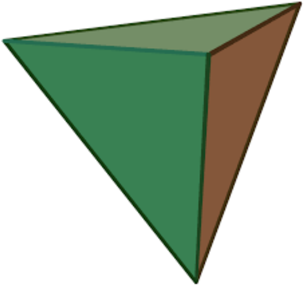

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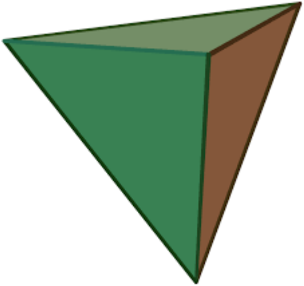

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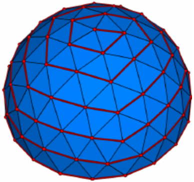
- 2-Spanning acycle : Any 3 of the 4 triangles/2-faces.

More Examples again



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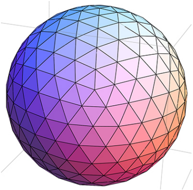
SA is itself.



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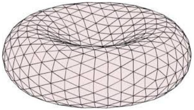
Remove any one triangle for SA.



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- Generalizes the connection between MST and H_0 persistence.

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- Corollary was proven by Hiraoka-Shirai '15 by different methods.

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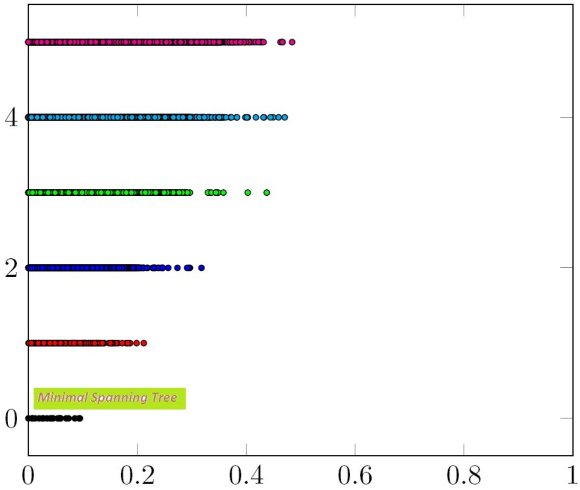
- MSA_d - d-Minimal spanning acycle i.e., minimal basis.

Simulation

- Point process of Weights of faces in MSA_d for different d's



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- Thresholds - Linial-Meshulam '06, Meshulam-Wallach '09. Marginal distributions - Kahle-Pittel '14.

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- Proof via Kruskal's algorithm.
- Useful in extending various results to models with "noisy" weights.

Some References

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