PERCOLATION AND CONNECTIVITY IN AB RANDOM GEOMETRIC GRAPHS

by

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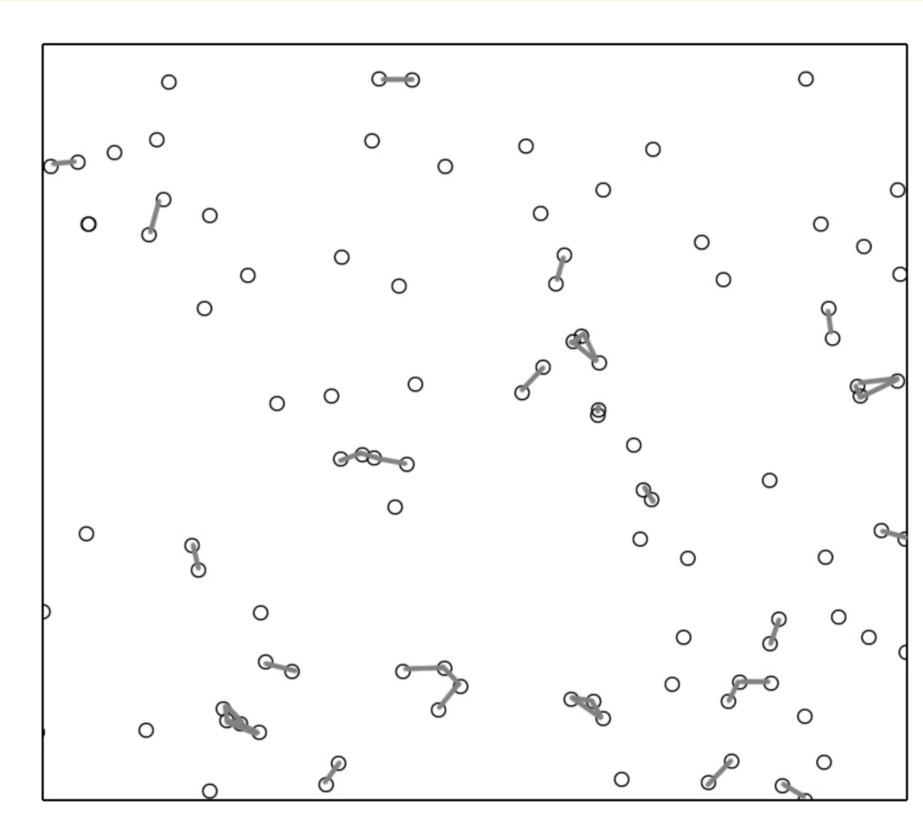
Random Geometric Graphs



Drop points on the plane ; Link any two points within a distance r.

AB Random Geometric Graphs

Drop two set of points on the plane.
Link any two points of the different type within a distance r.

Motivation

- Frequency division half duplex transmission scheme : Nodes have two choices of transmission-reception frequency - (f_1, f_2) or (f_2, f_1).
- Multi-level Node deployment : Air-borne nodes and ground-level nodes. Communication barred between nodes at same level.
- Secure communication : Tagged nodes broadcast a key and normal nodes which receive the same key can communicate.

AB Boolean model : Percolation

- $d \geq 2$. $\Phi^{(1)} = \{X_i\}_{i\geq 1}$ and $\Phi^{(2)} = \{Y_i\}_{i\geq 1}$ be independent Poisson point processes in $\mathbb{R}^d$ with intensities λ and μ respectively.
- Boolean Model: $G(\lambda, r) := (\Phi^{(1)}, E(\lambda, r))$; $\langle X_i, X_j\rangle \in E(\lambda, r)$ if $|X_i - X_j| \leq 2r$.
- Percolation in a graph $\Rightarrow$ existence of an infinite connected subset of points.
- For Boolean model, percolation $\equiv$ existence of unbounded connected (topological) subset in $\cup_i B_{X_i}(r)$.
- There exist $0 < \lambda_c(r) < \infty$ such that $G(\lambda, r)$ percolates a.s. iff $\lambda > \lambda_c(r)$.
- AB Boolean Model : $G(\lambda, \mu, r) := (\Phi^{(1)}, E(\lambda, \mu, r))$;

  $\langle X_i, X_j\rangle \in E(\lambda, \mu, r)$ if $|X_i - Y| \leq 2r, |X_j - Y| \leq 2r,$ for some $Y \in \Phi^{(2)}$.
- Critical Intensity : $\mu_c(\lambda, r) := \sup\{\mu : \mathsf{P}\left(G(\lambda, \mu, r) \text{ percolates}\right) = 0\}$.

Percolation Results

Simple bounds :

- $\mu_c(\lambda, r) \geq \lambda_c(r) - \lambda$.
- $\mu_c(\lambda, r) = \infty$, if $\lambda \leq \lambda_c(2r)$.

Theorem : $d = 2$. $\lambda > \lambda_c(2r)$ iff $\mu_c(\lambda, r) < \infty$.

Proposition $d \geq 2$.

1. For λ large, $\mu_c(\lambda, r) < \infty$. (i.e, Phase transition for all $d \geq 2$).
2. For λ large, there exists a $p(\lambda) < \frac{1}{2}$, such that $G(p\lambda, (1-p)\lambda, r)$ percolates a.s., for all $p \in (p(\lambda), 1 - p(\lambda))$.

Giant component is unique.

Part (1) of the Proposition holds true for more word percolation i.e, percolation models with k types of point processes.
AB Boolean model is equivalent to word percolation with $k = 2$.

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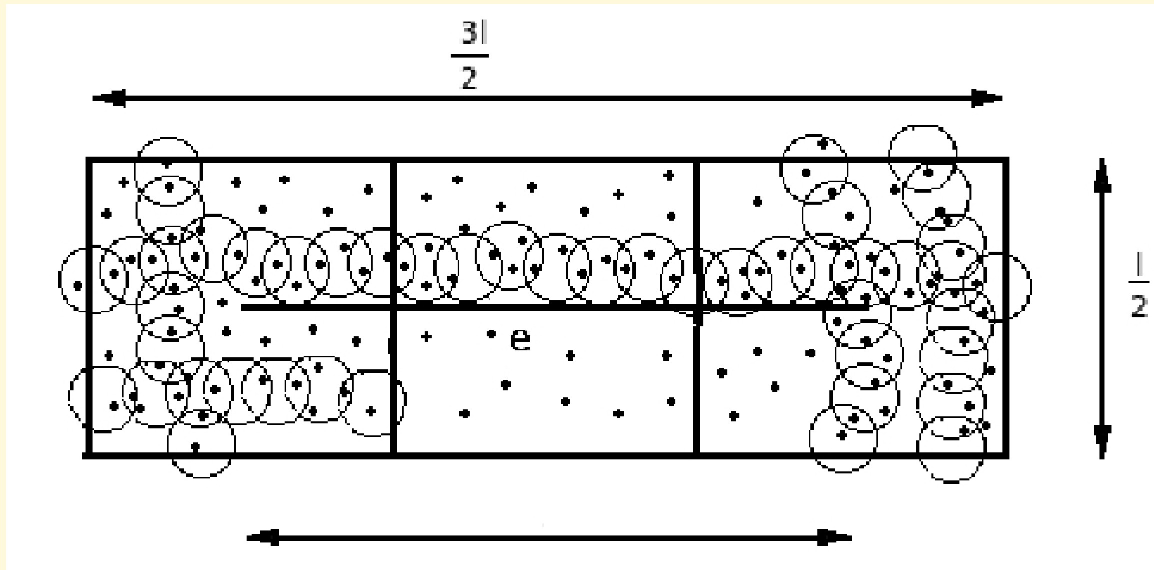
Proof



- Fir r_1 such that $\lambda > \lambda_c(2r_1)$.
- $A_e = \{\, G(\lambda, 2r_1)$ has left-right crossing and top-down crossing in the last boxes$\}$.
- B_e = each pair of balls with non-empty intersection in $G(\lambda, 2r_1)$, when expanded to balls of radius $2r$ contain atleast one point of $\Phi^{(2)}$.
- Note that $A_e B_e$ percolates $\Rightarrow G(\lambda, \mu, r)$ percolates.
- Prove percolation in the discrete model via Peierls argument and 1-dependence structure.

AB Random Geometric Graphs : Connectivity

- $d \geq 2$. $\mathcal{P}_n^{(1)}$ and $\mathcal{P}_n^{(2)}$ be independent homogenous Poisson point processes of intensity n in $U = [0,1]^d$ (Toroidal metric).

- AB Random geometric graph : $G_n(m,r) := (\mathcal{P}_n^{(1)}, E_n(m,r))$;

  $\langle X_i, X_j \rangle \in E_n(m,r)$ if $d(X_i,Y) \leq r, d(X_j,Y) \leq r,$ for some $Y \in \mathcal{P}_m^{(2)}$.

- Aim : Study connectivity threshold in $G_n(cn,r)$ as $n \to \infty$ for $c > 0$.

- Radius regime: $\theta_d = \|B_O(1)\|$, volume of unit ball. $\beta > 0$.

$$r_n(c,\beta) = \left(\frac{\log(n/\beta)}{cn\theta_d}\right)^{\frac{1}{d}}.$$

Lemma : $W_n(r_n(c,\beta))$ be the number of isolated nodes in $G_n(cn, r_n(c,\beta))$. There exists $1 < c_0(2) < 4$ and $c_0(d) = 1, d \geq 3$ such that

$$\mathsf{E}(W_n(r_n(c,\beta))) \to \beta \quad \text{for} \quad c < c_0(d),$$
$$\mathsf{E}(W_n(r_n(c,\beta))) \to \infty \quad \text{for} \quad c > 2^d.$$

Connectivity Threshold

Proposition : $M_n := \sup\{r \geq 0 : W_n(r) > 0\}$. For $0 < c < c_0(d)$,

$$W_n(r_n(c,\beta)) \stackrel{d}{\Rightarrow} Po(\beta),$$

$$\mathsf{P}\left(M_n \leq r_n(c,\beta)\right) \to e^{-\beta}.$$

Theorem : $\alpha_n(c) := \inf\{a : G_n(cn, a^{\frac{1}{d}} r_n(c,1))$ is connected$\}$. Then almost surely,

$$\liminf_{n\to\infty} \alpha_n(c) \geq 1,$$

for any $c < c_0(d)$, and for any $c > 0$,

$$\limsup_{n\to\infty} \alpha_n(c) \leq \alpha(c),$$

where $\alpha(c) \leq \left(1 + \frac{c^{\frac{1}{d}}}{2}\right)^d$ for $d \geq 2$ with equality for $d \geq 3$.

Random Geometric Graphs :

Random geometric graph : $G_n(R) := (\mathcal{P}_n^{(1)}, E_n(R))$; $\langle X_i, X_j \rangle \in E_n(R)$ if $d(X_i, X_j) \leq R$.

Radius Regime :

$$R_n(\beta) = \left(\frac{\log(n/\beta)}{n\theta_d}\right)^{\frac{1}{d}}.$$

Connectivity Threshold : $\alpha_n^* := \inf\{a : G_n(a^{\frac{1}{d}} R_n(1)) \text{ is connected}\}$.

$$\lim_{n\to\infty} \alpha_n^* = 1.$$

Proofs :

Proof of Lemma :

$$\mathsf{E}(W_n(r)) = n\,\mathsf{E}(\exp(-cn\|B_O(r) \cap \mathcal{C}(n,r)\|)) = n\,\mathsf{E}\Big(\exp(-cn\pi r^2(1 - V(r)))\Big)\,,$$

where $V(r) := 1 - \frac{\|B_O(r) \cap \mathcal{C}(n,r)\|}{\pi r^2}$ with $\mathcal{C}(n,r) := \cup_{X_i \in \mathcal{P}_n^{(1)}} B_{X_i}(r)$.

Estimate $\mathsf{P}\,(V(r) > 0)$; Better estimates in $d = 2$.

Stein-Chen method for Poisson approximation $\Rightarrow$ Proposition.

Proof of Theorem : For $a > \alpha(c)$, there exists $A > 1$ such that w.h.p. the following happens :

$X_1, X_2 \in \mathcal{P}_n^{(1)}, |X_1 - X_2| \leq AR_n(1)$, then $< X_1, X_2 > \in E_c(cn, ar_n(1))$.

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References

Srikanth K. Iyer and D. Yogeshwaran (2010).
Percolation and Connectivity in AB Random Geometric Graphs. arXiv : 0904.0223.

Meester, R. and Roy, R. (1996).
Continuum Percolation.

Penrose, M. (2002).
Random Geometric Graphs.

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