The forbidden region for random zeros: appearance of quadrature domains

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Joint work with Alon Nishry

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Overview

1. Statistical mechanics of random zeros
2. Some words on the Ginibre ensemble
3. Main result for general Jordan holes
4. Notions from potential theory and PDE
5. A family of examples

1. Statistical mechanics of zeros

Joint density of zeros

- Gaussian entire function F_R

$$F_R(z) = \sum_{n\geq 0} \xi_n \frac{(Rz)^n}{\sqrt{n!}}, \qquad \xi_n \text{ i.i.d. Gaussians.}$$

- Density of zeros $(z_j)_j$ of Taylor polynomial $P_{N,R}$:

$$f_{N,R}(z_1,\ldots,z_N) = \frac{1}{Z_{N,R}} \frac{\prod_{1\leq j<k\leq N} |z_j - z_k|^2}{\left(\|Q_z\|^2_{L^2(\mathrm{e}^{-R^2|w|^2})}\right)^{N+1}},$$

where $Q_z = \prod_j (z - z_j)$, and $Z_{N,R}$ explicit normalizing constant. Here scaling is $N = \alpha R^2$.

- Compare with two-dimensional Coulomb system/Ginibre

$$g_{N,R}(z_1,\ldots,z_N) = \frac{1}{Z_N} \prod_{1\leq j<k\leq N} |z_j - z_k|^2 \prod_{j=1}^{N} \mathrm{e}^{-R^2|z_j|^2}$$

Notation

The **logarithmic potential**

$$U^{\mu}(z) = \int \log|z - w|\mathrm{d}\mu(w).$$

In the sense of distributions

$$(2\pi)^{-1}\Delta U^{\mu} = \mu.$$

The **energy** (with discrete analogue) is

$$\Sigma(\mu) = \int \log\frac{1}{|z-w|}\mathrm{d}\mu(z)\mathrm{d}\mu(w) = -\int U^{\mu}\mathrm{d}\mu$$

$$\Sigma^{*}(\mu_{\mathsf{z}}) = N^{-1}\sum_{i\neq j}\log\frac{1}{|z_i - z_j|},$$

where $\mu_{\mathsf{z}} = \frac{1}{N}\sum_j \delta_{z_j}$.

Statistical mechanics of zeros

- **Zeitouni-Zelditch**: The logarithm of the confining term

$$\frac{1}{N^2}\log\left(\|Q_z\|^2_{L^2(\mathrm{e}^{-R^2|w|^2})}\right)^{N+1} \asymp \frac{1}{N}\log\int_{\mathbb{C}} \mathrm{e}^{2N\left(U_z^{\mu}(w)-\frac{|w|^2}{2\alpha}\right)}\mathrm{dA}(w)$$

where $\mu_z = N^{-1}\sum_j \delta_{z_j}$, can be approximated by

$$2B_\alpha(\mu) = 2\sup_{w\in\mathbb{C}}\left(U^{\mu}(w) - \frac{|w|^2}{2\alpha}\right).$$

- The Vandermonde determinant gives energy term $N^2\Sigma^*(\mu_z)$
- LDP with rate function (energy functional)

$$I_\alpha(\mu) = \Sigma(\mu) + 2B_\alpha(\mu).$$

The extremal problem on the hole event

Interested in spatial distribution of zeros of $P_{N,R}(z)$, given

$$\mathcal{H}_{N,R}(\mathcal{G}) = \left\{ P_{N,R}(z) \neq 0 \text{ for } z \in \mathcal{G} \right\}.$$

with $N = \alpha R^2$ and α large.

Problem. Find minimizer $\mu_0 = \mu_{\alpha,\mathcal{G}}$ of

$$I_\alpha(\mu) = \Sigma(\mu) + 2B_\alpha(\mu)$$

among all probability measures μ with $\mu(\mathcal{G}) = 0$ for large α.

A simple reformulation

The confining term

$$B_\alpha(\mu) = \sup_{z\in\mathbb{C}} \left(U^\mu(z) - \frac{|z|^2}{2\alpha} \right).$$

measures deviation from *unconstrained minimizer*.

Lemma. The constrained minimization of I_α is equivalent to: minimizing $\Sigma(\mu)$ under constraint $\mu(\mathcal{G}) = 0$ and

$$U^\mu(z) \leq \tfrac{1}{2\alpha}|z|^2 + c_0, \qquad \text{equality for } |z| \text{ large}$$

- The constant c_0 independent of $\mathcal{G}$ for α large enough.

3. Some words on the Ginibre ensemble

The functional for Ginibre

The functional for the Ginibre ensemble:

$$J_\alpha(\mu) = \Sigma(\mu) + \alpha^{-1} \int_{\mathbb{C}} |z|^2 \mathrm{dA}(z).$$

Minimizer μ_0 among μ with $\mu(\mathcal{G}) = 0$ (cf. Adhikari-Reddy).

Fact from classical potential theory.
Equivalent to find the unconstrained extremal measure

$$J_{Q,\alpha}(\mu) = \Sigma(\mu) + 2\alpha^{-1} \int_{\mathbb{C}} Q(z) \mathrm{dA}(z)$$

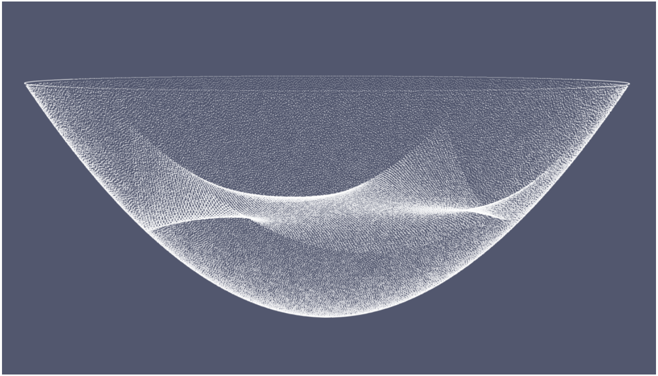
where Q is the potential

$$Q(z) = \begin{cases} \frac{1}{2}|z|^2, & z \in \mathcal{G}^c \\ +\infty, & z \in \mathcal{G}. \end{cases}$$

The hole event for Ginibre and Brownian motion

Suppose $\mathcal{G} \subset \sqrt{\alpha}\mathbb{D}$ (important!). Explicit modification of the unconstrained solution U_0:

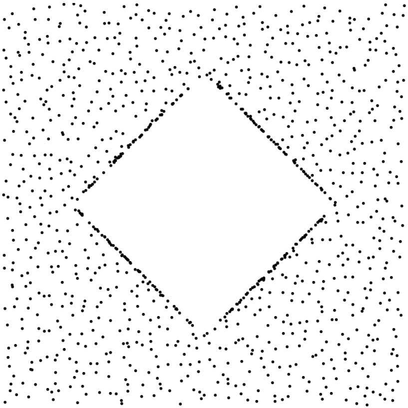
$$U(z) = 1_{\mathcal{G}} \mathsf{P}_{\mathcal{G}}[U_0] + 1_{\mathcal{G}^c} U_0.$$



The hole event for Ginibre and Brownian motion

The optimal measure takes the form

$$\mu_0 = \mu_{\mathrm{eq}} 1_{\mathcal{G}^c} + \int_{\mathcal{G}} \omega_{\mathcal{G}}(w, \cdot) \mathrm{d}\mu_{\mathrm{eq}}(w)$$



4. Main results

Subharmonic functions domains

Sub-mean value property. If u is subharmonic ($\Delta u \geq 0$) on a disk D around z_0, then

$$u(z_0) \leq \frac{1}{|D|} \int_D u(z) \mathrm{dA}(z)$$

If D_j are disjoint disks centered at z_j, then rearranging gives

$$\int_{\cup_j D_j} u(z) \mathrm{dA}(z) \geq \sum_j |D_j| u(z_j)$$

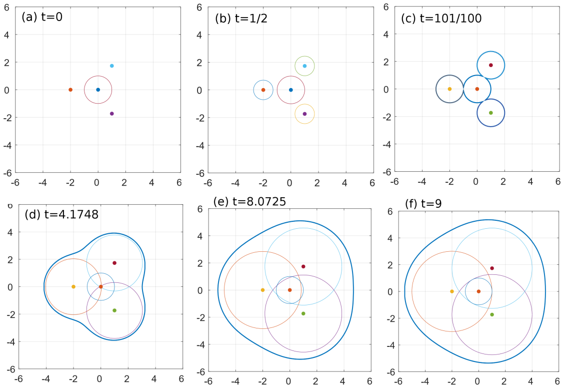
Quadrature domains

Definition. Ω is a quadrature domain with respect to $\nu = \sum_j \rho_j \delta_{\lambda_j}$, if for bounded subharmonic u,

$$\int_\Omega u(z) \mathrm{dA}(z) \geq \sum_j \rho_j u(\lambda_j)$$

- For each finitely supported ν, there exists a unique Ω_ν
- Also known as smash sum of $\cup_j \mathbb{D}(\lambda_j, \sqrt{\rho_j})$

Quadrature domains



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The forbidden region

Recall that whenever $\mathcal{G}$ is *disk-like*, the conditional limiting zero distribution on the hole event has a **circular forbidden region**.

Proposition

A general extremal measure $\mu_{\alpha,\mathcal{G}}$ *also has an associated forbidden region.*

Q. What is the shape?

General Jordan holes

Theorem

Assume that $\partial\mathcal{G}$ is a C^2-smooth simple Jordan curve. Then there exists a finitely supported measure $\nu = \sum_{\lambda\in\Lambda}\rho_\lambda\delta_\lambda$, such that the forbidden region is the quadrature domain $\Omega = \Omega_\nu$.

Specifically, the limiting conditional zero density is

$$\mathrm{d}\mu = \sum_{\lambda\in\Lambda}\rho_\lambda \mathrm{d}\omega(\lambda,\cdot,\mathcal{G}) + \chi_{\mathbb{C}\setminus\Omega_\nu}\mathrm{dA}. \tag{1}$$

- Algebraic boundary of Ω (a priori expect only piecewise C^ω)
- Geometric connection $\mathcal{G}\sim\Omega$ remains unknown

4. Notions from potential theory

Dirichlet energy

The Dirichlet energy is ($u \in H^1(\Omega) = W^{1,2}(\Omega)$)

$$\mathcal{D}(u) = \int_\Omega |\nabla u|^2 \mathrm{dA}.$$

Remark. Assume that U^μ and ∇U^μ are *fixed* on $\partial\Omega$. Then

$$\begin{aligned}\Sigma(\mu) &= -\int_\Omega U^\mu \mathrm{d}\mu = -\frac{1}{2\pi}\int_\Omega U^\mu \Delta U^\mu \\ &= \frac{1}{2\pi}\int_\Omega \nabla U^\mu \cdot \nabla U^\mu - \int_{\partial\Omega} U^\mu \partial_\mathrm{n} U^\mu \mathrm{d}\sigma \\ &= \frac{1}{2\pi}\mathcal{D}(U^\mu) + C\end{aligned}$$

The Poisson extension as an extremal function

The **Poisson extension** of f to Ω solves

$$\inf\left\{\mathcal{D}(u) : u \in H^1(\Omega), \qquad u = f \text{ on } \partial\Omega\right\}.$$

Indeed, let u_0 be the energy minimal solution. Comparing u_0 with $u_\epsilon = u_0 + \epsilon\varphi$ (φ test function) we see that

$$\mathcal{D}(u_\epsilon) = \mathcal{D}(u_0) + 2\epsilon\int \nabla u_0 \cdot \nabla\varphi + \mathrm{O}(\epsilon^2)$$

Hence we must have

$$\int \nabla u \cdot \nabla\varphi = 0,$$

which says $\Delta u = 0$ in the sense of distributions.

The obstacle problem

Denote by D a domain, ψ a function on D and f a boundary datum.

The obstacle problem. Minimize

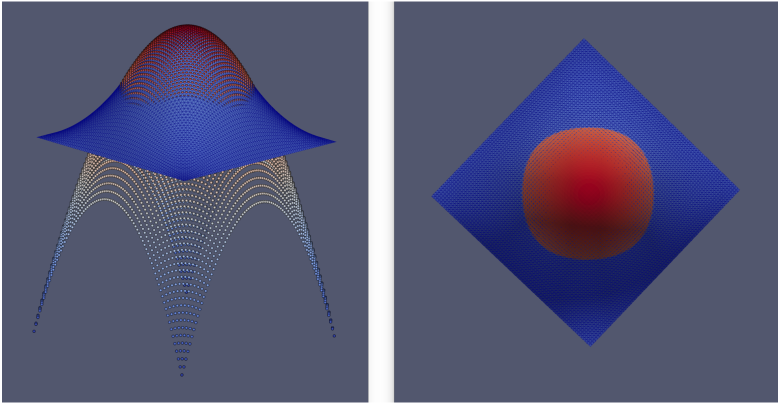
$$\mathcal{D}(u) = \int_D |\nabla u|^2 \mathrm{dA}$$

among all $u \in H^1(D)$ with

$$u \leq \psi, \qquad u = f \text{ on } \partial D.$$

Remark. u is subharmonic and $\Delta u = 0$ when $u < \psi$. In fact $(u - \psi)\Delta u = 0$ characterizes the solution ($\Delta u = \Delta\psi 1_{\{u=\psi\}}$).

The obstacle problem

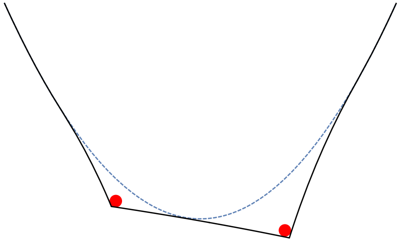


Solution to the *obstacle problem* (left) with obstacle $-|z|^2$ and Dirichlet boundary datum on a square, and the associated *coincidence set* (right)[1]

[1] FEniCS (numerics) and ParaView (graphics)

Mixed obstacle prolem

- Mixed obstacle: solution u_0 contrained by $u \leq \psi$ on D and $u \leq g$ on a curve Γ:



- Distributional Laplacian on Γ is given by the jump of normal derivative.
- If the thin constraint is restrictive enough on $\partial\mathcal{G}$, then u_0 is automatically harmonic inside.

Lemma (An implicit obstacle problem)

There exists a $g = g_{\alpha,\mathcal{G}} \in H^{\frac{1}{2}}(\partial\mathcal{G})$ *such that the minimizer* μ_0 *for the hole problem on* $\mathcal{G}$ *is the Laplacian of the solution* u_0 *to*

$$\inf \int_D |\nabla u|^2 \mathrm{d}A,$$

among all $u \in H^1(D)$ *with constraints*

$$u(z) \leq \frac{|z|^2}{2},\ u(z) = \frac{|z|^2}{2} \text{ on } \partial D,\ u \leq g \text{ on } \partial\mathcal{G}.$$

Remark

This implies existence of a forbidden region and gives the structure of μ_0

5. A family of examples

Neumann ovals

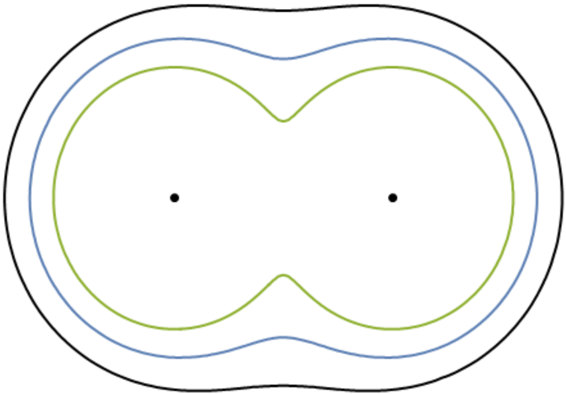


Figure: Nodes at ± 1, growing symmetric masses

A trivial example

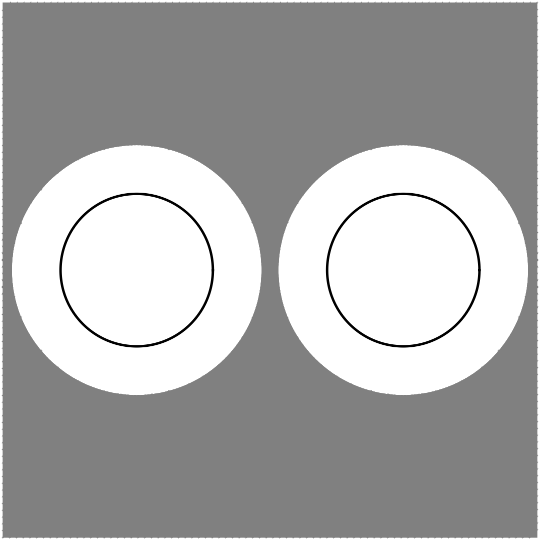


Figure: Two disjoint disks, forbidden regions disjoint disks

An example

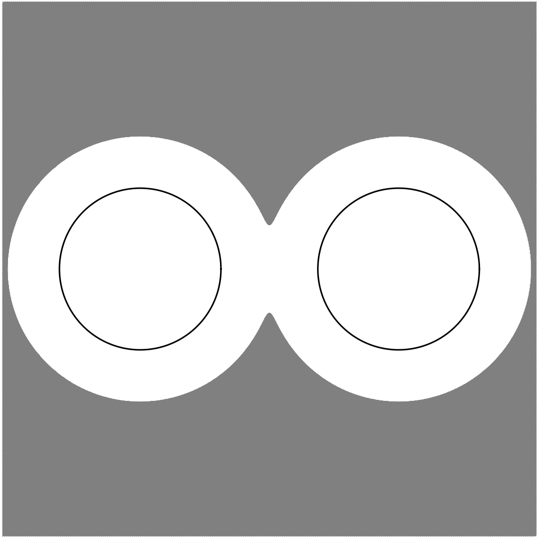


Figure: Two disjoint disks, forbidden region turns to Neumann oval

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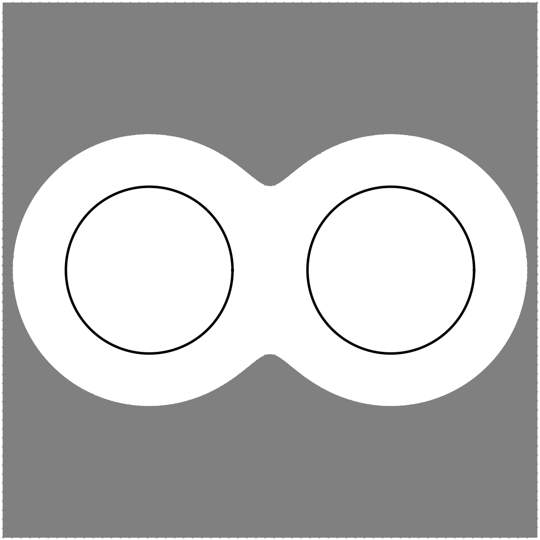

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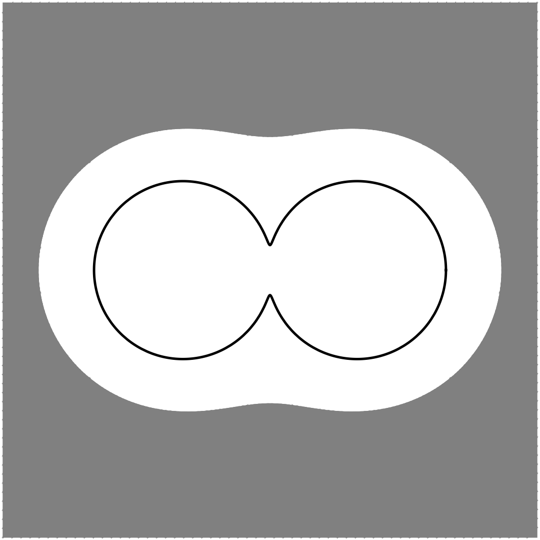

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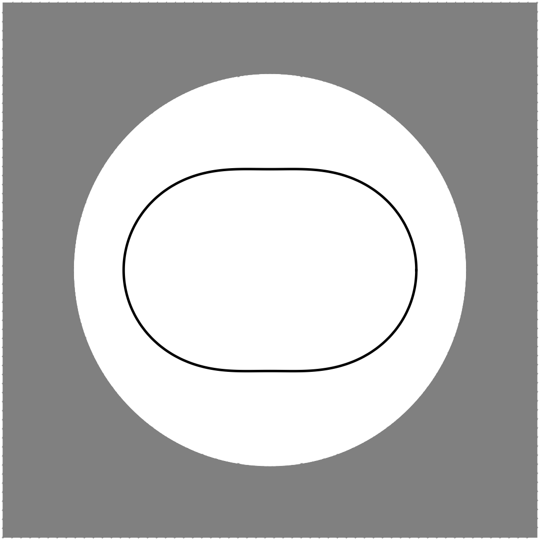

Figure: The inner oval is disk-like