SHARP PHASE-TRANSITION & NOISE SENSITIVITY IN CONTINUUM PERCOLATION

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JOINT WORK WITH

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CONTINUUM PERCOLATION,
SHARP PHASE-TRANSITION
&
OSSS VARIANCE INEQUALITY.

POISSON BOOLEAN MODEL

POISSON PROCESS : $\tilde{\eta} = \{X_1, X_2, \dots\} \subseteq \mathbb{R}^d$, $d \geq 2$, $\lambda \in (0,\infty)$.

$\Rightarrow \tilde{\eta} \cap W = \{X_1, \dots, X_{N_W}\}$, $W \subseteq \mathbb{R}^d$, bdd.

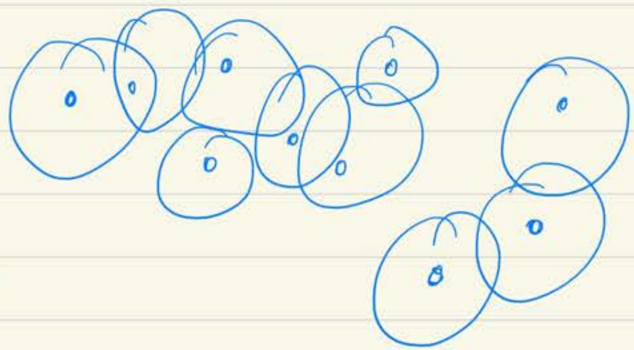
where N_W = Poisson$(\lambda|W|)$ r.v. & X_i i.i.d. Unif on W.

R_i, $i \geq 1$ i.i.d. with prob. distribn. $\mathbb{Q}$ on $[0, r_0]$, $r_0 \in (0,\infty)$.

$\eta = \{(X_i, R_i)\}$ - marked Poisson process

BOOLEAN MODEL

$C(\lambda) = \bigcup_i B_{R_i}(X_i)$ occupied region.



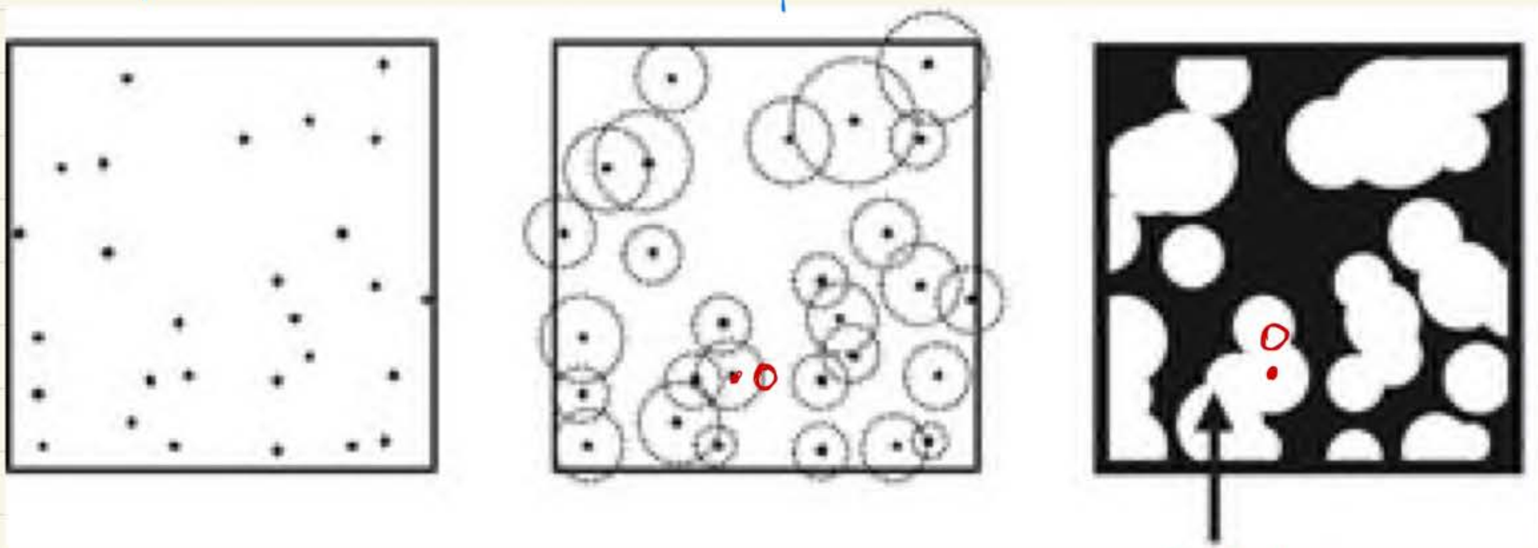
$B_r(x)$ - ball of radius r at x.

$C_0(\lambda)$ = Component of $C(\lambda)$ containing 0. ; $C_0(\lambda) = \phi$ if $0 \notin C(\lambda)$.

$\tilde{\eta}$ - Poisson point process.

$C(\lambda)$ - Boolean model

$C(\lambda)$ - union of white patches.



$C_0(\lambda)$

Applied Vegetation Science 14 (2011) 189–199

A hierarchical model for analysing the stability of vegetation patterns created by grazing in temperate pastures

Nicolas Rossignol, Joel Chadoeuf, Pascal Carrere & Bertrand Dumont

QUESTIONS

BOOLEAN MODEL

$$C(\lambda) = \bigcup_i B_{R_i}(x_i)$$ — occupied region.

Qn: Is $C_0(\lambda)$ unbounded? How large?

Simplest model of CONTINUUM PERCOLATION.

Introduced by E. Gilbert for communication networks in 1961

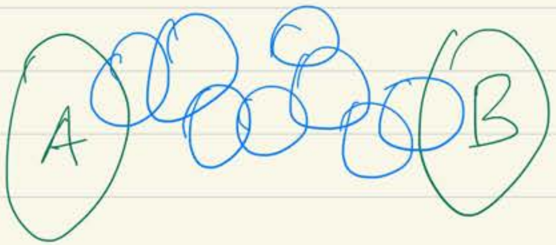
References:

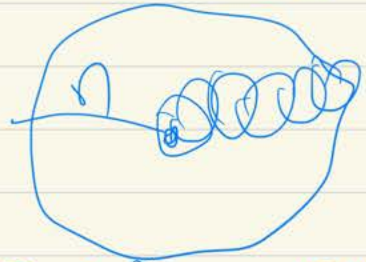
(1) R. Meester & R. Roy - "Continuum percolation"

(2) B. Bollobas & O. Riordan - "Percolation."

PERCOLATION PROBABILITY

$A \overset{C(\lambda)}{\longleftrightarrow} B$ — $\exists$ a path in $C(\lambda)$ from A to B





Boundary of ball

$$\theta_n(\lambda) = \mathbb{P}\left(0 \overset{C(\lambda)}{\longleftrightarrow} \partial B_n(0)\right) = \mathbb{P}\left(C_0(\lambda) \cap \partial B_n(0) \neq \phi\right)$$

$$\theta_n(\lambda) \downarrow \theta(\lambda) = \mathbb{P}\left(0 \overset{C(\lambda)}{\longleftrightarrow} \infty\right)$$

$$= \mathbb{P}\left(|C_0(\lambda)| = \infty\right)$$

[Percolation probability]

$\theta(\lambda)$ is $\uparrow$ in λ.

$$\theta(\lambda) = \begin{cases} 0 & \lambda < \lambda_c \\ > 0 & \lambda > \lambda_c \end{cases}$$

CRITICAL THRESHOLD

i.e., $\lambda_c := \inf\{\lambda > 0 : \theta(\lambda) > 0\} \in (0, \infty)$.

(Gilbert '61).

Zuyev / Molchanov :

[Margulis- Russo formula]

$$\frac{d\theta_n(\lambda)}{d\lambda} = \int \mathbb{E}\left[D_{(x,r)} f_n(\eta)\right] \mathbb{Q}(dr)\, dx \quad - (1)$$

$$D_{(x,r)} f_n(\eta) = f_n(\eta \cup (x,r)) - f_n(\eta) \geq 0 \quad \boxed{\text{add-one cost}}$$

$$\eta = \{(X_i, R_i)\}_{i \geq 1}.$$

$$f_n(\eta) = \mathbb{1}\left[0 \longleftrightarrow \partial B_n(0)\right] \; ; \; \theta_n(\lambda) = \mathbb{E}\left[f_n(\eta)\right]$$

Poincaré Inequality :

$$\theta_n(\lambda)(1-\theta_n(\lambda)) = Var(f_n) \leq \lambda \int \mathbb{E}\left[\left|D_{(x,r)} f_n(\eta)\right|^2\right] dx\, \mathbb{Q}(dr) \quad - (2)$$

$$(1) + (2) \Rightarrow \frac{\theta_n(\lambda)(1-\theta_n(\lambda))}{\lambda} \leq \frac{d\theta_n(\lambda)}{d\lambda}$$

$$\left[f_n \in \{0,1\} \; \& \; D_{(x,r)} f(\tilde{\eta}) \in \{0,1\}\right]$$

Not very useful !!

We need
$$\frac{\theta_n(\lambda)(1-\theta_n(\lambda))}{\lambda} \leq C \frac{S_n}{n} \frac{d\theta_n(\lambda)}{d\lambda} \quad - (*)$$

where $S_n = \int_0^n \theta_s(\lambda)\, ds$.

THM: If differential inequality (*) holds then

(1) $\lambda < \lambda_c$. $\theta_n(\lambda) \leq \exp(-c_\lambda n) \quad \forall\, n \geq 1$.

(2) $\lambda_c < \lambda < \lambda_c + b$. $\theta_n(\lambda) \geq c_b (\lambda - \lambda_c) \quad \forall\, n \geq 1$.

[No proof or explanation here.]

See THEOREM 3.1 of Duminil-Copin, Raoufi & Tassion (2019).

→ ① proved in Meester & Roy (Menshikov's argument)

→ ① proved in Ziesche (2018) for bdd shapes.

Continuum adaptation of Duminil-Copin & Tassion (2016)

→ ① proved in Duminil-Copin, Raoufi & Tassion (2020) for exp. decaying radii via renormalization techniques.

OUR STRATEGY :

Prove (*) by using variance inequalities involving "randomized algorithms" & come up with a "good algorithm". Inspired by Duminil-Copin, Raoufi & Tassion (2019, 2019, 2020).

Why care about exponential decay?

$$\tilde{\lambda}_c := \inf \{ \lambda > 0 : \mathbb{E}[|C_0(\lambda)|] = \infty \}$$

$$\hat{\lambda}_c := \inf \{ \lambda > 0 : \inf_s \mathbb{P}(B_s(0) \longleftrightarrow \partial B_{2s}(0)) > 0 \}$$

$$\lambda_c := \inf \{ \lambda > 0 : \mathbb{P}(|C_0(\lambda)| = \infty) > 0 \}$$

$$\tilde{\lambda}_c \leq \lambda_c \quad (\mathbb{P}(|C_0| = \infty) > 0 \Rightarrow \mathbb{E}[|C_0|] = \infty)$$

$$\Downarrow$$

$$\hat{\lambda}_c \leq \lambda_c \quad (\mathbb{P}(B_s(0) \longleftrightarrow \partial B_{2s}(0)) > 0 \quad \forall s > 0)$$

$$\boxed{\underline{\text{THM}} : \quad \lambda_c = \hat{\lambda}_c = \tilde{\lambda}_c .}$$

One more consequence in the next talk.

TO PROVE THAT

$$\theta_n(\lambda) = \mathbb{P}(0 \longleftrightarrow \partial B_n(0))$$

$$\frac{\theta_n(\lambda)(1-\theta_n(\lambda))}{\lambda} \leq C \frac{S_n}{n} \frac{d\theta_n(\lambda)}{d\lambda} \quad - \quad (*)$$

$$S_n = \int_0^n \theta_s(\lambda)\, ds.$$

ENOUGH TO PROVE

$$\mathrm{Var}(f_n) \leq \boxed{C\frac{S_n}{n}} \int \mathbb{E}\left[|D_{(x,r)} f(\eta)|^2 \right] \lambda\, dx\, \mathbb{Q}(dr).$$

$(**)$

additional term from Poincaré ineq.

$$= C\frac{S_n}{n} \lambda \int \mathbb{E}\left[|D_{(x,r)} f(\eta)| \right] dx\, \mathbb{Q}(dr)$$

$[f(\eta) \in \{0,1\}$ & $D_{(x,r)} f(\eta) \in \{0,1\}.]$

How can we improve Poincaré Inequality?

CONTINUOUS TIME DECISION TREE

1. Z - stopping set Z - random closed set.

$Z(\eta) \subseteq K$, a compact set $\iff Z(\eta \cap K) \subseteq K$.

i.e., to know that $Z \subseteq K$, we need to know configuration of η inside K only.

2. Z determines $f(\eta)$ if $f(\eta) = f(\eta \cap Z)$

3. CTDT (Continuous time Decision tree).

$Z_t \uparrow Z$, Z_t stopping sets. $Z_{t+s} \downarrow Z_t$ as $s \downarrow 0$.

+ some more regularity properties.

Revealment prob: $\delta(Z) := \sup_{(x,r) \in \mathbb{R}^d \times [0, r_0]} \mathbb{P}((x,r) \in Z)$.

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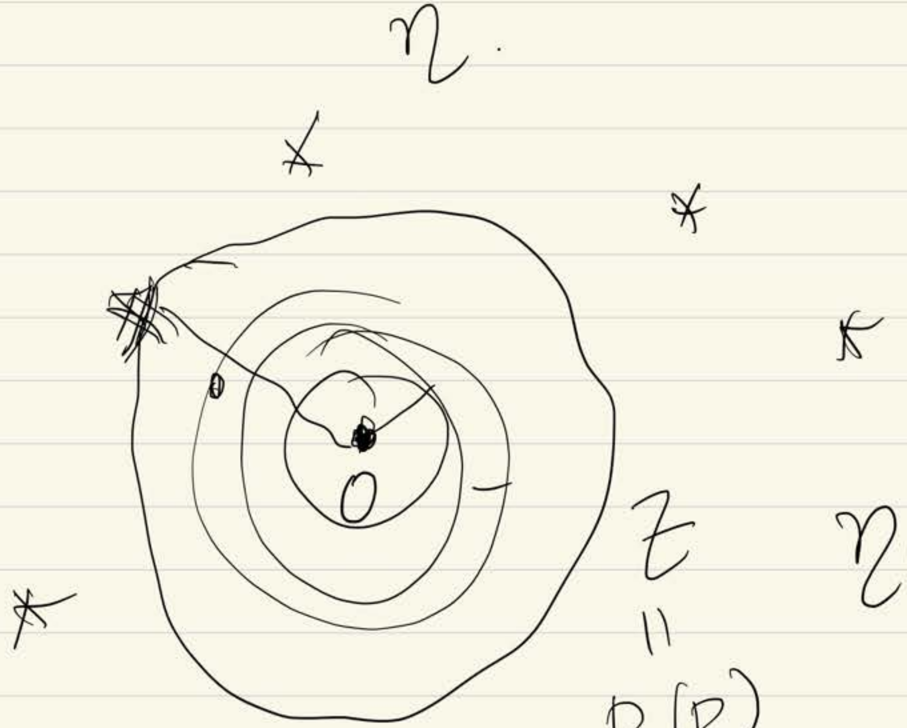


$Z = B_0(R)$ ↳ Random

$\eta \cap \mathring{Z} = \phi$

$$Z \subseteq K \Leftrightarrow Z(\underline{\eta \cap K}) \subseteq K$$

$$\Leftrightarrow \eta \cap K \neq \phi.$$

THM: POISSON-OSSS INEQUALITY

(G. Last, G. Peccati & D. Y., 2020(?))

Assume Z - CTDT determining $f(\eta)$ & $f(\eta) \in [-1,1]$.

Then we have that.

$$\boxed{\begin{aligned} \mathrm{Var}(f(\eta)) &\leq 2\lambda \int_{\mathbb{R}^d \times \mathbb{R}_+} \mathbb{P}((x,r)\in Z)\, \mathbb{E}\left[|D_{(x,r)} f(\eta)|\right] \mathbb{Q}(dr)\, dx \\ &\lesssim 2\, \delta(Z)\, \lambda \int_{\mathbb{R}^d \times \mathbb{R}_+} \mathbb{E}\left[|D_{(x,r)} f(\eta)|\right] \mathbb{Q}(dr)\, dx \end{aligned}}$$

Rem: (1) Analytic improvements of Poincaré.

— log-Sobolev & Talagrand's $L_1 - L_2$ inequalities are very restrictive in Poisson setting & not comparable to OSSS!

(2) Talagrand's inequality not good for all applications.
(even if proved fully)

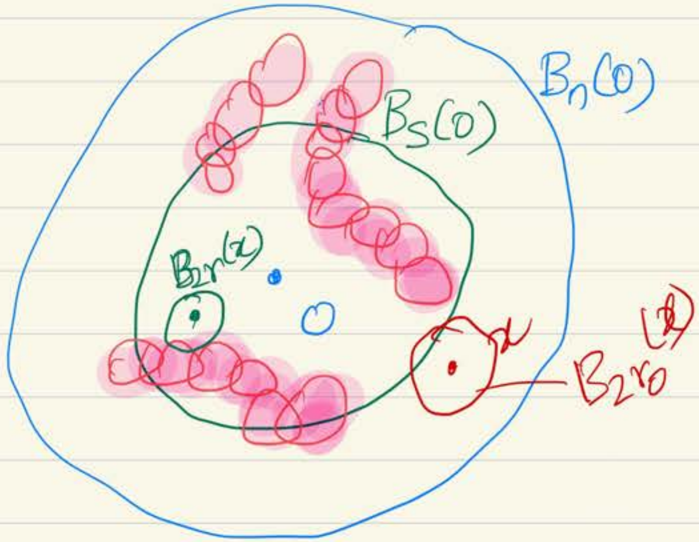
(3) Holds for Poisson process on Polish spaces with diffuse intensity measure.

(4)
OSSS INEQUALITY (O'Donnell, Saks, Schramm, Servedio '05)
proved in the context of complexity of learning Boolean functions.
For $f(X_1, \ldots, X_n)$ X_i indep. rand. variables.

Application to sharp phase transition
by Duminil-Copin, Raoufi & Tassion [2019, 2019, 2020]

APPLICATION TO PERCOLATION IN BOOLEAN MODEL

$$f_n = \mathbb{1}[0 \longleftrightarrow \partial B_n(0)]$$



What is Z ?

- Fix $s \in (0, n)$.
- Look at components of $C(\mathcal{X})$

intersecting $\partial B_s(0)$. Call their union to be S

- $Z^s = S \oplus B_{2r_0}(0) = \{y : d(y, S) \le 2r_0\}$
- Z^s can be constructed as CTDT.
- $\mathbb{P}(x \in Z^s) = \mathbb{P}(B_{2r_0}(x) \cap S \neq \emptyset)$

(FKG + ...) $\quad \le C a_{r_0} \mathbb{P}(0 \longleftrightarrow \partial B_{s-|x|}(0))$

$$= C a_{r_0} \theta_{|s-|x||}(\lambda)$$

- Choose $s \in (0,n)$ at random - Call Z^s as Z.

$$\mathbb{P}(x \in Z) \le \frac{C a_{r_0}}{n} \int_0^n \theta_{|s-|x||}(\lambda)\, ds$$

$$\le \frac{2C a_{r_0}}{n} \int_0^n \theta_s(\lambda)\, ds.$$

- $\delta(Z \times \mathbb{R}_+) = \sup_x \mathbb{P}((x,r) \in Z \times \mathbb{R}_+)$

$$\le \sup_x \mathbb{P}(x \in Z) \le 2C a_{r_0} \frac{S_n}{n}$$

P-OSSS :

$$\theta_n(\lambda)(1-\theta_n(\lambda)) \le 2C a_r \frac{S_n}{n} \lambda \int_{\mathbb{R}^d \times \mathbb{R}_+} \mathbb{E}[D_{(x,r)} f_n]\, dx\, \mathbb{Q}(dr)$$

POISSON - OSSS INEQUALITY

(G. Last, G. Peccati & D. Y., 2020(?))

Z - CTDT determining $f(\eta)$. $f(\eta) \in [-1,1]$.

$$\mathrm{Var}(f(\eta)) \leq 2\lambda \int_{\mathbb{R}^d} \mathbb{P}((x,r) \in Z)\, \mathbb{E}[|D_{(x,r)} f(\eta)|]\, dx\, \mathbb{Q}(dr)$$

PROOF SKETCH:

Suppose $Z_n \uparrow Z$, $n \uparrow \infty$. $Z_0 = \phi$.

$$\zeta_n = \eta_{Z \setminus Z_n} + \eta'_{Z_n} + \eta'_{Z^c}$$

indep. copy of η.

replace η by η' in the explored region.

$$f(\eta) = f(\eta \cap Z) = f(\zeta_0)$$

$$f(\eta) \overset{d}{=} f(\eta') = f(\zeta_\infty)$$

$$\mathbb{E}|f(\eta) - f(\eta')| \leq \sum_{n=1}^{\infty} \mathbb{E}|f(\zeta_n) - f(\zeta_{n-1})|$$

If $\eta \cap (Z_n \setminus Z_{n-1}) \neq \phi$, $\quad \eta' \cap (Z_n \setminus Z_{n-1}) = \phi$

& $\quad |Z_n \setminus Z_{n-1}| \approx 0$

$$\mathbb{E}\,|f(\zeta_n) - f(\zeta_{n-1})| \approx \lambda \int \mathbb{P}((x,r) \in Z_n \setminus Z_{n-1})\, \mathbb{E}\big[|D_{(x,r)} f(\eta)|\big]\, dx\, \mathbb{Q}(dr)$$

$$\mathbb{E}\,|f(\eta) - f(\eta')| \leq \sum_{n=1}^{\infty} \mathbb{E}\,|f(\zeta_n) - f(\zeta_{n-1})|$$

$$\approx 2\lambda \int \mathbb{P}((x,r) \in Z)\, \mathbb{E}\big[|D_{(x,r)} f(\eta)|\big]\, dx\, \mathbb{Q}(dr)$$

If $\eta' \cap (Z_n \setminus Z_{n-1}) \neq \phi$ & $\eta \cap (Z_n \setminus Z_{n-1}) = \phi$.

$$\operatorname{Var}(f(\eta)) \leq \mathbb{E}\big[|f(\eta)\,(f(\eta) - f(\eta'))|\big]$$

$$\leq \mathbb{E}\big[|f(\eta) - f(\eta')|\big] \qquad \blacksquare$$

New applications of Poisson OSSS inequality?

→ k-percolation in Boolean model

→ Confetti percolation.

More details at the end of next talk.

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NOISE SENSITIVITY IN CONTINUUM PERCOLATION AND SCHRAMM-STEIF INEQUALITY.

SPATIAL BIRTH DEATH PROCESS.

$\eta = \{(X_i, R_i)\}_{i \geq 1}$ – Poisson process with i.i.d. radii.

$R_i \sim \mathbb{Q}$, $\mathrm{Supp}(\mathbb{Q}) \subseteq [0, r_0]$.

$(1-e^{-t}) \circ \eta'$ = Keep each (X_i, R_i) independently of others with probability $1-e^{-t}$ – Poisson $(\lambda(1-e^{-t}))$

η' independent copy of η.

$e^{-t} \circ \eta$ = Keep each (X_i', R_i') independently of others with prob. e^{-t} Poisson (λe^{-t})

Poisson(λ)

$$\eta^t \stackrel{d}{=} (1-e^{-t}) \circ \eta' + e^{-t} \circ \eta \rightarrow \text{Superposition.}$$

$\eta^0 = \eta$, $\eta^\infty = \eta'$, $\eta^t \stackrel{d}{=} \eta \ \forall t$.

$\{\eta^t\}_{t \geq 0}$ – Spatial birth-death process.

One can realize η^t as a Markov process.

Let $K \subseteq \mathbb{R}^d$ be a compact set.

Given $\eta^t \cap K = \{(X_i, R_i)\}_{i \geq 1}$

$\longrightarrow$ Each (X_i, R_i) dies at EXP. RATE 1.

$\longrightarrow$ New particle (X_i', R_i') born at EXP. RATE $\lambda|K|$

$$X_i' \stackrel{d}{=} \mathrm{Unif}(K), \quad R_i' \stackrel{d}{=} \mathbb{Q}(\cdot)$$

NOISE SENSITIVITY QN:

$f_n(\eta) \in \{0, 1\}$ - Boolean functions, $n \geq 1$.

Are $f_n(\eta)$ & $f_n(\eta^t)$ indep. asymptotically for all $t > 0$?

i.e., is "re-sampling" a small proportion of points making $f_n(\eta^t)$ indep. of $f_n(\eta)$?

NOISE SENSITIVITY

A sequence of functions $f_n(\eta) \in \{0,1\}$ is NOISE-SENSITIVE if $\forall\, t > 0$

$$NS_n(t) := \mathbb{E}[f_n(\eta^t) f_n(\eta)] - \mathbb{E}[f_n(\eta)]^2 \longrightarrow 0 \text{ as } n \to \infty.$$

EXCEPTIONAL TIMES

$$S_n := \{ t \in [0,1] : f_n(\eta^{t-}) \neq f_n(\eta^t) \}$$

THM: (G. Last, G. Peccati & D. Y., 2020(?))

If $\mathbb{E} f_n(\eta) \in [a, 1-a]$ for all $n \geq 1$ & $a \in (0,1)$

non-degenerate

& $NS_n(t_n) \longrightarrow 0$ for some bdd sequence t_n,

$|S_n \cap [0, b_n]| \xrightarrow{P} \infty$ where $\frac{t_n}{b_n} \longrightarrow 0$.

[VOLATILITY]

$NS_n(t_n) \to 0$ + non-degeneracy $\Rightarrow$ Volatility

Adaptation of proof of Benjamini, Kalai & Schramm '99

One more consequence

$NS(t_n) \to 0$ & non-degeneracy of f_n

$\Rightarrow$ "Sharp phase-transition"!

We'll not discuss these consequences of noise sensitivity here.

Ref: C. Garban & J. Steif:

"Noise sensitivity of Boolean functions & Percolation".

C. Garban: "Oded Schramm's contributions to Noise Sensitivity".

STOPPING SETS & NOISE SENSITIVITY

Z - stopping set Z - random closed set.

$Z(\eta) \subseteq K$, a compact set $\iff Z(\eta \cap K) \subseteq K$.

i.e., to know that $Z \subseteq K$, we need to know configuration of η inside K only.

Z determines $f(\eta)$ if $f(\eta) = f(\eta \cap Z)$

Revealment prob: $\delta(Z) := \sup_{(x,r)} \mathbb{P}((x,r) \in Z)$.

"Stopping Sets" are continuum analogue of set of bits "revealed by an algorithm".

THM (G. Last, G. Peccati, D. Y., 2020(?))

Let $f_n(\eta) \in \{0,1\}$ be a sequence of functions for $n \geq 1$.

Assume that f_n is non-degenerate i.e.,

$\forall n \geq 1$, $\mathbb{E} f_n(\eta) \in [a, 1-a]$ for some $a \in (0,1)$.

Let Z_n be stopping sets determining f_n, $n \geq 1$.

Assume $\delta_n = \delta(Z_n) \longrightarrow 0$.

Then for $t_n \ni \delta_n << (1-e^{-t_n})^2 = O(t_n^2)$

$$NS_n(t_n) = \mathbb{E}[f_n(\eta^{t_n}) f_n(\eta)] - \mathbb{E}[f_n(\eta)]^2 \longrightarrow 0.$$

Rem: (1) Non-degeneracy + "Smart stopping set" $\Rightarrow$ Quantitative noise sensitivity.

(2) Thm is corollary of Poisson Schramm-Steif inequality. Sketch of proof later.

POISSON BOOLEAN MODEL

POISSON PROCESS : $\tilde{\eta} = \{X_1, X_2, \dots\} \subseteq \mathbb{R}^d$, $d \geq 2$, $\lambda \in (0, \infty)$

$\Rightarrow \tilde{\eta} \cap W = \{X_1, \dots, X_{N_W}\}$, $W \subseteq \mathbb{R}^d$, bdd.

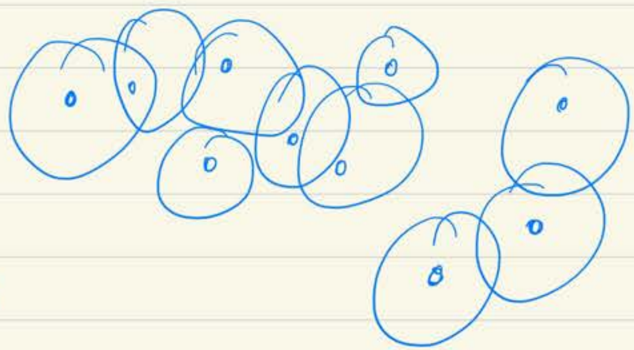
where N_W = Poisson$(\lambda|W|)$ r.v. & X_i i.i.d. Unif on W.

R_i, $i \geq 1$ i.i.d. with prob. distribn. $\mathbb{Q}$ on $[0, r_0]$ $r_0 \in (0, \infty)$.

$$\eta = \{(X_i, R_i)\}_{i \geq 1}.$$

BOOLEAN MODEL

$$C(\lambda) = \bigcup_i B_{R_i}(X_i)$$ occupied region.

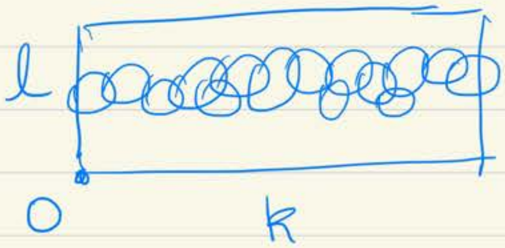


$B_r(x)$ - ball of radius r at x.

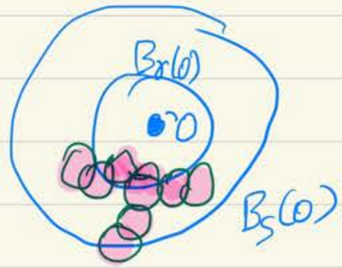
$$\theta_s(\lambda) = \mathbb{P}\left(0 \xleftrightarrow{C(\lambda)} \partial B_s(0)\right) \underset{s \to \infty}{\downarrow} \theta(\lambda) = \mathbb{P}(0 \leftrightarrow \infty).$$

$$\theta(\lambda) = \begin{cases} 0 & \lambda < \lambda_c \\ > 0 & \lambda > \lambda_c \end{cases}$$

CROSSING PROBABILITY, $\boxed{d=2.}$

$\mathrm{Cross}_{k,l}(\lambda) = \Big\{$  $\Big\}$

$= \Big\{ \{0\}\times[0,l] \xleftrightarrow{C(\lambda)} \{k\}\times[0,l] \Big\}$

$\mathrm{Arm}_{r,s}(\lambda) = \Big\{ B_r(0) \xleftrightarrow{C(\lambda)} \partial B_s(0) \Big\} =$ 

$r < s$

From Talk 1, $\theta_s(\lambda) \le \exp(-c_\lambda s)$, $s > 0$, $\lambda < \lambda_c$.

$\Rightarrow \mathbb{P}(\mathrm{Cross}_{kn,n}(\lambda)) \longrightarrow 0$ as $n \to \infty$ for $\lambda < \lambda_c, k > 0$.

ARM PROBABILITY & NOISE SENSITIVITY. $d=2$.

Ahlberg, Tassion & Texeira (2018): $\exists\, a \in (0,1) \ni$

$\kappa > 0$. $\mathbb{P}(\mathrm{Cross}_{\kappa n, n}(\lambda_c)) \in [a, 1-a] \quad \forall\ n \geq 1.$

i.e., non-degenerate.

$$\mathbb{P}(\mathrm{Arm}_{r,s}(\lambda_c)) \leq C_\lambda \left(\frac{r}{s}\right)^\alpha, \quad \alpha > 0.$$

Earlier works

Roy (1990)

Alexander (1996)

G. Last, G. Peccati & D.Y. (2020(?)) Let $f_n(\eta) = \mathbb{1}[\mathrm{Cross}_{\kappa n, n}(\lambda_c)]$.

Then $\delta_n := \delta(Z_n) \leq \frac{c}{\kappa n} \int_0^n \mathbb{P}(\mathrm{Arm}_{r_0,s}(\lambda))\, ds.$

Thus

$$NS(t_n) = \mathbb{E}[f_n(\eta^{t_n}) f_n(\eta)] - \mathbb{E}[f_n(\eta)]^2 \longrightarrow 0$$

for $t_n = n^{-\frac{\alpha}{2} + \varepsilon}$, $\varepsilon \in (0, \frac{\alpha}{2})$.

$\rightarrow$ Settles a question of Ahlberg, Broman, Griffiths & Morris (2014).

Remarks:

(1) $d=2$ & assumption of balls needed to apply RSW-type estimates of Ahlberg, Tassion & Texeira (2018)

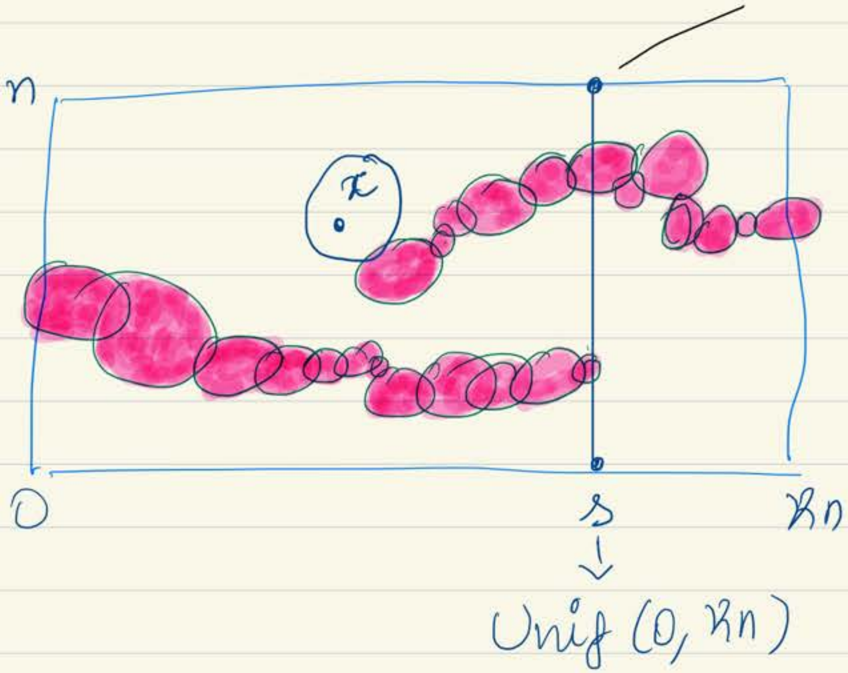
(2) In general (i.e., $d \geq 2$ & random bdd shapes) non-degeneracy + Arm decay probabilities $\Rightarrow$ Quant. noise sens.

(3) ATT(2018) ideas should extend to many planar percolation models !!!

(4) Quant. noise sensitivity also holds for Confetti percolation - ATT(2018) & our theorem.

RANDOMIZED STOPPING SET.

Proof Sketch:



Set. S = Union of connected components intersecting the line $\{s\} \times [0,n]$; $Z^s = (S \oplus B_r(0)) \times [0, r_0]$

$(x,r) \in Z^s \implies B_r(x) \cap S \neq \phi \implies B_r(x) \longleftrightarrow \{s\} \times [0,n]$

$\mathbb{P}((x,r) \in Z^s) \leq \mathbb{P}(B_r(x) \longleftrightarrow \{s\} \times [0,n]) \leq \mathbb{P}(Arm_{r_0, s-|x|}(x))$

o o o o . . . .

CHAOS EXPANSION.

η - Poisson process ; $\mathbb{E}[f(\eta)^2] < \infty$.

Chaos Expansion : $f(\eta) = \sum_{k=0}^{\infty} I_k(u_k(x_1, \ldots, x_k))$

Kernels : $(k!)\, u_k(x_1, \ldots, x_k) = \mathbb{E}\left[D^{(k)}_{x_1, \ldots, x_k} f(\eta) \right]$; $u_0 = \mathbb{E}[f(\eta)]$

$D_x f(\eta) = f(\eta \cup x) - f(\eta)$; $D^{(k)}_{x_1 \ldots x_k} = D_{x_1}\left(D^{(k-1)}_{x_2 \ldots x_k} \right)$

Multiple Wiener-Itô integrals : $I_k(u_k) = \int u_k(x_1, \ldots, x_k)\, \hat{\eta}(dx_1) \cdots \hat{\eta}(dx_k)$

$I_0(u_0) = u_0$ $\qquad \hat{\eta}(dx) = \eta(dx) - \lambda dx$ - compensated measure

KEY PROPERTY : ORTHOGONALITY

$$\mathbb{E}[I_k(u) I_l(v)] = \mathbb{1}[k = l]\, k!\, \lambda^k \langle u, v \rangle_{L^2((dx)^{\otimes k})}$$

$$\Rightarrow \mathrm{COV}(f(\eta), g(\eta)) = \sum_{k=1}^{\infty} \mathbb{E}[I_k(u_k) I_k(v_k)]$$

POISSON SCHRAMM-STEIF INEQUALITY

THM: (G. Last, G. Peccati & D. Y., 2020(?))

Let $f(\eta)$ be $\ni$ $\mathbb{E}[f(\eta)^2] < \infty$ and f be determined by a stopping set Z. Then

$$\mathbb{E}[I_k(u_k)^2] \leq k\,\delta(Z)\,\mathbb{E}[f(\eta)^2].$$

Proof Sketch:

$$\mathbb{E}[I_k(u_k)^2] = \mathbb{E}[f(\eta)\, I_k(u_k)] \quad \text{(orthogonality)}$$

$$= \mathbb{E}\Big[f(\eta)\, \mathbb{E}[I_k(u_k) \mid \eta \cap Z]\Big]$$

$$\leq \mathbb{E}[f(\eta)^2]^{1/2}\ \mathbb{E}\Big[\mathbb{E}[I_k(u_k) \mid \eta \cap Z]^2\Big]^{1/2}$$

$$\overset{\wedge\!\!|}{\big(k\,\delta(Z)\,\mathbb{E}[I_k(u_k)^2]\big)^{1/2}}$$

Apply chaos expansion to condnl expectation & analyse

(Conditional W-I formula)

$\blacksquare$

SCHRAMM-STEIF INEQUALITY & NS.

Mehler's Formula (Last, Peccati & Schulte, 2016)

$$NS_n(t) = \mathrm{COV}(f_n(\eta), f_n(\eta^t)) = \sum_{k=1}^{\infty} e^{-tk} \mathbb{E}[I_k(u_{k,n})^2]$$

(Poisson Schramm-Steif)
$$\leq \sum_{k=1}^{\infty} k e^{-tk} \delta(Z_n) \mathbb{E}[f_n(\eta)^2]$$

$[f(\eta) \in \{0,1\}]$
$$\leq \delta(Z_n) \sum_{k=1}^{\infty} k e^{-tk}$$

$$\leq \delta(Z_n) \frac{e^{-t}}{(1-e^{-t})^2} \quad \blacksquare$$

Schramm-Steif $\Rightarrow$ L^2 OSSS with $\sqrt{\delta(Z)}$

$$\mathrm{Var}(f(\eta)) \leq 3 \sqrt{\delta(Z)}\, \lambda \int \mathbb{E}[|D_x f(\eta)|^2] \mathbb{Q}(dr)dx$$

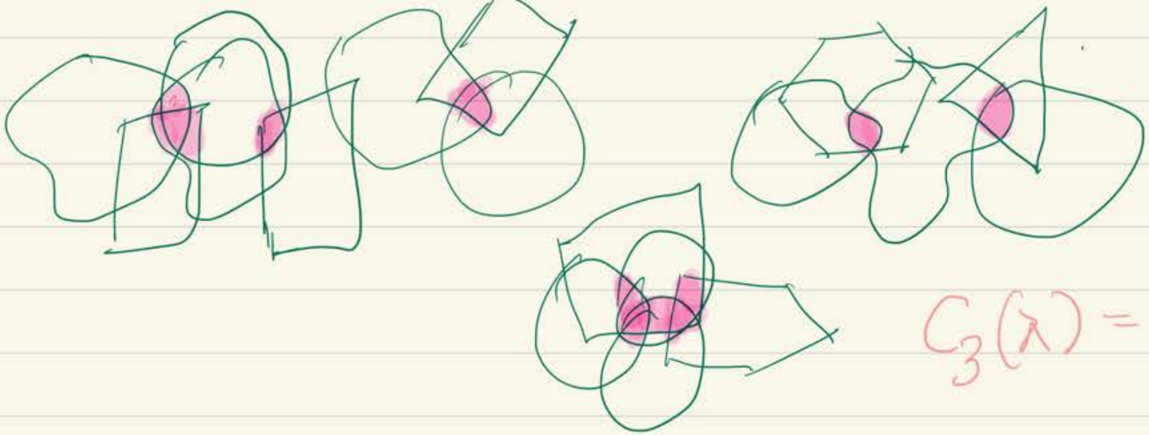
Any new applications?

<u>k- percolation:</u> $\eta = (x_i, M_i)_{i \geq 1}$ M_i - bounded random shapes.

$$C_k(\lambda) = \bigcup_{i_1 < \cdots < i_k} (x_{i_1} + M_{i_1}) \cap \cdots \cap (x_{i_k} + M_{i_k})$$

↳ k- covered region.

$$C_1(\lambda) = C(\lambda)$$



$C_3(\lambda)$ = Shaded region.

→ Poisson OSSS gives sharp phase-transition here.

→ To prove NS, we need non-degeneracy & arm probabilities

ATT (2018) methods should work for $d=2$ & random balls.

Confetti Percolation : $\eta = (X_i, M_i, T_i, 0/1)$; $X_i \in \mathbb{R}^d$

M_i - Shaped cloud hanging above X_i at height T_i

0 - black colour cloud with prob. 'p' . 1 - white cloud.

Colour $x \in \mathbb{R}^d$ black if first cloud above is black.

O_p - black coloured region of $\mathbb{R}^d$.

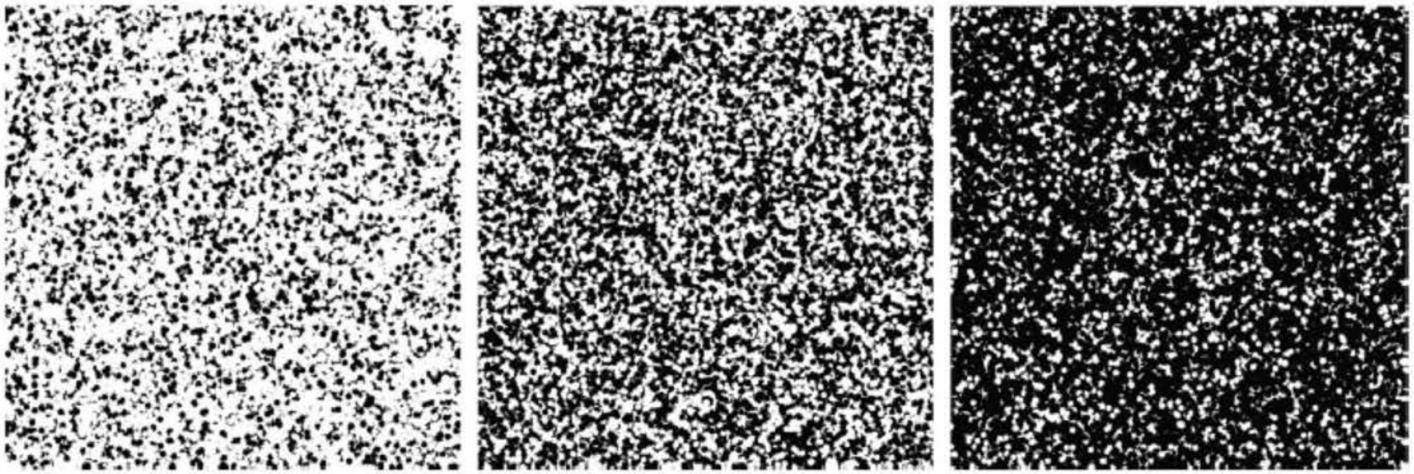


Fig. 1. Simulations of confetti percolation with $p = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. A square of dimensions 200×200 is shown.

Confetti with discs.

The critical probability for confetti percolation equals 1/2. Tobias Muller
Random structures and Algorithms. 2017.

→ Same proof strategy works for $\mathcal{O}_p$.

<u>COROLLARY:</u> $d = 2$.

M_i - bdd random shapes invariant under $\pi/2$ - rotations.

$$\mathbb{P}(\mathcal{O}_p \text{ percolates}) = \mathbb{P}(0 \xleftrightarrow{\mathcal{O}_p} \infty) = \begin{cases} 0 & p \leq \frac{1}{2} \\ 1 & p > 1/2. \end{cases}$$

- Conjectured by Benjamini & Schramm (1998) for case of fixed balls.
- Proven by Hirsch (2015) for boxes, Müller (2017) for balls, Ghosh & Roy (2018) for random boxes.

Ghosh & Roy use discrete OSSS + discretization.

ARM PROBABILITY & NOISE SENSITIVITY.

Ahlberg, Tassion & Texeira (2018) : $\exists\ a \in (0,1) \ni$

$\lambda > 0$. $\quad \mathbb{P}(\mathrm{Cross}_{\lambda n, n}(p_c)) \in [a, 1-a] \quad \forall\ n \geq 1.$

i.e., non-degenerate.

$$\mathbb{P}(\mathrm{Arm}_{r,s}(p_c)) \leq C_\lambda \left(\frac{r}{s}\right)^\alpha, \quad \alpha > 0.$$

$d=2$, Confetti model with discs.

G. Last, G. Peccati & D. Y. (2020(?)) Let $f_n(\eta) = \mathbb{1}[\mathrm{Cross}_{\lambda n, n}(p_c)]$.

Then $\quad \delta_n := \delta(Z_n) \leq \dfrac{c}{n} \displaystyle\int_0^n \mathbb{P}(\mathrm{Arm}_{r,s}(p_c))\, ds.$

Thus

$$NS(t_n) = \mathbb{E}[f_n(\eta^{t_n}) f_n(\eta)] - \mathbb{E}[f_n(\eta)]^2 \longrightarrow 0$$

for $t_n = n^{-\frac{\alpha}{2} + \varepsilon}, \quad \varepsilon \in (0, \frac{\alpha}{2})$.

Thank You