Distributions of maxima of quadratic forms and homogeneous functions of Gaussian random vectors

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Let $f_1, \ldots, f_p$ be random variables with nice distributions. When does

$$f = \max(f_1, \ldots, f_p)$$

have a bounded distribution density ϱ_f and when is ϱ_f of bounded variation, i.e., $\varrho_f \in BV$?
The latter is slightly weaker than $\varrho'_f \in L^1(\mathbb{R})$.

Of particular interest: f_i linear, quadratic forms, second order polynomials in Gaussian r.v.
TRIVIAL:

$$h^{-1}P(f \in [t, t+h)) \leq h^{-1}P(\cup_i \{f_i \in [t, t+h)\})$$

$$\leq \sum_i h^{-1}P(f_i \in [t, t+h)) \leq pM$$

if the distribution density of f_i is bounded by M.
OF INTEREST: bounds independent of p.

Set $\mu \circ f^{-1}(B) := \mu(f^{-1}(B))$

Theorem 1. Let f be a Borel function on $\mathbb{R}^d$ homogeneous of order $\alpha > 0$ (i.e., $f(tu) = t^\alpha f(u)$, $t > 0$), $f(u) \geq m > 0$ on the unit sphere, μ a Borel probability measure on $\mathbb{R}^d$ represented as the product of a probability measure σ on the unit sphere and a probability measure ν on $(0, +\infty)$. If the function t^α has a bounded distribution density $\varrho_{\nu,\alpha}$ with respect to ν, then $\mu \circ f^{-1}$ has a bounded distribution density ϱ_f and

$$\varrho_f \leq m^{-1} \sup_t \varrho_{\nu,\alpha}(t).$$

Corollary 2 The hypotheses of the theorem about μ are fulfilled if μ is the standard Gaussian measure γ_d on $\mathbb{R}^d$ (with density $(2\pi)^{-d/2}\exp(-|x|^2/2)$) and $\alpha \le d$. In this case

$$\varrho_f \le C(\alpha) m^{-1} d^{(1-\alpha)/2},$$

where $C(\alpha)$ depends only on α.

Corollary 3 Let $\mu = \gamma_d$ and

$$f = \max(Q_1, \ldots, Q_p),$$

where Q_j are positive definite quadratic forms on $\mathbb{R}^d$ with $d > 1$ such that the maximum of their minimal eigenvalues is $m > 0$. Then the distribution density ϱ_f of f does not exceed $C(d)/m$, where $C(d)$ depends only on d and is the maximum of the distribution density of χ_d^2. In particular, $\varrho_f \leq Cd^{-1/2}$ with some absolute constant C.

BOUNDEDNESS OF VARIATION AND SOBOLEV REGULARITY

Let $W^{1,k}(\mathbb{R})$ denote the Sobolev class of functions $u \in L^1(\mathbb{R})$ on the real line such that the derivative of u of order $k-1$ is absolutely continuous and $u^{(k)} \in L^1(\mathbb{R})$. The usual norm on $W^{1,k}(\mathbb{R})$: by

$$\|u\|_{W^{1,k}} = \|u\|_{L^1} + \cdots + \|u^{(k)}\|_{L^1}.$$

The class BV consists of functions $u \in L^1(\mathbb{R})$ on the real line such that the generalized derivative of u is a bounded measure Du. The natural norm on BV is defined by

$$\|u\|_{BV} = \|u\|_{L^1} + \|Du\|.$$

The class $V^k(\mathbb{R})$ with $k > 1$ consists of functions $u \in L^1(\mathbb{R})$ such that $u', \dots, u^{(k-1)}$ are in $L^1(\mathbb{R})$ and $u^{(k-1)} \in BV$. Its natural norm is

$$\|u\|_{L^1} + \cdots + \|u^{(k-1)}\|_{L^1} + \|Du^{(k-1)}\|.$$

Let f be a Borel function on $\mathbb{R}^d$ homogeneous of order α.

Theorem 4. If $f \neq 0$ a.e., then $\gamma_d \circ f^{-1}$ has a density ϱ_f with

$$\varrho_f(t) \leq \frac{C_1(d,\alpha)}{|t|}.$$

If $\alpha < d/k$ and

$$\int_{S^{d-1}} \frac{d\theta}{|f(\theta)|^k} < \infty,$$

then $\varrho_f \in W^{1,k}$. If $\alpha = d/k$, then $\varrho_f \in V^k(\mathbb{R})$.

Theorem 5. Let μ be a measure on $\mathbb{R}^d$ with density $\varrho \in W^{1,1}(\mathbb{R}^d)$. Assume that

$$\frac{1}{f} \in L^1(\mu) \quad \text{and} \quad \frac{1}{f}\left\langle \frac{\nabla \varrho}{\varrho}, x \right\rangle \in L^1(\mu).$$

Then $\varrho_f \in BV$.

Corollary 6. Let f be as in the previous theorem and $\mu = \gamma_d$. Let $\alpha < d$ and $m = \inf\{|f(\theta)|\colon |\theta| = 1\} > 0$. Then

$$\|D\varrho_f\| \leq c(d, \alpha) m_Q^{-1}.$$

Theorem 7. Let $f = \max\{f_1, \ldots, f_p\}$, where $f_j(x) = \langle A_j x, x\rangle + \langle x_j, x\rangle + c_j$ is a second order polynomial with $A_j \geq a \cdot \mathrm{I}$. Then

$$\|\varrho_f'\|_{\mathrm{TV}} \leq 15p^2 a^{-1}.$$

Theorem 8. Let γ be a centered Radon Gaussian measure on a locally convex space X and H its Cameron–Martin space. One can assume that γ is the countable power of the one-dimensional standard Gaussian measure and $X = \mathbb{R}^\infty$ the space of all sequences, then $H = l^2$.
Let $Q_1, \dots, Q_p$ be γ-measurable quadratic forms on X such that there is a 3-dimensional subspace L in the Cameron–Martin space H of γ for which $Q_j(h) \geq |h|_H^2$ for all $h \in L$. Then $Q = \max(Q_1, \dots, Q_p)$ has a density of bounded variation.