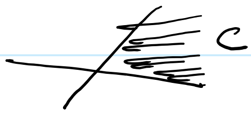
Topics in spherical stochastic geometry.

$\mathbb{R}^n$.

Def. Hyperplane: $H_u = \{x \in \mathbb{R}^n : \langle x, u \rangle = 0\}$, $u \in \mathbb{S}^{n-1}$

Halfspace $H_u^- = \{x \in \mathbb{R}^n : \langle x, u \rangle \le 0\}$. H_u^+.

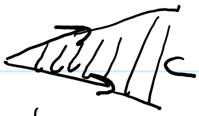
Cone $C = \bigcap_{i=1}^{m} H_{u_i}^-$, $u_1, \dots, u_m \in \mathbb{S}^{n-1}$.



Spherically convex set: $C \cap \mathbb{S}^{n-1}$.

Equiv. Def of cone: $C = \mathrm{pos}(v_1, \dots, v_k) = \{\sum_{i=1}^{k} \lambda_i v_i : \lambda_i \ge 0\}$.

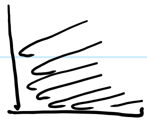
Ex. Weyl chamers:



of type A: $W_A^n = \{x_1 \ge \dots \ge x_n\}$.

of type B: $W_B^n = \{x_1 \ge \dots \ge x_n \ge 0\}$.

Ex. Orthant: $\mathbb{R}_+^n = \{x_i \ge 0 : \forall i = 1, \dots, n\}$.



Def. Let $C \subset \mathbb{R}^n$ cone. Projection on C:

$$\Pi_C : \mathbb{R}^n \to C \qquad \Pi_C(x) = \operatorname*{argmin}_{y \in C} \|x - y\|.$$

Ex. $x \in C$: $\Pi_C(x) = x$.

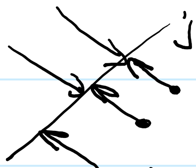
Def. Conic intrinsic volumes of C:

$$v_k(C) = \mathbb{P}[\Pi_C(U) \in \text{relint of some } k\text{-dim face of } C]$$

$U \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.

$k = 0, \dots, n$.

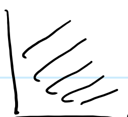
Rem. $\sum_{k=0}^{n} v_k(C) = 1$. $\qquad v_k(C) \ge 0$.



Ex. $C = L_j$ = lin. subspace of dim j. $\quad v_k(L_j) = \begin{cases} 1, & \text{if } k = j \\ 0, & \text{if } k \ne j. \end{cases}$

Ex $C = \mathbb{R}_+^n$. Let $(\xi_1, \dots, \xi_n)$ be st. Gauss on $\mathbb{R}^n$.

$$\Pi_C(\xi_1, \dots, \xi_n) = (\xi_i \mathbb{1}_{\xi_i > 0})_{i=1}^{n}$$



$x_1, \dots, x_n \ge 0$ $\qquad$ k-dim faces: $x_{i_1} \ge 0, \dots, x_{i_k} \ge 0$, all other are $= 0$.

$$v_k(\mathbb{R}_+^n) = \frac{1}{2^n}\binom{n}{k}, \quad k = 0, \dots, n. \qquad \mathrm{Bin}(n, 1/2).$$

Def. Grassmann angles of C:

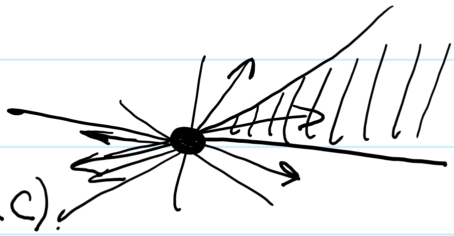
$$\gamma_k(C) = \mathbb{P}[C \cap W_{n-k} \neq \{0\}], \qquad k=0,\dots,n.$$

where W_{n-k} is $(n-k)$-dim. uniform lin. subspace in $\mathbb{R}^n$.

Ex. $k=n-1$: $\gamma_{n-1}(C) = \mathbb{P}[W_1 \cap C \neq \{0\}] = 2\alpha(C)$,
if $\dim C = n$, $C \neq \mathbb{R}^n$.

$\alpha(C) = \mathbb{P}[U \in C]$,

$U \sim \mathrm{Unif}(\mathbb{S}^{n-1} \cap \mathrm{lin}\, C)$.

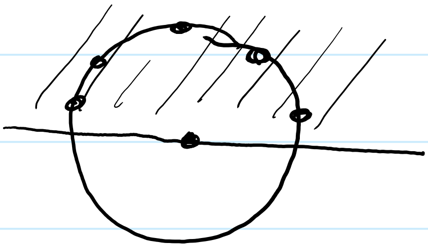


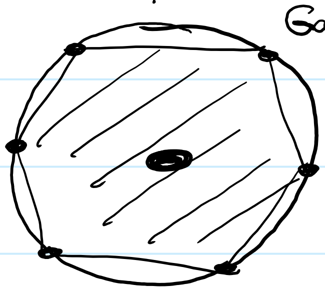
Crofton formula. $\gamma_k(C) = 2\left[\upsilon_{k+1}(C) + \upsilon_{k+3}(C) + \dots\right]$

if C is not lin subspace.

Q1 Wendel (1962) Let $z_1, \dots, z_n \sim \mathrm{Unif}(\mathbb{S}^{d-1})$, $n \geq d$.

Gauss $(\mathbb{R}^d)$





Actually true under ass: $(\pm z_1, \pm z_2, \dots, \pm z_n) \stackrel{d}{=} (z_1, \dots, z_n)$.

$p = \mathbb{P}[\exists \text{ Halfspace containing } z_1, \dots, z_n]$ $\qquad p+q=1$.

$q = \mathbb{P}[\mathrm{conv}[z_1, \dots, z_n] \ni 0]$ $\qquad p, q = ?$

Solution. $q = \mathbb{P}[\exists \lambda_1, \dots, \lambda_n \geq 0,\ \lambda_1 + \dots + \lambda_n = 1 \text{ s.t. } \lambda_1 z_1 + \dots + \lambda_n z_n = 0]$

$= \mathbb{P}[\exists \underbrace{\lambda_1, \dots, \lambda_n}_{\lambda \in \mathbb{R}^n_+ \setminus \{0\}} \geq 0, \text{ not all zero s.t. } \lambda_1 z_1 + \dots + \lambda_n z_n = 0]$

$$A = \underbrace{\begin{bmatrix} z_1 & z_2 & \dots & z_n \end{bmatrix}}_{n} \Big\} d$$

$= \mathbb{P}[\exists \lambda \in \mathbb{R}^n_+ \setminus \{0\} : A\lambda = 0]$

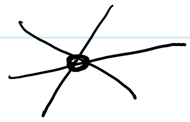
$= \mathbb{P}[\mathrm{Ker}\, A \cap \mathbb{R}^n_+ \neq \{0\}]$ $\qquad A: \mathbb{R}^n \to \mathbb{R}^d$.

$= \gamma_d(\mathbb{R}^n_+) \stackrel{\text{Crofton}}{=} 2\left[\upsilon_{d+1} + \upsilon_{d+3} + \dots\right]$

Ker A has dim $n-d$.
Ker A is unif. distr.

$= \frac{2}{2^n}\left(\binom{n}{d+1} + \binom{n}{d+3} + \dots\right).$

$\boxed{Q2}$ Schläfli, 1850



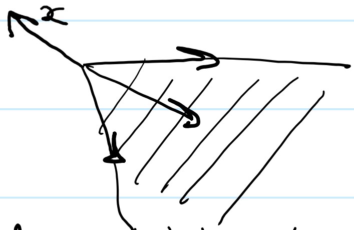
$n \geq d$.

In $\mathbb{R}^d$ n hyperplanes through 0 ($H_{u_1}, \dots, H_{u_n}$). Assume: general position, that is every d normal vectors among $u_1, \dots, u_n$ are lin. indep.

Number of parts?

<u>Solution.</u> $H_{u_1}^- \cap \dots \cap H_{u_n}^- \neq \{0\} \iff \exists x \in \mathbb{R}^d \setminus \{0\}$: $\langle u_i, x \rangle \leq 0 \ \forall i = 1, \dots, n$.

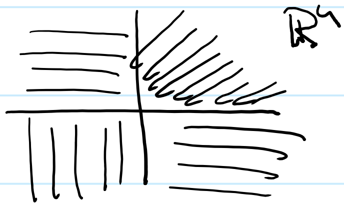
$\iff \operatorname{pos}(u_1, \dots, u_n) \neq \mathbb{R}^d$ $\qquad A = (u_1 \dots u_n)$



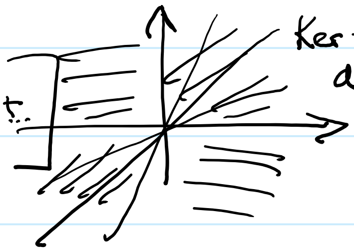
$\iff$ (as in Q1) $\operatorname{Ker} A \cap \mathbb{R}^n_+ = \{0\}$.

$\varepsilon_i \in \{\pm 1\}$

${}^{\varepsilon_1}H_{u_1} \cap \dots \cap H_{u_n}^{\varepsilon_n} \neq \{0\} \iff \operatorname{Ker} A$ does not intersect the orthant $\mathbb{R}_{\varepsilon_1} \times \dots \times \mathbb{R}_{\varepsilon_n}$



Q: Compute the number of orthants not intersected by Ker A.



It is $= 2^n \cdot p = 2 \left[\binom{n}{d-1} + \binom{n}{d-3} + \dots \right]$

<u>Q.</u> $\mathbb{R}^d$, d very large.

n points on S^{d-1}, unif.

How large should be n (appr), so that $\operatorname{conv}(\dots) \ni 0$?