Limit theorems for random geometric complexes in the thermodynamic regime

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ISID, Nov 2015.

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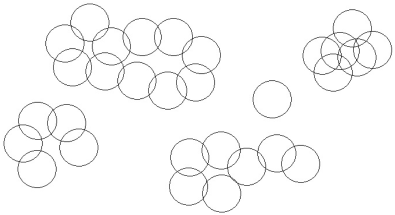

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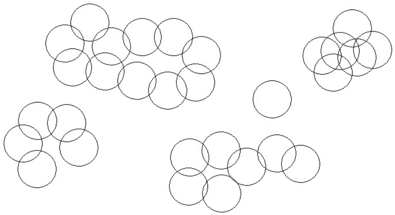

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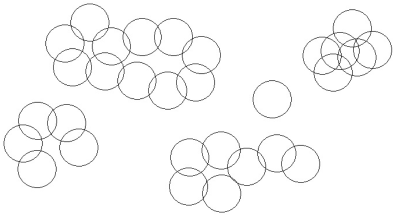

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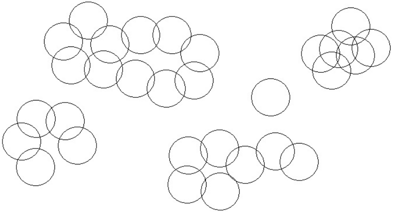
- Combinatorial properties : Subgraph counts, Component counts.

Algebraic Topoogy Viewpoint



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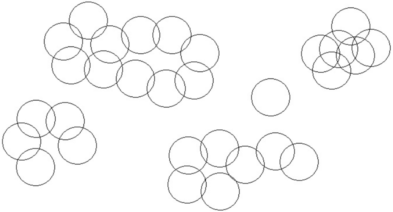

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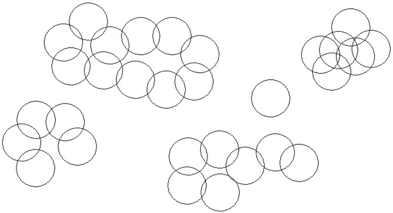

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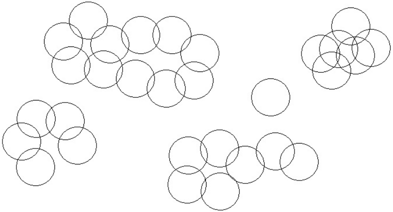

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- Euler-Poincaré Characteristic: $\chi = 2$.
- What about the holes ?

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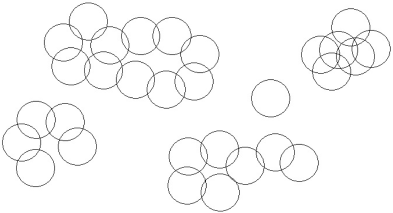
- Alexander's duality : $\beta_{d-1}(A) = \beta_0(\mathbb{R}^d \setminus A) - 1.$

The example of Boolean Model



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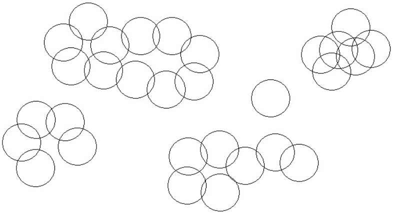

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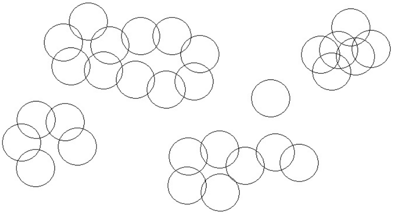

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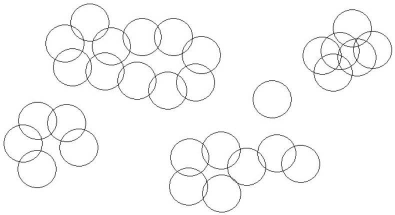

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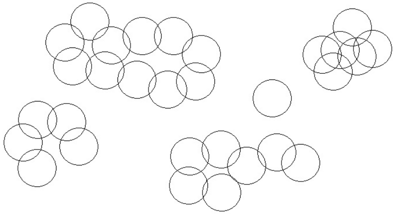

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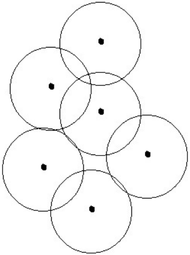
- $\chi(C_B(\mathcal{X}, r)) := \sum_{k \geq 0}(-1)^k \beta_k(C_B(\mathcal{X}, r)) = 2$.

Geometric complexes



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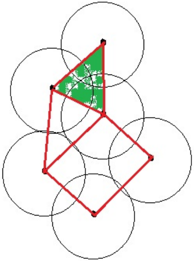

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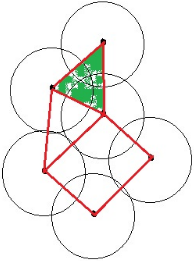

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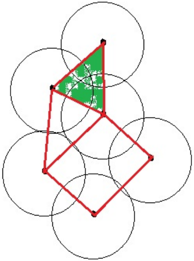

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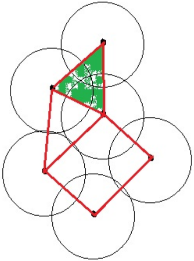

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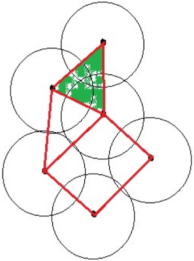

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Cech complex $C(\Phi, r) := \cup_{k=0}^{\infty} C_K(\Phi, r) \subset 2^{\Phi}$.

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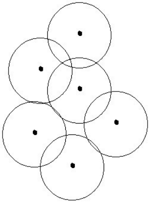

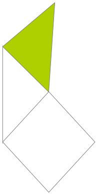

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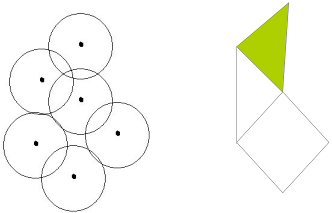
Nerve theorem : Cech complex $\equiv \cup_i B_r(X_i)$, Boolean model.



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- $\beta_k(C(\mathcal{P}_n, r_n)), \beta_k(C(\mathcal{X}_n, r_n))$ as $n \to \infty, nr_n^d \to r$?

Variance Bounds

- Add-one cost or difference operator : $\mathcal{X}$ finite set, $x \in \mathbb{R}^d$

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- Poincaré or Efron-Stein inequalities

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- Variance bounds of Last-Peccati-Schulte (2015):

$$\mathrm{VAR}(\beta_{d-1}(C(\mathcal{P}_n, r_n))) = \Omega(n).$$

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$$\mathsf{P}\Big(\Big|\beta_k(C(\mathcal{X}_n, r_n)) - \mathsf{E}(\beta_k(.))\Big| \geq \epsilon n^a\Big) \;\leq\; \frac{C}{\epsilon} n^{2k+2-a} \exp(-n^{\gamma}).$$

$$\lim_{n\to\infty} n^{-1}\left[\beta_k(C(\mathcal{P}_n, r_n)) - \mathsf{E}(\beta_k(C(\mathcal{P}_n, r_n)))\right] \;=\; 0,$$

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- Strong and weak laws hold for Ergodic point processes under appropriate asymptotics. Proof via sub-additive arguments.

Central Limit Theorem

- $\mathcal{P}$ - Poisson (1) pp in $\mathbb{R}^d$; $\mathcal{P}_n = \mathcal{P} \cap W_n$, $W_n := [-\frac{n^{1/d}}{2}, \frac{n^{1/d}}{2}]^d$.

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- Weakly stabilizing: $D_O\beta_k(C(\mathcal{P}_n, r)) \to D_\infty\beta_k$ a.s..

Poisson and Binomial CLT

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Theorem

$$\frac{\beta_k(C(\mathcal{X}_n, r)) - \mathrm{E}(\beta_k(C(\mathcal{X}_n, r)))}{\sqrt{n}} \Rightarrow N(0, \tau^2), \; 0 < \tau^2 \leq \sigma^2.$$

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