Comparison of point processes and applications to random geometric networks

D. Yogeshwaran
Indian Statistical Institute Bangalore.

Joint work with: B. Błaszczyszyn.
ENS-INRIA, Paris.

EVA, Ann Arbor, June 2015.

Point Processes

- **Point process** - locally finite random collection of points in $\mathbb{R}^d$.
- $\Phi = \{X_i\}_{i \geq 1} \subset \mathbb{R}^d$, such that no: of points within a bounded Borel subset (bBS) B,$\Phi(B) < \infty$ *a.s.*.

Point Processes

- **Point process** - locally finite random collection of points in $\mathbb{R}^d$.
- $\Phi = \{X_i\}_{i \geq 1} \subset \mathbb{R}^d$, such that no: of points within a bounded Borel subset (bBS) B,$\Phi(B) < \infty$ *a.s.*.
- **simple** - points are a.s. distinct.

Point Processes

- **Point process** - locally finite random collection of points in $\mathbb{R}^d$.
- $\Phi = \{X_i\}_{i \geq 1} \subset \mathbb{R}^d$, such that no: of points within a bounded Borel subset (bBS) B,$\Phi(B) < \infty$ *a.s.*.
- **simple** - points are a.s. distinct.
- Φ is a Po(λ) pp if

Point Processes

- **Point process** - locally finite random collection of points in $\mathbb{R}^d$.
- $\Phi = \{X_i\}_{i \geq 1} \subset \mathbb{R}^d$, such that no: of points within a bounded Borel subset (bBS) B,$\Phi(B) < \infty$ *a.s.*.
- **simple** - points are a.s. distinct.
- Φ is a $\text{Po}(\lambda)$ pp if
  - For disjoint bBs $B_1, \ldots, B_n$, $\Phi(B_1), \ldots, \Phi(B_n)$ are independent rvs.

Point Processes

- **Point process** - locally finite random collection of points in $\mathbb{R}^d$.
- $\Phi = \{X_i\}_{i \geq 1} \subset \mathbb{R}^d$, such that no: of points within a bounded Borel subset (bBS) B,$\Phi(B) < \infty$ $a.s.$.
- **simple** - points are a.s. distinct.
- Φ is a $\mathrm{Po}(\lambda)$ pp if
  - For disjoint bBs $B_1, \ldots, B_n$, $\Phi(B_1), \ldots, \Phi(B_n)$ are independent rvs.
  - $\mathrm{P}(\Phi(B) = k) = e^{-\lambda \|B\|}(\lambda \|B\|)^k / k$ for any bBs B.

Point Processes

- **Point process** - locally finite random collection of points in $\mathbb{R}^d$.
- $\Phi = \{X_i\}_{i \geq 1} \subset \mathbb{R}^d$, such that no: of points within a bounded Borel subset (bBS) B,$\Phi(B) < \infty \ \ a.s.$.
- **simple** - points are a.s. distinct.
- Φ is a $\text{Po}(\lambda)$ pp if
  - For disjoint bBs $B_1, \ldots, B_n$, $\Phi(B_1), \ldots, \Phi(B_n)$ are independent rvs.
  - $\text{P}(\Phi(B) = k) = e^{-\lambda \|B\|}(\lambda \|B\|)^k / k$ for any bBs B.
- **Stationary :** $\Phi + x \overset{d}{=} \Phi$.

Point Processes

- **Point process** - locally finite random collection of points in $\mathbb{R}^d$.
- $\Phi = \{X_i\}_{i \geq 1} \subset \mathbb{R}^d$, such that no: of points within a bounded Borel subset (bBS) B,$\Phi(B) < \infty$ *a.s.*.
- **simple** - points are a.s. distinct.
- Φ is a Po(λ) pp if
    - For disjoint bBs $B_1, \ldots, B_n$, $\Phi(B_1), \ldots, \Phi(B_n)$ are independent rvs.
    - $\mathrm{P}(\Phi(B) = k) = e^{-\lambda \|B\|}(\lambda \|B\|)^k / k$ for any bBs B.
- **Stationary :** $\Phi + x \stackrel{d}{=} \Phi$.
- $\Rightarrow \mathrm{E}(\Phi(B)) = \lambda |B|$. Assume $\lambda = 1$.

Poissonian World

- Analysis of local/global functionals of Poisson or Bernoulli pp. cf. e.g.

Poissonian World

- Analysis of local/global functionals of Poisson or Bernoulli pp. cf. e.g.
  - R. Meester & R. Roy Continuum Percolation,
  - M. Penrose Random Geometric Graphs,
  - J. Yukich Limit theorems in discrete stochastic geometry,
  - G. Peccati & M. Reitzner Stochastic analysis for Poisson point processes **upcoming**
  - P. Calka Tessellation

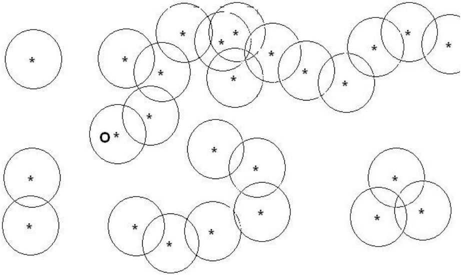
Poissonian World

- Analysis of local/global functionals of Poisson or Bernoulli pp. cf. e.g.
  - R. Meester & R. Roy Continuum Percolation,
  - M. Penrose Random Geometric Graphs,
  - J. Yukich Limit theorems in discrete stochastic geometry,
  - G. Peccati & M. Reitzner Stochastic analysis for Poisson point processes **upcoming**
  - P. Calka Tessellation
- What about non-Poisson point processes ? Eg. Cox point processes, perturbed lattices, Determinantal point processes et al.

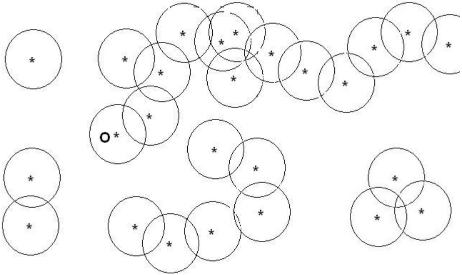
Poissonian World

- Analysis of local/global functionals of Poisson or Bernoulli pp. cf. e.g.
  - R. Meester & R. Roy Continuum Percolation,
  - M. Penrose Random Geometric Graphs,
  - J. Yukich Limit theorems in discrete stochastic geometry,
  - G. Peccati & M. Reitzner Stochastic analysis for Poisson point processes **upcoming**
  - P. Calka Tessellation
- What about non-Poisson point processes ? Eg. Cox point processes, perturbed lattices, Determinantal point processes et al.
- Generic methods ?

Percolation

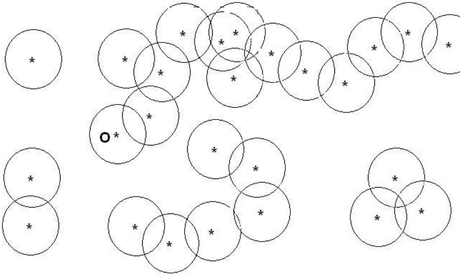


Percolation



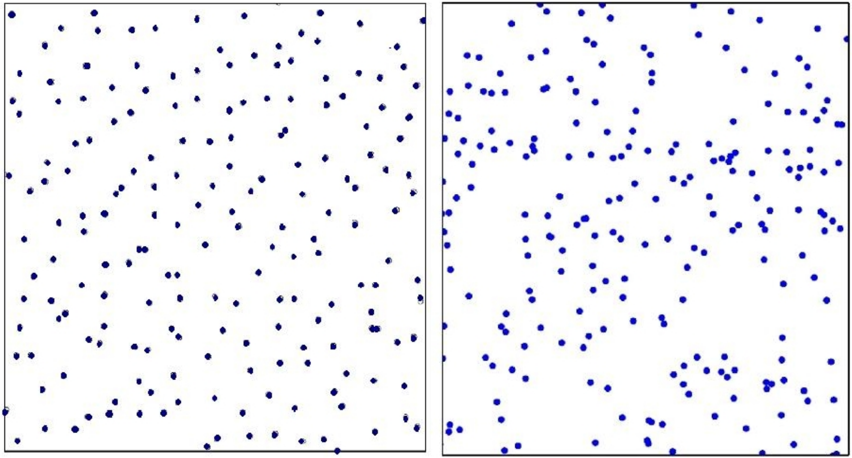
- O percolates if there exists unbounded connected subset such that $O \in W \subset C(\Phi, r) := \cup_{X \in \Phi} B_r(X)$.

Percolation

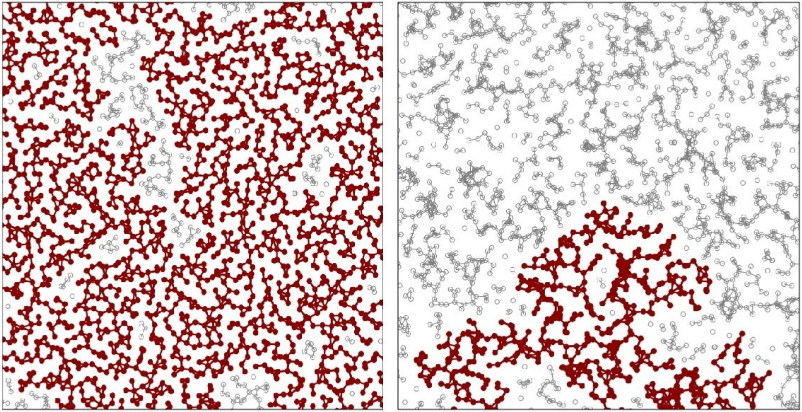


- O percolates if there exists unbounded connected subset such that $O \in W \subset C(\Phi, r) := \cup_{X \in \Phi} B_r(X)$.
- $r_c(\Phi) := \inf\{r : \mathsf{P}(0 \text{ percolates in } C(\Phi, r)) > 0\}$.

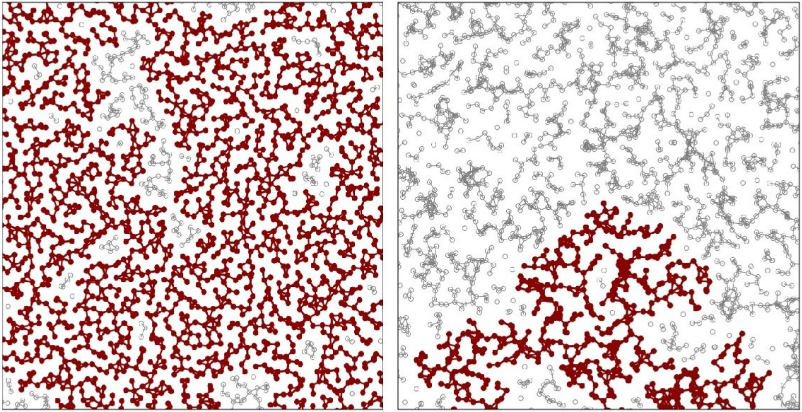
Percolation and clustering



Percolation and clustering

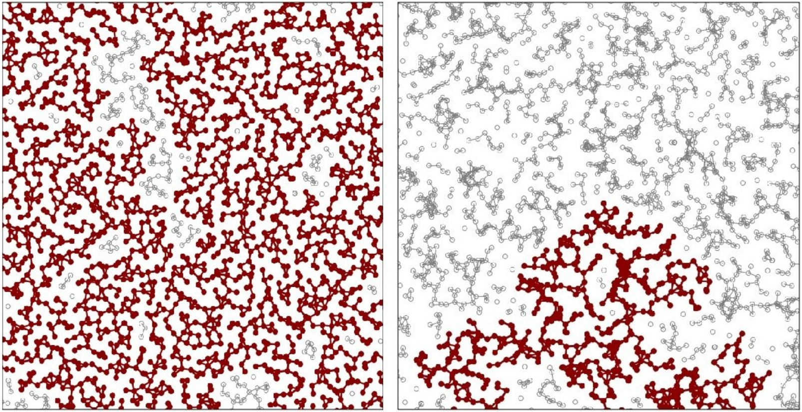


Percolation and clustering



- $G(\Phi, r)$: (X, Y) an edge if $|X - Y| \leq 2r$. Percolation = infinite component.

Percolation and clustering



- $\Phi_1 \leq_{\text{clustering}} \Phi_2 \Rightarrow r_c(\Phi_1) \leq r_c(\Phi_2)$!

Percolation and clustering

- **HEURISTIC :** Clustering worsens Percolation.

Percolation and clustering

- **HEURISTIC :** Clustering worsens Percolation.

- How do we measure clustering ?

Percolation and clustering

- **HEURISTIC :** Clustering worsens Percolation.
- How do we measure clustering ?
- Statistical descriptors: Ripley's K-function, pair correlation function et al.

Percolation and clustering

- **HEURISTIC :** Clustering worsens Percolation.
- How do we measure clustering ?
- Statistical descriptors: Ripley's K-function, pair correlation function et al.
- Does there exist a partial order $\leq$ on space of point processes such that

Percolation and clustering

- **HEURISTIC :** Clustering worsens Percolation.
- How do we measure clustering ?
- Statistical descriptors: Ripley's K-function, pair correlation function et al.
- Does there exist a partial order $\leq$ on space of point processes such that
  - $\Phi_1 \leq \Phi_2 \Rightarrow \Phi_1$ "clusters less" than Φ_2.

Percolation and clustering

- **HEURISTIC :** Clustering worsens Percolation.
- How do we measure clustering ?
- Statistical descriptors: Ripley's K-function, pair correlation function et al.
- Does there exist a partial order $\leq$ on space of point processes such that
  - $\Phi_1 \leq \Phi_2 \Rightarrow \Phi_1$ "clusters less" than Φ_2.
  - $\Phi_1 \leq \Phi_2 \Rightarrow F(\Phi_1) \leq F(\Phi_2)$ for many F.

Percolation and clustering

- **HEURISTIC :** Clustering worsens Percolation.
- How do we measure clustering ?
- Statistical descriptors: Ripley's K-function, pair correlation function et al.
- Does there exist a partial order $\leq$ on space of point processes such that
  - $\Phi_1 \leq \Phi_2 \Rightarrow \Phi_1$ "clusters less" than Φ_2.
  - $\Phi_1 \leq \Phi_2 \Rightarrow F(\Phi_1) \leq F(\Phi_2)$ for many F.
  - And of course, many examples !

Percolation and clustering

- **HEURISTIC :** Clustering worsens Percolation.
- How do we measure clustering ?
- Statistical descriptors: Ripley's K-function, pair correlation function et al.
- Does there exist a partial order $\leq$ on space of point processes such that
  - $\Phi_1 \leq \Phi_2 \Rightarrow \Phi_1$ "clusters less" than Φ_2.
  - $\Phi_1 \leq \Phi_2 \Rightarrow F(\Phi_1) \leq F(\Phi_2)$ for many F.
  - And of course, many examples !
  - In other words, a simple but not simple enough partial order !

Weak sub-Poisson

- Poisson point process: Two characterizations -

Weak sub-Poisson

- Poisson point process: Two characterizations -
  - $\mathsf{E}(\Phi(A_1)\ldots\Phi(A_k)) = \prod_i |A_i|$;

Weak sub-Poisson

- Poisson point process: Two characterizations -
  - $\mathsf{E}(\Phi(A_1)\ldots\Phi(A_k)) = \prod_i |A_i|$;
  - $\mathsf{P}(\Phi(B) = 0) = e^{-|B|}$.

Weak sub-Poisson

- Poisson point process: Two characterizations -
  - $\mathsf{E}(\Phi(A_1)\ldots\Phi(A_k)) = \prod_i |A_i|$;
  - $\mathsf{P}(\Phi(B) = 0) = e^{-|B|}$.
- Weak sub-Poisson process: For $B_1, \ldots, B_k$ disjoint bBs

Weak sub-Poisson

- Poisson point process: Two characterizations -
  - $\mathsf{E}(\Phi(A_1)\ldots\Phi(A_k)) = \prod_i |A_i|$;
  - $\mathsf{P}(\Phi(B) = 0) = e^{-|B|}$.
- Weak sub-Poisson process: For $B_1, \ldots, B_k$ disjoint bBs
  - $\mathsf{E}\Big(\prod_{i=1}^{k} \Phi(B_i)\Big) \leq \prod_{i=1}^{k} \mathsf{E}(\Phi(B_i)) = \mathsf{E}\Big(\prod_{i=1}^{k} \Phi_{(1)}(B_i)\Big)$.
    AND

Weak sub-Poisson

- Poisson point process: Two characterizations -
  - $\mathsf{E}(\Phi(A_1)\ldots\Phi(A_k)) = \prod_i |A_i|$;
  - $\mathsf{P}(\Phi(B) = 0) = e^{-|B|}$.

- Weak sub-Poisson process: For $B_1, \ldots, B_k$ disjoint bBs
  - $\mathsf{E}\Big(\prod_{i=1}^k \Phi(B_i)\Big) \leq \prod_{i=1}^k \mathsf{E}(\Phi(B_i)) = \mathsf{E}\Big(\prod_{i=1}^k \Phi_{(1)}(B_i)\Big)$.

    AND
  - $\mathsf{P}(\Phi(B) = 0) \leq e^{-|B|} = \mathsf{P}\big(\Phi_{(1)}(B) = 0\big)$.

Weak sub-Poisson

- Poisson point process: Two characterizations -
  - $\mathsf{E}(\Phi(A_1)\dots\Phi(A_k)) = \prod_i |A_i|$;
  - $\mathsf{P}(\Phi(B) = 0) = e^{-|B|}$.
- Weak sub-Poisson process: For $B_1, \dots, B_k$ disjoint bBs
  - $\mathsf{E}\Big(\prod_{i=1}^k \Phi(B_i)\Big) \leq \prod_{i=1}^k \mathsf{E}(\Phi(B_i)) = \mathsf{E}\Big(\prod_{i=1}^k \Phi_{(1)}(B_i)\Big)$.

    AND
  - $\mathsf{P}(\Phi(B) = 0) \leq e^{-|B|} = \mathsf{P}\big(\Phi_{(1)}(B) = 0\big)$.
- Can be extended to a Partial order on point processes.

Ordering of Laplace transforms

- For $f \geq 0$ or $f \leq 0$,

$$\mathrm{E}\left(\exp\left[\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right]\right) \leq \exp\left[\int_{\mathbb{R}^d} (e^{f(x)} - 1)\,\mathrm{d}x\right].$$

Ordering of Laplace transforms

- For $f \geq 0$ or $f \leq 0$,

$$\mathrm{E}\left(\exp\left[\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right]\right) \leq \exp\left[\int_{\mathbb{R}^d} (e^{f(x)} - 1)\,\mathrm{d}x\right].$$

- Chernoff's bound :

$$\mathrm{P}\left(\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x) \geq t\right) \leq e^{-bt}\mathrm{E}\left(\exp\left[b\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right]\right)$$

.

Ordering of Laplace transforms

- For $f \geq 0$ or $f \leq 0$,

$$E\left(\exp\left[\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right]\right) \leq \exp\left[\int_{\mathbb{R}^d} (e^{f(x)} - 1)\,\mathrm{d}x\right].$$

- Chernoff's bound :

$$P\left(\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x) \geq t\right) \leq e^{-bt} E\left(\exp\left[b\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right]\right)$$

.

- Setting $|B_n| = n$, $f(x) = \mathbf{1}[x \in B_n] \Rightarrow$

Ordering of Laplace transforms

- For $f \geq 0$ or $f \leq 0$,

$$\mathrm{E}\left(\exp\left[\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right]\right) \leq \exp\left[\int_{\mathbb{R}^d} (e^{f(x)} - 1)\,\mathrm{d}x\right] .$$

- Chernoff's bound :

$$\mathrm{P}\left(\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x) \geq t\right) \leq e^{-bt}\mathrm{E}\left(\exp\left[b\int_{\mathbb{R}^d} f(x)\,\Phi(\mathrm{d}x)\right]\right)$$

.

- Setting $|B_n| = n$, $f(x) = \mathbf{1}[x \in B_n] \Rightarrow$
- Concentration inequality :
  $\mathrm{P}(|\Phi(B_n) - n| \geq n^a) \leq 2\exp\left[-n^{2a-1}/9\right], \; a > \frac{1}{2}.$

Directionally Convex order

- Introduced in connection with Ross-type conjectures in Queueing theory.

Directionally Convex order

- Introduced in connection with Ross-type conjectures in Queueing theory.
- For $f : \mathbb{R}^d \to \mathbb{R}$, $x \in \mathbb{R}^d$, $1 \leq i \leq d$, $\delta > 0$, define

$$\Delta_i^\delta f(x) := f(x + \delta e_i) - f(x) \quad \text{(difference operator).}$$

Directionally Convex order

- Introduced in connection with Ross-type conjectures in Queueing theory.
- For $f : \mathbb{R}^d \to \mathbb{R}$, $x \in \mathbb{R}^d$, $1 \le i \le d$, $\delta > 0$, define

$$\Delta_i^\delta f(x) := f(x + \delta e_i) - f(x) \quad \text{(difference operator)}.$$

- f is *dcx* if $\forall x \in \mathbb{R}^d$, $i, j \in \{1, \ldots, d\}$, $\delta > 0, \epsilon > 0$,

$$\Delta_i^\epsilon \Delta_j^\delta f(x) \ge 0.$$

Directionally Convex order

- Introduced in connection with Ross-type conjectures in Queueing theory.
- For $f : \mathbb{R}^d \to \mathbb{R}$, $x \in \mathbb{R}^d$, $1 \leq i \leq d$, $\delta > 0$, define

$$\Delta_i^\delta f(x) := f(x + \delta e_i) - f(x) \quad \text{(difference operator)}.$$

- f is *dcx* if $\forall x \in \mathbb{R}^d$, $i, j \in \{1, \ldots, d\}$, $\delta > 0, \epsilon > 0$,

$$\Delta_i^\epsilon \Delta_j^\delta f(x) \geq 0.$$

- $\Phi_1 \leq_{dcx} \Phi_2$ if for all bBs $B_1, \ldots, B_n$,

$$(\Phi_1(B_1), \ldots, \Phi_1(B_n)) \leq_{dcx} (\Phi_2(B_1), \ldots, \Phi_2(B_n)),$$

Directionally Convex order

- Introduced in connection with Ross-type conjectures in Queueing theory.
- For $f : \mathbb{R}^d \to \mathbb{R}$, $x \in \mathbb{R}^d$, $1 \leq i \leq d$, $\delta > 0$, define

$$\Delta_i^\delta f(x) := f(x + \delta e_i) - f(x) \quad \text{(difference operator)}.$$

- f is *dcx* if $\forall x \in \mathbb{R}^d$, $i, j \in \{1, \ldots, d\}$, $\delta > 0, \epsilon > 0$,

$$\Delta_i^\epsilon \Delta_j^\delta f(x) \geq 0.$$

- $\Phi_1 \leq_{dcx} \Phi_2$ if for all bBs $B_1, \ldots, B_n$,

$$(\Phi_1(B_1), \ldots, \Phi_1(B_n)) \leq_{dcx} (\Phi_2(B_1), \ldots, \Phi_2(B_n)),$$

- i.e, $\forall f$ dcx,

$$\mathsf{E}(f(\Phi_1(B_1), \ldots, \Phi_1(B_n))) \leq \mathsf{E}(f(\Phi_2(B_1), \ldots, \Phi_2(B_n))).$$

Sub-Poisson

- $\Phi_1 \leq_{dcx} \Phi_2 \rightarrow \mathsf{E}(\Phi_1(B)) = \mathsf{E}(\Phi_2(B))$

Sub-Poisson

- $\Phi_1 \leq_{dcx} \Phi_2 \rightarrow \mathsf{E}(\Phi_1(B)) = \mathsf{E}(\Phi_2(B))$
- Φ is sub-Poisson if $\Phi \leq_{dcx} \Phi_{(\lambda)}$.

Sub-Poisson

- $\Phi_1 \leq_{dcx} \Phi_2 \rightarrow \mathsf{E}(\Phi_1(B)) = \mathsf{E}(\Phi_2(B))$
- Φ is sub-Poisson if $\Phi \leq_{dcx} \Phi_{(\lambda)}$.
- Examples of *dcx* functions :

  $f(x) = (1-x) \vee 0; \;\; f(x) = \prod_i (x_i \vee 0); \;\; f(x) = e^{-\sum_i x_i}.$

Sub-Poisson

- $\Phi_1 \leq_{dcx} \Phi_2 \rightarrow \mathsf{E}(\Phi_1(B)) = \mathsf{E}(\Phi_2(B))$
- Φ is sub-Poisson if $\Phi \leq_{dcx} \Phi_{(\lambda)}$.
- Examples of *dcx* functions :

  $$f(x) = (1-x) \vee 0; \;\; f(x) = \prod_i (x_i \vee 0); \;\; f(x) = e^{-\sum_i x_i}.$$
- sub-Poisson $\Rightarrow$ weak sub-Poisson.

Sub-Poisson

- $\Phi_1 \leq_{dcx} \Phi_2 \to \mathsf{E}(\Phi_1(B)) = \mathsf{E}(\Phi_2(B))$
- Φ is sub-Poisson if $\Phi \leq_{dcx} \Phi_{(\lambda)}$.
- Examples of *dcx* functions :

  $f(x) = (1-x) \vee 0$; $f(x) = \prod_i (x_i \vee 0)$; $f(x) = e^{-\sum_i x_i}$.
- sub-Poisson $\Rightarrow$ weak sub-Poisson.
- Super-Poisson point processes :

  Poisson-Poisson cluster point processes, Lévy based Cox point processes, Shot-Noise Cox point processes etc.

Sub-Poisson

- $\Phi_1 \leq_{dcx} \Phi_2 \to \mathsf{E}(\Phi_1(B)) = \mathsf{E}(\Phi_2(B))$
- Φ is sub-Poisson if $\Phi \leq_{dcx} \Phi_{(\lambda)}$.
- Examples of *dcx* functions :

  $f(x) = (1-x) \vee 0; \;\; f(x) = \prod_i (x_i \vee 0); \;\; f(x) = e^{-\sum_i x_i}$.
- sub-Poisson $\Rightarrow$ weak sub-Poisson.
- Super-Poisson point processes :

  Poisson-Poisson cluster point processes, Lévy based Cox point processes, Shot-Noise Cox point processes etc.
- sub-Poisson : Perturbed lattices, Ginibre Ensemble: Eigenvalues of $N \times N$ matrix with i.i.d. standard complex Gaussian entries as $N \to \infty$.

Negative Association

- Φ - negatively associated if for f, g increasing functions and $A_1, \ldots, A_n$ disjoint

$$\mathrm{Cov}\left(f(\Phi(A_1), \ldots, \Phi(A_m))g(\Phi(A_{m+1}, \ldots, \Phi(A_n))\right) \leq 0$$

Negative Association

- Φ - negatively associated if for f, g increasing functions and $A_1, \ldots, A_n$ disjoint

$$\mathrm{Cov}\left(f(\Phi(A_1), \ldots, \Phi(A_m))g(\Phi(A_{m+1}, \ldots, \Phi(A_n))\right) \leq 0$$

- **NA $\Rightarrow$ weak sub-Poisson**

Negative Association

- Φ - negatively associated if for f, g increasing functions and $A_1, \ldots, A_n$ disjoint

$$\mathrm{Cov}\left(f(\Phi(A_1), \ldots, \Phi(A_m))g(\Phi(A_{m+1}, \ldots, \Phi(A_n))\right) \leq 0$$

- **NA $\Rightarrow$ weak sub-Poisson**
- **Φ is NA, $\Phi(A) \leq_{convex} Poi(\lambda|A|) \Rightarrow \Phi \leq_{dcx} \Phi_{(\lambda)}$.**

Negative Association

- Φ - negatively associated if for f, g increasing functions and $A_1, \ldots, A_n$ disjoint

$$\mathrm{Cov}\,(f(\Phi(A_1), \ldots, \Phi(A_m))g(\Phi(A_{m+1}, \ldots, \Phi(A_n))) \leq 0$$

- **NA $\Rightarrow$ weak sub-Poisson**
- **Φ is NA,** $\Phi(A) \leq_{convex} Poi(\lambda|A|) \Rightarrow \Phi \leq_{dcx} \Phi_{(\lambda)}$.
- NA point processes : Simple perturbed lattice, Some determinantal point processes.

Recap

- $V_\Phi(y) := \sum_{X \in \Phi} h(X, y) = \int_{\mathbb{R}^d} h(x, y)\Phi(dx), y \in S.$

Recap

- $V_\Phi(y) := \sum_{X\in\Phi} h(X,y) = \int_{\mathbb{R}^d} h(x,y)\Phi(dx), y \in S.$
- Laplace transform ordering : For any $b \in \mathbb{R}$, Φ weak sub-Poisson -

$$\mathsf{E}\left(e^{b\sum_i V_\Phi(y_i)}\right) = \mathcal{L}_\Phi(b\sum_i h(.,y_i)) \leq \mathcal{L}_{Poi}(b\sum_i h(.,y_i)).$$

Recap

- $V_\Phi(y) := \sum_{X\in\Phi} h(X,y) = \int_{\mathbb{R}^d} h(x,y)\Phi(dx), y \in S.$
- Laplace transform ordering : For any $b \in \mathbb{R}$, Φ weak sub-Poisson -

$$\mathsf{E}\left(e^{b\sum_i V_\Phi(y_i)}\right) = \mathcal{L}_\Phi(b\sum_i h(.,y_i)) \leq \mathcal{L}_{Poi}(b\sum_i h(.,y_i)).$$

- $NA+ \Rightarrow DCX$ sub-Poisson $\Rightarrow$ weak sub-Poisson $\Rightarrow$ Laplace transform ordering.

Applications : Percolation

- $V_\Phi(y) := \sum_{X\in\Phi} 1[|X - y| \leq r]\ , y \in \mathbb{R}^d$.

Applications : Percolation

- $V_\Phi(y) := \sum_{X \in \Phi} 1[|X - y| \leq r]$, $y \in \mathbb{R}^d$.
- Level-sets : $V^r_{\geq k} := V_\Phi^{-1}[k, \infty]$ - *k-covered* region.

Applications : Percolation

- $V_\Phi(y) := \sum_{X \in \Phi} 1[|X - y| \leq r]$, $y \in \mathbb{R}^d$.
- Level-sets : $V^r_{\geq k} := V_\Phi^{-1}[k, \infty]$ - *k-covered* region.
- $r_c(k) := \inf\{r : \mathsf{P}\big(V^r_{\geq k} \text{ has an unbounded component}\big) > 0\}$.

Applications : Percolation

- $V_\Phi(y) := \sum_{X\in\Phi} 1[|X-y| \leq r]$, $y \in \mathbb{R}^d$.
- Level-sets : $V^r_{\geq k} := V_\Phi^{-1}[k,\infty]$ - *k-covered* region.
- $r_c(k) := \inf\{r : \mathsf{P}\big(V^r_{\geq k}$ has an unbounded component$\big) > 0\}$.
- $r_c(k)$ - critical radius for percolation of k-covered region.

Applications : Percolation

- $V_\Phi(y) := \sum_{X \in \Phi} 1[|X - y| \leq r]$, $y \in \mathbb{R}^d$.
- Level-sets : $V^r_{\geq k} := V_\Phi^{-1}[k, \infty]$ - *k-covered* region.
- $r_c(k) := \inf\{r : \mathrm{P}\big(V^r_{\geq k}$ has an unbounded component$\big) > 0\}$.
- $r_c(k)$ - critical radius for percolation of k-covered region.
- **Is** $0 < r_c(k) < \infty$ **?**

Applications : Percolation

- $V_\Phi(y) := \sum_{X\in\Phi} 1[|X-y|\le r]$, $y\in\mathbb{R}^d$.
- Level-sets : $V^r_{\ge k} := V_\Phi^{-1}[k,\infty]$ - *k-covered* region.
- $r_c(k) := \inf\{r : \mathsf{P}\big(V^r_{\ge k}$ has an unbounded component$\big) > 0\}$.
- $r_c(k)$ - critical radius for percolation of k-covered region.
- **Is** $0 < r_c(k) < \infty$ **?**
- Φ weak sub-Poisson $\Rightarrow$

$$0 < c_* < r_c(k) < c_k < \infty.$$

Applications : Percolation

- $V_\Phi(y) := \sum_{X\in\Phi} 1[|X-y| \le r]$, $y \in \mathbb{R}^d$.
- Level-sets : $V^r_{\ge k} := V_\Phi^{-1}[k,\infty]$ - *k-covered* region.
- $r_c(k) := \inf\{r : \mathsf{P}\big(V^r_{\ge k}$ has an unbounded component$\big) > 0\}$.
- $r_c(k)$ - critical radius for percolation of k-covered region.
- **Is** $0 < r_c(k) < \infty$ **?**
- Φ weak sub-Poisson $\Rightarrow$

$$0 < c_* < r_c(k) < c_k < \infty.$$

- Boolean model = 1-covered region.

Applications : Percolation

- $V_\Phi(y) := \sum_{X\in\Phi} 1[|X-y|\leq r]$, $y \in \mathbb{R}^d$.
- Level-sets : $V^r_{\geq k} := V_\Phi^{-1}[k,\infty]$ - *k-covered* region.
- $r_c(k) := \inf\{r : \mathsf{P}\big(V^r_{\geq k}$ has an unbounded component$\big) > 0\}$.
- $r_c(k)$ - critical radius for percolation of k-covered region.
- **Is** $0 < r_c(k) < \infty$ **?**
- Φ weak sub-Poisson $\Rightarrow$

$$0 < c_* < r_c(k) < c_k < \infty.$$

- Boolean model $=$ 1-covered region.
- There exists Φ super-Poisson such that $r_c(1) = 0$.

Other Applications

- Other Applications : SINR Percolation, AB percolation, Lilypond models (Daley-Last), near-Minimal spanning trees (Hirsch-Neuhaueser-Schmidt).

Other Applications

- Other Applications : SINR Percolation, AB percolation, Lilypond models (Daley-Last), near-Minimal spanning trees (Hirsch-Neuhaueser-Schmidt).
- Key Step : Chernoff bounds and then - Branching process argument, Peierl's argument.

Other Applications

- Other Applications : SINR Percolation, AB percolation, Lilypond models (Daley-Last), near-Minimal spanning trees (Hirsch-Neuhaueser-Schmidt).
- Key Step : Chernoff bounds and then - Branching process argument, Peierl's argument.
- $\mathsf{E}\big(V^r_{\geq 1} \cap W\big) \geq \mathsf{E}\Big(\widehat{V}^r_{\geq 1} \cap W\Big)$.

Other Applications

- Other Applications : SINR Percolation, AB percolation, Lilypond models (Daley-Last), near-Minimal spanning trees (Hirsch-Neuhaueser-Schmidt).
- Key Step : Chernoff bounds and then - Branching process argument, Peierl's argument.
- $\mathsf{E}\big(V^r_{\geq 1} \cap W\big) \geq \mathsf{E}\Big(\widehat{V}^r_{\geq 1} \cap W\Big)$.
- weak sub-Poisson + ... $\Rightarrow$ inequality for $\mathsf{E}\big(V^r_{\geq k} \cap W\big)$ reverses for $k > k_0$.

Other Applications

- Other Applications : SINR Percolation, AB percolation, Lilypond models (Daley-Last), near-Minimal spanning trees (Hirsch-Neuhaueser-Schmidt).
- Key Step : Chernoff bounds and then - Branching process argument, Peierl's argument.
- $\mathsf{E}\big(V^r_{\geq 1} \cap W\big) \geq \mathsf{E}\Big(\widehat{V}^r_{\geq 1} \cap W\Big)$.
- weak sub-Poisson $+$ .... $\Rightarrow$ inequality for $\mathsf{E}\big(V^r_{\geq k} \cap W\big)$ reverses for $k > k_0$.
- Comparison of SINR coverage and capacity under *dcx* assumption.

What else ?

- Asymptotics for "local" U-Statistics of Φ - $\frac{1}{k!}\sum_{X_1,\ldots,X_k\in\Phi} h(X_1,\ldots,X_k)$. R. J. Adler-D.Y.

What else ?

- Asymptotics for "local" U-Statistics of Φ - $\frac{1}{k!}\sum_{X_1,\ldots,X_k \in \Phi} h(X_1,\ldots,X_k)$. R. J. Adler-D.Y.
- Clique number, Chromatic number of random geometric graphs.

What else ?

- Asymptotics for "local" U-Statistics of Φ - $\frac{1}{k!}\sum_{X_1,\ldots,X_k \in \Phi} h(X_1,\ldots,X_k)$. R. J. Adler-D.Y.
- Clique number, Chromatic number of random geometric graphs.
- Thresholds for Betti numbers of $\cup_{X \in \Phi} B_X(r)$. R. J. Adler-D.Y.

What else ?

- Asymptotics for "local" U-Statistics of Φ - $\frac{1}{k!}\sum_{X_1,\ldots,X_k \in \Phi} h(X_1,\ldots,X_k)$. R. J. Adler-D.Y.
- Clique number, Chromatic number of random geometric graphs.
- Thresholds for Betti numbers of $\cup_{X \in \Phi} B_X(r)$. R. J. Adler-D.Y.
- CLT for "local" functionals - Eg. Intrinsic volumes, edge lenths in k-nearest neighbour graphs. *ongoing work* with B.B. and J. Yukich.

What else ?

- Asymptotics for "local" U-Statistics of Φ - $\frac{1}{k!}\sum_{X_1,\ldots,X_k \in \Phi} h(X_1,\ldots,X_k)$. R. J. Adler-D.Y.
- Clique number, Chromatic number of random geometric graphs.
- Thresholds for Betti numbers of $\cup_{X \in \Phi} B_X(r)$. R. J. Adler-D.Y.
- CLT for "local" functionals - Eg. Intrinsic volumes, edge lenths in k-nearest neighbour graphs. *ongoing work* with B.B. and J. Yukich.
- More concentration inequalities ?
- Other functionals, Eg. Capacity ?
- Where do Gibbs point processes fit in ?

Shot-noise field ordering

- $V_\Phi(y) := \sum_{X \in \Phi} h(X, y) = \int_{\mathbb{R}^d} h(x, y)\Phi(dx), y \in S.$

Shot-noise field ordering

- $V_\Phi(y) := \sum_{X \in \Phi} h(X, y) = \int_{\mathbb{R}^d} h(x, y)\Phi(dx), y \in S.$
- $\Phi_1 \leq_{dcx} \Phi_2 \Rightarrow \{V_{\Phi_1}(y)\}_{y \in S} \leq_{dcx} \{V_{\Phi_2}(y)\}_{y \in S}.$

Shot-noise field ordering

- $V_\Phi(y) := \sum_{X\in\Phi} h(X,y) = \int_{\mathbb{R}^d} h(x,y)\Phi(dx), y \in S.$
- $\Phi_1 \leq_{dcx} \Phi_2 \Rightarrow \{V_{\Phi_1}(y)\}_{y\in S} \leq_{dcx} \{V_{\Phi_2}(y)\}_{y\in S}.$
- i.e, for all $y_1,\ldots,y_n$,

$$(V_{\Phi_1}(y_1),\ldots,V_{\Phi_1}(y_n)) \leq_{dcx} (V_{\Phi_2}(y_1),\ldots,V_{\Phi_2}(y_n)).$$

Some References

- Clustering Comparison of Point Processes, with Applications to Random Geometric Models - Book Chapter in *Springer Lecture Notes in Mathematics* on **Stochastic Geometry, Spatial Statistics and Random Fields.** B. B., D. Y.
- More detailed slides on Bartek Błaszczyszyn's webpage.