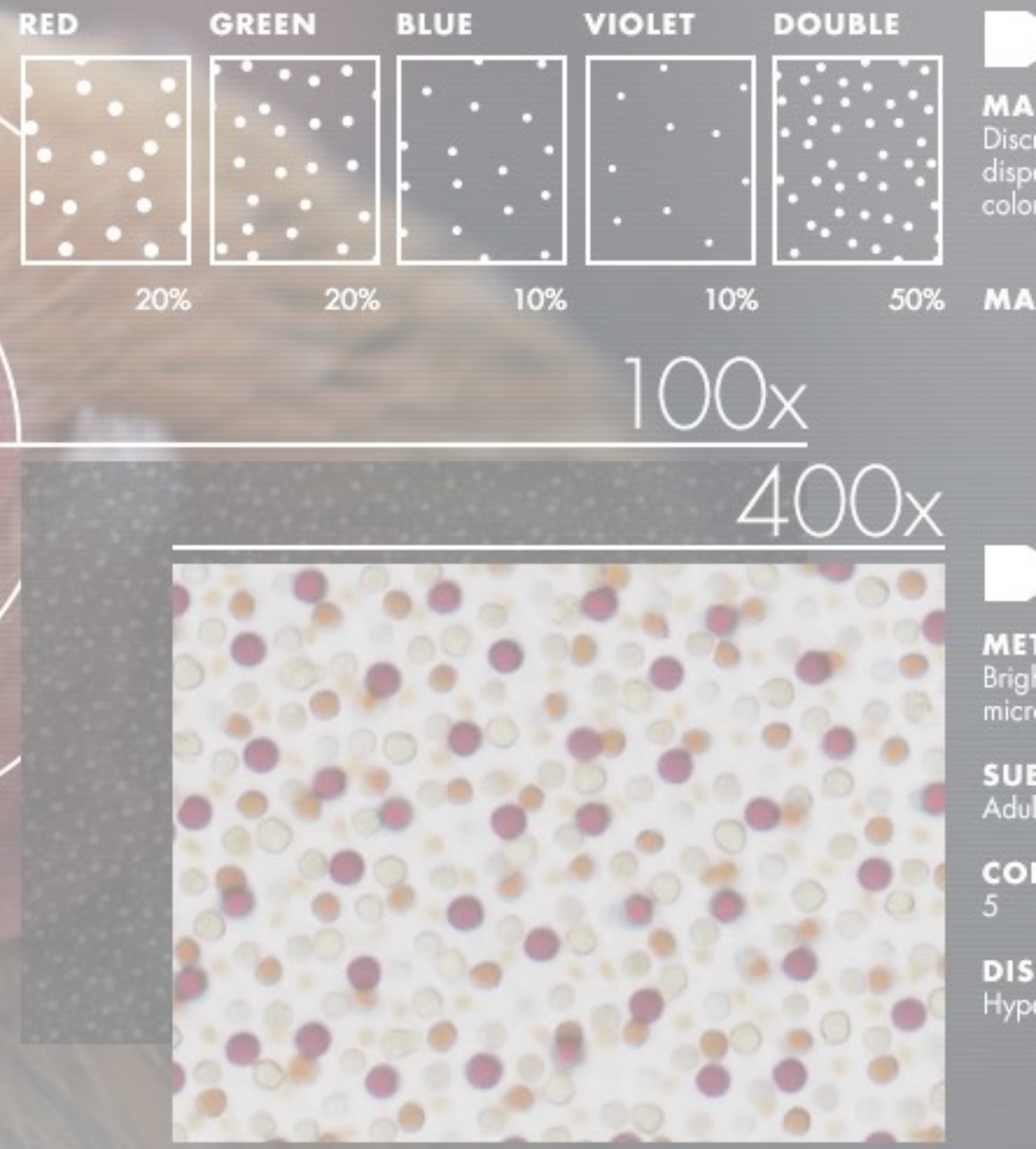
INVARIANT MATCHINGS OF RANDOM POINTS

AND

HYPERUNIFORMITY



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TIFR, Bombay ; October 2025

Joint projects with

Michael Klatt; Ulm. Guenter Last and Luca Lotz ; Karlsruhe.
arXiv:2506.05907

Raphael Lachieze-Rey; Paris. arXiv:2402.13705

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Disclaimer: Figures are taken from various sources without explicit credits.

Some Point Process Notions

- Point process $\mu = \{x_i\}_{i\geq 1} = \sum_{i\geq 1} \delta_{x_i} \subset \mathbb{R}^d \; ; d \geq 1$

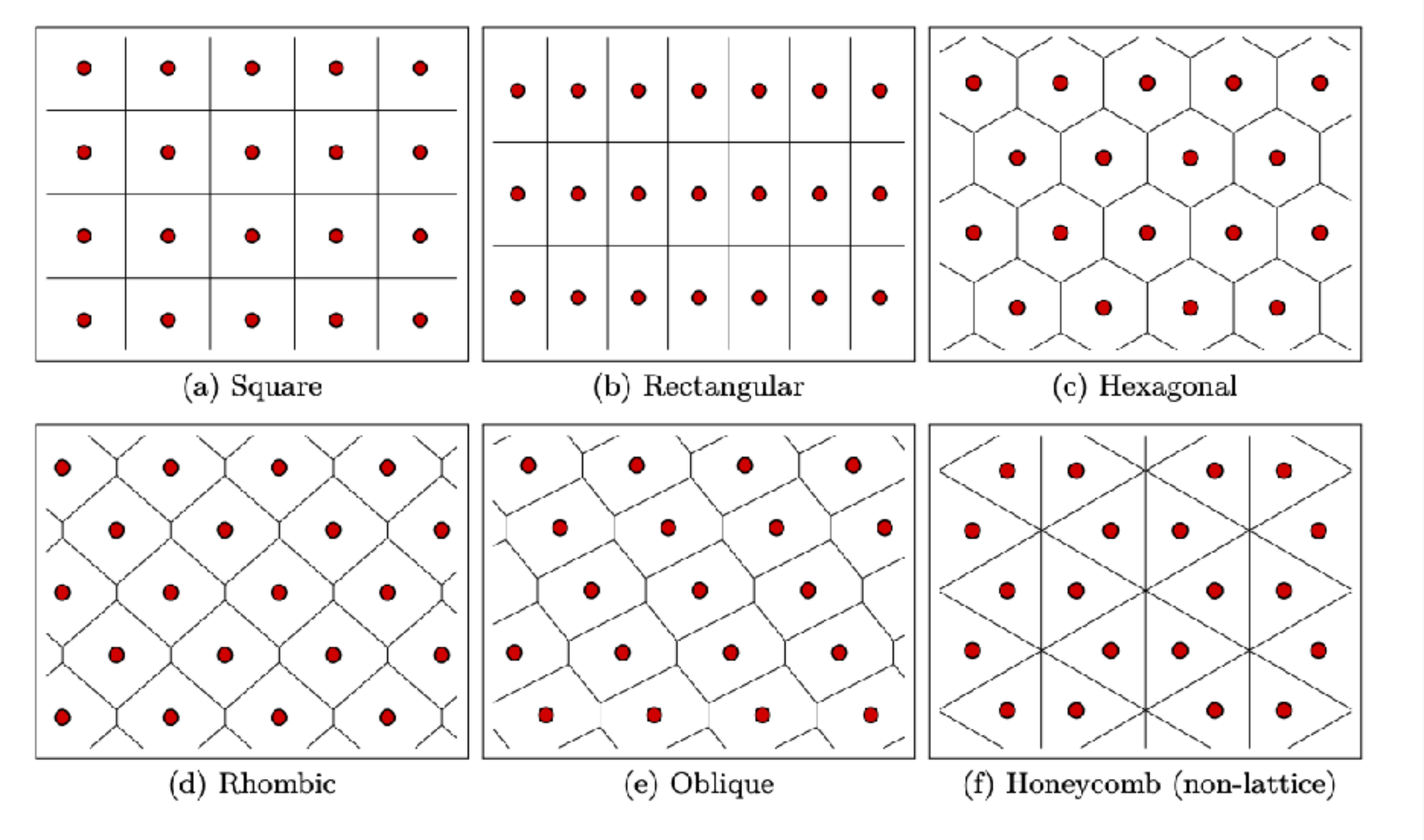
  locally-finite random point set / random counting measures.

- Stationarity / Invariance : $\forall z \in \mathbb{R}^d, \, \mu \overset{d}{=} \mu + z = \{x_i + z\}_i \, .$ $\quad \mu(A) = \# \mu \cap A$

- $\mu = \sum_{i \geq 1} \delta_{x_i}$ - Stationary point process in $\mathbb{R}^d$ with unit intensity ; $\mathbb{E}\mu(A) = |A| \, .$

Examples of Point Processes

Lattice / Crystal - $\mathbb{L}^d$: $\mu = \sum_{z \in \mathbb{L}^d} \delta_{z+U}$ - $U =$ Uniform r.v.



Examples of Point Processes

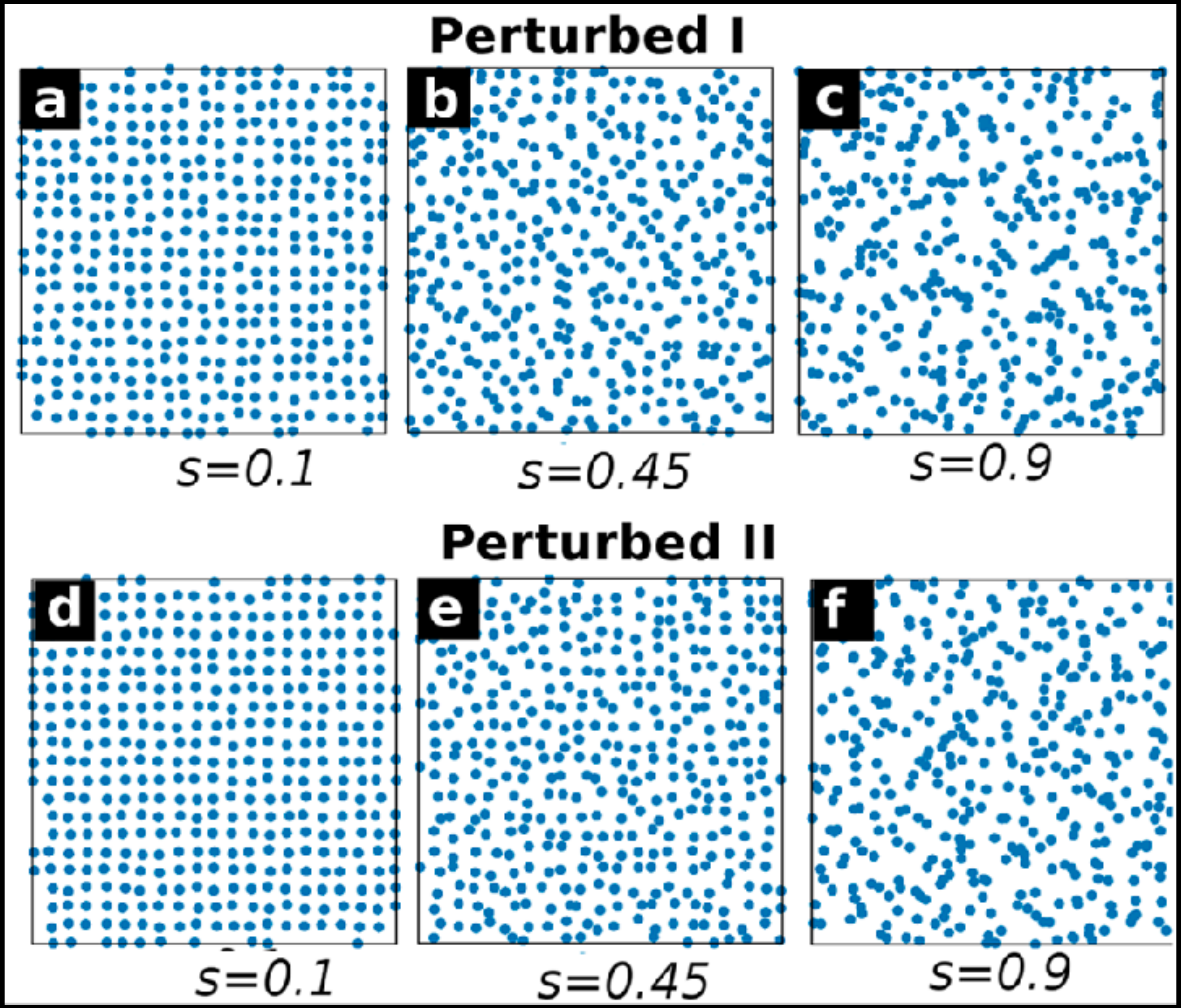
Perturbed Lattices

$$\mu = \sum_{z \in \mathbb{Z}^d} \delta_{z+U+T(z)}$$

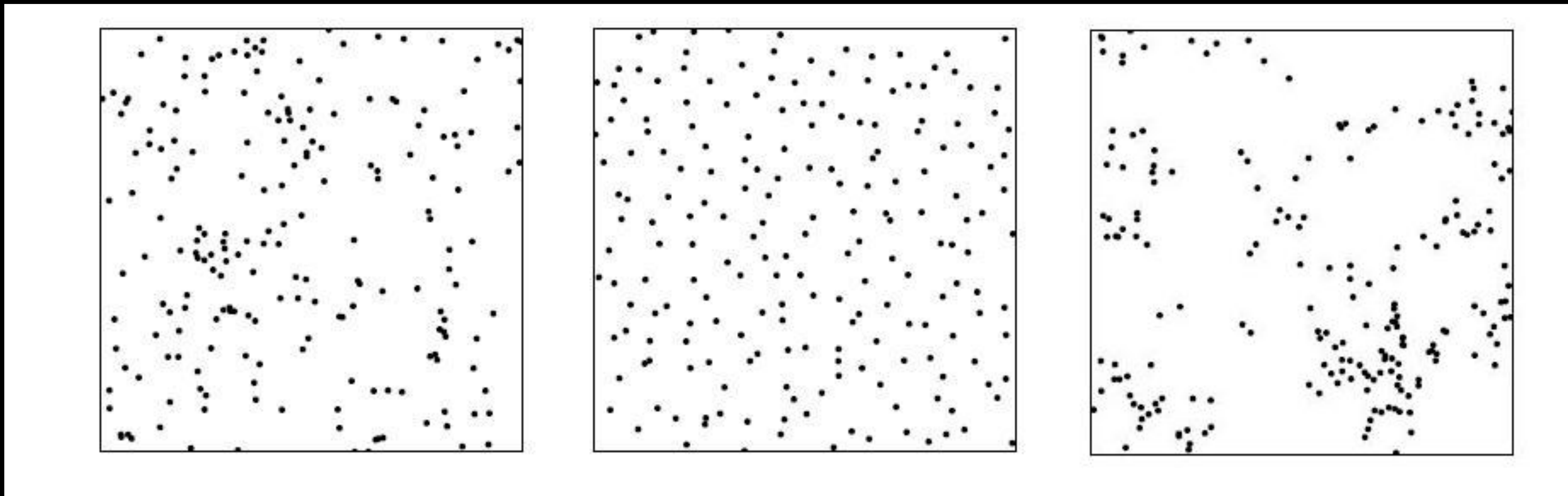
U- Uniform r.v.

$T(z), z \in \mathbb{Z}^d$ - i.i.d. random vectors.

Uniform in $B(0,s)$



Examples of Point Processes



Poisson process | Ginibre process | Cox process

Poisson process: Poisson $|W_n|$ points uniformly and independently in $W_n = [-n, n]^d$

Infinite version of i.i.d random points

Ginibre process: Eigenvalues of $n \times n$ random matrices as $n \to \infty$.

Cox process: Poisson points with randomized intensity measure.

Hyperuniformity

- Point process $\mu = \{x_i\}_{i \geq 1} = \sum_{i \geq 1} \delta_{x_i} \subset \mathbb{R}^d$, $d \geq 1$; locally-finite random point set

- Stationarity / Invariance : $\mu \stackrel{d}{=} \mu + z = \{x_i + z\}_i \; \forall z \in \mathbb{R}^d$. $\quad \mu(A) = \#\mu \cap A$

- $\mu = \sum_{i \geq 1} \delta_{x_i}$ - Stationary point process in $\mathbb{R}^d$ with unit intensity; $\mathbb{E}\mu(A) = |A|$.

- FOCUS OF TALK: Is $\lim_{R \to \infty} R^{-d} \, VAR \, \mu(B_R)$ = 0 ? $\quad B_R = B_R(0) \quad ; \quad |B_R| = \pi_d R^d$

- For many point processes, $\sigma_\mu^2 := \lim_{n \to \infty} R^{-d} \, VAR \, \mu(B_R) < \infty$. Is the limit zero ?

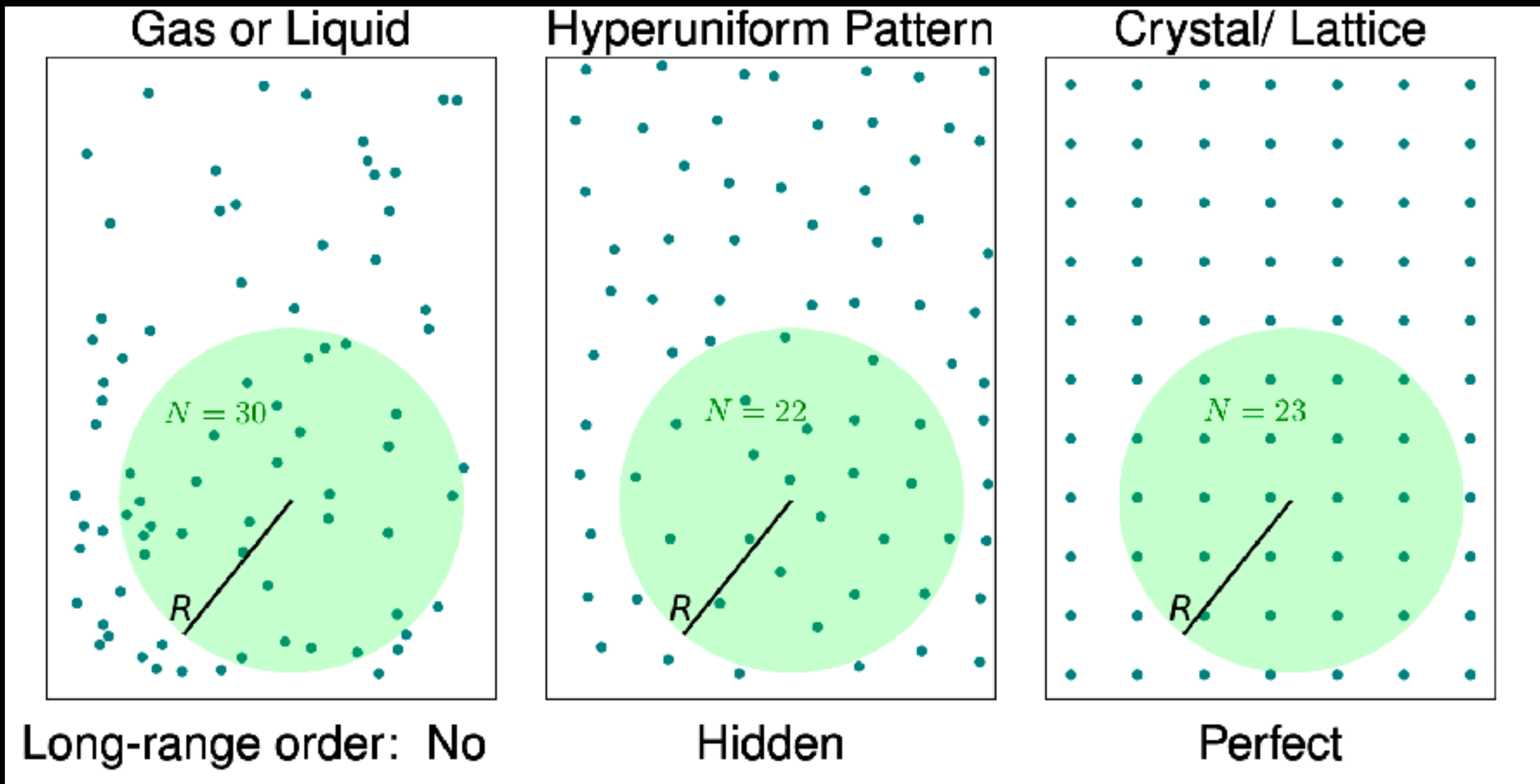
- HYPERUNIFORMITY (HU) / SUPER-HOMOGENEITY : $\sigma_\mu^2 = 0$.

Indicates "regularity of point patterns", "Long-range order and hidden short-range disorder"

Gabrielli, Joyce, Labini (2002) ; Torquato-Stillinger (2003) ; Torquato (2018)

Hyperuniformity (HU) - $\mathbb{V}AR\big(\mu(B_R)\big) = o(R^d)$

'Large-scale suppression of fluctuations' or `local disorder and hidden long-range order'



$VAR(\mu(B_R)) \approx R^d$

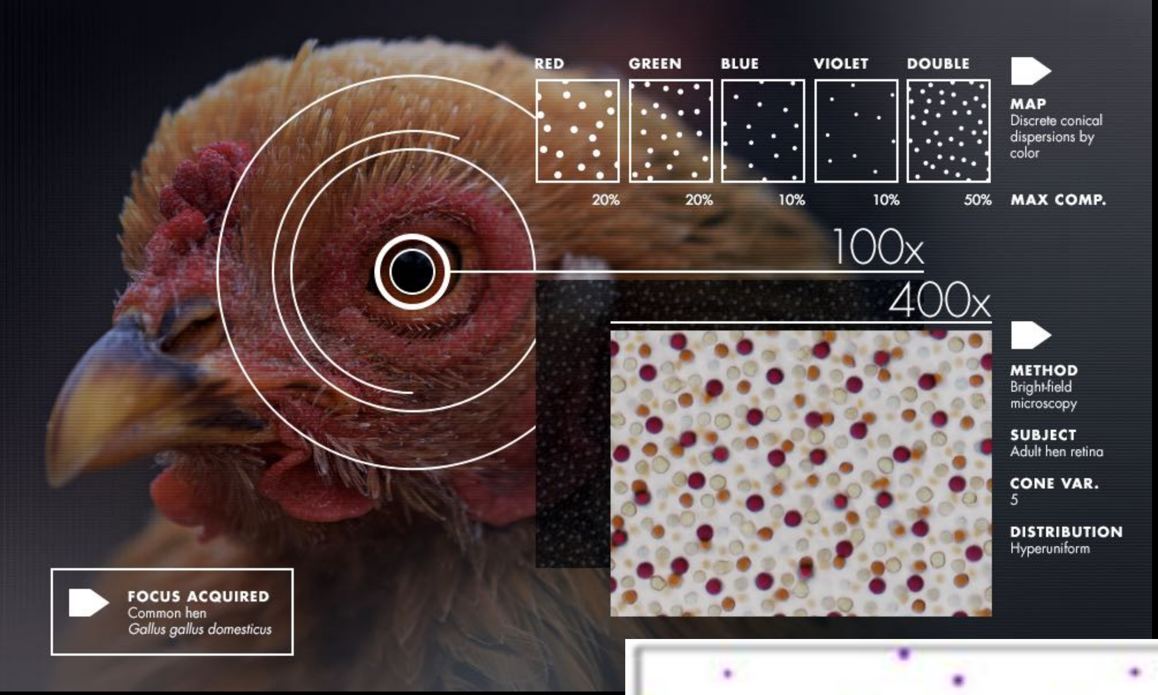
$VAR(\mu(B_R)) = o(R^d)$

$VAR(\mu(B_R)) = O(R^{d-1})$

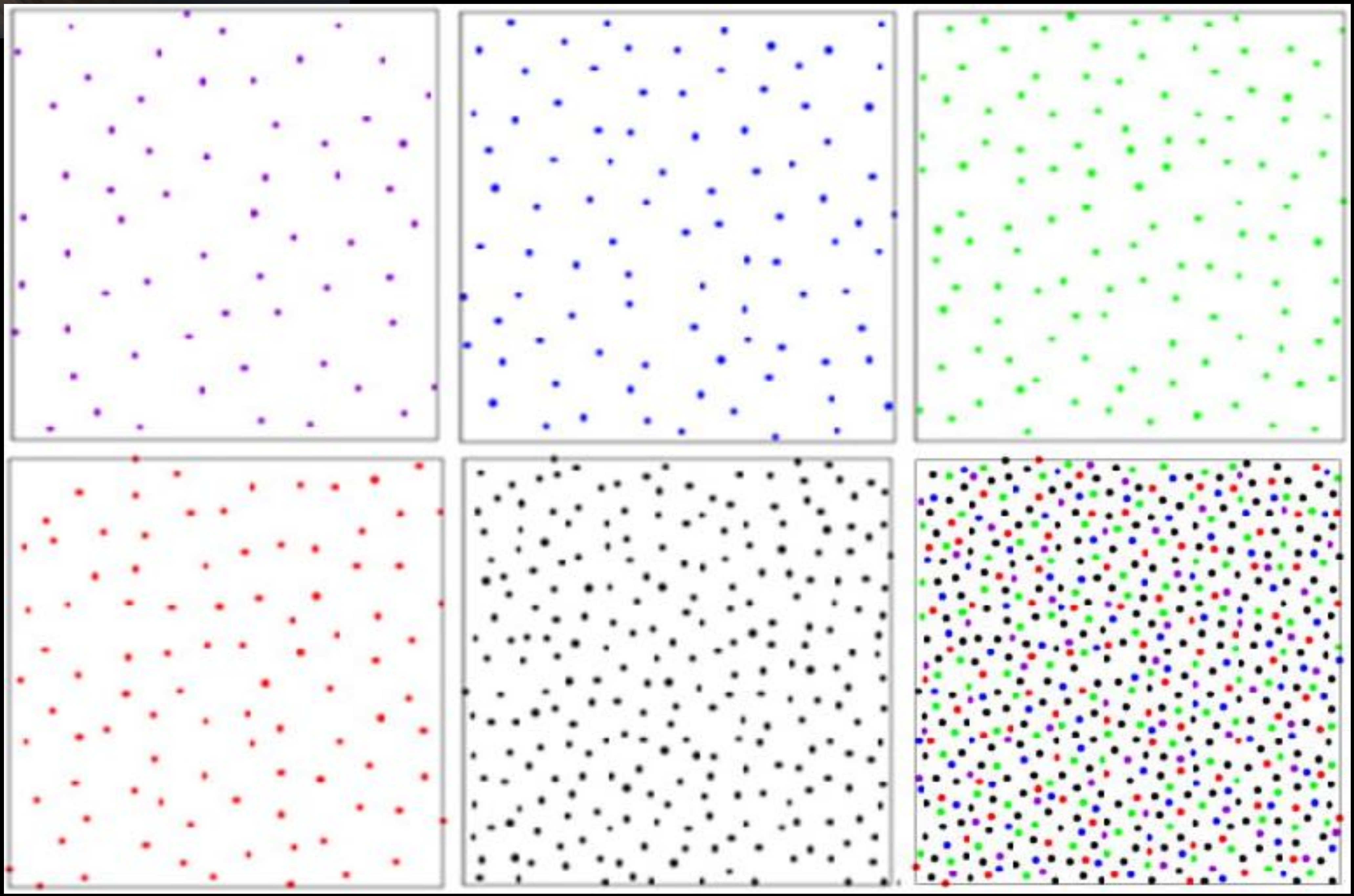
Volume-order variance

Lower than
Volume-order variance

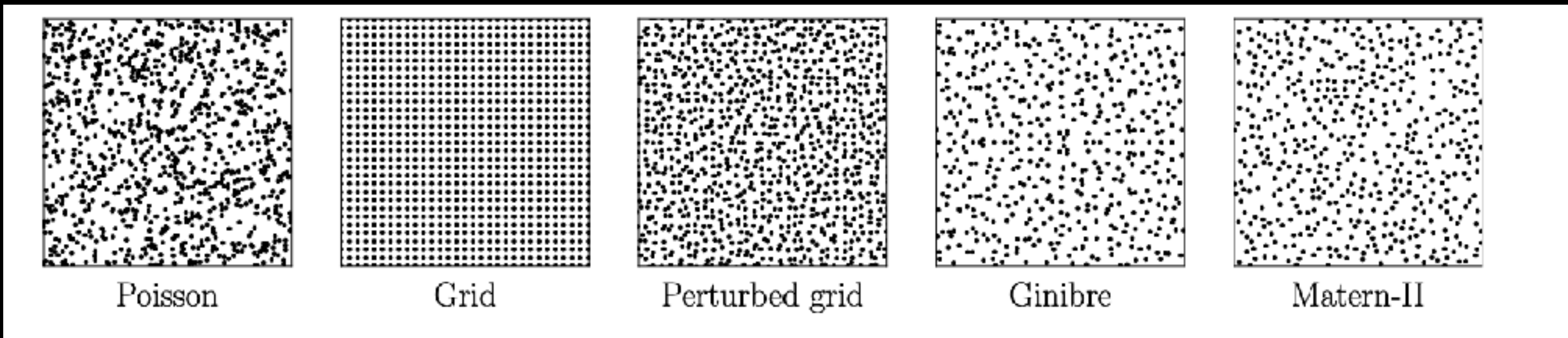
Surface-order variance



Simulated point configurations representing the spatial arrangements of chicken cone photoreceptor



Hyperuniform Point process



NOT HU

Total

Randomness

$$\mu = \sum_{z \in \mathbb{Z}^d} \delta_{z+U}$$

$$\mu = \sum_{z \in \mathbb{Z}^d} \delta_{z+U+T(z)}$$

Gacs-Sasz ‘ 75.

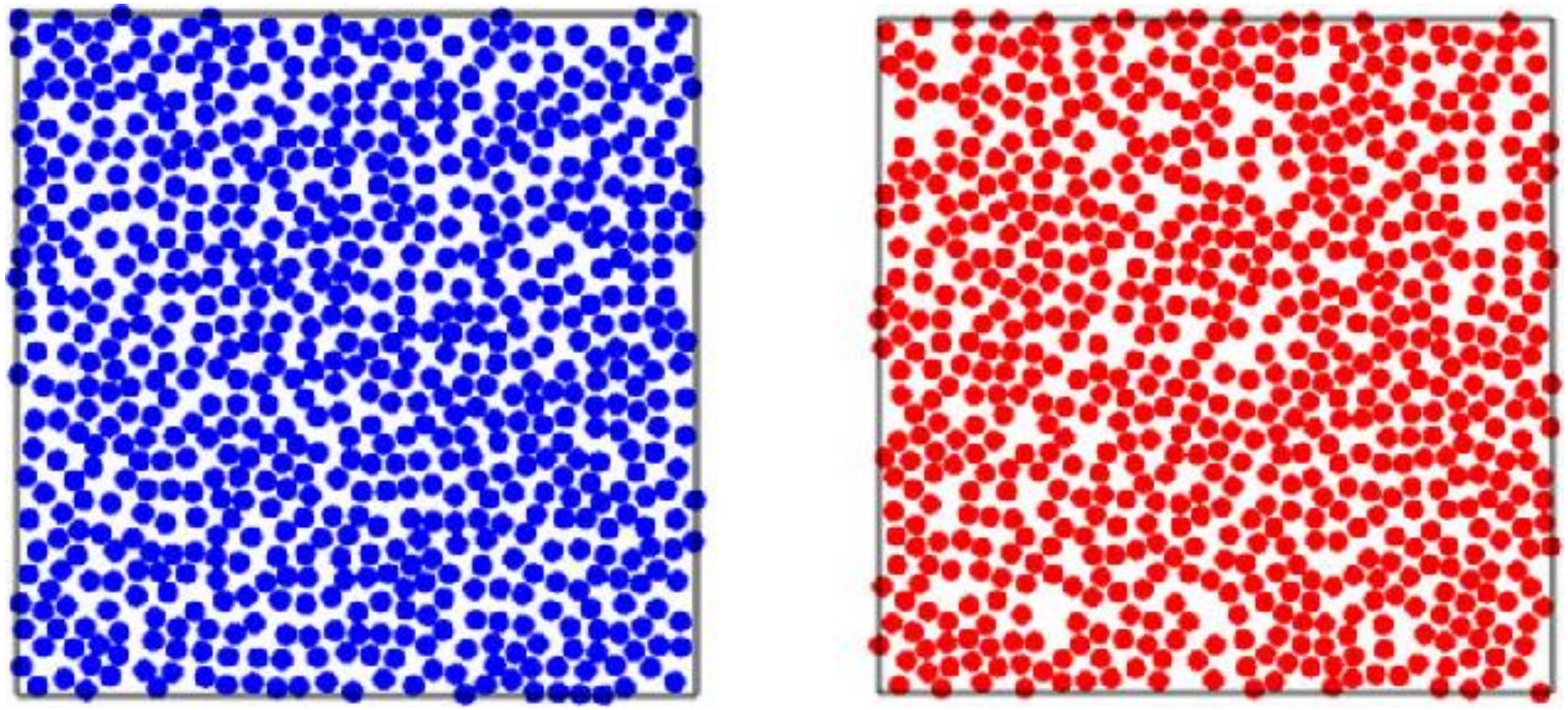
Q theory problem

of Cox.

Eigenvalues

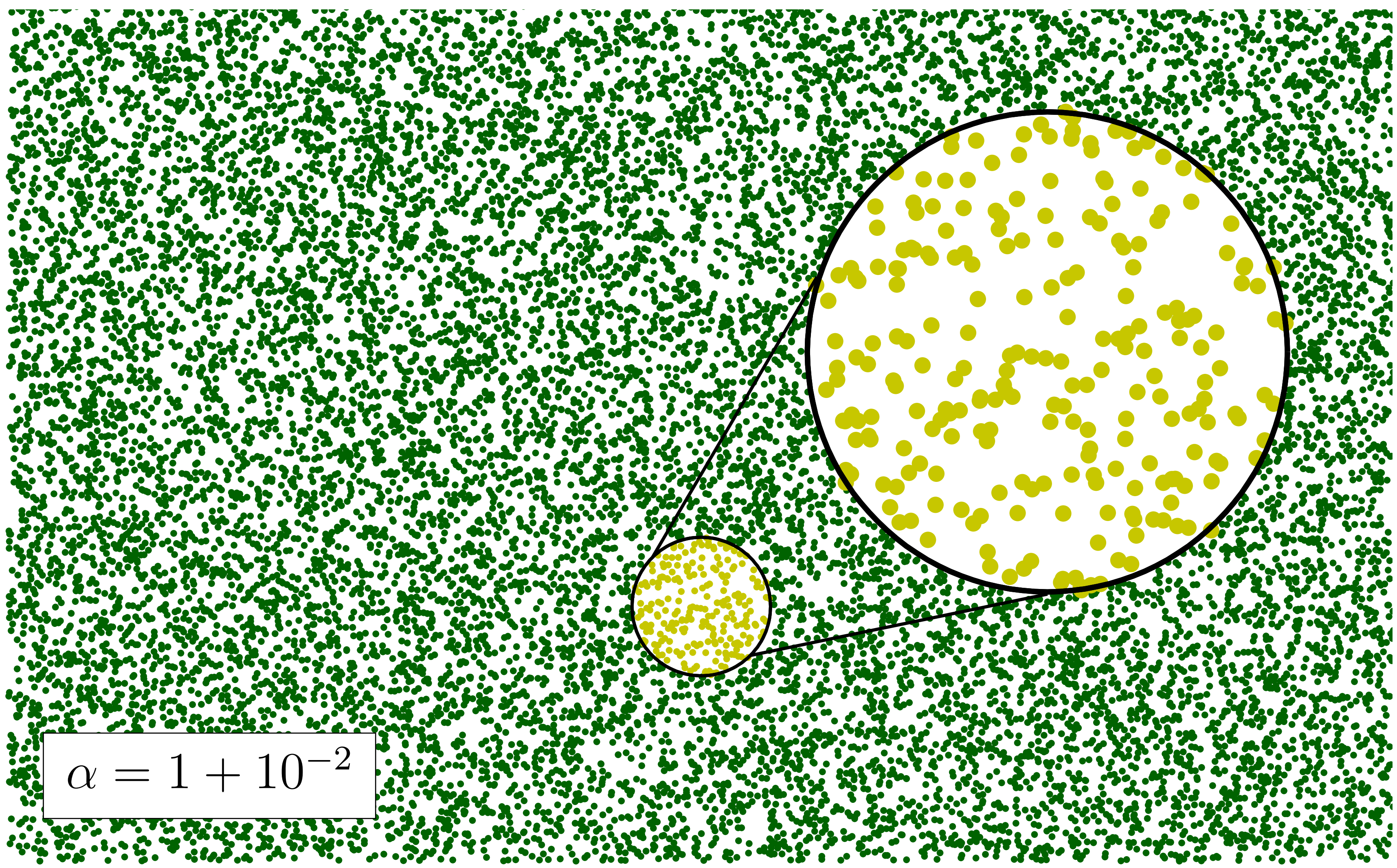
of complex

Random matrix.



Random sequential adsorption models close to saturation

Not HU



Stable partial matching of a Poisson process with lattice.

HU

Some Point Process Notions

- $\mu = \sum_{i \geq 1} \delta_{x_i}$ - Stationary point process in $\mathbb{R}^d, (d \geq 1)$ with unit intensity; $\mathbb{E}\mu(A) = |A|$.

- Reduced Pair Correlation Measure (RPCM)

$$\beta_\mu(\mathrm{d}x) := \alpha_\mu(\mathrm{d}x) - 1 \approx \frac{\mathbb{P}\big(\mathrm{d}x \in \mu \mid 0 \in \mu)}{\mathrm{d}x} - 1 \ ;$$

$\beta \equiv 0$ for Poisson

$\beta = \sum_{z \in \mathbb{L}^d} \delta_z - 1$ for lattices

Formally, for compactly supported φ

$$\mathbb{E}\Big[\sum_{x \neq y \in \mu} \varphi(x)\psi(y)\Big] = \int \varphi(x)\,\psi(y)\,\mathrm{d}x\,\mathrm{d}y + \int \varphi(x)\,\psi(x+z)\,\beta(\mathrm{d}z)\,\mathrm{d}x$$

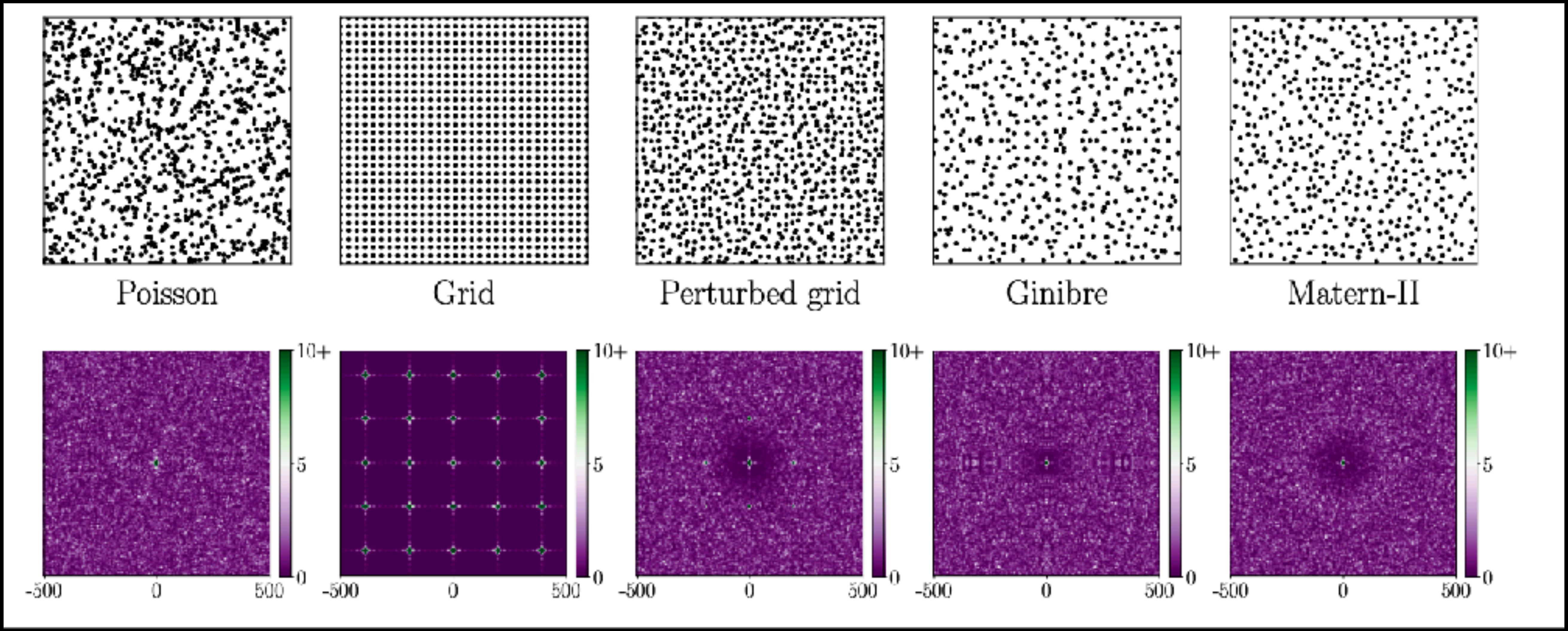
- Variance formula - If β is integrable, $\mathbb{V}AR\big(\mu(B_R)\big) = \pi_d R^d\big(1 + \beta(\mathbb{R}^d)\big) + o(R^d)$.

- Structure Factor - $S(k) = 1 + \int e^{-ik\cdot x}\,\beta(\mathrm{d}x),\ k \in \mathbb{R}^d$; well-defined if β is integrable.

- Hyperuniformity (HU) - $\mathbb{V}AR\big(\mu(B_R)\big) = o(R^d)$ iff $\beta(\mathbb{R}^d) = -1$ iff $S(0) = 0$. Integrable β

Coste (2023) ; Torquato (2018) ; Bjorklund - Bylehene (2024) ;

Hyperuniform Point process



Many examples of stationary HU processes are $\mu = \{z + U + T(z) : z \in \mathbb{Z}^d\}$ where $T : \mathbb{Z}^d \to \mathbb{R}^d$ is a $\mathbb{Z}^d-$invariant random field with $|T(0)|$ having good moments and mixing properties.

T - Perturbation of $\mathbb{Z}^d$ or Matching between $\mathbb{Z}^d$ to μ

$|T(0)| -$ Typical matching cost or perturbation distance.

Invariant Matchings and Hyperuniformity

Many examples of stationary HU processes are $\{z + U + T(z) : z \in \mathbb{Z}^d\}$ where $T : \mathbb{Z}^d \to \mathbb{R}^d$ is a $\mathbb{Z}^d-$invariant random field with $T(0)$ having good moments and mixing properties.

$|T(0)| -$ Typical matching cost or perturbation distance.

Are all HU processes Perturbed lattices or

Invariant matchings of a lattice ?

Do `GOOD' invariant matchings of lattice / HU processes

give rise to HU processes ?

Perturbed Lattices and Hyperuniformity : Mixed News

Examples of perturbed lattices $\mu = \{z + U + T(z) : z \in \mathbb{Z}^d\}$

- If $T(z), z \in \mathbb{Z}^d$ are i.i.d. then μ is Hyperuniform. Gacs-Sasz '75
- $\mu-$ Poisson in $d \geq 3$; Not Hyperuniform but there exist T with $|T(0)|$ having exponential moments.
  Shor, Yukich, Talagrand '80s, '90s ….. Holroyd, Peres, Pemantle, Schramm '06
- Zeros of GAF in d = 2 ; Hyperuniform and $|T(0)|$ has exponential moments.
  Sodin, Tsirelson '10
- Most 'nice' point processes in $d \geq 3$ have $\mathbb{E}|T(0)|^2 < \infty$; Lachieze-Rey, Y. '24
- In d = 2, if μ is Hyperuniform and $\mathbb{V}AR\, \mu(B_R) = o(R^2/\log R)$
  then $\mathbb{E}|T(0)|^2 < \infty$.

Lachieze-Rey, Y. '24 ; Butez, Dallaporta, Garcia-Zallada '24; Huesmann, Leble '24

'Most' Planar HU processes are invariant matchings of the lattice.

But not so in higher dimensions !

How do we construct invariant matchings

that give rise to HU processes ?

Good Moments on $|T(0)|$ suffice ???

AND / OR

Mixing / Asymptotic Independence of $T(z)$ suffice ???

Transport of lattice

- Stable partial matching of Lattice to Poisson of higher intensity for all dimensions ; Hyperuniform and $|T(0)|$ with exponential moments. Klatt, Last, Y. '20.

- In d = 2, if $\mathbb{E}|T(0)|^2 < \infty$ then $\mu = \{z + U + T(z) : z \in \mathbb{Z}^2\}$ is Hyperuniform. Dereudre, Flimmel, Huesmann, Leble '23.

- $\kappa(z) := \| \mathbb{P}[(T(0), T(z)) \in \cdot\ ] - \mathbb{P}[T(0) \in \cdot\ ]^{\otimes 2} \|_{TV} \quad z \in \mathbb{Z}^d, \mathbb{R}^d$.

- $\|\nu\|_{TV} := \sup\{ \int f \,\mathrm{d}\nu \ : \ |f| \leq 1\}$ $\nu-$signed finite measure.

- $T(\ \cdot\ )$ - Gaussian random field. Then $\kappa(z) \leq C_d \|\mathrm{COV}(T(0), T(z))\|$

- KLLY '25: If $\sum_{z \in \mathbb{Z}^d} \kappa(z) < \infty$ then μ is Hyperuniform.

- Works in all dimensions ; No moment assumptions only WEAK MIXING

  i.e., Asymptotic independence

General Transport

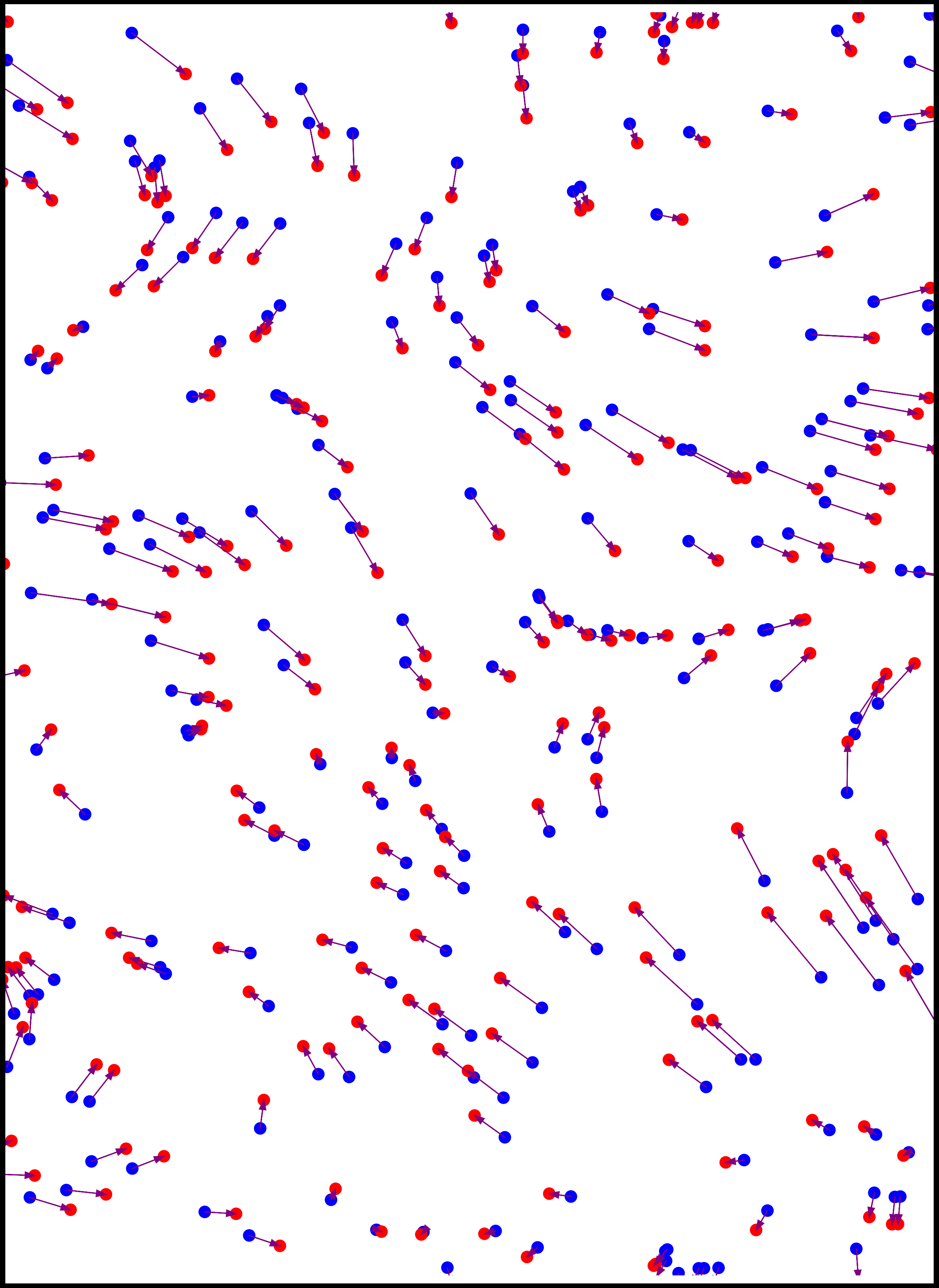
Invariant point process Φ, SOURCE and Independent Transport map T,

TARGET
$T\Phi := \{X + T(X) \ : X \in \Phi\}$

- KLLY '25 : If Φ is locally square integrable and $\int \kappa(z)\, \alpha_\Phi(\mathrm{d}z) < \infty$

Then $\sigma^2_{T\Phi} = \sigma^2_\Phi$

where $\sigma^2_\Phi := \lim R^{-d}\, \mathbb{V}AR\, \Phi(B_R)$



Blue points shifted by a short-range Gaussian field

Hyperuniformerer

SOURCE Invariant point process Φ ; FAIR PARTITION $C(x, \Phi), x \in \Phi$
i.e., disjoint interiors, $\mathbb{R}^d = \cup_{x \in \Phi} C(x, \Phi)$, $|C(x, \Phi)| = 1$ and $x \in C(x, \Phi)$.

Transport Map

$T(x), x \in \Phi$ are i.i.d. Uniform in

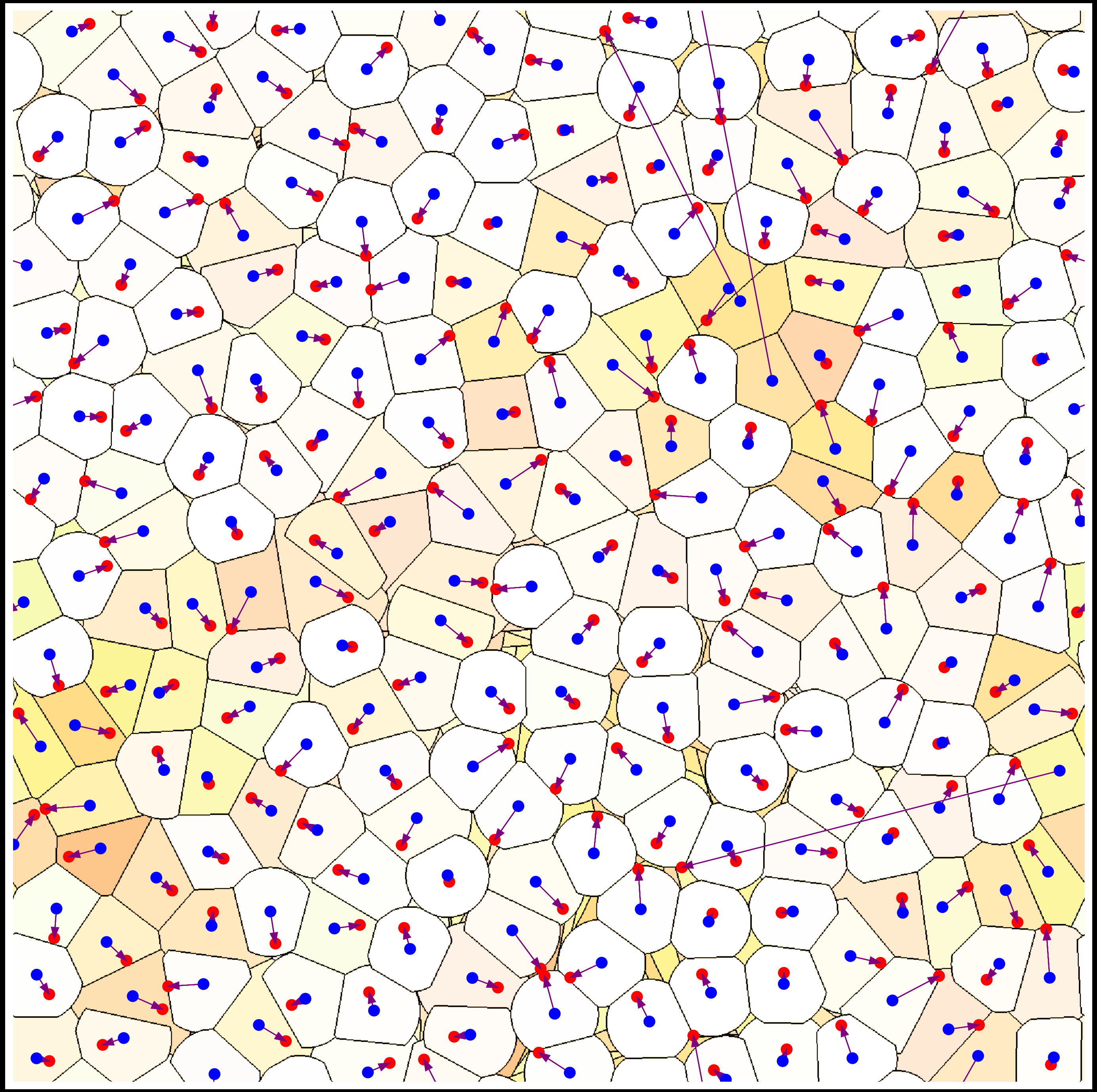
$C(x, \Phi) - x$, given the partition.

TARGET
$T\Phi := \{x + T(x) \ : x \in \Phi\}$

KLLY '25 : $T\Phi$ is Hyperuniform.

How ? $\mathbb{E}[\delta_{T(x)} \mid T] = \lambda_d(C(x) \cap \cdot\ )$

and HU of $\sum_{x \in \Phi} \lambda_d(C(x) \cap \cdot\ ) = \lambda_d$



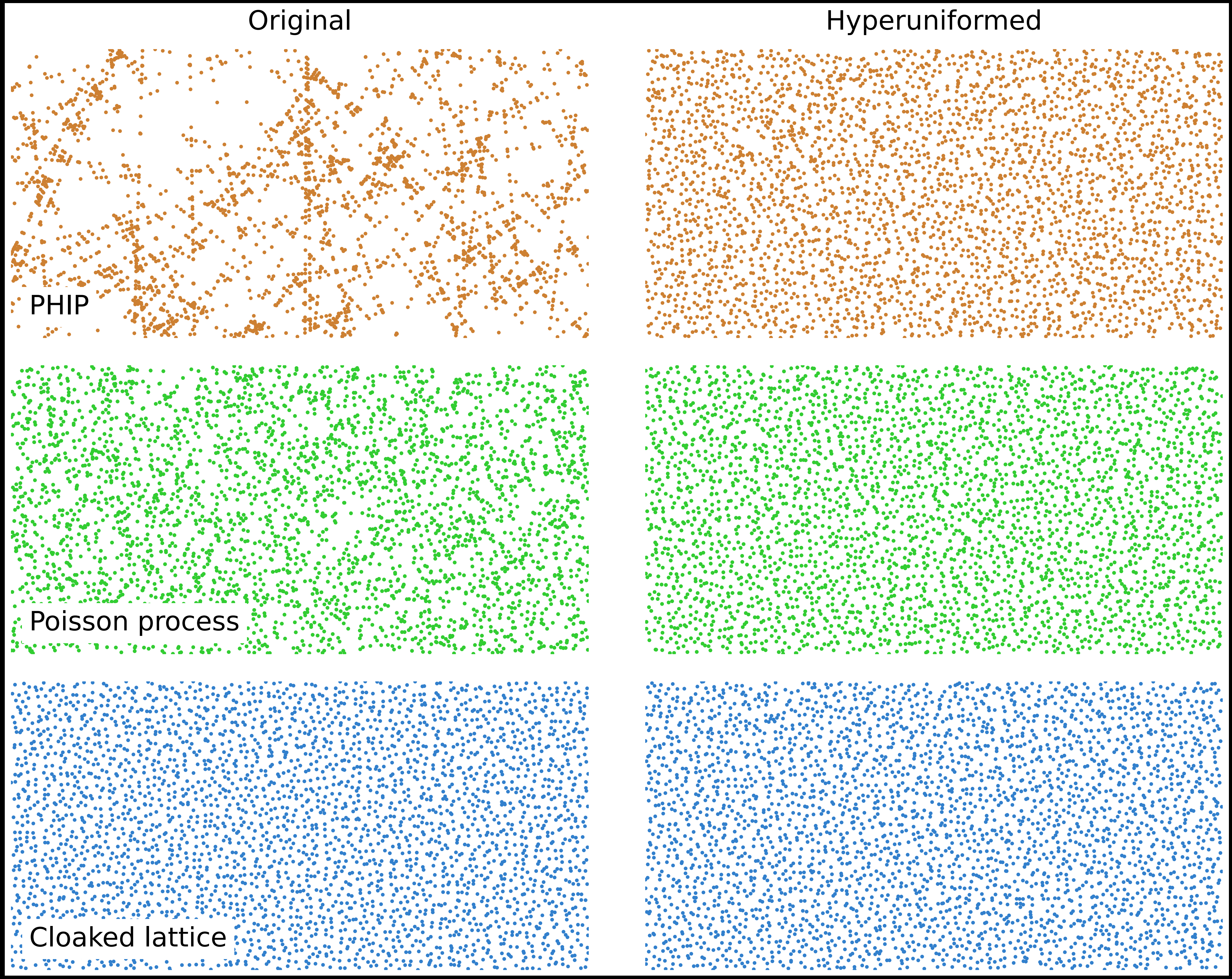
Fair partition via stable matching.

Hyperuniformed Samples

$\sigma_\Phi^2 = \infty$

$\sigma_\Phi^2 = 1$

$\sigma_\Phi^2 = 0$



The key driving principle

Invariant point process Φ SOURCE ; Transport map T ;

TARGET $\quad T\Phi := \{X + T(X) \ : X \in \Phi\}$

- Reduced Pair Correlation Measure (RPCM)

$$\beta_\Phi(\mathrm{d}x) := \alpha_\Phi(\mathrm{d}x) - 1 \ \approx \ \frac{\mathbb{P}\big(\mathrm{d}x \in \Phi \mid 0 \in \Phi\big)}{\mathrm{d}x} - 1 \ ;$$

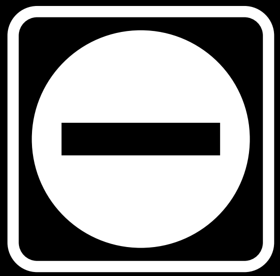
$\beta \equiv 0$ for Poisson

$\beta = \sum_{z \in \mathbb{L}^d} \delta_z - 1$ for lattices

$$\mathbb{V}AR\, T\Phi(W) = \mathbb{V}AR\, \Phi(W) + \int_W \eta(W - x)\, \mathrm{d}x \ , \quad W \text{ - convex window}$$

and where $\quad \eta := \alpha_{T\Phi} - \alpha_\Phi,$ signed measure.

If $\dfrac{1}{|B_R|}\displaystyle\int_{B_R} \eta(B_R - x)\, \mathrm{d}x \to 0$ then $\sigma^2_{T\Phi} = \sigma^2_\Phi$.



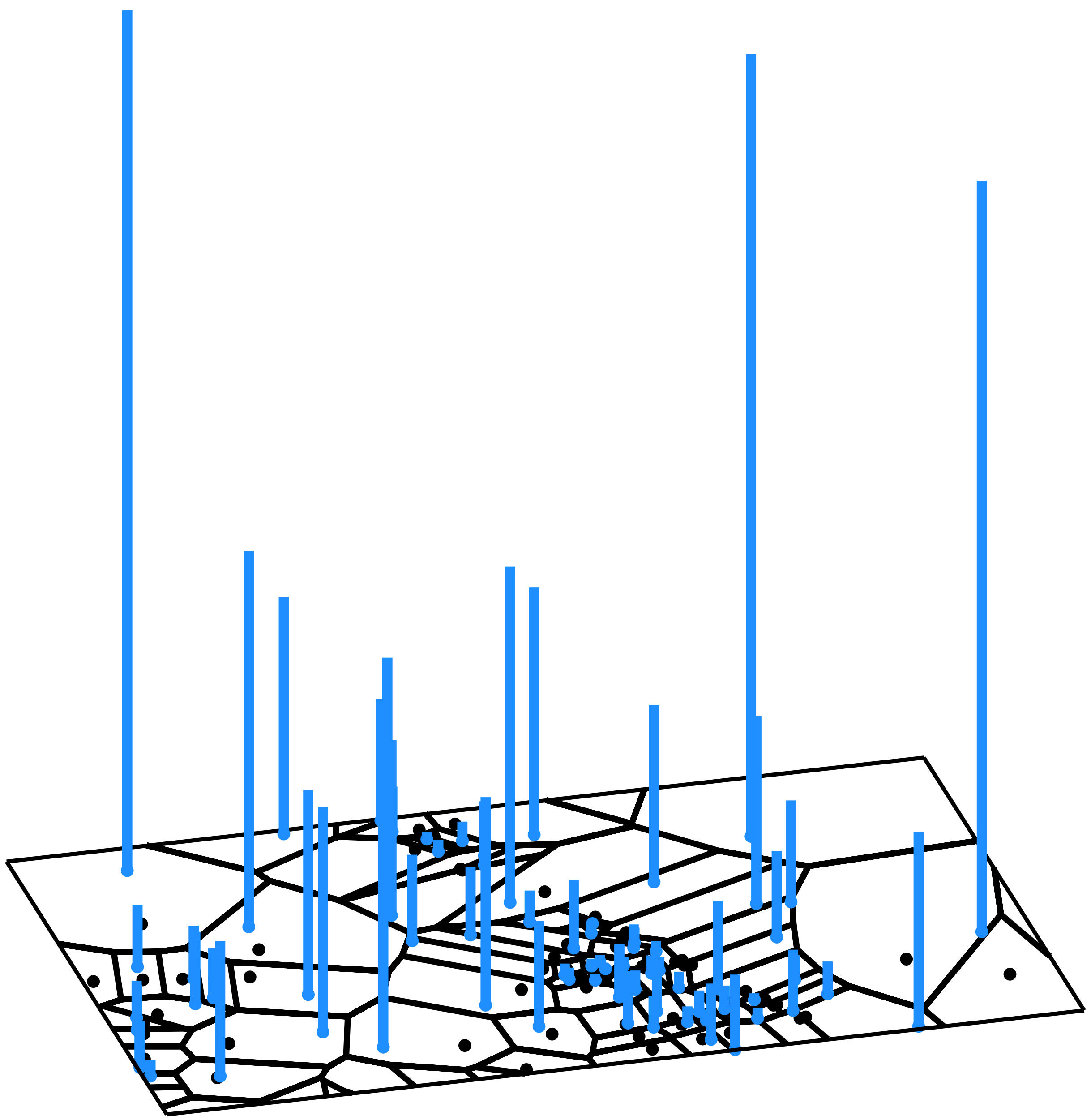
- Two targets: T_1, T_2 two transport maps. We apply the above idea to show that $\sigma^2_{T_1\Phi} = \sigma^2_{T_2\Phi}$ with $\eta = \alpha_{T_1\Phi} - \alpha_{T_2\Phi}$.

Hyperuniform weighted point processes

Invariant point process Φ SOURCE ; Partition $V(x, \Phi), x \in \Phi$ with $x \in V(x, \Phi)$ i.e., disjoint interiors, $\mathbb{R}^d = \cup_{x \in \Phi} V(x, \Phi)$, $\mathbb{E}_x | V(x, \Phi) | = 1$

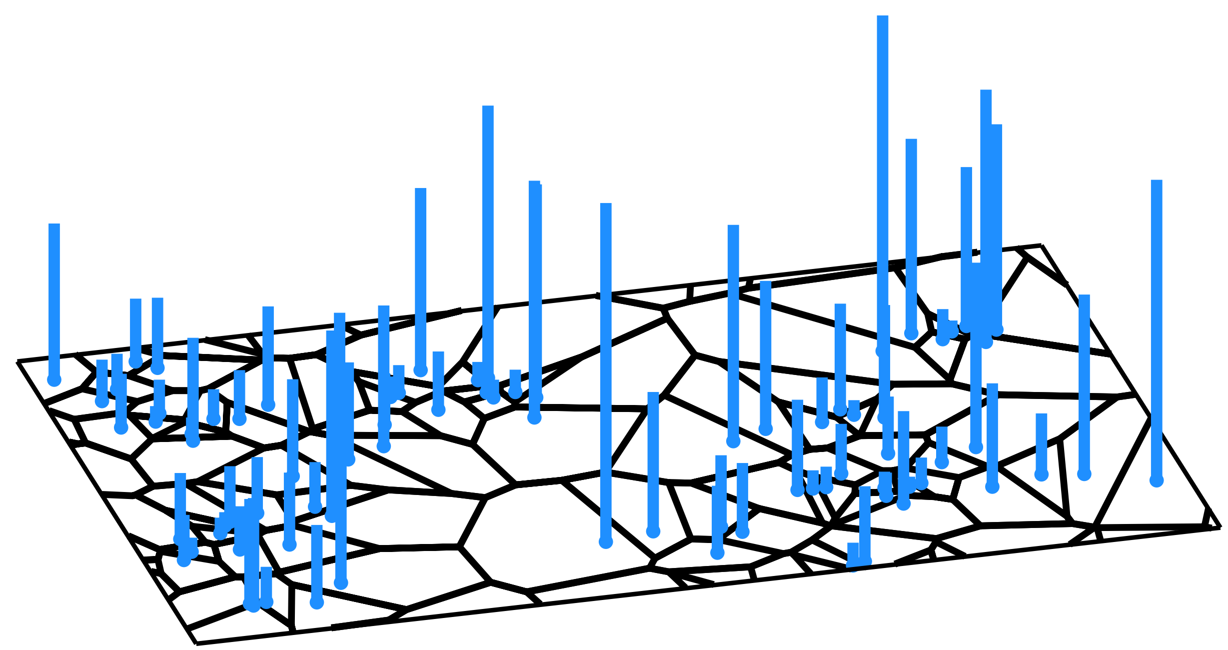
$T_0 \Phi = \sum_{x \in \Phi} | V(x, \Phi) | \delta_{U(x)}$ is

Hyperuniform if $U(x)$ - Uniform in $V(x, \Phi)$



$T_1 \Phi = \sum_{x \in \Phi} | V(x, \Phi) | \delta_x$ is

Hyperuniform if partition is formed by some local rule, Φ has good mixing properties, exponential hole probabilities.



Lloyd's Algorithm

Poisson point process Φ SOURCE ; Partition $V(x,\Phi), x \in \Phi$

i.e., disjoint interiors, $\mathbb{R}^d = \cup_{x\in\Phi} V(x,\Phi)$, $\mathbb{E}_x |V(x,\Phi)| = 1$ and $x \in V(x,\Phi)$.

Partition is using some 'nice' local rule - For eg., Voronoi partition.

$$\Phi_0 := \Phi \ ; \ \Phi_1 = \sum_{x\in\Phi} \delta_{Ce(x,\Phi)} \quad Ce(x,\Phi) \text{ - Centroid of } V(x,\Phi)$$

$$\Phi_k = \sum_{x\in\Phi_{k-1}} \delta_{Ce(x,\Phi_{k-1})} \quad Ce(x,\Phi_{k-1}) \text{ - Centroid of } V(x,\Phi_{k-1})$$

Then $\sigma^2_{\Phi_k} = \sigma^2_{\Phi} = 1$, i.e., the original variance is preserved.

In contrast to the weighted Voronoi tessellation.

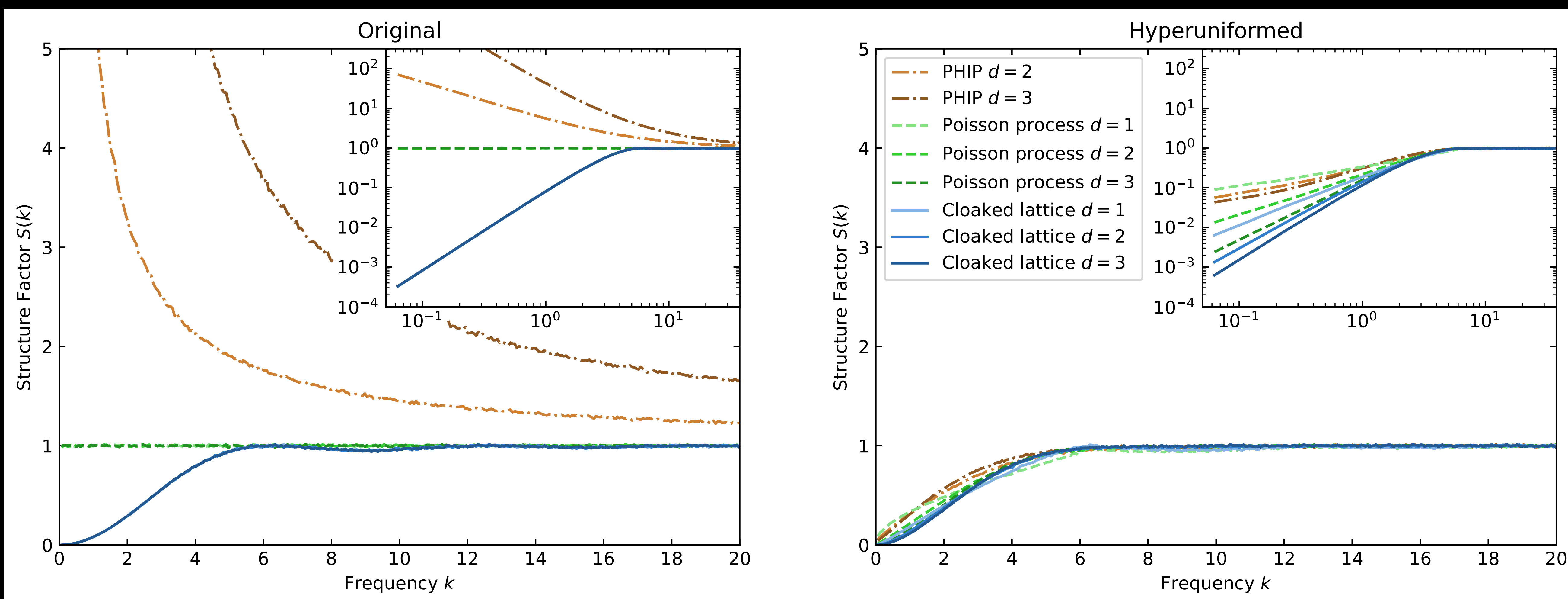
Infinite iterations of Lloyd's algorithm conjectured to give a lattice-like structure and hence hyperuniform !!!

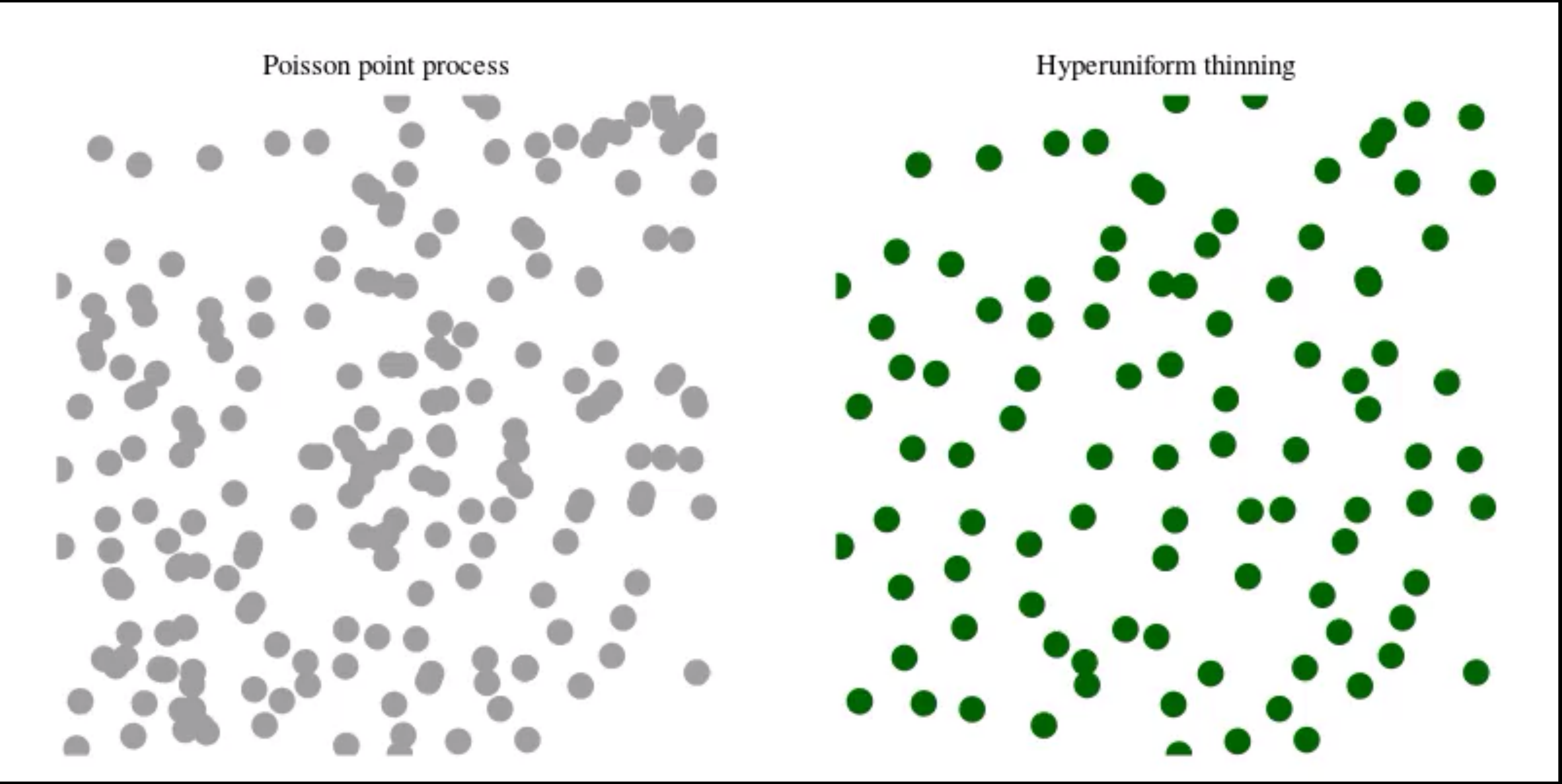
What else ?

- The general principles and many examples work for more general random measures.

- Rates of hyperuniformity - Better measured via Structure factor !

Results on rates of target vs source.





NEVER NEGLECT ZERO VARIANCE RANDOM MEASURES